Nonlinear Dynamics and Systems Theory, **22** (2) (2022) 155–168

Existence and Uniqueness of Solutions for a Semilinear Functional Dynamic Equation with Infinite Delay and Impulses

C. Duque [1], H. Leiva [2] and A. Tridane [3*]

[1] *Department of Mathematics, Faculty of Sciences, Los Andes University, Mérida, Venezuela.*
[2] *School of Mathematical and Computational Science, Yachay Tech. University, San Miguel de Urcuqui, Imbabura, Ecuador.*
[3] *Department of Mathematical Sciences, United Arab Emirates University, Al Ain, UAE.*

Received: November 15, 2021; Revised: March 20, 2022

Abstract: This work is devoted to the prove of the existence of solutions for a semilinear retarded differential equation with infinite delay and impulses on time-scales, which is done by using a version of the Arzela-Ascoli theorem on time-scales, and applying the Leray-Schauder alternative. After that, the uniqueness of solutions is proved by applying a version of Gronwall's inequality for impulsive differential equations, and finally, the continuation of solutions is proved.

Keywords: *semilinear functional dynamic equations; infinite delay; infinite impulses; Leray-Schauder alternative; existence; uniqueness; continuation; time-scales.*

Mathematics Subject Classification (2010): 93C10, 93C23, 34N05, 34K45.

1 Introduction

In the last decades, the theory of time scales has occupied an important space within the mathematical community, attracting the interest of many researchers since it is a powerful tool for continuous and discrete analysis from a unified point of view (see, for instance, [1–3] and references therein).

The time scales theory has made possible to create models in population dynamics, physics, chemical technology, economics, control theory, among others, that allow the study of certain phenomena and processes where the temporal variable can vary both continuously and discretely (see [3, 6–9] and references therein). However, there exists

* Corresponding author: mailto:a-tridane@uaeu.ac.ae

the possibility that these processes and phenomena on time scales could undergo drastic changes of their states at given times. These alterations in state might be due to certain external factors and these changes can be represented in mathematical notation in the form of impulses, which cannot be well described by pure time scales models, therefore, the influence of these impulses on the system could be investigated by introducing impulses effects, see, for instance, [10–12] and references therein.

In these order of ideas, in this paper we are going to study the existence, uniqueness and continuation of solutions for the following semilinear functional dynamic equation with infinite delay and impulses:

$$\begin{cases} z^{\Delta}(t) = A(t)z(t) + f(t, z_t), \quad t \in [0,\infty)_{\mathbb{T}} \setminus \bigcup_{k=1}^{\infty}\{t_k\}, \\ z(s) = \phi(s), \quad s \in (-\infty, 0]_{\mathbb{T}}, \\ z(t_k^+) = z(t_k^-) + J_k(t_k, z(t_k^-)), \quad k = 1, 2, \ldots. \end{cases} \tag{1}$$

For system (1), we are assuming that $0 \in \mathbb{T}$, $\inf \mathbb{T} = -\infty$, $\sup \mathbb{T} = \infty$ and $t + \tau \in \mathbb{T}$ if $t, \tau \in \mathbb{T}$. $0 < t_1 < t_2 < t_3 \cdots < t_k \to +\infty$, $t_k \in \mathbb{T}$. Here $z(t_k^+)$ and $z(t_k^-)$ represent the right and left limits with respect to the time scale, and, in addition, if t_k is right-scattered, then $z(t_k^+) = z(t_k)$, whereas if t_k is left-scattered, then $z(t_k^-) = z(t_k)$. Moreover, it is usually assumed that the solution z should be left-continuous (see [10]), in this case $z(t_k^+) = z(t_k) + J_k(t_k, z(t_k))$, $k = 1, 2, \ldots$. On the other hand, if t_k is right-scattered, then $J(t_k, z(t_k)) = 0$, in other words, it makes sense to consider impulses at right-dense points only (see [11]). Here $A(t) \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and $\phi \in \mathscr{C}_{hp}$, where $\mathscr{C}_{hp}$ is called the phase space that will be defined later. For this type of problems, the phase space for initial functions plays an important role in the study of both qualitative and quantitative theory, for more details, in the continuous case and without impulses, we refer to Hale and Kato [13], Hino et al. [14] and Shin [15]. In the case of functional dynamic equations on time scales with and without impulses, there are a few works in this directions, we can cite Benchohra et al. [16] and Li et al. [17]. Particularly in this work we will use a modified version of the phase space defined in [17] since the initial function $\phi : (-\infty, 0]_{\mathbb{T}} \longrightarrow \mathbb{R}^n$ has a fixed number of points of discontinuity, where the side limits exist and the function ϕ is left-continuous at such points. The function $z_t(\theta) = z(t + \theta)$ for $\theta \in (-\infty, 0]_{\mathbb{T}}$ illustrates the history of the state up to the time t, and also remembers much of the historical past of ϕ, carrying part of the present to the past. $f : [0,\infty)_{\mathbb{T}} \times \mathscr{C}_{hp} \longrightarrow \mathbb{R}^n$ is an rd-continuous function on t and continuous on $\mathscr{C}_{hp}$, $J_k : [0,\infty)_{\mathbb{T}} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ are rd-continuous on t and continuous on $\mathbb{R}^n$.

The paper is organized as follows. In Section 2, we present a summary on dynamical systems on time scale, particularly the concept of rd-continuity, the exponential function, the variation of constants formula and a generalization of Gronwall's inequality to be applied to impulsive differential equations. In Section 3, we define the phase space for our problem, which satisfies the Hale and Kato Axiomatic Theory for Retarded Differential Equations with Infinite Delay. Section 4 is devoted to the proof of our main results, the existence and the uniqueness of solutions, which is done in two theorems, one for the existence using the Arzela-Ascoli theorem on time-scale (see [18]) and applying the Leray-Shauder alternative; and the other theorem for the uniqueness of solutions. Section 5 is dedicated to the study of the continuation of the solutions of our system, introducing the concept of maximal interval of existence of solutions on time scale and applying the generalization of Gronwall's inequality. Section 6 is devoted to an example, where we can apply our results. Finally, Section 7 presents the conclusion and final remark, where we formulate future problems to investigate.

2 Preliminaries

In this section, we will make a brief introduction to the calculus on time scales, especially to clarify the notations and definitions, for a better understanding by the reader. For more details about time scales theory, we recommend the excellent monograph [3].

