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Type-II Left Censoring of Some Finite Support Family Lifetime Distributions

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Abstract: Reliability systems are uncertain random systems, where the failure times are random variables that follow some probability distributions. Due to the difficulty of having complete failure times data for different units in a certain given test, different censoring schemes are proposed and studied in the literature. This paper considers Type-II left censoring of certain popular members of finite support family distributions, namely, the J-family distributions, regular power function distribution, and generalized uniform distribution. The maximum likelihood estimators (MLEs) for these distributions parameters were derived under the Type-II left censoring scheme. A comprehensive simulation study was performed using different sample sizes, parameter values, and censored proportions to investigate the behavior of the estimators via bias and root mean square error (RMSE) criteria. Two lifetime data sets from engineering were analyzed to illustrate the Type-II left censoring scheme which prevailed appropriate results.

Keywords: *generalized uniform distribution; J-family distributions; power function distribution; type-II left censoring.*

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1 Introduction

There are widespread applications of different censoring schemes in life-testing and reliability experiments in reliability systems, where it is impossible to follow the lifetime of the units till the end of their lifetimes. Several statistical parametric approaches and scenarios of censoring are considered in the literature based on the selected models and the available information [1]. The most popular censoring schemes are the conventional Type-I and Type-II censoring. Type-I censoring describes the situation when a test is terminated at a particular point in time in one direction (left censoring or right censoring) or two directions (interval censoring). However, the Type-II censoring scheme requires fixing the number of failures to be observed. Progressive and hybrid censoring have also been studied in the literature [2–4]. Another censoring scheme is random censoring which is used in life testing experiments and clinical trials, where both the survival and the censoring times are random. Different studies related to random censoring have been conducted [5, 6].

One of the main censoring schemes is the left censoring, which is an appropriate one when the event of interest has already occurred for the individual before the observation time. Applications involving left censoring may include survival analysis and reliability engineering. Coburn et al. [7] studied the patterns of health insurance coverage among rural and urban children with the incidence of a higher proportion of rural children whose spells were "left censored" in the sample. Also, a job duration might be incomplete because the beginning of the job spells is not observed, which is an incidence of left censoring [8]. Jiang et al. [9] conducted a semiparametric analysis on survival data with left truncation and right censoring dependent. Robert et al. [10] presented a method of handling left-censored data in quantitative microbial risk assessment. Yoshinari et al. [11] studied the Bayesian estimation using left-censored data via Markov Chain Monte Carlo simulation.

Survival analysis using various parametric models under the left censoring scheme has been considered extensively in the literature [12, 13]. Mira and Kundu [14] studied the left censored data using the generalized exponential distribution. Sindhu et al. [15] considered the Bayesian estimation of the left censored data using the inverse Rayleigh distribution. Asgharzadeh et al. [16] performed estimation and reconstruction based on the left censored data using the Pareto model. Sindhu et al. [17] applied the Gumbel Type II distribution under the Bayesian approach to the left censored data.

The J-shaped family distributions were introduced by Toop and Lone [18]. The applications of the J-shaped family distributions were considered by Nadarajah and Kotz [19] who showed that the hazard rate function is bathtub shaped. An advantage of the J-shaped family distributions, which have a bathtub shaped hazard rate function, is attributed to the possession of only two parameters, whereas other distributions with a bathtub shaped hazard function involve three or four parameters. Bathtub shaped hazard rate functions have a wide range of applications in reliability engineering and reliability analysis. The bathtub shaped hazard rate function can be applied to human populations. For example, at the infant age, the death rate is high due to birth defects or infant diseases, then the death rate remains constant up to the age of thirty, then it increases again. Also, some manufactured items such as televisions, handheld calculators, and microprocessors follow this pattern.

The power function distribution is commonly used in survival analysis. It is a flexible distribution as it can be used to model various types of data. Different versions of the power function distribution were reported in the literature [20]. In this work, we consider two versions of power function distribution, namely, the regular power function distribution, and the generalized uniform distribution; both distributions have two parameters: a scale parameter, and a shape parameter. Meniconi and Barry [21] compared the power function distribution with the exponential, lognormal, and Weibull distributions to measure the reliability. They concluded that the power function distribution is the best one to model such types of data. The power function distribution is characterized by the simplicity of its mathematical form and can be handled easily by medical researchers and reliability engineers to obtain failure rates and reliability data. The generalized uniform distribution was used as a model of plant growth [22]. Lee [23] studied the estimation of the generalized uniform distribution (GUD). Bhatt [22] discussed the consistent characterization of the GUD through expectation. Khan and Khan [24] obtained the characterization of the GUD based on lower record values.

