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On the Existence of Periodic Solutions of a Degenerate Parabolic Reaction-Diffusion Model

Amer Mesbahi [1] and Salim Mesbahi [2*]

[1] *LIM Laboratory, Department of Mathematics and Computer Science, Faculty of Sciences and Technology, Mohamed-Cherif Messaidia University, Souk Ahras, Algeria.*
[2] *LMFN Laboratory, Department of Mathematics, Faculty of Sciences, Ferhat Abbas University, Setif, Algeria.*

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Abstract: The aim of this paper is to study a degenerate parabolic reaction-diffusion model with nonlinear boundary conditions. Its specificity lies in the introduction of degenerate diffusion. We prove the existence of maximal and minimal periodic solutions, including the uniqueness of the solution. This model appears in the modeling of many periodic diffusion phenomena in various sciences. Our approach towards our goal is through the method of upper and lower solutions.

Keywords: *reaction-diffusion systems; degenerate parabolic systems; nonlinear dynamics; upper and lower solutions.*

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1 Introduction

Many problems arise in biology, chemistry, applied science and engineering in the form of periodic reaction-diffusion models. This has been observed in recent scientific studies. Different models can be found in Murray [12,13]. As for the mathematical methods used, some of them are found in the works of Alaa and Mesbahi et *al.* [2,3,10,11,17], and also in Pao [16].

In recent years, special attention has been paid to degenerate reaction-diffusion systems with specific diffusion coefficients and reaction functions, either in the elliptical or parabolic case, as it is in our work. This is due to their wide applications in various sciences. Our work will be in this context; we will prove the existence of periodic maximal

* Corresponding author: mailto:salimbra@gmail.com

and minimal solutions for a class of degenerate quasilinear parabolic reaction-diffusion systems, including the uniqueness of the positive solution.

Degenerate reaction-diffusion systems appear naturally in the mathematical modeling of a wide variety of diffusion phenomena, not only in the natural sciences but also in engineering, chemistry and economics, as, for example, the dynamics of gas, population dynamics, dynamic systems, fusion process, certain biological models, valuation of assets in economy, composite media. We find many models and applications in Abuweden [1], Alaa et *al.* [3], Anderson [4], Bouzelmate and Gmira [5], Carrillo [6], Holden et *al.* [8], Saffidine and Mesbahi [17], and Zhang and Lin [18], where we also find well-known techniques and methods which are frequently used to study such a problem.

These systems are of great importance from the point of view of applications and also from the point of view of analysis, as they require the design of new technologies and the development of known techniques to study them. Therefore, this topic is of great and growing importance in science and engineering.

The introduction of degenerate diffusion leads to difficulties in the mathematical analysis of the model. For this, we will use a successful technique described by Pao based on the method of upper and lower solutions and its associated monotone iterations. The basic idea of this method is that when using an upper solution or a lower solution as the initial iteration in a suitable iterative process, the resulting sequence of iterations is monotone and converges to a solution of the problem. For more details on this technique, see Pao's works [14–16]. We will therefore pay special attention to a model that has several applications which all have in common that they are modeled by the following nonlinear degenerate parabolic reaction-diffusion system:

$$\begin{cases} (u_j)_t - d_j \mathbf{div}\left(D_j\left(u_j\right)\nabla u_j\right) = f_j\left(t,x,\mathbf{u}\right) & \text{in } \Gamma, \\ D_j\left(u_j\right)\dfrac{\partial u_j}{\partial \eta} = \beta_j\left(t,x\right)u_j + \varphi_j\left(t,x,\mathbf{u}\right) & \text{on } \Sigma, \\ u_j\left(0,x\right) = u_j\left(T,x\right) & \text{in } \Omega, \\ \text{for all } 1 \le j \le m, & \end{cases} \tag{1}$$

where $\mathbf{u} = \mathbf{u}(t,x) = (u_1(t,x),\ldots,u_m(t,x))$, Ω is a bounded domain subset of $\mathbb{R}^n$ $(n \ge 1)$ with the smooth boundary $\partial\Omega$, $\Gamma = \mathbb{R}^+ \times \Omega$, $\bar{\Gamma} = \mathbb{R}^+ \times \bar{\Omega}$, $\Sigma = \mathbb{R}^+ \times \partial\Omega$, η denotes the unit normal vector to the boundary $\partial\Omega$, $\dfrac{\partial}{\partial \eta}$ denotes the outward normal derivative on $\partial\Omega$. For each $1 \le j \le m$, $d_j > 0$ and D_j, f_j, φ_j, β_j are prescribed functions satisfying the conditions in hypothesis (H), which we will mention in the next section.