The time scales theory was introduced by Stefan Hilger (see [4]), and defined a time scale as any arbitrary nonempty closed subset of $\mathbb{R}$, this set is denoted by $\mathbb{T}$. For every $t \in \mathbb{T}$, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \longrightarrow \mathbb{T}$ are defined, respectively, as $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$. A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$, left-scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$. The function $\mu : \mathbb{T} \longrightarrow [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$ is known as the graininess function. It is assumed that $\mathbb{T}$ has the topology inherited from standard topology on the real numbers. The time scale interval $[a, b]_{\mathbb{T}}$ is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$, with $a, b \in \mathbb{T}$, and is similarly defined by open intervals and open neighborhoods.

Definition 2.1 [3] A function $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ is said to be right-dense continuous or just rd-continuous if f is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim_{s\to t^-} f(s)$ exists (finite) for every left-dense point $t \in \mathbb{T}$.

The class of all rd-continuous functions $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^n)$. If $f : \mathbb{T} \to \mathbb{R}^n$ is a function, then we define the function $f \circ \sigma : \mathbb{T} \to \mathbb{R}^n$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, i.e., $f^\sigma = f \circ \sigma$. We define the set $\mathbb{T}^\kappa$ by $\mathbb{T}^\kappa = \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ if $\mathbb{T}$ has a left-scattered maximum, and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise.

Definition 2.2 [3] A function $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ is called delta differentiable (or simply Δ-differentiable) at $t \in \mathbb{T}^\kappa$ provided there exists $f^\Delta(t)$ with the property that given $\varepsilon > 0$, there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ for some $\delta > 0$ such that

$$\left\| f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s) \right\| \leq |\sigma(t) - s| \,, \text{ for all } s \in U.$$

In this case, $f^\Delta(t)$ will be call the Δ-derivative of f in t.

If f is Δ-differentiable at $t \in \mathbb{T}^\kappa$, then it is easy to show that (see [3], Thm. 1.16)

$$f^\Delta(t) = \begin{cases} \dfrac{f^\sigma(t) - f(t)}{\sigma(t) - t} & \text{if} \quad \sigma(t) > t, \\[2ex] \lim\limits_{s\to t} \dfrac{f(t) - f(s)}{t - s} & \text{if} \quad \sigma(t) = t. \end{cases}$$

Definition 2.3 [3] A function $F : \mathbb{T} \longrightarrow \mathbb{R}^n$ is called an antiderivative of $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ if $F^\Delta(t) = f(t)$ for $t \in \mathbb{T}^\kappa$. The Cauchy integral is defined by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \quad t, s \in \mathbb{T},$$

where F is an antiderivative of f.

A function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$, $t \in \mathbb{T}$, and positively regressive if $1 + \mu(t)p(t) > 0$, $t \in \mathbb{T}$. We will denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions and by $\mathcal{R}^+$ the set of all positive regressive and rd-continuous functions.

Definition 2.4 [3] If $p \in \mathcal{R}$, then the generalized exponential function is defined by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right),$$

where

$$\xi_\mu(z) := \begin{cases} \frac{1}{\mu}\mathrm{Log}(1+\mu z) & \text{if} \quad \mu > 0, \\ z, & \text{if} \quad \mu = 0. \end{cases}$$

Here $z \in \mathbb{C}_\mu := \{z \in \mathbb{C} : z \neq 1/\mu\}$ and $\mathrm{Log} z = \log|z| + i \arg z$, $-\pi < \arg z \leq \pi$.
Let A be an $n \times n$-matrix valued function on $\mathbb{T}$.

Definition 2.5 [3] We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$, and the class of all such rd-continuous $n \times n$ matrix-valued functions on $\mathbb{T}$ is denoted by $C_{rd}(\mathbb{T}, \mathbb{R}^{n\times n})$. A is called regressive (with respect to $\mathbb{T}$) provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, and the class of all such regressive and rd-continuous functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Let $t_0 \in \mathbb{T}$ and A be an $n \times n$ regressive matrix-valued function defined on $\mathbb{T}$. Then the unique solution of the initial value problem

$$X^\Delta = A(t)X, \quad X(t_0) = I,$$

is called the matrix exponential function and it is denoted by $e_A(t, t_0)$. The matrix exponential function has the following properties.

Theorem 2.1 *([3], Thm. 5.24) Let $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$ and suppose that $f : \mathbb{T} \longrightarrow \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x^0 \in \mathbb{R}^n$. Then the initial value problem*

$$\begin{cases} x^\Delta(t) = A(t)x(t) + f(t), \\ x(t_0) = x^0 \end{cases} \tag{2}$$

has a unique solution $x : \mathbb{T} \longrightarrow \mathbb{R}^n$. Moreover, this solution is given by

$$x(t) = e_A(t, t_0)x^0 + \int_{t_0}^t e_A(t, \sigma(s))f(s)\Delta s.$$

We will need the following fixed theorem to prove the existence of solutions of system (1).

Theorem 2.2 (Leray-Schauder alternative ([5], Thm. 5.4)) *Let $\mathscr{D}$ be a closed convex subset of a Banach space $\mathscr{X}$ with $0 \in \mathscr{D}$. Let $\mathcal{P} : \mathscr{D} \to \mathscr{D}$ be a completely continuous operator. Then either $\mathcal{P}$ has a fixed point in $\mathscr{D}$ or the set*

$$\{z \in \mathscr{D} : z = \lambda\mathcal{P}(z), \quad 0 < \lambda < 1\}$$

is unbounded.

Following Corollary 6.7 in [3] and Theorem 1.5.1 in [10], it is possible to prove the following Gronwall's inequality with impulses on time scales.

Theorem 2.3 (Gronwall's inequality) *Assume that*

1. *the sequence* $\{t_k\}$ *satisfies* $0 \leq t_0 < t_1 < \cdots < t_k \ldots$, $\lim_{k\to\infty} t_k = \infty$,
2. $u \in C_{rd}(\mathbb{T}, \mathbb{R})$ *and* u *is left continuous at* t_k, $k = 1, 2, \ldots$,
3. $p \in \mathcal{R}^+$, $p \geq 0$, $\beta_k \geq 0$, *and* $\alpha \in \mathbb{R}$.