This paper considers Type-II left censoring of some popular finite support family distributions, namely, the J-family distributions, regular power function distribution, and generalized uniform distribution. The maximum likelihood estimators (MLEs) for the model parameters were derived. A simulation study was performed using different sample sizes, parameter values, and censored proportions to observe the behavior of the estimators in terms of bias and root mean square error (RMSE) criteria. Finally, two real lifetime data sets from engineering were analyzed to illustrate the derived results.

2 Finite Family Support Distributions

2.1 J-Family distribution

The cumulative distribution function (CDF) of the J-shaped family of distributions is given by

$$F(x;\theta,\beta) = \begin{cases} 0, & x < 0, \\ (\frac{x}{\theta}(2-\frac{x}{\theta}))^{\beta}, & 0 \le x < \theta,\ 0 < \beta_0 < 1, \\ 1, & \theta \le x, \end{cases} \tag{1}$$

with the corresponding probability density function (PDF) given by

$$f(x;\theta,\beta) = \frac{2\beta}{\theta}\Big(1-\frac{x}{\theta}\Big)\Big(\frac{x}{\theta}\Big(2-\frac{x}{\theta}\Big)\Big)^{\beta-1} \quad ;0 < x \le \theta,\ 0 < \beta < 1, \tag{2}$$

where θ is the scale parameter and β is the shape parameter. The reliability function of the distribution is given by

$$R(t) = P(T > t) = 1 - \Big(\frac{t}{\theta}\Big(2-\frac{t}{\theta}\Big)\Big)^{\beta}$$

and the hazard rate function is given by

$$h(t) = \frac{f(t)}{R(t)} = \frac{\frac{2\beta}{\theta}\Big(1-\frac{t}{\theta}\Big)\Big(\frac{t}{\theta}\Big(2-\frac{t}{\theta}\Big)\Big)^{\beta-1}}{1-\Big(\frac{t}{\theta}\Big(2-\frac{t}{\theta}\Big)\Big)^{\beta}},$$

2.2 Regular power function distribution

The cumulative distribution function (CDF) of the regular power function distribution is given by

$$F(x;\theta,\beta) = \left(\frac{x}{\theta}\right)^p \quad ,0 < x \leq \theta \ ,p > 0, \theta > 0 \tag{3}$$

with the corresponding probability density function (PDF) given by

$$f(x;\theta,\beta) = \frac{p}{\theta^p} x^{p-1} \quad ,0 < x \leq \theta \ ,p > 0, \theta > 0, \tag{4}$$

where θ is the scale parameter and p is the shape parameter. It is denoted by $X \sim PFF(p,\theta)$. The reliability function of the distribution can be expressed as

$$R(t) = P(T > t) = 1 - \left(\frac{t}{\theta}\right)^p$$

and the hazard rate function is given by

$$h(t) = \frac{f(t)}{R(t)} = \frac{pt^{p-1}}{\theta^p - t^p}.$$

2.3 Generalized uniform distribution

The cumulative distribution function (CDF) of the generalized uniform distribution is given by Lee [23]

$$F(x;\theta,\beta) = \left(\frac{x}{\theta}\right)^{p+1} \ ,0 < x \leq \theta, \ \ -1 < p, \ \theta > 0 \tag{5}$$

with the corresponding probability density function (PDF) given by

$$f(x,\theta,\beta) = \left(\frac{p+1}{\theta}\right)\left(\frac{x}{\theta}\right)^p \ ,0 < x \leq \theta, \ \ -1 < p, \ \theta > 0, \tag{6}$$

where θ is the scale parameter, and p is the shape parameter. It is denoted by $X \sim GUD(p,\theta)$. The generalized uniform distribution is a uniform distribution over $(0,\theta)$ if $p = 0$. It should be noted that the density function (6) is decreasing with x if $-1 < p < 0$, and constant if $p = 0$, and increasing if $p > 0$.