The rest of this paper is organized as follows. In the next section, we present the assumptions under which we will study our problem. Next, we give some results regarding the approximate problem. In the fourth section, we state our main result and also present its proof in detail. The penultimate section is devoted to an application of the obtained result. Finally, we conclude with some remarks and perspectives.

2 Assumptions and Notations

In all that follows, we denote $\tilde{\mathbf{u}} \equiv (\tilde{u}_1,\ldots\tilde{u}_m)$, $\hat{\mathbf{u}} \equiv (\hat{u}_1,\ldots\hat{u}_m)$. The inequality $\hat{\mathbf{u}} \le \tilde{\mathbf{u}}$ means that $\hat{u}_j \le \tilde{u}_j$ for all $1 \le j \le m$. Below, we will denote $\mathbf{E}$ to one of the sets Γ, $\bar{\Gamma}$, Σ or Ω, $\mathbf{C}^{\ell}(\mathbf{E})$ to the space of all continuous functions whose partial derivatives up to the m-th order are continuous in $\mathbf{E}$, $\mathbf{C}^{\ell+\alpha}(\mathbf{E})$ to the space of functions in $\mathbf{C}^{\ell}(\mathbf{E})$ that are Hölder continuous in $\mathbf{E}$ with exponent $\alpha \in (0,1)$. Let, also, $\mathbf{C}^{\ell,m}(\mathbf{E})$ be the space of functions whose ℓ -times derivatives in t and m -times derivatives in x are continuous in $\mathbf{E}$. In particular, the space $\mathbf{C}^{1,2}(\mathbf{E})$ consists of all functions that are once continuously differentiable in t and twice continuously differentiable in x for $(t,x) \in \mathbf{E}$. When $\ell = 0$, we denote by $\mathbf{C}(\mathbf{E})$ the set of continuous functions in $\mathbf{E}$.

Now, we have to clarify in which sense we want to solve our problem.

Definition 2.1 A pair of vector functions $\tilde{\mathbf{u}} \equiv (\tilde{u}_1, \dots \tilde{u}_m)$, $\hat{\mathbf{u}} \equiv (\hat{u}_1, \dots \hat{u}_m)$ in $\boldsymbol{C}\left(\bar{\Gamma}\right) \cap \boldsymbol{C}^{1,2}(\Gamma)$ are called ordered upper and lower solutions of (1) if $\hat{\mathbf{u}} \le \tilde{\mathbf{u}}$ and if $\hat{\mathbf{u}}$ satisfies the relations

$$\begin{cases} (\hat{u}_j)_t - d_j \mathbf{div}\left(D_i(\hat{u}_j) \nabla \hat{u}_j\right) \le f_j(t,x,\hat{\mathbf{u}}) & \text{in } \Gamma, \\ D_j(\hat{u}_j) \dfrac{\partial \hat{u}_j}{\partial \eta} \le \beta_j(t,x)\hat{u}_j + \varphi_j(t,x,\hat{\mathbf{u}}) & \text{on } \Sigma, \\ \hat{u}_j(0,x) \le \hat{u}_j(T,x) & \text{in } \Omega, \end{cases} \tag{2}$$

for all $1 \le j \le m$, and $\tilde{\mathbf{u}}$ satisfies (2) with inequalities reversed.

Now, we make the following assumption:

(H) For each $1 \le j \le m$, the following conditions hold:

(i) $f_j(t,x,\cdot) \in \mathbf{C}^{\frac{\alpha}{2},\alpha}\left(\bar{\Gamma}\right)$, $0 \le \beta_j \in \mathbf{C}^1(\Sigma)$, $\varphi_j(t,x,\cdot) \in \mathbf{C}^{1+\frac{\alpha}{2},2+\alpha}(\Sigma)$, and they are all T-periodic in t.

(ii) $D_j(u_j) \in \mathbf{C}^{1+\alpha}(\mathbf{Q}_j)$, $D_j(u_j) > 0$ for $u_j > 0$ and $D_j(0) \ge 0$.

(iii) $f_j(\cdot,\mathbf{u})$, $\varphi_j(\cdot,\mathbf{u}) \in \mathbf{C}^1(\mathbf{Q})$ such that

$$\begin{aligned} \frac{\partial f_j}{\partial u_i}(\cdot,\mathbf{u}) &\ge 0, \ \frac{\partial \varphi_j}{\partial u_j}(\cdot,\mathbf{u}) = 0, \\ \frac{\partial \varphi_j}{\partial u_i}(\cdot,\mathbf{u}) &\ge 0 \ \text{ for all } j \ne i, \ \mathbf{u} \in \mathbf{Q}. \end{aligned}$$

In the above hypothesis, the subsets $\mathbf{Q}_j$ and $\mathbf{Q}$ are given by the sectors between a pair of upper and lower solutions.