Then

$$u(t) \leq \alpha + \int_{t_0}^{t} p(s)u(s)\Delta s + \sum_{t_0<t_k<t} \beta_k u(t_k), \quad t \geq t_0,$$

implies

$$u(t) \leq \alpha \prod_{t_0<t_k<t} (1 + \beta_k) e_p(t, t_0), \quad t \geq t_0.$$

3 The Phase Space

In this section, we will introduce an adequate phase space that will permit us to solve our problem. This phase space is a modification of the phase space presented in [17].

We denote by $\mathbb{T}^- = (-\infty, 0]_{\mathbb{T}}$. Now, we shall define the functions space

$$\mathscr{PW}_p = \{\phi : \mathbb{T}^- \longrightarrow \mathbb{R}^n : \phi \text{ is rd-continuous except on } s_k \in \mathbb{T}^-, k = 1, 2, \ldots, \text{ and such that } \phi(s_k^-), \phi(s_k^+) \text{ exist with } \phi(s_k^-) = \phi(s_k)\}.$$

Following [17], we consider $h \in C_{rd}(\mathbb{T}^-, \mathbb{R}^n)$, $h(s) > 0$ for all $s \in \mathbb{T}^-$ and

$$\int_{-\infty}^{0} h(s)\Delta s = 1.$$

Now, we define the following space of functions:

$$\mathscr{C}_{hp} = \left\{ \phi \in \mathscr{PW}_p : \int_{-\infty}^{0} h(s) \left|\phi\right|^{[s,0]_{\mathbb{T}}} \Delta s < \infty \right\},$$

where $|\phi|^{[a,b]_{\mathbb{T}}} = \sup_{a\leq\theta\leq b} |\phi(\theta)|$, and $|\cdot|$ is a norm in $\mathbb{R}^n$.

It is clear that $\mathscr{C}_{hp}$ is a linear subspace of $\mathscr{PW}_p$, and for $\phi \in \mathscr{C}_{hp}$,

$$\|\phi\|_{\mathscr{C}_{hp}} = \int_{-\infty}^{0} h(s) \left|\phi\right|^{[s,0]_{\mathbb{T}}} \Delta s.$$

Define a norm on $\mathscr{C}_{hp}$. Furthermore, analogously to Theorem 3.1 in [17], the space $(\mathscr{C}_{hp}, \|\cdot\|_{\mathscr{C}_{hp}})$ is a Banach space.

Next, for $\tau \in (0, \infty)_{\mathbb{T}}$ being arbitrary but fixed, we consider the space

$$\mathscr{PW}_{h\tau} = \mathscr{PW}_{h\tau}((-\infty, \tau]_{\mathbb{T}}, \mathbb{R}^n)$$

given by

$$\mathscr{PW}_{h\tau} = \{z : (-\infty, \tau]_{\mathbb{T}} \longrightarrow \mathbb{R}^n : z\big|_{\mathbb{T}^-} \in \mathscr{C}_{hp} \text{ and } z\big|_{[0,\tau]_{\mathbb{T}}} \text{ is rd-continuous except at } t_k, k = 1, \dots, p \text{ with } t_p < \tau, \text{ where } z(t_k^+), z(t_k^-) \text{ exist and } z(t_k^-) = z(t_k)\}.$$

Note that $\mathscr{PW}_{h\tau}$ is a Banach space endowed with the norm

$$\|z\|_{\mathscr{PW}_{h\tau}} = \left\|z\big|_{\mathbb{T}^-}\right\|_{\mathscr{C}_{hp}} + |z|^{[0,\tau]_{\mathbb{T}}}.$$

By using Theorem 3.2 in [17], it is possible to show that if $\phi \in \mathscr{C}_{hp}$, then

P1) If $z \in \mathscr{PW}_{h\tau}$ and $z_0 = \phi$, then for every $t \in [0, \tau]_{\mathbb{T}}$ we have that

- i) z_t is in $\mathscr{C}_{hp}$,
- ii) z_t is rd-continuous with respect to t,
- iii) there exists $H > 0$ such that $|z(t)| \leq H \|z_t\|_{\mathscr{C}_{hp}}$.

P2) $\|z_t\|_{\mathscr{C}_{hp}} \leq 2 \|z\|_{\mathscr{PW}_{h\tau}}$.

4 Main Result

In this section we will show the existence of solutions for system (1). In order to accomplish this, we shall assume the following hypotheses:

H1) $|f(t, \phi) - f(t, \varphi)| \leq \eta(t) \|\phi - \varphi\|_{\mathscr{C}_{hp}}$, for all $\phi, \varphi \in \mathscr{C}_{hp}$ and $t \in [0, \tau]_{\mathbb{T}}$, where $\eta \in C_{rd}([0, \tau]_{\mathbb{T}}, \mathbb{R}^+)$.

H2) $|f(t, \phi)| \leq \nu(t)(1 + \|\phi\|_{\mathscr{C}_{hp}})$, for $\phi \in \mathscr{C}_{hp}$ and $t \in [0, \tau]_{\mathbb{T}}$, $\nu \in C_{rd}([0, \tau]_{\mathbb{T}}, \mathbb{R}^+)$.

H3) $|J_k(t, x) - J_k(t, y)| \leq d_k |x - y|$, $J_k(t, 0) = 0$, $k = 1, 2, \dots$ and $\displaystyle\sum_{k \geq 1} d_k < \infty$.

A straightforward computation shows that

Theorem 4.1 *$z(\cdot)$ is a solution of system* (1) *on $(-\infty, \tau]_{\mathbb{T}}$ if and only if $z(\cdot)$ satisfies*

$$z(t) = \begin{cases} \phi(t), & t \in \mathbb{T}^-, \\ e_A(t, 0)\phi(0) + \displaystyle\int_0^t e_A(t, \sigma(s)) f(s, z_s) \Delta s + \sum_{0 < t_k < t} e_A(t, t_k) J_k(t_k, z(t_k)), & t \in [0, \tau]_{\mathbb{T}}. \end{cases} \tag{3}$$

Now, for a given $\phi \in \mathscr{C}_{hp}$ being arbitrary but fixed, define $\phi^* : (-\infty, \tau]_{\mathbb{T}} \longrightarrow \mathbb{R}^n$ by

$$\phi^*(t) = \begin{cases} \phi(t), & t \in \mathbb{T}^-, \\ e_A(t, 0)\phi(0), & t \in [0, \tau]_{\mathbb{T}}. \end{cases} \tag{4}$$