The reliability function of the distribution can be expressed as

$$R(t) = P(T > t) = 1 - (\frac{t}{\theta})^{p-1}$$

and the hazard rate function is given by

$$h(t) = \frac{f(t)}{R(t)} = \frac{\left(\frac{p+1}{\theta}\right)\left(\frac{t}{\theta}\right)^p}{1 - \left(\frac{t}{\theta}\right)^{p-1}}.$$

3 Maximum Likelihood Estimation

The Type-II left censoring scheme is considered. Suppose the initial r observations are censored or unobserved and the largest $n-r$ lifetimes $X_{(r+1)} < X_{(r+2)} < \ldots < X_{(n)}$ have only been observed. Then the joint probability density function of $X_{(r+1)}, X_{(r+2)}, ..., X_{(n)}$ is given by

$$f(x_{(r)}, ..., x_{(n)}; \theta, p) = \frac{n!}{r!}\Big((F(x_{(r+1)})\Big)^r f(x_{(r+1)})...f(x_{(n)}). \tag{7}$$

3.1 Maximum likelihood estimation of J-family distribution

When using equation (7), the joint probability density function of $X_{(r+1)}, ..., X_{(n)}$ is given by

$$f(x_{(r)}, ..., x_{(n)}; \theta, \beta) =$$
$$\frac{n!}{r!}\left(\left(\frac{x_{(r)}}{\theta}\left(2-\frac{x_{(r)}}{\theta}\right)\right)^{\beta}\right)^{r} \quad \prod_{i=r+1}^{n}\left[\frac{2\beta}{\theta}\left(1-\frac{x_{(i)}}{\theta}\right)\left(\frac{x_{(i)}}{\theta}\left(2-\frac{x_{(i)}}{\theta}\right)\right)^{\beta-1}\right],$$

where

$$0 < x_{(r)} \leq x_{(r+1)} \leq ... \leq x_{(n)} \leq \theta, \quad 0 < \beta < 1.$$

The likelihood function is given by

$$L(x_{(r+1)}, ..., x_{(n)}, \theta, \beta) =$$
$$\frac{n!}{r!}\frac{x_{(r)}^{r\beta}}{\theta^{r\beta}}\left(2-\frac{x(r)}{\theta}\right)^{r\beta}\frac{(2\beta)^{n-r}}{n-r} \quad \prod_{i=r+1}^{n}\left(1-\frac{x_{(i)}}{\theta}\right)\left(\frac{x_{(i)}}{\theta}\right)^{\beta-1}\left(2-\frac{x_{(i)}}{\theta}\right)^{\beta-1},$$

where

$$0 < x_{(r)} \leq x_{(r+1)} \leq ... \leq x_{(n)} \leq \theta,$$

It is noticed that for fixed $0 < \beta < 1$,

$$\lim_{\theta \to x_{(n)}} L(\theta, \beta | x_{(r+1)}, ..., x_{(n)}) = \lim_{\theta \to \infty} L(\theta, \beta | x_{(r+1)}, ..., x_{(n)}).$$

Thus, for a fixed value of β, the value of θ that maximizes the likelihood function lies in the interval $(x_{(n)}, \infty)$. Therefore, the MLE of (θ, β) is the solution of the likelihood equations, such that

$$\frac{\partial L}{\partial \beta} = 0 \text{ and } \frac{\partial L}{\partial \theta} = 0$$

or, equivalently, $\frac{\partial \log L(\beta,\theta)}{\partial \beta} = 0$ and $\frac{\partial \log L(\beta,\theta)}{\partial \theta} = 0$.

The log-likelihood function can be expressed as

$$\log L(\theta, \beta; x_{(r+1)}, ..., x_{(n)}) =$$
$$\log\frac{n!}{r!} + r\beta\log x_{(r)} \quad -r\beta\log\theta + r\beta\log\left(2-\frac{x_{(r)}}{\theta}\right) + (n-r)\log 2\beta$$
$$-(n-r)\log\theta \quad + \sum_{i=r+1}^{n}\log\left(1-\frac{x_{(i)}}{\theta}\right) + (\beta-1)\sum_{i=r+1}^{n}\log\left(\frac{x_{(i)}}{\theta}\right)$$
$$+(\beta-1)\sum_{i=r+1}^{n}\log\left(2-\frac{x_{(i)}}{\theta}\right).$$