Remark 2.1 In the above hypothesis, we allow $D_j(u_j) > 0$ for $u_j > 0$ and $D_j(0) \ge 0$. This is why we say that system (1) is degenerate, this is our main point of research. For more information on degenerate parabolic problems, see DiBenedetto [7].

3 Approximating Scheme

To simplify our study, we perform the following change of variables:

$$w_j = I_j(u_j) = \int_0^{u_j} D_j(s)\,ds \ \text{ for } u_j \ge 0,\, 1 \le j \le m.$$

Note that this is a continuous change, where $I_j'(u_j) = D_j(u_j)$, and therefore its inverse $u_j = q_j(w_j)$ exists and is an increasing function of $w_j > 0$ for all $1 \le j \le m$. We have

$$(w_j)_t = D_j(u_j)(u_j)_t, \quad \nabla w_j = D_j(u_j)\nabla u_j, \quad \frac{\partial w_j}{\partial \eta} = D_j(u_j)\frac{\partial u_j}{\partial \eta},$$

then system (1) is equivalent to the following:

$$\begin{cases} (D_j(u_j))^{-1}(w_j)_t - d_j\Delta w_j = f_j(t,x,\mathbf{u}) & \text{in } \Gamma, \\ \dfrac{\partial w_j}{\partial \eta} = \beta_j(t,x).q_j(w_j) + \varphi_j(t,x,\mathbf{u}) & \text{on } \Sigma, \\ w_j(0,x) = w_j(T,x) & \text{in } \Omega, \\ u_j = q_j(w_j) & \text{in } \bar{\Gamma}, \\ \text{for all } 1 \le j \le m. \end{cases} \tag{3}$$

Let $\tilde{w}_j = I_j(\tilde{u}_j)$, $\hat{w}_j = I_j(\hat{u}_j)$, $\tilde{\mathbf{w}} = (\tilde{w}_1, \ldots, \tilde{w}_m)$ and $\hat{\mathbf{w}} = (\hat{w}_1, \ldots, \hat{w}_m)$. It is easy to verify that $(\tilde{\mathbf{u}}, \tilde{\mathbf{w}})$ and $(\hat{\mathbf{u}}, \hat{\mathbf{w}})$ are ordered upper and lower solutions of (3). We set

$$\begin{aligned} \mathbf{Q}_j &= \left\{u_j \in \mathbf{C}\left(\bar{\Gamma}\right) : \hat{u}_j \le u_j \le \tilde{u}_j\right\},\ 1 \le j \le m, \\ \mathbf{Q} &= \left\{\mathbf{u} \in \boldsymbol{C}\left(\bar{\Omega}\right) : \hat{\mathbf{u}} \le \mathbf{u} \le \tilde{\mathbf{u}}\right\}, \\ \mathbf{Q}_j \times \mathbf{Q} &= \left\{(\mathbf{u}, \mathbf{w}) \in \boldsymbol{C}\left(\bar{\Omega}\right) \times \boldsymbol{C}\left(\bar{\Omega}\right) : (\hat{\mathbf{u}}, \hat{\mathbf{w}}) \le (\mathbf{u}, \mathbf{w}) \le (\tilde{\mathbf{u}}, \tilde{\mathbf{w}})\right\}. \end{aligned}$$

Now, we define the modified functions $\bar{D}_j(u_j)$, $1 \le j \le m$, by

$$\bar{D}_j(u_j) = \begin{cases} D_j(u_j) + (u_j - \tilde{u}_j)\ , \text{ if } u_j > \tilde{u}_j, \\ D_j(u_j)\ , \text{ if } \hat{u}_j \le u_j \le \tilde{u}_j, \\ D_j(u_j) + (\hat{u}_j - u_j)\ , \text{ if } u_j < \hat{u}_j. \end{cases}$$

It is clear that $\bar{D}_j(0) > 0$ if either $D_j(0) > 0$ and $\hat{u}_j \ge 0$ or $D_j(0) = 0$ and $\hat{u}_j \ge \delta_j > 0$. This implies the existence of nonnegative functions $\lambda_j^{(1)}$, $\lambda_j^{(2)} \in \boldsymbol{C}^\alpha\left(\bar{\Gamma}\right)$ such that

$$\lambda_j^{(1)}\bar{D}_j(u_j) + \frac{\partial f_j}{\partial u_j}(\cdot, \mathbf{u}) \ge 0\ ,\ \ \lambda_j^{(2)}\bar{D}_j(u_j) + \beta_j \ge 0 \ \text{ for } \mathbf{u} \in \mathbf{Q}. \tag{4}$$