Note that $\phi_0^* = \phi$. Let $x(t) = z(t) - \phi^*(t)$, then $x(t)$ satisfies

$$x(t)=\begin{cases}0, \quad t\in\mathbb{T}^-,\\ \displaystyle\int_0^t e_A(t,\sigma(s))f(s,x_s+\phi_s^*)\Delta s+\sum_{0<t_k<t}e_A(t,t_k)J_k(t_k,x(t_k)+\phi^*(t_k)), \ \ t\in[0,\tau]_{\mathbb{T}}.\end{cases} \tag{5}$$

Finding a solution of system (1) on $(-\infty,\tau]_{\mathbb{T}}$ is equivalent to solving the integral equation (5), and this is equivalent to finding a fixed point of the operator

$$\mathcal{T}:\mathscr{PW}^0_{h\tau}\longrightarrow\mathscr{PW}^0_{h\tau}$$

defined by

$$(\mathcal{T}x)(t)=\begin{cases}0, \quad t\in\mathbb{T}^-,\\ \displaystyle\int_0^t e_A(t,\sigma(s))f(s,x_s+\phi_s^*)\Delta s+\sum_{0<t_k<t}e_A(t,t_k)J_k(t_k,x(t_k)+\phi^*(t_k)), \ t\in[0,\tau]_{\mathbb{T}},\end{cases} \tag{6}$$

where $\mathscr{PW}^0_{h\tau}=\{x\in\mathscr{PW}_{h\tau}:x_0=0\}$, with $\|x\|_{\mathscr{PW}^0_{h\tau}}=\left\|x\big|_{\mathbb{T}^-}\right\|_{\mathscr{C}_{hp}}+|x|^{[0,\tau]_{\mathbb{T}}}=|x|^{[0,\tau]_{\mathbb{T}}}$. Notice that $(\mathscr{PW}^0_{h\tau},\|\cdot\|_{\mathscr{PW}^0_{h\tau}})$ is a Banach space.

Theorem 4.2 *Suppose that H1), H2) and H3) hold, then system* (1) *has at least one solution on* $(-\infty,\tau]_{\mathbb{T}}$.

Proof. To prove that the operator (6) has a fixed point, we will use the Leray-Schauder alternative. We denote by $M=\sup\{\|e_A(t,\xi)\|:t,\xi\in[0,\tau]_{\mathbb{T}}\}$, $\eta^*=\sup\{\eta(t):t\in[0,\tau]_{\mathbb{T}}\}$ and $\nu^*=\sup\{\nu(t):t\in[0,\tau]_{\mathbb{T}}\}$.

First, we will show that in three steps the operator $\mathcal{T}$ is completely continuous.

Step 1: $\mathcal{T}$ is continuous. If $t\in[0,\tau]_{\mathbb{T}}$, then

$$\begin{aligned}|\mathcal{T}x)(t)-(\mathcal{T}y)(t)|\le&\int_0^t\|e_A(t,\sigma(s))\|\,|f(s,x_s+\phi_s^*)-f(s,y_s+\phi_s^*)|\,\Delta s\\ &+\sum_{0<t_k<t}\|e_A(t,s(s))\|\,|J_k(t_k,x(t_k)+\phi^*(t_k))-J_k(t_k,y(t_k)+\phi^*(t_k))|\\ \le&M\Bigg\{\int_0^t|f(s,x_s+\phi_s^*)-f(s,y_s+\phi_s^*)|\,\Delta s\\ &+\sum_{0<t_k<t}|J_k(t_k,x(t_k)+\phi^*(t_k))-J_k(t_k,y(t_k)+\phi^*(t_k))|\Bigg\}\\ \le&M\Bigg\{\int_0^t\eta(s)\,\|x_s-y_s\|_{\mathscr{C}_{hp}}\,\Delta s+\sum_{k=1}^p d_k\,|x(t_k)-y(t_k)|\Bigg\}\\ \le&M\Bigg\{2\eta^*\int_0^\tau\|x-y\|_{\mathscr{PW}_{h\tau}}\,\Delta s+\sum_{k=1}^p d_k\,|x(t_k)-y(t_k)|\Bigg\}\\ \le&M\Bigg\{2\eta^*\tau+\sum_{k=1}^\infty d_k\Bigg\}\|x-y\|_{\mathscr{PW}^0_{h\tau}}.\end{aligned}$$

Therefore,

$$\|\mathcal{T}x-\mathcal{T}y\|_{\mathscr{PW}^0_{h\tau}} \leq M\left\{2\eta^*\tau+\sum_{k=1}^{\infty}d_k\right\}\|x-y\|_{\mathscr{PW}^0_{h\tau}}.$$

So, we have proved that $\mathcal{T}$ is locally Lipschitz and therefore it is continuous.

Step 2: $\mathcal{T}$ maps bounded sets of $\mathscr{PW}^0_{h\tau}$ into bounded sets of $\mathscr{PW}^0_{h\tau}$. It is enough to show that for any $R>0$, there exists $r>0$ such that for each $x \in B_R=\{x \in \mathscr{PW}^0_{h\tau} : \|x\|_{\mathscr{PW}^0_{h\tau}} \leq R\}$, we have that $\|\mathcal{T}x\|_{\mathscr{PW}^0_{h\tau}} \leq r$. Indeed,

$$\begin{aligned}
|(\mathcal{T}x)(t)| &\leq M\left\{\int_0^t |f(s,x_s+\phi_s^*)|\,\Delta s+\sum_{k=1}^{p}d_k\,|x(t_k)+\phi^*(t_k)|\right\}\\
&\leq M\left\{\int_0^t \nu(s)(1+\|x_s+\phi_s^*\|_{\mathscr{C}_{hp}})\Delta s+\sum_{k=1}^{p}d_k(|x(t_k)|+|\phi^*(t_k)|)\right\}\\
&\leq M\left\{\int_0^t \nu(s)(1+\|x_s\|_{\mathscr{C}_{hp}}+\|\phi_s^*\|_{\mathscr{C}_{hp}})\Delta s+\sum_{k=1}^{p}d_k(|x(t_k)|+\|e_A(t_k,0)\|\,|\phi(0)|)\right\}\\
&\leq M\left\{\int_0^\tau \nu^*(1+2\,\|x\|_{\mathscr{PW}^0_{h\tau}}+2\,\|\phi^*\|_{\mathscr{PW}_{h\tau}})\Delta s+\sum_{k=1}^{p}d_k(\|x\|_{\mathscr{PW}^0_{h\tau}}+M|\phi(0)|)\right\}\\
&\leq M\left\{\nu^*(1+2R+2\,\|\phi^*\|_{\mathscr{PW}_{h\tau}})\tau+(R+M\,|\phi(0)|)\sum_{k=1}^{\infty}d_k\right\}=r,
\end{aligned}$$