The derivative of the log-likelihood function for β gives the following normal equation:

$$r\log x_{(r)} - r\log\theta + r\log\left(2-\frac{x_{(r)}}{\theta}\right) + \frac{(n-r)}{\beta} + \sum_{i=r+1}^{n}\log\left(\frac{x_{(i)}}{\theta}\right) + \sum_{i=r+1}^{n}\log\left(2-\frac{x_{(i)}}{\theta}\right) = 0, \tag{8}$$

while the derivative of the log-likelihood function for θ results in the following normal equation

$$\frac{r\beta x_{(r)}}{\theta(2\theta - x_{(r)})} + \sum_{i=r+1}^{n}\frac{x_{(i)}}{\theta(\theta - x_{(i)})} + (\beta-1)\sum_{i=r+1}^{n}\frac{x_{(i)}}{\theta(2\theta - x_{(i)})} - \frac{n\beta}{\theta} = 0. \tag{9}$$

The maximum likelihood estimates $\hat{\beta}$ and $\hat{\theta}$ of the unknown parameters β and θ can be obtained by solving equations 8 and 9 numerically.

3.2 Maximum likelihood estimation of power function distribution

When using equation (7), the joint probability density function of $x_{(r+1)}, ..., x_{(n)}$ and the likelihood function can be, respectively, expressed as

$$f(x_{(r)}, ..., x_{(n)}; \theta, \beta) = \frac{n!}{r!}\left(\left(\frac{x_{(r+1)}}{\theta}\right)^p\right)^r \prod_{i=r+1}^{n} \frac{p}{\theta^p} x_{(i)}^{p-1},$$

$$L(x_{(r+1)}, ..., x_{(n)}; \theta, \beta) = \frac{n!}{r!}\frac{(x_{(r+1)})^{rp}}{\theta^{rp}}\frac{p^{n-r}}{\theta^{p(n-r)}} \prod_{i=r+1}^{n} x_{(i)}^{p-1}. \quad (10)$$

The MLEs of p and θ can be derived by maximizing the function L in equation 10. Since this likelihood function is a decreasing function of θ, the MLE of θ is

$$\hat{\theta} = X_{(n)} = \max(X_1, X_2, ..., X_n),$$

while the MLE of p can be obtained by solving

$$\frac{d \log L_1(p, \hat{\theta})}{dp}.$$

The log-likelihood function in this case is given by

$$\ln L(\hat{\theta}, p; (r+1), ..., x_{(n)}) = \log \frac{n!}{r!} + rp \ln x_{(r+1)} + (n-r)\ln p - np \ln \hat{\theta} + (p-1)\sum_{i=r+1}^{n} \ln x_{(i)}.$$

The derivative of the log-likelihood function for p gives the following normal equation: $r \ln x_{(r+1)} - n \ln \hat{\theta} + \sum_{i=r+1}^{n} \ln x_{(i)} + \frac{n-r}{p} = 0$. Thus, the maximum likelihood estimator (MLE) of p can be derived:

$$\hat{p} = \frac{n-r}{n \log \hat{\theta} - r \log x_{(r+1)} - \sum_{i=r+1}^{n} \ln x_{(i)}}.$$

3.3 Maximum likelihood estimation of generalized uniform distribution

When using equation 7, the joint probability density function of $X_{(r+1)}, X_{(r+2)}, ..., X_{(n)}$ and the likelihood function, respectively, can be expressed as

$$f(x_{(r)}, ..., x_{(n)}; \theta, p) = \frac{n!}{r!}\left(\left(\frac{x_{(r+1)}^{p+1}}{\theta}\right)^{p+1}\right)^r \prod_{i=r+1}^{n} \frac{p+1}{\theta^{p+1}} x_{(i)}^{p},$$

$$L(x_{(r+1)}, ..., x_{(n)}; \theta, \beta) = \frac{n!}{r!}\frac{(x_{(r+1)})^{r(p+1)}}{\theta^{r(p+1)}}\frac{p^{n-r}}{\theta^{p(n-r)}} \prod_{i=r+1}^{n} x_{(i)}^{p-1},$$

$$L(x_{(r+1)}, ..., x_{(n)}; \theta, p) = \frac{n!}{r!}\frac{(x_{(r+1)})^{r(p+1)}((p+1)^{n-r}}{\theta^{r(p+1)}\theta^{(n-r)(p+1)}} \prod_{i=r+1}^{n} x_{(i)}^{p}. \quad (11)$$