System (3) directly implies

$$\begin{cases} (D_j(u_j))^{-1}(w_j)_t - \left(d_j\Delta w_j - \lambda_j^{(1)} w_j\right) = f_j(t,x,\mathbf{u}) + \lambda_j^{(1)} w_j & \text{in } \Gamma, \\ \dfrac{\partial w_j}{\partial \eta} + \lambda_j^{(2)} w_j = \beta_j(t,x)\ q_j(w_j) + \varphi_j(t,x,\mathbf{u}) + \lambda_j^{(2)} w_j & \text{on } \Sigma, \\ w_j(0,x) = w_j(T,x) & \text{in } \Omega, \\ u_j = q_j(w_j) & \text{in } \bar{\Gamma}, \\ \text{for all } 1 \le j \le m. \end{cases}$$

For all $1 \le j \le m$, we denote

$$\begin{aligned} F_j(t,x,\mathbf{u}) &= f_j(t,x,\mathbf{u}) + \lambda_j^{(1)} w_j = f_j(t,x,\mathbf{u}) + \lambda_j^{(1)} \bar{I}_j(u_j), \\ \Psi_j(t,x,\mathbf{u}) &= \beta_j(t,x) u_j + \varphi_j(t,x,\mathbf{u}) + \lambda_j^{(2)} \bar{I}_j(u_j), \\ L_j w_j &= d_j\Delta w_j - \lambda_j^{(1)} w_j, \\ B_j w_j &= \frac{\partial w_j}{\partial \eta} + \lambda_j^{(2)} w_j, \end{aligned}$$

where

$$\bar{I}_j(u_j) = \int_0^{u_j} \bar{D}_j(s)\,ds, \text{ for } u_j \geq 0,\ 1 \leq j \leq m.$$

According to (4), $F_j(\cdot, \mathbf{u})$ and $G_j(\cdot, \mathbf{u})$ are nondecreasing, i.e.,

$$F_j(\cdot, \mathbf{v}) \leq F_j(\cdot, \mathbf{u})\ ,\ \Psi_j(\cdot, \mathbf{v}) \leq \Psi_j(\cdot, \mathbf{u}), \text{ where } \hat{\mathbf{u}} \leq \mathbf{v} \leq \mathbf{u} \leq \tilde{\mathbf{u}}. \tag{5}$$

Consequently, system (3) can be reformulated as follows:

$$\begin{cases} (D_j(u_j))^{-1}(w_j)_t - L_j w_j = F_j(t,x,\mathbf{u}) & \text{in } \Gamma, \\ B_j w_j = \Psi_j(t,x,\mathbf{u}) & \text{on } \Sigma, \\ w_j(0,x) = w_j(T,x) & \text{in } \Omega, \\ u_j = q_j(w_j) & \text{in } \bar{\Gamma}, \\ \text{for all } 1 \leq j \leq m. \end{cases} \tag{6}$$

It is clear that systems (1) and (6) are equivalent, therefore the existence of a periodic solution to the equivalent system(6) leads to the existence of that to system (1).

We recall the following important lemma, which will be used to construct monotone convergent sequences. In Pao and Ruan [14], we find a detailed proof of this lemma.

Lemma 3.1 *Let $\sigma(t,x) > 0$ in Γ, $C^{(2)}(t,x) \geq 0$ on Σ, and let either (i) $C^{(1)}(t,x) > 0$ in Γ or (ii) $\left(\frac{-C^{(1)}}{\sigma}\right)$ be bounded on $\bar{\Gamma}$. If $z \in \boldsymbol{C}^{2,1}(\bar{\Gamma}) \cap \boldsymbol{C}(\bar{\Gamma})$ and satisfies the following inequalities:*

$$\begin{cases} \sigma(t,x) z_t - \mathbf{div}(a\nabla z) + b.\nabla z + C^{(1)} z \geq 0 & \text{in } \Gamma, \\ \dfrac{\partial z}{\partial \eta} + C^{(2)} z \geq 0 & \text{on } \Sigma, \\ z(0,x) \geq 0 & \text{in } \Omega, \end{cases}$$

then $z \geq 0$ in $\bar{\Gamma}$.

