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Asymptotic Analysis of a Nonlinear Elliptic Equation with a Gradient Term

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Abstract: The main purpose of the present paper is to study the equation

$$div(|\nabla u|^{p-2}\nabla u) + \alpha u + \beta x.\nabla u + |u|^{q-1}u = 0, \quad x \in \mathbb{R}^N,$$

where $p > 2$, $q > 1$, $N \geqslant 1$, $\alpha > 0$ and $\beta > 0$. We investigate the structure of radial solutions and we present the asymptotic behavior of positive solutions near infinity. The study depends strongly on the sign of $N\beta - \alpha$ and the comparison between the three determining values $\dfrac{\alpha}{\beta}$, $\dfrac{p}{q+1-p}$ and $\dfrac{N-p}{p-1}$. More precisely, we prove under some assumptions that there exists a positive solution u which has the following behavior near infinity:

$$u(r) \underset{+\infty}{\sim} \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}} r^{-\alpha/\beta}.$$

Keywords: *nonlinear elliptic equation; radial self-similar solution; global existence; energy function; asymptotic behavior; equilibrium point; nonlinear dynamical systems.*

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1 Introduction

The main purpose of this paper is to study the nonlinear elliptic equation

$$div(|\nabla U|^{p-2}\nabla U) + \alpha U + \beta x.\nabla U + |U|^{q-1}U = 0, \quad x \in \mathbb{R}^N, \tag{1}$$

where $p > 2$, $q > 1$, $N \geqslant 1$, $\alpha > 0$ and $\beta > 0$. The equation (1) is derived from the self-similar solutions of the nonlinear parabolic equation

$$v_t - \Delta_p v - |v|^{q-1}v = 0, \quad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{2}$$

These particular solutions are of the form

$$v(t, x) = t^{-\alpha}U(t^{-\beta}|x|), \tag{3}$$

where

$$\alpha = \frac{1}{q-1} \quad \text{and} \quad \beta = \frac{q+1-p}{p(q-1)}.$$

If $p = 2$, $\alpha = 0$ and $\beta = 0$, the equation (1) is due to Emden-Fowler and plays an important role in astrophysics, this motivates many researchers to be interested in the study of this case, the examples include (but are not limited to) [3, 7–10, 12, 17, 18]. In the case $p = 2$, $\alpha > 0$ and $\beta > 0$, the equation (1) was studied in [6, 14–16, 19, 20, 22–24]. In the case $p > 2$, $\alpha = 0$ and $\beta = 0$, (1) was investigated in [2], [13] and [21]. When $p > 2$, $\alpha > 0$ and $\beta = 1$, equation (1) was studied in [1]. When $p > 2$, $\alpha = \dfrac{1}{q-1}$ and $\beta = \dfrac{q+1-p}{p(q-1)}$, equation (1) was studied in [11]. In the case $p > 2$, $\alpha < 0$ and $\beta < 0$, we have studied an equation similar to (1) but with the term $|U|^{q-1}U$ weakened by its multiplication by the function $|x|^l$ with $l < 0$ that tends to 0 at infinity. This study was carried out in [4] and gave the existence and asymptotic behavior of unbounded solutions near infinity using nonlinear dynamical systems theory. In this paper, we consider the case where $\alpha > 0$, $\beta > 0$ and $l = 0$. It is also a generalization of the study carried out in [11]. We will present a result that improves asymptotic behavior near infinity of positive solutions, we investigate the structure of solutions of problem (P) in the cases $\dfrac{\alpha}{\beta} \geq N$ and $\dfrac{\alpha}{\beta} < N$ and we give an important relation between the solutions of the problem (P) and those of a nonlinear dynamical system obtained by using the logarithmic change.

If we put $U(x) = u(|x|)$, it is easy to see that u satisfies the equation

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u(r) + \beta r u'(r) + |u|^{q-1}u(r) = 0, \quad r > 0. \tag{4}$$

Since we are interested in radial regular solutions, we impose the condition $u'(0) = 0$. Thus we consider the following Cauchy problem.