Step 3: $\mathcal{T}$ maps bounded sets into equicontinuous sets. Let us consider B_R as in step 2. We shall prove that $\mathcal{T}(B_R)$ is equicontinuous on the interval $[0,\tau]_{\mathbb{T}}$. If $t',t'' \in [0,\tau]_{\mathbb{T}}$ with $t'<t''$, then

$$\begin{aligned}
\left|(\mathcal{T}x)(t'')-(\mathcal{T}x)(t')\right| &\leq \int_0^{t'}\|e_A(t'',\sigma(s))-e_A(t',\sigma(s))\|\,|f(s,x_s+\phi_s^*)|\,\Delta s\\
&\quad+\sum_{0<t_k<t'}\|e_A(t'',t_k)-e_A(t',t_k)\|\,|J_k(t_k,x(t_k)+\phi^*(t_k))|\\
&\quad+\sum_{t'<t_k<t''}\|e_A(t'',t_k)\|\,|J_k(t_k,x(t_k)+\phi^*(t_k))|\\
&\leq \int_0^{t'}\|e_A(t'',\sigma(s))-e_A(t',\sigma(s))\|\,\nu(s)(1+\|x_s+\phi_s^*\|_{\mathscr{C}_{hp}})\Delta s\\
&\quad+\int_{t'}^{t''}\|e_A(t'',\sigma(s))\|\,\nu(s)(1+\|x_s+\phi_s^*\|_{\mathscr{C}_{hp}})\Delta s\\
&\quad+\sum_{0<t_k<t'}d_k\,\|e_A(t'',t_k)-e_A(t',t_k)\|\,|x(t_k)+\phi^*(t_k)|\\
&\quad+\sum_{t'<t_k<t''}d_k\,\|e_A(t'',t_k)\|\,|x(t_k)+\phi^*(t_k)|
\end{aligned}$$

$$\begin{aligned}
&\leq \int_0^\tau \|e_A(t'',\sigma(s)) - e_A(t',\sigma(s))\| \, \nu(s)(1+ 2\|x\|_{\mathscr{PW}_{h\tau}} + 2\|\phi^*\|_{\mathscr{PW}_{h\tau}})\Delta s \\
&\quad + \int_{t'}^{t''} \|e_A(t'',\sigma(s))\| \, \nu(s)(1 + 2\|x\|_{\mathscr{PW}_{h\tau}} + 2\|\phi^*\|_{\mathscr{PW}_{h\tau}})\Delta s \\
&\quad + \sum_{0<t_k<t'} d_k \|e_A(t'',t_k) - e_A(t',t_k)\| \left(\|x\|_{\mathscr{PW}^0_{h\tau}} + M|\phi(0)|\right) \\
&\quad + \sum_{t'<t_k<t''} d_k \|e_A(t'',t_k)\| \left(\|x\|_{\mathscr{PW}^0_{h\tau}} + M|\phi(0)|\right) \\
&\leq \nu^*(1 + 2R + 2\|\phi^*\|_{\mathscr{PW}_{h\tau}}) \int_0^\tau \|e_A(t'',\sigma(s)) - e_A(t',\sigma(s))\| \Delta s \\
&\quad + M\nu^*(1 + 2R + 2\|\phi^*\|_{\mathscr{PW}_{h\tau}}) |t'' - t'| \\
&\quad + (R + M|\phi(0)|) \sum_{0<t_k<t'} d_k \|e_A(t'',t_k) - e_A(t',t_k)\| \\
&\quad + M(R + M|\phi(0)|) \sum_{t'<t_k<t''} d_k.
\end{aligned}$$

Since $e_A(\cdot,\sigma(s))$ is continuous, we have $|(\mathcal{T}x)(t'') - (\mathcal{T}x)(t')| \longrightarrow 0$ as $t' \to t''$, independently of $x \in B_R$.

Therefore, $\mathcal{T}(B_R)$ is equicontinuous. From the Arzéla-Ascoli theorem we have that $\mathcal{T}(B_R)$ is relatively compact, so $\mathcal{T}$ is completely continuous.

Now, let us consider the set

$$\mathscr{D} = \{x \in \mathscr{PW}^0_{h\tau} : x = \lambda \mathcal{T}x, \quad 0 < \lambda < 1\}.$$

If $x \in \mathscr{D}$, then for $t \in [0,\tau]_{\mathbb{T}}$, we get

$$\begin{aligned}
|x(t)| =& \lambda \left| \int_0^t e_A(t,\sigma(s)) f(s, x_s + \phi^*_s)\Delta s + \sum_{0<t_k<t} e_A(t,t_k) J_k(t_k, x(t_k) + \phi(t_k)^*) \right| \\
\leq& \int_0^t \|e_A(t,\sigma(s))\| \, |f(s, x_s + \phi^*_s)| \, \Delta s + \sum_{0<t_k<t} \|e_A(t,t_k)\| \, |J_k(t_k, x(t_k) + \phi^*(t_k))| \\
\leq& M\nu^* \int_0^t (1 + \|x_s\|_{\mathscr{C}_{hp}} + \|\phi^*_s\|_{\mathscr{C}_{hp}}))\Delta s + M \sum_{0<t_k<t} d_k(|x(t_k)| + |\phi(t_k)^*|) \\
\leq& M \left(\nu^*(1 + 2\|\phi^*\|_{\mathscr{PW}_{h\tau}})\tau + M|\phi(0)| \sum_{k=1}^\infty d_k \right) + M\nu^* \int_0^t \|x_s\|_{\mathscr{C}_{hp}} \Delta s \\
&+ M \sum_{0<t_k<t} d_k |x(t_k)|.
\end{aligned}$$