The MLEs of p and θ can be derived by maximizing the function L in equation 11. Since this likelihood function is a decreasing function of θ, the MLE of θ is

$$\hat{\theta} = X_{(n)} = \max(X_1, X_2, ..., X_n),$$

while the MLE of p can be obtained by solving

$$\frac{d \log L_1(p, \hat{\theta})}{dp} = 0.$$

In this case, the log-likelihood function is given by

$$\log L(\hat{\theta}, p; x_{(r+1)}, ..., x_{(n)}) = \log \frac{n!}{r!} - n(p+1)\ln \hat{\theta} \quad + r(p+1)\ln x_{(r+1)} + (n-r)\ln p + 1 + p\sum_{i=r+1}^{n} \ln x_{(i)}.$$

The derivative of the log-likelihood function for p gives the following normal equation:

$$-n \ln x_{(n)} + r \ln x_{(r+1)} + \frac{n-r}{p+1} + \sum_{i=r+1}^{n} \ln x_{(i)} = 0.$$

Thus, the maximum likelihood estimator (MLE) of p can be derived:

$$\hat{p} = \frac{n-r}{n \ln x_{(n)} - r \ln x_{(r+1)} - \sum_{i=r+1}^{n} \ln x_{(i)}} - 1.$$

4 Simulation Study

A simulation study was performed to deduce the behavior of the estimators. Different sample sizes, namely, $n = 25, 50$ and 100, different combinations of the parameter values and different censored proportions were considered. The simulation results were based on 1000 replicates. The means and root mean square errors (RMSE) of the maximum likelihood estimators of the shape parameters were calculated. The simulation results for the J-shaped family, power function, and generalized uniform distribution are displayed in Tables 1-3, respectively.

The following remarks can be drawn based on the simulation results:

a. The performance of the estimators improves in terms of bias and RMSE due to the increase in the sample size.

b. As the number of censored observations increases, the biases and RMSEs increase and vice versa.

c. The bias and RMSE increase with increasing values of the shape parameter.

5 Applications

In this section, two applications of Type-II left censoring lifetime data sets are presented. The first application is related to the J-shaped family distributions and the second application is related to the power function distribution.

Table 1: Mean and RMSE for MLE of β for different combinations of r, n, β and θ of the J-shaped family distribution.

θ	β	$n = 25$			$n = 50$			$n = 100$		
		r	Mean	RMSE	r	Mean	RMSE	r	Mean	RMSE
2	0.5	3	0.62421	0.16763	5	0.56925	0.09498	10	0.54283	0.06246
		5	0.63776	0.17955	10	0.57903	0.10371	20	0.54937	0.06819
	0.7	3	0.88971	0.24339	5	0.80039	0.13669	10	0.76625	0.08940
		5	0.91057	0.26119	10	0.82138	0.14954	20	0.77623	0.09778
	1.0	3	1.29608	0.36162	5	1.16690	0.20129	10	1.10455	0.13093
		5	1.32866	0.38876	10	1.19022	0.22061	20	1.11998	0.14342
4	0.5	3	0.62421	0.16763	5	0.56925	0.09498	10	0.54283	0.06246
		5	0.63776	0.17955	10	0.57903	0.10371	20	0.54937	0.06819
	0.7	3	0.88971	0.24339	5	0.80039	0.13669	10	0.76625	0.08940
		5	0.91057	0.26119	10	0.82138	0.14954	20	0.77622	0.09777
	1.0	3	1.29608	0.36162	5	1.16690	0.20129	10	1.10455	0.09777
		5	1.32866	0.38876	10	1.19022	0.22061	20	1.11998	0.14342

Table 2: Mean and RMSE for MLE of β for different combinations of r, n, β and θ of the power function distribution.