Problem (P): Find a function u defined on $[0, +\infty[$ such that $|u'|^{p-2}u' \in C^1([0, +\infty[)$ and satisfying

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u(r) + \beta r u'(r) + |u|^{q-1}u(r) = 0, \quad r > 0 \tag{5}$$

and

$$u(0) = A > 0, \quad u'(0) = 0, \tag{6}$$

where $p > 2$, $q > 1$, $N \geqslant 1$, $\alpha > 0$ and $\beta > 0$.

By reducing the problem (P) to a fixed point for a suitable integral operator (see for example [5]), we prove that for each $A > 0$, the problem (P) has a unique global solution $u(., A, \alpha, \beta)$.

The main results are the following.

Theorem 1.1 *Problem (P) has a unique solution $u(., A)$. Moreover,*

$$(|u^{'}|^{p-2}u^{'})^{'}(0) = \frac{-A}{N}\left(\alpha + A^{q-1}\right). \tag{7}$$

Theorem 1.2 *Problem (P) has no positive solutions in the following cases:*
(i) $\dfrac{\alpha}{\beta} \geq N$.
(ii) $\dfrac{N-p}{p-1} \leqslant \dfrac{\alpha}{\beta} < N$.
(iii) $q \leqslant p-1$ *and* $\dfrac{\alpha}{\beta} < \dfrac{N-p}{p-1}$.
(iv) $q > p-1$ *and* $\dfrac{\alpha}{\beta} \neq \dfrac{p}{q+1-p} < \dfrac{N-p}{p-1}$.

Theorem 1.3 *Assume* $\dfrac{\alpha}{\beta} < N$. *Then the solution $u(., A)$ of problem (P) is strictly positive in the following cases:*
(i) $0 < A < (\beta N - \alpha)^{\frac{1}{q-1}}$.
(ii) $\dfrac{\alpha}{\beta} = \dfrac{p}{q+1-p} < \min\left(\dfrac{N-p}{p}, \dfrac{p}{2}\right)$.

Theorem 1.4 *Assume* $\dfrac{\alpha}{\beta} = \dfrac{p}{q+1-p} < \dfrac{N-p}{p-1}$. *Let u be a strictly positive solution of problem (P). Then*

$$\lim_{r \to +\infty} r^{\frac{\alpha}{\beta}} u(r) = \Gamma > 0$$

and

$$\lim_{r \to +\infty} r^{\frac{\alpha}{\beta}+1} u'(r) = \frac{-\alpha}{\beta}\Gamma,$$

where

$$\Gamma = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}.$$

The rest of the paper is organized as follows. In the second section, we present basic tools for the study of the problem (P). The third section concerns asymptotic behavior near infinity of solutions of problem (P); more precisely, we give explicit equivalents of solutions and their derivatives near infinity. The fourth section concerns the structure of solutions of problem (P). The last section, in the form of a conclusion, presents the asymptotic behavior of the solution of a nonlinear dynamical system around its equilibrium point and explains its relation with the asymptotic behavior of the solution of the problem (P).

2 Preliminaries and Basic Tools

In this section, we give existence of global solutions of problem (P) and we present the necessary basic tools that will be useful to us in the rest of the work.

Theorem 2.1 *Problem (P) has a unique solution $u(.,A)$. Moreover,*

$$(|u'|^{p-2}u')'(0) = \frac{-A}{N}\left(\alpha + A^{q-1}\right). \tag{8}$$

Proof. The proof of theorem is divided into three steps.
Step 1: Existence and uniqueness of a local solution.
Multiply equation (5) by r^{N-1}, we obtain

$$\left(r^{N-1}|u'|^{p-2}u'(r) + \beta r^N u(r)\right)' = (\beta N - \alpha)\, r^{N-1}u(r) - r^{N-1}|u|^{q-1}u(r). \tag{9}$$