If we put $\alpha = M \left(\nu^*(1 + 2\|\phi^*\|_{\mathscr{PW}_{h\tau}})\tau + M|\phi(0)| \sum_{k=1}^\infty d_k \right)$, then

$$|x(t)| \leq \alpha + M\nu^* \int_0^t \|x_s\|_{\mathscr{C}_{hp}} \Delta s + MH \sum_{0<t_k<t} d_k \|x_{t_k}\|_{\mathscr{C}_{hp}}.$$

Thus

$$\|x_t\|_{\mathscr{C}_{hp}} \leq \alpha + M\nu^* \int_0^t \|x_s\|_{\mathscr{C}_{hp}} \Delta s + MH \sum_{0<t_k<t} d_k \|x_{t_k}\|_{\mathscr{C}_{hp}}.$$

By applying Gronwall's inequality with impulses on time scales, we get that

$$\|x_t\|_{\mathscr{C}_{hp}} \leq \alpha \prod_{0<t_k<t} (1 + MHd_k) e_{M\nu^*}(t,0) \leq \alpha \prod_{k=1}^{p} (1 + MHd_k) e_{M\nu^*}(t,0).$$

Then

$$\|x\|_{\mathscr{P}\mathscr{W}^0_{h\tau}} \leq \alpha H \prod_{k=1}^{p} (1 + MHd_k) e_{M\nu^*}(t,0).$$

Therefore, $\mathscr{D}$ is a bounded set, and by the Leray-Schauder alternative, the operator $\mathcal{T}$ has a fixed point.

Theorem 4.3 *Under the conditions of Theorem 4.2, the solution of system* (1) *on* $(-\infty, \tau]_{\mathbb{T}}$ *is unique.*

Proof. Let $\phi \in \mathscr{C}_{hp}$, and suppose that for some $\tau_0 \in (0, \tau]_{\mathbb{T}}$, there are two solutions z and $\tilde{z}$ mapping $(-\infty, \tau_0]_{\mathbb{T}} \longrightarrow \mathbb{R}^n$ with $z \neq \tilde{z}$. Let

$$\tau^* = \inf\{t \in (0, \tau_0)_{\mathbb{T}} : z(t) \neq \tilde{z}(t)\}.$$

Then, for $-\infty < t < \tau^*$, $z(t) = \tilde{z}(t)$. On the other hand

$$z(t) = e_A(t,0)\phi(0) + \int_0^t e_A(t,\sigma(s)) f(s, z_s) \Delta s + \sum_{0<t_k<t} e_A(t,t_k) J_k(t_k, z(t_k))$$

and

$$\tilde{z}(t) = e_A(t,0)\phi(0) + \int_0^t e_A(t,\sigma(s)) f(s, \tilde{z}_s) \Delta s + \sum_{0<t_k<t} e_A(t,t_k) J_k(t_k, \tilde{z}(t_k)).$$

Therefore,

$$\begin{aligned}
|z(t) - \tilde{z}(t)| \leq & \int_{\tau^*}^t \|e_A(t,\sigma(s))\| \, |f(s,z_s) - f(s,\tilde{z}_s)| \, \Delta s \\
& + \sum_{\tau^*<t_k<t} \|e_A(t,t_k)\| \, |J_k(t_k, z(t_k)) - J_k(t_k, \tilde{z}(t_k))| \\
\leq & \int_{\tau^*}^t M\eta(s) \|z_s - \tilde{z}_s\|_{\mathscr{C}_{hp}} \Delta s + \sum_{\tau^*<t_k<t} Md_k \, |z(t_k) - \tilde{z}(t_k)| \\
\leq & \varepsilon + \int_{\tau^*}^t M\eta^* \|z_s - \tilde{z}_s\|_{\mathscr{C}_{hp}} \Delta s + \sum_{\tau^*<t_k<t} Md_k H \|z_{t_k} - \tilde{z}_{t_k}\|_{\mathscr{C}_{hp}},
\end{aligned}$$

for $\varepsilon > 0$ being arbitrary. So,

$$\|z_t - \tilde{z}_t\|_{\mathscr{C}_{hp}} \leq \varepsilon + \int_{\tau^*}^t M\eta^* \|z_s - \tilde{z}_s\|_{\mathscr{C}_{hp}} \Delta s + \sum_{\tau^*<t_k<t} Md_k H \|z_{t_k} - \tilde{z}_{t_k}\|_{\mathscr{C}_{hp}}.$$

By using Gronwall's inequality, we get that

$$\|z_t - \tilde{z}_t\|_{\mathscr{C}_{hp}} \leq \varepsilon \prod_{\tau^*<t_k<t} (1 + MHd_k) e_{M\eta^*}(t, \tau^*) \leq \varepsilon \prod_{\tau^*<t_k<\tau} (1 + MHd_k) e_{M\eta^*}(\tau, \tau^*).$$

Therefore,

$$|z(t) - \tilde{z}(t)| \leq \varepsilon H \prod_{\tau^*<t_k<\tau} (1 + MHd_k) e_{M\eta^*}(\tau, \tau^*).$$

Since ε is arbitrary, one has $|z(t) - \tilde{z}(t)| = 0$ for $t \in (\tau^*, \tau)_{\mathbb{T}}$, contradicting the definition of τ^*.

5 Continuation of Solutions

In this section, we will show that $z(t)$ is defined on $(-\infty, \infty)_{\mathbb{T}}$.

Definition 5.1 We shall say that $(-\infty, \tau)_{\mathbb{T}}$ is a maximal interval of existence of the solution $z(\cdot)$ of system (1) if there is no solution of (1) on $(-\infty, \tau^*)_{\mathbb{T}}$ with $\tau^* > \tau$.

Theorem 5.1 *Suppose that the conditions of existence and uniqueness hold. If z is a solution of problem (1) on $(-\infty, \tau)_{\mathbb{T}}$ and τ is maximal, then either $\tau = +\infty$ or $z(t)$ is not bounded in any neighborhood of τ.*

Proof. Suppose that $\tau < \infty$ and there is a neighborhood U of τ such that $|z(t)| \leq R$ for $t \in U \cap (-\infty, \tau)_{\mathbb{T}}$, then we can suppose that $|z(t)| \leq R$ for all $t \in (-\infty, \tau)_{\mathbb{T}}$. Let t_p be such that $t_p \leq \tau$. Suppose first that $t_p < \tau$.