Assume that a pair of ordered upper and lower solutions $\tilde{\mathbf{u}}$, $\hat{\mathbf{v}}$ exist and hypothesis (H) holds, using either $\mathbf{u}^{(0)} = \tilde{\mathbf{u}}$ or $\mathbf{u}^{(0)} = \hat{\mathbf{u}}$ as the initial iteration, we can construct a sequence $\{\mathbf{u}^{(k)}, \mathbf{w}^{(k)}\}$ from the linear iteration process

$$\begin{cases} \left(\bar{D}_j\left(u_j^{(k)}\right)\right)^{-1}\left(w_j^{(k)}\right)_t - L_j w_j^{(k)} = F_j\left(t,x,\mathbf{u}^{(k-1)}\right) & \text{in } \Gamma, \\ B_j w_j^{(k)} = \Psi_j\left(t,x,\mathbf{u}^{(k-1)}\right) & \text{on } \Sigma, \\ w_j^{(k)}(0,x) = w_j^{(k-1)}(T,x) & \text{in } \Omega, \\ u_j^{(k)} = q_j\left(w_j^{(k-1)}\right) & \text{in } \bar{\Gamma} \\ \text{for all } 1 \leq j \leq m, \end{cases} \tag{7}$$

where $\mathbf{u}^{(k)} = \left(u_1^{(k)}, \ldots, u_m^{(k)}\right)$ and $\mathbf{w}^{(k)} = \left(w_1^{(k)}, \ldots, w_m^{(k)}\right)$. It is clear that this sequence is well defined, see Ladyženskaja et *al.* [9]. Denote the sequence by $\{\bar{\mathbf{u}}^{(k)}, \bar{\mathbf{w}}^{(k)}\}$ if $\mathbf{u}^{(0)} = \tilde{\mathbf{u}}$, and by $\{\underline{\mathbf{u}}^{(k)}, \underline{\mathbf{w}}^{(k)}\}$ if $\mathbf{u}^{(0)} = \hat{\mathbf{u}}$, and refer to them as the maximal and minimal sequences, respectively.

Lemma 3.2 *The maximal and minimal sequences* $\left\{\bar{\mathbf{u}}^{(k)},\bar{\mathbf{w}}^{(k)}\right\}$, $\left\{\underline{\mathbf{u}}^{(k)},\underline{\mathbf{w}}^{(k)}\right\}$ *possess the monotone property, i.e., for* $k\geq 1$,

$$(\hat{\mathbf{u}},\hat{\mathbf{w}})\leq\left(\underline{\mathbf{u}}^{(k)},\underline{\mathbf{w}}^{(k)}\right)\leq\left(\underline{\mathbf{u}}^{(k+1)},\underline{\mathbf{w}}^{(k+1)}\right)\leq\left(\bar{\mathbf{u}}^{(k+1)},\bar{\mathbf{w}}^{(k+1)}\right)\leq\left(\bar{\mathbf{u}}^{(k)},\bar{\mathbf{w}}^{(k)}\right)\leq(\tilde{\mathbf{u}},\tilde{\mathbf{w}}).$$

Proof. Let $\underline{z}_j^{(1)}=\underline{w}_j^{(1)}-\underline{w}_j^{(0)}=\underline{w}_j^{(1)}-\hat{w}_j$, $1\leq j\leq m$. Then by (7) and the property of a lower solution stipulated in the previous Definition 2.1, we obtain

$$\begin{cases}\left(\bar{D}_j\left(u_j^{(1)}\right)\right)^{-1}\left(\underline{z}_j^{(1)}\right)_t-L_j\underline{z}_j^{(1)}+\gamma_j^{(0)}\underline{z}_j^{(1)}\geq 0 & \text{in }\Gamma,\\ B_j\underline{z}_j^{(1)}=\Psi_j\left(\cdot,\underline{\mathbf{u}}^{(0)}\right)-B_j\hat{w}_j\geq 0 & \text{on }\Sigma,\\ \underline{z}_j^{(1)}(0,x)=\underline{w}_j^{(0)}(T,x)-\underline{w}_j^{(0)}(0,x)=\hat{w}_j(T,x)-\hat{w}_j(0,x) & \text{in }\Omega,\end{cases}$$

where $\gamma_j^{(0)}$ is a bounded function on $\bar{\Gamma}$ given in the form

$$\gamma_j^{(0)}=-\frac{\bar{D}_j'\left(\xi_j^{(0)}\right)}{\left(\bar{D}_j\left(\xi_j^{(0)}\right)\right)^3}\left(\underline{w}_j^{(0)}\right)_t\quad\text{with}\quad\underline{u}_j^{(0)}\leq\xi_j^{(0)}\equiv\xi_j^{(0)}(t,x)\leq\underline{u}_j^{(1)}.$$