Integrating (9) twice from 0 to r and taking into account (6), we see that problem (P) is equivalent to the equation

$$u(r) = A - \int_0^r G(F[u](s))\, ds, \tag{10}$$

where

$$G(s) = |s|^{(2-p)/(p-1)}s, \qquad s \in \mathbb{R}, \tag{11}$$

and the nonlinear mapping F is given by the formula

$$F[u](s) = \beta s u(s) + s^{1-N}\int_0^s \sigma^{N-1}u(\sigma)\left((\alpha - \beta N) + |u(\sigma)|^{q-1}\right)\, d\sigma. \tag{12}$$

Now, we consider for $A > M > 0$, the complete metric space

$$E_A = \{\varphi \in C\left([0,R]\right) \ \ \textit{such that} \ \ ||\varphi - A||_0 \leqslant M\}. \tag{13}$$

Next, we define the mapping Ψ on E_A by

$$\Psi[\varphi](r) = A - \int_0^r G(F[\varphi](s))\, ds. \tag{14}$$

Claim 1: Ψ maps E_A into itself for some small M and $R > 0$.
Obviously, $\Psi[\varphi] \in C([0,R])$. From the definition of the space E_A, $\varphi(r) \in [A-M, A+M]$, for any $r \in [0,R]$. It is easy to prove that $F[\varphi]$ has a constant sign in $[0,R]$ for every $\varphi \in E_A$. Moreover, there exists a constant $K > 0$ such that

$$F[\varphi](s) \geq Ks \qquad \textit{for all} \ \ s \in [0,R], \tag{15}$$

where $K = \dfrac{A}{2N}\left(\alpha + A^{q-1}\right)$.
Taking into account that the function $r \to \dfrac{G(r)}{r}$ is decreasing on $(0,+\infty)$, we have

$$|\Psi[\varphi](r) - A| \leq \int_0^r \frac{G(F[\varphi](s))}{F[\varphi](s)}|F[\varphi](s)|\, ds \leq \int_0^r \frac{G(Ks)}{Ks}|F[\varphi](s)|\, ds$$

for $r \in [0, R]$. On the other hand,

$$|F[\varphi](s)| \leq Cs, \; where \quad C = [\beta + |\frac{\alpha}{N} - \beta| + (A+M)^{q-1}](A+M).$$

We thus get

$$|\Psi[\varphi](r) - A| \leq \frac{p-1}{p} C K^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}}$$

for every $r \in [0, R]$. Choose R small enough such that

$$|\Psi[\varphi](r) - A| \leq M, \quad \varphi \in E_A.$$

And thereby $\Psi[\varphi] \in E_A$. The first claim is thus proved.
Claim 2: Ψ is a contraction in some interval $[0, r_A]$.
According to Claim 1, if r_A is small enough, the space E_A applies into itself. For any $\varphi, \psi \in E_A$, we have

$$|\Psi[\varphi](r) - \Psi[\psi](r)| \leq \int_0^r |G(F[\varphi](s)) - G(F[\psi](s))| \, ds, \tag{16}$$

where $F[\varphi]$ is given by (12. Next, let

$$\Phi(s) = \min(F[\varphi](s), F[\psi](s)).$$

As a consequence of estimate (15), we have

$$\Phi(s) \geq Ks \qquad for \;\; 0 \leq s \leq r < r_A$$

and then

$$\begin{aligned} |G(F[\varphi](s)) - G(F[\psi](s))| &\leq \frac{G(\Phi(s))}{\Phi(s)} |F[\varphi](s) - F[\psi](s)| \\ &\leq \frac{G(Ks)}{Ks} |F[\varphi](s) - F[\psi](s)|. \end{aligned} \tag{17}$$

Moreover,

$$|F[\varphi](s) - F[\psi](s)| \leq C' ||\varphi - \psi||_0 s, \tag{18}$$

where $C' = [\beta + |\frac{\alpha}{N} - \beta| + (A+M)^{q-1}](A+M)$. Combining (16), (17) and (18), we have

$$|\Psi[\varphi](s) - \Psi[\psi](s)| \leq \frac{p-1}{p} C' K^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}} ||\varphi - \psi||_0 \tag{19}$$

for any $r \in [0, r_A]$. When choosing r_A small enough, Ψ is a contraction. This proves the second claim.