θ	β	$n = 25$			$n = 50$			$n = 100$		
		r	Mean	RMSE	r	Mean	RMSE	r	Mean	RMSE
2	0.5	3	0.55191	0.12795	5	0.53020	0.08585	10	0.51294	0.05625
		5	0.55363	0.13522	10	0.52976	0.08607	20	0.51495	0.06071
	0.7	3	0.77271	0.17914	5	0.74220	0.12021	10	0.71811	0.07875
		5	0.77516	0.18931	10	0.74167	0.12050	20	0.72093	0.08500
	1.0	3	1.10399	0.25591	5	1.06040	0.17173	10	1.02588	0.11249
		5	1.10737	0.27044	10	1.05953	0.17214	20	1.02990	0.12142
4	0.5	3	0.55191	0.12795	5	0.53020	0.08586	10	0.51294	0.05625
		5	0.55368	0.13522	10	0.52976	0.08607	20	0.51495	0.06071
	0.7	3	0.77210	0.17914	5	0.74228	0.12021	10	0.71811	0.07875
		5	0.77516	0.18931	10	0.74167	0.12050	20	0.72093	0.08500
	1.0	3	1.10310	0.25591	5	1.06040	0.17173	10	1.02509	0.11249
		5	1.10737	0.27044	10	1.05953	0.17214	20	1.02990	0.12142

5.1 Application (1)

This application considers the use of the Type-II left censored J-shaped family distributions to fit a real-life data set which represents the number of cycles to failure for a group of 60 electrical appliances [25]. The failure times are

14	34	61	69	80	123	165	210	381	464	479	556
574	839	917	969	991	1064	1088	1091	1174	1270	1275	1355
1397	1477	1578	1649	1702	1893	1932	2011	2161	2292	2326	2337
2628	2785	2811	2886	2993	3122	3248	3715	3790	3857	3912	4100
4106	4116	4315	4510	4584	5267	5299	5583	6065	9701.		

Table 3: Mean and RMSE for MLE of β for different combinations of r, n, β and θ of the generalized uniform distribution.

θ	β	$n = 25$			$n = 50$			$n = 100$		
		r	Mean	RMSE	r	Mean	RMSE	r	Mean	RMSE
2	0.5	3	0.65210	0.38329	5	0.57583	0.23810	10	0.53565	0.1684
		5	0.67379	0.41918	10	0.58752	0.26122	20	0.54136	0.17896
	0.7	3	0.87339	0.43439	5	0.78594	0.27084	10	0.74041	0.19055
		5	0.89696	0.47507	10	0.79919	0.29604	20	0.74876	0.20284
	1.0	3	1.20399	0.51105	5	1.10110	0.31864	10	1.04754	0.22417
		5	1.23171	0.55891	10	1.11670	0.34829	20	1.05515	0.23861
4	0.5	3	0.65299	0.38329	5	0.57587	0.23900	10	0.53565	0.16814
		5	0.67379	0.41918	10	0.58752	0.26122	20	0.54136	0.17896
	0.7	3	0.87339	0.43439	5	0.78534	0.27084	10	0.74041	0.19055
		5	0.89696	0.47507	10	0.79919	0.29604	20	0.74876	0.20281
	1.0	3	1.20399	0.51105	5	1.10110	0.31864	10	1.04754	0.22418
		5	1.23172	0.55891	10	1.11669	0.34829	20	1.05515	0.23861

The last observation was ignored as it is about 4 standard deviations above the mean and thus can be considered as an outlier. Thus, the data was rescaled by dividing each observation by 7000 [26]. The maximum likelihood estimates for θ and β were found to be 0.8664 and 0.8425, respectively. The Kolmogorov-Smirnov ($K - S$) test was used for this data set. The Kolmogorov-Smirnov test statistic value was found to be 0.11 and the theoretical critical value at $\alpha = 0.05$ was 0.17. Thus, fitting the J-shaped family distribution is adequate for the above data set. In the reliability analysis, the first 10 observations were censored, i.e., $r = 10$. The maximum likelihood estimates using the remaining data were 0.8664 for θ and 0.9086 for β. The estimated hazard function of the J-shaped family distribution using the complete and censored samples is shown in Figure 1. It is seen that the estimated hazard functions for the complete and censored samples are very close.

5.2 Application (2)

This application considers the use of the Type-II left censored power function distribution to fit a real-life data set which represents the failure times (in minutes) for a sample of 15 electronic components in an accelerated life test [25]. The failure times were analyzed to illustrate the Type-II left censoring scheme. The failure times are

1.4 5.1 6.3 10.8 12.1 18.5 19.7 22.2 23.0
30.6 37.3 46.3 53.9 59.8 66.2.