By the hypothesis $\bar{D}_j(0)>0$ or $D_j(0)=0$ and $\hat{u}_j\geq\delta_j>0$, the function $\left(\bar{D}_j\left(u_j^{(1)}\right)\right)^{-1}$ is also bounded in $\bar{\Gamma}$. By Lemma 3.1, we find $\underline{z}_j^{(1)}\geq 0$. This proves $\underline{w}_j^{(1)}\geq\underline{w}_j^{(0)}$ and $\underline{u}_j^{(1)}\geq\underline{u}_j^{(0)}$. In the same way, but with the upper solution, we find $\overline{w}_j^{(1)}\leq\overline{w}_j^{(0)}$ and $\overline{u}_j^{(1)}\leq\overline{u}_j^{(0)}$. In the following, we prove that $\overline{u}_j^{(1)}\geq\underline{u}_j^{(1)}$. Let $z_j^{(1)}=\overline{w}_j^{(1)}-\underline{w}_j^{(1)}$, then by (5) and (7), we have

$$\begin{cases}\left(\bar{D}_j\left(u_j^{(1)}\right)\right)^{-1}\left(z_j^{(1)}\right)_t-L_jz_j^{(1)}+\gamma_j^{(0)}z_j^{(1)}=F_j\left(\cdot,\bar{\mathbf{u}}^{(0)}\right)-F_j\left(\cdot,\underline{\mathbf{u}}^{(0)}\right) & \text{in }\Gamma,\\ B_jz_j^{(1)}=\Psi_j\left(\cdot,\bar{\mathbf{u}}^{(0)}\right)-\Psi_j\left(\cdot,\underline{\mathbf{u}}^{(0)}\right)\geq 0 & \text{on }\Sigma,\\ z_j^{(1)}(0,x)=\overline{w}_j^{(1)}(0,x)-\underline{w}_j^{(1)}(0,x)=\overline{w}_j^{(0)}(T,x)-\underline{w}_j^{(0)}(T,x)\geq 0 & \text{in }\Omega.\end{cases}$$

By Lemma 3.1, we have $z_j^{(1)}\geq 0$. This is what gives

$$\underline{\mathbf{u}}^{(0)}\leq\underline{\mathbf{u}}^{(1)}\leq\bar{\mathbf{u}}^{(1)}\leq\bar{\mathbf{u}}^{(0)}.$$

By induction, we can easily have the monotone property.

According to Lemma 3.2, the pointwise limits

$$\lim_{k\to\infty}\left(\bar{\mathbf{u}}^{(k)},\bar{\mathbf{w}}^{(k)}\right)=(\bar{\mathbf{u}},\bar{\mathbf{w}}),\quad\lim_{k\to\infty}\left(\underline{\mathbf{u}}^{(k)},\underline{\mathbf{w}}^{(k)}\right)=(\underline{\mathbf{u}},\underline{\mathbf{w}})\tag{8}$$

exist and verify the relation $(\bar{\mathbf{u}},\bar{\mathbf{w}})\geq(\underline{\mathbf{u}},\underline{\mathbf{w}})$ in $\bar{\Gamma}$. It results from (8) and $\mathbf{u}^{(k)}(0,x)=\mathbf{u}^{(k)}(T,x)$ that $\bar{\mathbf{u}}(0,x)=\bar{\mathbf{u}}(T,x)$ and $\underline{\mathbf{u}}(0,x)=\underline{\mathbf{u}}(T,x)$ on $\bar{\Omega}$.

We will show that $\bar{\mathbf{u}}$ and $\underline{\mathbf{u}}$ are, respectively, the maximal and minimal periodic solutions of (1). In other words, we will prove that if $\mathbf{u}$ is another periodic solution of (1) in $(\hat{\mathbf{u}},\tilde{\mathbf{u}})$, then $\underline{\mathbf{u}}\leq\mathbf{u}\leq\bar{\mathbf{u}}$.

4 The Main Result

Now, we can state the main result of this paper, it is the following theorem.

Theorem 4.1 *Let* $\tilde{\mathbf{u}}$, $\hat{\mathbf{u}}$ *be a pair of ordered upper and lower solutions of* (1)*, and let hypothesis* (H) *hold with* $D_j(0) > 0$ *or* $D_j(0) = 0$ *and* $\hat{u}_j \geq \delta > 0$*. Then the sequences* $\left\{\bar{\mathbf{u}}^{(k)}, \bar{\mathbf{w}}^{(k)}\right\}$, $\left\{\underline{\mathbf{u}}^{(k)}, \underline{\mathbf{w}}^{(k)}\right\}$ *obtained from* (7) *converge monotonically from above to a maximal periodic solution* $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$ *and from below to a minimal periodic solution* $(\underline{\mathbf{u}}, \underline{\mathbf{w}})$ *in* $\mathbf{Q} \times \bar{\mathbf{Q}}$*, respectively, and satisfy the following inequalities for* $k \geq 1$:

$$\begin{aligned}&(\hat{\mathbf{u}}, \hat{\mathbf{w}}) \leq \left(\underline{\mathbf{u}}^{(k)}, \underline{\mathbf{w}}^{(k)}\right) \leq \left(\underline{\mathbf{u}}^{(k+1)}, \underline{\mathbf{w}}^{(k+1)}\right) \leq (\underline{\mathbf{u}}, \underline{\mathbf{w}}) \\ &\leq (\bar{\mathbf{u}}, \bar{\mathbf{w}}) \leq \left(\bar{\mathbf{u}}^{(k+1)}, \bar{\mathbf{w}}^{(k+1)}\right) \leq \left(\bar{\mathbf{u}}^{(k)}, \bar{\mathbf{w}}^{(k)}\right) \leq (\tilde{\mathbf{u}}, \tilde{\mathbf{w}}).\end{aligned}$$

Moreover, $\bar{\mathbf{u}}$ *and* $\underline{\mathbf{u}}$ *are the maximal and minimal periodic solutions of* (1)*, respectively. If, in addition,* $\bar{\mathbf{u}}(0, x) = \underline{\mathbf{u}}(0, x)$*, then* $\bar{\mathbf{u}}(t, x) = \underline{\mathbf{u}}(t, x) (\equiv \mathbf{u}^*(t, x))$ *and* $\mathbf{u}^*(t, x)$ *is the unique solution of* (1)*.*

Proof. As in Theorem 2.1 in Pao and Ruan [14], using the standard regularity argument for the equivalent quasilinear parabolic equations and Schauder estimates, we can conclude that the limits $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$ and $(\underline{\mathbf{u}}, \underline{\mathbf{w}})$ are the solutions of (6), and therefore $\bar{\mathbf{u}}$, $\underline{\mathbf{u}}$ are the solutions of (1). We next show the periodic property of solutions $(\bar{\mathbf{u}}, \bar{\mathbf{w}})$ and $(\underline{\mathbf{u}}, \underline{\mathbf{w}})$. We let $z_j(t, x) = w_j(t, x) - w_j(t + T, x)$, where w_j stands for either $\overline{w}_j$ or $\underline{w}_j$ for $1 \leq j \leq m$. By hypothesis (H) and the mean-value theorem, we have

$$\begin{aligned}&(D_j(u_j))^{-1}(z_j)_t - L_j(t) z_j \\ =\;& [(D_j(u_j(t, x)))^{-1}(w_j)_t(t, x) \\ &-L_j(t) w_j(t, x)] - [(D_j(u_j(t, x)))^{-1}(w_j)_t(t + T, x) \\ &-L_j(t + T) w_j(t + T, x)] \\ =\;& F_j(t, x, \mathbf{u}(t, x)) - F_j(t, x, \mathbf{u}(t + T, x)) \\ &+ (D_j(u_j(t + T, x)))^{-1}(w_j)_t(t + T, x) \\ &- (D_j(u_j(t, x)))^{-1}(w_j)_t(t + T, x) \\ =\;& \sum_{k=1}^{m} \frac{\partial F_j}{\partial u_k}(t, x, \xi) z_j(t, x) + \frac{D_j'(\eta_j)}{(D_j(\eta_j))^3}(w_j)_t(t + T, x) z_j(t, x),\end{aligned}$$

which gives us

$$(D_j(u_j))^{-1}(z_j)_t - L_j(t) z_j + \gamma_j z_j = \sum_{k=1}^{m} \frac{\partial F_j}{\partial u_k}(t, x, \xi) z_j(t, x) \quad \text{in } \bar{\Gamma}, \tag{9}$$

where $\gamma_j = \frac{D_j'(\eta_j)}{(D_j(\eta_j))^3}(w_j)_t(t + T, x)$ is bounded in $\bar{\Gamma}$, $\xi \equiv \xi(t, x)$ is the different intermediate value in $\mathbf{Q}$. We can also get

$$\begin{aligned}B_j z_j &= B_j(t) w_j(t, x) - B_j(t + T) w_j(t + T, x) \\ &= \Psi_j(t, x, \mathbf{u}(t, x)) - \Psi_j(t, x, \mathbf{u}(t + T, x)) \\ &= \sum_{k=1}^{m} \frac{\partial \Psi_j}{\partial u_k}(t, x, \zeta) z_j(t, x) \ \text{ on } \Sigma,\end{aligned} \tag{10}$$

and

$$z_j(0,x) = w_j(0,x) - w_j(T,x) \quad \text{in } \Omega, \tag{11}$$

where $\zeta = \zeta(t,x)$ is the different intermediate value in $\mathbf{Q}$. Using relation (5) and Lemma 10.9.1 in Pao [16], we obtain $z_j(t,x) \geq 0$ in $\bar{\Gamma}$ for $1 \leq j \leq m$. Replacing z_j by $-z_j$ in (9)-(11) leads to $z_j(t,x) \leq 0$ in $\bar{\Gamma}$, this yields $z_j(t,x) = 0$, which proves $w_j(t,x) = w_j(t+T,x)$ for $1 \leq j \leq m$; hence the periodicity of w_j. Therefore $\mathbf{w}(t,x) = \mathbf{w}(t+T,x)$ and then $\mathbf{u}(t,x) = \mathbf{u}(t+T,x)$.