The Banach Fixed Point Theorem then implies the existence of a unique fixed point of Ψ in E_A, which is a solution of (10) and consequently, of problem (P). As usual, this solution can be extended to a maximal interval $[0, r_{max}[, \; 0 < r_{max} \leq +\infty$.

Step 2: Existence of a global solution.
Define the energy function

$$E(r) = \frac{p-1}{p}|u'|^p + \frac{\alpha}{2}u^2(r) + \frac{1}{q+1}|u|^{q+1}. \tag{20}$$

Then by equation (5), the energy function satisfies

$$E'(r) = -\left(\frac{N-1}{r}|u'|^p + \beta r u'^2\right). \tag{21}$$

Then E is decreasing, hence it is bounded. Consequently, u and u' are also bounded and the local solution constructed above can be extented to $\mathbb{R}^+$.
Step 3: $(|u^{'}|^{p-2}u^{'})^{'}(0) = \frac{-A}{N}\left(\alpha + A^{q-1}\right)$.
Integrating (9) between 0 and r, we get

$$\frac{|u^{'}|^{p-2}u^{'}}{r} = -\beta u(r) + (\beta N - \alpha)r^{-N}\int_0^r s^{N-1}u(s)\,ds - r^{-N}\int_0^r s^{N-1}|u|^{q-1}u(s)\,ds.$$

Hence, using L'Hospital's rule and letting $r \to 0$, we obtain the desired result. The proof of the theorem is complete. □

Proposition 2.1 *Let u be a solution of problem (P) and let $S_u := \{r > 0,\ u(r) > 0\}$. Then $u'(r) < 0$ for any $r \in S_u$.*

Proof. We argue by contradiction. Let $r_0 > 0$ be the first zero of u'. Since by (8) $u'(r) < 0$ for $r \sim 0$, we have by continuity and the definition of r_0, there exists a left neighborhood $]r_0 - \varepsilon, r_0[$ (for some $\varepsilon > 0$), where u' is strictly increasing and strictly negative, that is, $(|u'|^{p-2}u')'(r) > 0$ for any $r \in]r_0 - \varepsilon, r_0[$, hence, by letting $r \to r_0$, we get $(|u'|^{p-2}u')'(r_0) \geq 0$. But by equation (5), we have $(|u'|^{p-2}u')'(r_0) = -\alpha u(r_0) - |u|^{q-1}u(r_0) < 0$ since $u(r_0) > 0$, $u'(r_0) = 0$ and $\alpha > 0$. This is a contradiction. The proof is complete. □

Proposition 2.2 *Assume $N > 1$. Let u be a solution of problem (P). Then*

$$\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u^{'}(r) = 0. \tag{22}$$

Proof. Since $E^{'}(r) \leqslant 0$ and $E(r) \geqslant 0$ for all $r > 0$, there exists a constant $l \geqslant 0$ such that $\lim_{r\to+\infty} E(r) = l$. Suppose $l > 0$. Then there exists $r_1 > 0$ such that

$$E(r) \geqslant \frac{l}{2} \quad \text{for } r \geqslant r_1. \tag{23}$$

Now consider the function

$$D(r) = E(r) + \frac{N-1}{2r}|u'|^{p-2}u'(r)u(r) + \frac{\beta(N-1)}{4}u^2(r) + \beta\int_0^r s u'^2(s)\,ds.$$

Then

$$D'(r) = -\frac{N-1}{2r}\left[|u'(r)|^p + \frac{N}{r}|u'|^{p-2}u'u(r) + |u(r)|^{q+1} + \alpha u^2(r)\right].$$

Recall that u and u' are bounded (because E is bounded), then

$$\lim_{r\to+\infty} \frac{|u^{'}|^{p-2}u^{'}u(r)}{r} = 0.$$

Moreover, by (20) and (23), we have for $r \geqslant r_1$,

$$\alpha u^2(r) + |u'(r)|^p + |u(r)|^{q+1} \geqslant \frac{p-1}{p}|u'(r)|^p + \frac{1}{q+1}|u(r)|^{q+1} + \frac{\alpha}{2}u^2(r) = E(r) \geqslant \frac{l}{2}.$$