If τ is left-dense, then there is a sequence $\{\tau_n\}$ such that $t_p < \tau_1 < \tau_2 < \cdots < \tau_n < \cdots$, $\lim_{n\to\infty} \tau_n = \tau$ and $\lim_{n\to\infty} z(\tau_n) = z^*$ for some $z^* \in \mathbb{R}^n$. We shall see that $\lim_{t\to\tau^-} z(t) = z^*$.

Since $\lim_{n\to\infty} \tau_n = \tau$, then there is $\tau_N \in (\tau - \varepsilon, \tau)_{\mathbb{T}}$ such that $|z(\tau_N) - z^*| < \varepsilon$. So, for $t \in (\tau - \varepsilon, \tau)_{\mathbb{T}}$ with $t > \tau_N$, we have that $|z(t) - z^*| \leq |z(t) - z(\tau_N)| + |z(\tau_N) - z^*|$. Now

$$\begin{aligned}
|z(t) - z(\tau_N)| &\leq \|e_A(t,0) - e_A(\tau_N,0)\| \, |\phi(0)| + \int_0^{\tau_N} \|e_A(t,\sigma(s)) - e_A(\tau_N,\sigma(s))\| \, |f(s,z_s)| \, \Delta s \\
&+ \int_{\tau_N}^t \|e_A(\tau_N,\sigma(s))\| \, |f(s,z_s)| \, \Delta s + \sum_{k=1}^{p} d_k \, \|e_A(t,t_k) - e_A(\tau_N,\sigma(s))\| \, |z(t_k)| \\
&\leq \|e_A(t,0) - e_A(\tau_N,0)\| \, |\phi(0)| + \int_0^{\tau_N} \|e_A(t,\sigma(s)) - e_A(\tau_N,\sigma(s))\| \, \nu(s)(1 + \|z_s\|_{\mathscr{C}_{hp}}) \Delta s \\
&+ \int_{\tau_N}^t M\nu(s)(1 + \|z_s\|_{\mathscr{C}_{hp}}) \Delta s + \sum_{k=1}^{p} d_k \, \|e_A(t,t_k) - e_A(\tau_n,t_k)\| \, R
\end{aligned}$$

$$\leq \|e_A(t,0)-e_A(\tau_N,0)\| \, |\phi(0)|+\int_0^{\tau_N}\|e_A(t,\sigma(s))-e_A(\tau_N,\sigma(s))\| \, \nu(s)(1+2\,\|z\|_{\mathscr{PW}_{h\tau}})\Delta s$$

$$+\int_{\tau_N}^{t} M\nu(s)(1+2\,\|z\|_{\mathscr{PW}_{h\tau}})\Delta s+\sum_{k=1}^{p} d_k \, \|e_A(t,t_k)-e_A(\tau_N,t_k)\| \, R$$

$$\leq \|e_A(t,0)-e_A(\tau_N,0)\| \, |\phi(0)|+\int_0^{\tau}\|e_A(t,\sigma(s))-e_A(\tau_N,\sigma(s))\| \, \nu(s)(1+2R)\Delta s$$

$$\int_{\tau_N}^{\tau} M\nu(s)(1+2R)\Delta s+\sum_{k=1}^{p} d_k \, \|e_A(t,t_k)-e_A(\tau_N,t_k)\| \, R.$$

Hence, we get that if $\tau_N \longrightarrow \tau$, then $|z(t)-z(\tau_N)| \longrightarrow 0$, so $\lim\limits_{t\to\tau^-} z(t) = z^*$ and therefore $z(t)$ can be continued beyond τ, contradicting our assumption.

If τ is left-scattered, then $\rho(\tau)\in(0,\tau)_{\mathbb{T}}$ and since t_p is right-dense, we have $t_p<\rho(\tau)$, then the solution z exists also at τ, namely, by putting

$$z(\tau)=z(\rho(\tau))+\mu(\rho(\tau))[A(\rho(\tau))z(\rho(\tau))+f(\rho(\tau),z_{\rho(\tau)})],$$

we get a contradiction.

Now, if $\tau=t_p$ and t_p is left-dense, then we set $z^+=z^*+J_p(t_p,z^*)$. By using the same argument as previously, we can show that $\lim\limits_{t\to\tau^-} z(t)=z^*$, and therefore $z(t)$ can be continued beyond τ.

If $\tau=t_p$ is left-scattered, then

$$\begin{aligned} z(t_p) =& z(\rho(t_p))+\mu(\rho(t_p))[A(\rho(t_p))z(\rho(t_p))+f(\rho(t_p),z_{\rho(t_p)})],\\ z(t_p^+) =& z(t_p)+J_p(t_p,z(t_p)) \end{aligned}$$

and therefore $z(t)$ can be extended beyond τ to the right. This is a contradiction.

Corollary 5.1 *If hypothesis H2) is replaced by*

$$|f(t,\phi)|\leq \nu(t)(1+|\phi(0)|),\quad \phi\in\mathscr{C}_{hp}, t\in\mathbb{T},$$

then the system (1) *has a unique solution defined on all* $\mathbb{T}$.

Proof. Suppose that $z(t)$ is defined on $(-\infty,\tau)_{\mathbb{T}}$ with $\tau<\infty$, then

$$\begin{aligned} |z(t)| \leq& \left| e_A(t,0)\phi(0)+\int_0^t e_A(t,\sigma(s))f(s,z_s)\Delta s+\sum_{0<t_k<t} e_A(t,t_k)J_k(t_k,z(t_k))\right|\\ \leq& M\,|\phi(0)|+\int_0^t M\nu(s)(1+|z(s)|)\Delta s+M\sum_{k=1}^{p} d_k\,|z(t_k)|\\ \leq& M(|\phi(0)|+\nu^*\tau)+M\nu^*\int_0^t |z(s)|\,\Delta s+M\sum_{k=1}^{p} d_k\,|z(t_k)|\,. \end{aligned}$$

So

$$\begin{aligned} |z(t)| \leq& M(|\phi(0)|+\nu^*\tau)\prod_{k=1}^{p}(1+Md_k)e_{M\nu^*}(t,0)\\ \leq& M(|\phi(0)|+\nu^*\tau)\prod_{k=1}^{p}(1+Md_k)e_{M\nu^*}(\tau,0). \end{aligned}$$

This implies that $|z(t)|$ stays bounded in any neighborhood of τ. So, for Theorem 5.1 we have that $\tau = \infty$.