By (5), we observe that every solution $\mathbf{u}$ of (1) in $(\hat{\mathbf{u}}, \tilde{\mathbf{u}})$ is an upper solution as well as a lower solution. The argument in the proof of Lemma 3.2 yields $\mathbf{u} \geq \underline{\mathbf{u}}^{(k)} \geq \hat{\mathbf{u}}$ for every k. Letting $k \to \infty$ gives $\mathbf{u} \geq \underline{\mathbf{u}}$. A similar argument using $\mathbf{u}$ and $\hat{\mathbf{u}}$ as ordered upper and lower solutions leads to $\mathbf{u} \leq \bar{\mathbf{u}}$. The same work, taking $\mathbf{u}$ and $\hat{\mathbf{u}}$ as ordered upper and lower solutions, leads to $\mathbf{u} \leq \bar{\mathbf{u}}$.

Finally, if $\bar{\mathbf{u}}(0,x) = \underline{\mathbf{u}}(0,x)\,(\equiv \mathbf{u}_0(x))$, we have $\bar{\mathbf{w}}(0,x) = \underline{\mathbf{w}}(0,x)$, then when considering problem (6) with the initial condition $(\mathbf{u}(0,x), \mathbf{w}(0,x)) = (\mathbf{u}_0(x), \mathbf{w}_0(x))$, the well-known existence-uniqueness result for parabolic systems implies that $(\bar{\mathbf{u}}(t,x), \underline{\mathbf{w}}(t,x)) = (\underline{\mathbf{u}}(t,x), \underline{\mathbf{w}}(t,x))$, and $\bar{\mathbf{u}}(t,x) = \underline{\mathbf{u}}(t,x)$ on $\bar{\Gamma}$. With this we end the proof of Theorem 4.1.

5 Application

As an application of the obtained result, we give the following growth Lotka-Volterra competition model with two competing species, where the reaction rates of the competition follow the hypothesis of the Holling-Tanner interaction mechanism

$$\begin{cases} (u_1)_t - \mathbf{div}\left(D_1(u_1)\nabla u_1\right) = u_1\left(a_1 - b_1 u_1 - c_1 \dfrac{u_2}{1+\sigma_1 u_1}\right) & \text{in } \Gamma, \\ (u_2)_t - \mathbf{div}\left(D_2(u_2)\nabla u_2\right) = u_2\left(a_2 - b_2 \dfrac{u_1}{1+\sigma_2 u_1} - c_2 u_2\right) & \text{in } \Gamma, \\ D_1(u_1)\dfrac{\partial u_1}{\partial \eta} = \beta_1(x)\, u_1 \ , \ D_2(u_2)\dfrac{\partial u_2}{\partial \eta} = \beta_2(x)\, u_2 & \text{on } \Sigma, \\ u_1(0,x) = u_1(T,x) \ , \ u_2(0,x) = u_2(T,x) & \text{in } \Omega, \end{cases} \tag{12}$$

where for each $j \in \{1,2\}$, a_j, b_j, c_j are positive constants and $\beta_1(x) \geq 0$ on $\partial\Omega$, σ_j is nonnegative function. This system is discussed in Pao [15, 16], where there are also several other applications. One of the main concerns for problem (12) is whether, and when the two competing species can coexist. The coexistence problem is ensured if the system has a positive periodic solution.

6 Concluding Remarks and Perspectives

The fruit of this work is a result of existence and positivity of periodic solutions for a class of degenerate parabolic reaction-diffusion models. Despite some difficulties, we succeeded in obtaining several important results. It is clear from Theorem 4.1 that under hypothesis (H), system (1) admits at least one periodic solution if there exists a pair of ordered upper and lower solutions.

The results of this research paper will motivate the development of the implemented methods to different open problems in several scientific fields, such as the anisotropic system, which consists in adding diffusion coefficients to the studied system depending on (t, x) or, more generally, depending on $(t, x, u, \nabla u)$. Moreover, we can study our problem numerically using one of the well known methods.

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