Consequently, there exist two constants $c > 0$ and $r_2 \geqslant r_1$ such that

$$D'(r) \leqslant -\frac{c}{r} \quad \text{for } r \geqslant r_2.$$

Integrating the last inequality between r_2 and r, we get

$$D(r) \leqslant D(r_2) - c\ln(\frac{r}{r_2}) \quad \text{for } r \geqslant r_2.$$

In particular, we obtain $\lim_{r\to+\infty} D(r) = -\infty$. Since

$$E(r) + \frac{N-1}{2r}|u'|^{p-2}u'(r)u(r) \leqslant D(r),$$

we get $\lim_{r\to+\infty} E(r) = -\infty$. This is impossible, hence the conclusion. □

Proposition 2.3 *Let u be a strictly positive solution of problem (P), then u and u' have the same behavior (22).*

Proof. If $N > 1$, then by Proposition 2.2, $\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u'(r) = 0$. If $N = 1$, let

$$J(r) = |u'|^{p-2}u'(r) + \beta r\, u(r). \tag{24}$$

Then by equation (5),

$$J'(r) = (\beta - \alpha)u - |u|^{q-1}u(r). \tag{25}$$

Since u is strictly positive, it is strictly decreasing by Proposition 2.1. Therefore $\lim_{r\to+\infty} u(r) \in [0, +\infty[$. Since the energy function E given by (20) converges (because it is positive and decreasing), u' also necessarily converges and $\lim_{r\to+\infty} u'(r) = 0$. Suppose by contradiction that $\lim_{r\to+\infty} u(r) = L > 0$. Therefore $\lim_{r\to+\infty} J(r) = +\infty$.
Using L'Hospital's rule, we have

$$\lim_{r\to+\infty} J'(r) = \lim_{r\to+\infty} \frac{J(r)}{r}.$$

That is,

$$(\beta - \alpha)L - L^q = \beta\, L.$$

Therefore $-\alpha L - L^q = 0$. But this contradicts the fact that $L > 0$ and $\alpha > 0$. Hence $\lim_{r\to+\infty} u(r) = 0$. □

Proposition 2.4 *Let $0 < c \neq \dfrac{\alpha}{\beta}$. Let u be a strictly positive solution of problem (P). Then the function $r^c u(r)$ is strictly monotone for large r.*

Proof. For any $c > 0$, we consider the function

$$g_c(r) = cu(r) + ru'(r), \quad r > 0. \tag{26}$$

It is clear that

$$(r^c u(r))' = r^{c-1} g_c(r), \quad r > 0. \tag{27}$$

The monotonicity of the function $r^c u(r)$ can be obtained by the sign of the function $g_c(r)$. Using (5), we have for any $r > 0$ such that $u'(r) \neq 0$,

$$\begin{aligned}(p-1)|u'(r)|^{p-2} g_c'(r) = {} & (p-1)(c - \frac{N-p}{p-1})|u'|^{p-2}u'(r) \\ & - \beta r^2 u'(r) - \alpha r\, u(r) - r|u|^{q-1}u(r).\end{aligned} \tag{28}$$

Consequently, if $g_c(r_0) = 0$ for some $r_0 > 0$, we obtain by (26) and (28),

$$\begin{aligned}(p-1)|u'|^{p-2}(r_0) g_c'(r_0) = {} & r_0 u(r_0)\Big[(\beta c - \alpha) - |u(r_0)|^{q-1} \\ & + (p-1)c^{p-1}\left(\frac{N-p}{p-1} - c\right)\frac{|u(r_0)|^{p-2}}{r_0^p}\Big].\end{aligned} \tag{29}$$