The validity of the power function distribution was checked. Based on the maximum likelihood estimates of θ and p, the parameters of 66.2 and 0.792, respectively, were obtained. The Kolmogorov-Smirnov ($K-S$) test was used for this data set. It is observed that the K-S distance between the fitted and the empirical distribution functions, and the corresponding critical value at $\alpha = 0.05$ are 0.167 and 0.33, respectively. Thus, the fit of power function distribution fits the above data reasonably well.

In the analysis, the first three observations, $r = 3$, were censored, namely, $x_{(1)} = 1.4, x_{(2)} = 5.1$ and $x_{(3)} = 6.3$. The maximum likelihood estimate of θ was 66.2 and p

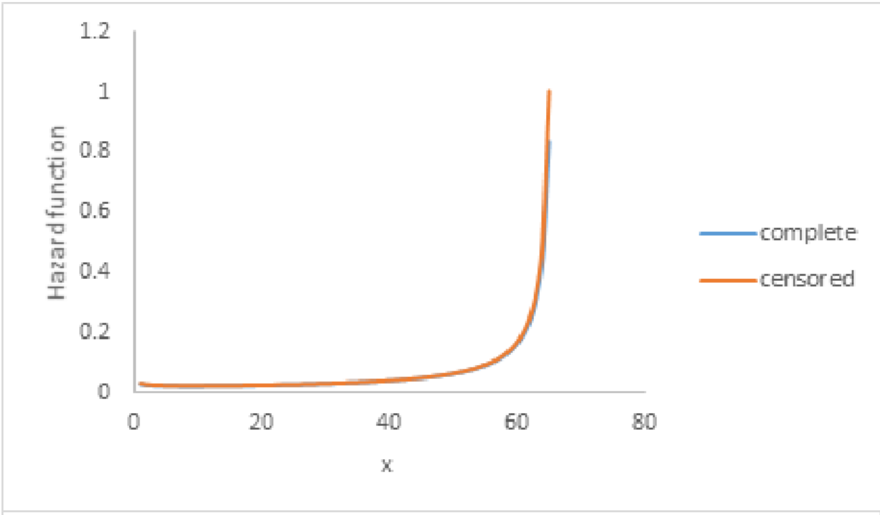

Figure 1: Hazard function of J-shaped distribution using complete and censored sample.

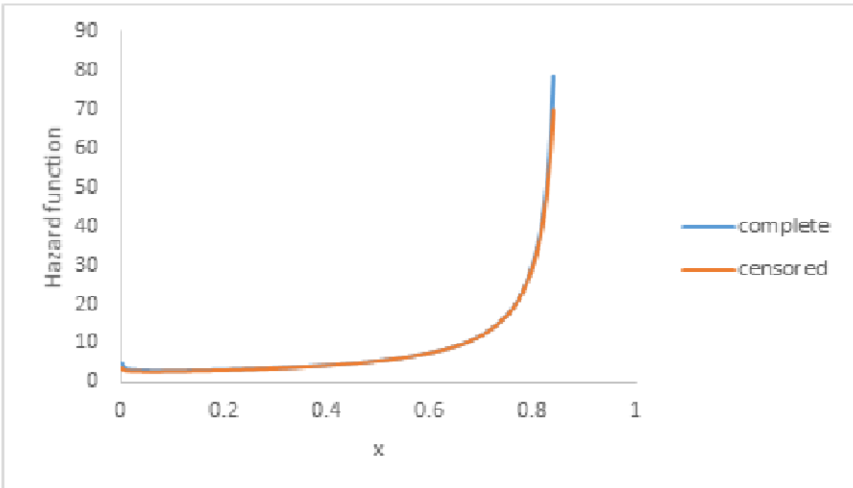

Figure 2: Hazard function of power function distribution using complete and censored samples.

was 0.7693. The estimated hazard functions for the complete and censored samples are shown in Figure 2. It is seen that the estimated hazard functions for the complete and censored samples are very close.

6 Conclusion

Type-II left censoring of three popular finite support family distributions, namely, the J-family distributions, the regular power function distribution and the generalized uniform distribution have been considered. The maximum likelihood estimators (MLEs) were derived for these distributions. A comprehensive simulation study was conducted for different sample sizes, parameter values, and censored proportions. Two lifetime data sets were analyzed to illustrate the Type-II left censoring scheme under the power function distribution and J-shaped family distributions and showed appropriate results.

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