6 Example

Consider the following semilinear functional dynamic equation with infinite delay and impulses on time scales:

$$\begin{cases} z^{\Delta}(t) = a(t)z(t) + b(t)\tanh(z_t) + c(t), \quad t \in [0,\infty)_{\mathbb{T}} \setminus \bigcup_{k=1}^{\infty}\{t_k\}, \\ z(s) = \phi(s), \quad s \in (-\infty, 0]_{\mathbb{T}}, \\ z(t_k^+) = z(t_k^-) + \frac{1}{2^k}\sin(z(t_k^-)), k = 1, 2, \ldots, \end{cases} \tag{7}$$

with $a \in \mathcal{R}(\mathbb{T},\mathbb{R})$ and $b, c \in C_{rd}(\mathbb{T},\mathbb{R})$. Then we have that

i) $|f(t,\phi) - f(t,\varphi)| = |b(t)|\,|\tanh(\phi) - \tanh(\varphi)| \le |b(t)|\,\|\phi - \varphi\|_{\mathscr{C}_{hp}}$;

ii) $|f(t,\phi)| = |b(t)\tanh(\phi) + c(t)| \le \nu(t)(1 + \|\phi\|_{\mathscr{C}_{hp}})$, where $\nu(t) = \max\{|b(t)|, |c(t)|\}$;

iii) $|J_k(t,x) - J_k(t,y)| \le \frac{1}{2^k}|x - y|$, $J_k(t,0) = 0$, for $k = 1, 2, \ldots$ and $\sum_{k=1}^{\infty}\frac{1}{2^k} < \infty$.

Therefore hypotheses H1), H2) and H3) hold, so, by Theorems 4.2 and 4.3, we get that the problem (7) has a unique solution $z(t)$ defined on $(-\infty, \tau]_{\mathbb{T}}$.

7 Conclusion and Final Remarks

In this work, first of all, we prove the existence of solutions for a semilinear retarded differential equation with infinite delay and impulses on time-scale, by using a version of the Arzela-Ascoli theorem on time-scale and applying the Leray-Schauder alternative. Secondly, we prove the uniqueness of solutions by applying a version of Gronwall's inequality for impulsive differential equations, and finally, we study the continuation of solutions. Of course, once we have an Arzela-Ascoli version on time-scale (see [18]), we can apply other fixed point theorems to prove the existence of solutions for such equations, perhaps one can apply Karakosta's fixed point theorem like in [19]. Our next work is devoted to the study of the exact controllability for this type of equations on time-scales by using Rothe's fixed point theorem like in [20].

References

[1] R. Agarwal, D. O'Regan and S. Saker. *Dynamic Inequalities on Time Scales.* Springer International Publishing, Switzerland, 2014.

[2] S. G. Georgiev, *Functional Dynamic Equations on Time Scales.* Springer Nature, Switzerland, AG 2019.

[3] M. Bohner and A. Peterson. *Dynamics Equations on Time Scales. An Introduction with Applications*, Birkähuser Boston Inc., Boston MA, 2001.

[4] S. Hilger. *Ein Maßkettenkalkül mit Anwendung auf Zentrumsmanningfaltigkeiten.* PhD thesis, Universität Würzburg, 1988.

[5] A. Granas and J. Dugundji. *Fixed point Theory.* Springer Monographs Mathematics, New York, 2003.

[6] C. Tisdell and A. Zaidi. Basic qualitative and quantitative results for solutions to nonlinear dynamic equations on time scales with application to economic modeling. *Nonlinear Anal.* **68** (11) (2008) 3504–3524.

[7] M. Bohner, M. Fan and J. Zhang. Existence of periodic solutions in predator-prey and competition dynamic systems. *Nonlinear Analysis: Real World and Applications* **7** (2006) 1193–1204.

[8] C. Duque, J. Uzcátegui and H. Leiva. Approximate controllability of semilinear dynamic equations on time scale. *Asian Journal of Control* **21** (5) (2019) 1–7.

[9] O. Lavrova, V. Mogylova, O. Stanzhytskyi and O. Misiats. Approximation of the optimal control problem on an interval with a family of optimization problems on time scales. *Nonlinear Dynamics and Systems Theory* **17** (3) (2017) 303–314.

[10] V. Lakshmikantham, D. Bainov and P. Simeonov. *Theory of Impulsive Differential Equations.* Series in Modern Applied Mathematics, Vol 6, World Scientific, 1989.

[11] M. Federson, J. G. Mesquita and A. Slavík. Basic results for functional differential and dynamic equations involving impulses. *Math. Nachr.* **286** (2-3) (2013) 181–204.

[12] T.A. lukyanova and A.A. Martynyuk. Stability analysis of impulsive Hopfield-type neuron system on time scale. *Nonlinear Dynamics and Systems Theory* **1**3 (3) (2017) 315–326.

[13] J. Hale and J. Kato. Phase space for retarded equations with infinite delay. *Funkcial. Ekvac.* **1** (1978) 11–41.

[14] Y. Hino, S. Murakami and T. Naito. *Functional Differential Equations with Infinite Delay*, Lecture Notes in Mathematics, Vol. 1473, Springer-Verlag, Berlin, 1991.

[15] J. S. Shin. On the uniqueness of solutions for functional differential equations with infinite delay. *Funkcial Ekvac.* **30** (1987) 225–236.

[16] M. Benchohra, J. Henderson, S. Ntouyas and A. Ouahab. Impulsive functional dynamic equations on time scales. *Dynamic Systems and Applications* **15** (2006) 43–52.

[17] L. Bi, M. Bohner and M. Fan. Periodic solutions of functional dynamic equations with infinite delay. *Nonlinear Analysis.* **68** (2008) 1226–1245.

[18] Z. Zhu and Q. Wang. Existence of nonoscillatory solutions to neutral dynamic equations on time scales. *J. Math. Anal. Appl.* **335** (2007) 751–762.

[19] M. Ayala, H. Leiva and D. Tallana. Existence of solutions for retarded equations with infinite delay, impulses, and nonlocal conditions. *Journal of Mathematical Control Science and Applications* **6** (1) (2020) 43–61.

[20] D. Cabada, R. Gallo and H. Leiva. Roughness of the controllability for time varying systems under the influence of impulses, delays and nonlocal conditions. *Nonauton. Dyn. Syst.* **7** (2020) 126–139.