Suppose that there exists a large r_0 such that $g_c(r_0) = 0$. Since $\lim\limits_{r\to+\infty} u(r) = 0$ and according to (29), we have for $c > \dfrac{\alpha}{\beta}$ (respectively, $c < \dfrac{\alpha}{\beta}$), $g_c'(r_0) > 0$ (respectively, $g_c'(r_0) < 0$) and thereby $g_c(r) \neq 0$ for large r if $c \neq \dfrac{\alpha}{\beta}$. Consequently, the function $r^c u(r)$ is strictly monotone for large r if $c \neq \dfrac{\alpha}{\beta}$. □

Proposition 2.5 *Let u be a strictly positive solution of problem (P). Then for any $0 < c < \dfrac{\alpha}{\beta}$, we have $g_c(r) < 0$ for large r and $\lim\limits_{r\to+\infty} r^c u(r) = 0$.*

Proof. We know by Proposition 2.4 that if $0 < c < \dfrac{\alpha}{\beta}$, $g_c(r) \neq 0$ for large r. Suppose that $g_c(r) > 0$ for large r. Then, by (26) and the fact that $u'(r) < 0$, we get

$$|u'(r)| < \frac{cu(r)}{r} \qquad for\ large\ r. \tag{30}$$

This gives by equation (5),

$$(|u'|^{p-2}u')'(r) < u(r)\left[(\beta c - \alpha) + (N-1)c^{p-1}\frac{u^{p-2}(r)}{r^p}\right]. \tag{31}$$

As $0 < c < \dfrac{\alpha}{\beta}$, $u(r) > 0$ and $\lim\limits_{r\to+\infty} u(r) = 0$, then $(|u'|^{p-2}u')'(r) < 0$ for large r. Combining with $u' < 0$, we get $\lim\limits_{r\to+\infty} u'(r) \in [-\infty, 0[$, which is impossible. Hence, $g_c(r) < 0$ for large r and by (27), $\lim\limits_{r\to+\infty} r^c u(r) \in [0, +\infty[$. Suppose that $\lim\limits_{r\to+\infty} r^c u(r) = L > 0$. Then $\lim\limits_{r\to+\infty} r^{c+\varepsilon} u(r) = +\infty$ for $0 < c + \varepsilon < \dfrac{\alpha}{\beta}$, but this contradicts the fact that $g_{c+\varepsilon}(r) < 0$ for large r. Consequently, $\lim\limits_{r\to+\infty} r^c u(r) = 0$. □

Proposition 2.6 *Let u be a strictly positive solution of problem (P). Then for any $\dfrac{\alpha}{\beta} < c \leqslant N$, we have $g_c(r) > 0$ for large r and $\lim\limits_{r\to+\infty} r^c u(r) = +\infty$.*

Proof. Let $\dfrac{\alpha}{c} < k < \beta$. We introduce the following energy function:

$$\phi(r) = r^{c-1}|u'|^{p-2}u' + kr^c u(r). \tag{32}$$

Using equation (5), we have

$$\begin{aligned}\phi'(r) =&(c-N)r^{c-2}|u'|^{p-2}u' + (k-\beta)r^c u'(r) + \\ &(kc-\alpha)r^{c-1}u(r) - r^{c-1}|u|^{q-1}u(r).\end{aligned} \tag{33}$$

As $u' < 0$, $c \leqslant N$ and $k < \beta$, then

$$\phi'(r) > r^{c-1}u\left[kc - \alpha - |u|^{q-1}\right]. \tag{34}$$

As $kc - \alpha > 0$ and $\lim\limits_{r\to+\infty} u(r) = 0$, then $\phi'(r) > 0$ for large r, therefore $\phi(r) \neq 0$ for large r. Suppose that $\phi(r) < 0$ for large r, then

$$|u'|^{p-2}u' < -k\,r\,u(r) \quad \text{for large } r. \tag{35}$$

Therefore

$$u'u^{\frac{-1}{p-1}} < -k^{\frac{1}{p-1}}r^{\frac{1}{p-1}} \quad \text{for large } r. \tag{36}$$

Integrating this last inequality on (R, r) for large R, we obtain

$$u^{\frac{p-2}{p-1}}(r) < u^{\frac{p-2}{p-1}}(R) - \frac{p-2}{p}k^{\frac{1}{p-1}}r^{\frac{p}{p-1}} + \frac{p-2}{p}k^{\frac{1}{p-1}}R^{\frac{p}{p-1}}.$$

Letting $r \to +\infty$, we obtain $\lim\limits_{r\to+\infty} u(r) = -\infty$, which is a contradiction. Consequently, $\phi(r) > 0$ for large r. Since ϕ is strictly increasing for large r, we have $\lim\limits_{r\to+\infty} \phi(r) \in]0, +\infty]$, so there exists $C_1 > 0$ such that $\phi(r) > C_1$ for large r. This gives by (32) and the fact that $u'(r) < 0$,

$$r^c u(r) > \frac{C_1}{k} \quad \text{for large } r.$$

On the other hand, using (34) and the fact that $\lim\limits_{r\to+\infty} u(r) = 0$, we obtain

$$r\phi'(r) > \frac{kc-\alpha}{2} r^c u(r) \quad \text{for large } r. \tag{37}$$

This implies that

$$r\phi'(r) > C \quad \text{for large } r, \tag{38}$$

where $C = \dfrac{C_1(kc-\alpha)}{2k} > 0$. Integrating this last inequality on (R, r) for large R, we obtain $\lim\limits_{r\to+\infty} \phi(r) = +\infty$. Consequently, by (32) and the fact that $u'(r) < 0$, we have $\lim\limits_{r\to+\infty} r^c u(r) = +\infty$. Moreover, since $g_c(r) \neq 0$ for large r, using (27), we have necessarily $g_c(r) > 0$ for large r. $\square$

Proposition 2.7 *Assume* $\dfrac{\alpha}{\beta} < N$. *Let* u *be a strictly positive solution of problem* (P). *Then the function* $r^{\alpha/\beta}u(r)$ *is not strictly monotone for large* r.

Proof. Assume by contradiction that $r^{\alpha/\beta}u(r)$ is strictly monotone for large r. Then by (27), $g_{\frac{\alpha}{\beta}}(r) \neq 0$ for large r. We distinguish two cases.
Case 1: $g_{\frac{\alpha}{\beta}}(r) < 0$ for large r.
We set

$$V(r) = u(r) - r^{p-1}|u'|^{p-1}. \tag{39}$$

Then by equation (5),

$$V'(r) = r^{p-1}u\left[-\alpha - u^{q-1}\right] + r^p u'\left[-\beta + r^{-p} + (p-N)r^{-2}|u'|^{p-2}\right]. \tag{40}$$

Using Proposition 2.6, we have $g_N(r) > 0$ for large r. Then

$$0 < r|u'(r)| < Nu(r) \quad \text{for large } r, \tag{41}$$

so $\lim\limits_{r\to+\infty} ru'(r) = 0$ and therefore

$$\lim_{r\to+\infty} V(r) = 0.$$

Using again inequality 41, we have

$$V(r) > u(r)\left(1 - N^{p-1}u^{p-2}(r)\right) \quad \text{for large } r. \tag{42}$$

Since $\lim\limits_{r\to+\infty} u(r) = 0$, one has $V(r) > 0$ for large r.
On the other hand, since $\lim\limits_{r\to+\infty} u(r) = 0$, $\lim\limits_{r\to+\infty} u'(r) = 0$ and $g_{\frac{\alpha}{\beta}}(r) < 0$ for large r, one has by (40),

$$V'(r) \underset{+\infty}{\sim} -\alpha r^{p-1}u(r) - \beta r^p u'(r) = -\beta r^{p-1} g_{\frac{\alpha}{\beta}}(r) > 0 \quad \text{for large } r. \tag{43}$$

But this contradicts the fact that $V(r) > 0$ for large r and $\lim\limits_{r\to+\infty} V(r) = 0$.
Case 2: $g_{\frac{\alpha}{\beta}}(r) > 0$ for large r.
Using equation (5), we obtain

$$(|u'|^{p-2}u')'(r) = -r\,u'(r)\left[\beta + \frac{N-1}{r^2}|u'|^{p-2}\right] - u(r)\left[\alpha + |u|^{q-1}\right]. \tag{44}$$

Since $\lim\limits_{r\to+\infty} u(r) = 0$, $\lim\limits_{r\to+\infty} u'(r) = 0$ and $g_{\frac{\alpha}{\beta}}(r) > 0$ for large r, we have

$$(|u'|^{p-2}u')'(r) \underset{+\infty}{\sim} -\beta r u'(r) - \alpha u(r) = -\beta g_{\frac{\alpha}{\beta}}(r) < 0 \quad \text{for large } r. \tag{45}$$

But this contradicts the fact that $u'(r) < 0$ and $\lim\limits_{r\to+\infty} u'(r) = 0$. $\square$

3 Asymptotic Behavior Near Infinity

In this section, we give explicit equivalents of the strictly positive solutions of the problem (P) and their derivatives near infinity.

Theorem 3.1 *Assume* $\dfrac{\alpha}{\beta}=\dfrac{p}{q+1-p}<\dfrac{N-p}{p-1}$. *Let* u *be a strictly positive solution of problem* (P). *Then*

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r)=\Gamma>0 \tag{46}$$

and

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}+1}u'(r)=\frac{-\alpha}{\beta}\Gamma, \tag{47}$$

where

$$\Gamma=\left(N-p-\frac{\alpha}{\beta}\,(p-1)\right)^{\frac{1}{q+1-p}}\left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}. \tag{48}$$

Proof. We consider the following function:

$$h(r)=r^{\frac{\alpha}{\beta}}u(r)\left[\beta+\frac{|u'|^{p-2}u'(r)}{r\,u}\right]. \tag{49}$$

Using equation (5), we have

$$h'(r)=\left(\frac{\alpha}{\beta}-N\right)r^{\frac{\alpha}{\beta}-2}|u'|^{p-2}u'(r)-r^{\frac{\alpha}{\beta}-1}u^{q}(r). \tag{50}$$

The proof will be done in four steps.
Step 1: $h(r)\underset{+\infty}{\sim}\beta r^{\alpha/\beta}u(r)$.
We know by Proposition 2.6 that $g_N(r)>0$ for large r, then using (41), we get

$$0<\frac{|u'(r)|^{p-1}}{ru(r)}<N^{p-1}\frac{u^{p-2}(r)}{r^{p}}\quad\text{for large } r. \tag{51}$$

As $p>2$ and $\lim\limits_{r\to+\infty}u(r)=0$, we get $\lim\limits_{r\to+\infty}\dfrac{|u'(r)|^{p-1}}{ru(r)}=0$. Consequently, by (49), we get $h(r)\underset{+\infty}{\sim}\beta r^{\alpha/\beta}u(r)$.
Step 2: $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r)$ exists and is finite. By Proposition 2.5, we have for any $\sigma>0$, $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}-\sigma}u(r)=0$. In particular, for

$$0<\sigma<\min\left(\frac{\alpha}{\beta}\frac{(q-1)}{q},\frac{1}{p-1}\left(\frac{\alpha}{\beta}(p-2)+p\right)\right)<\frac{\alpha}{\beta}, \tag{52}$$

there exists a constant $M>0$ such that

$$u(r)\leqslant M\,r^{\sigma-\frac{\alpha}{\beta}}\quad\text{for large } r. \tag{53}$$

We have also by (41),

$$|u'(r)|^{p-1}<\frac{N^{p-1}u^{p-1}(r)}{r^{p-1}}\quad\text{for large } r. \tag{54}$$

Combining (53) and (54), we obtain

$$r^{\frac{\alpha}{\beta}-1}u^q(r) < M^q r^{q(\sigma-\frac{\alpha}{\beta})+\frac{\alpha}{\beta}-1} \quad \text{for large } r \tag{55}$$

and

$$r^{\frac{\alpha}{\beta}-2}|u'(r)|^{p-1} < (MN)^{p-1} r^{\frac{\alpha}{\beta}(2-p)+\sigma(p-1)-p-1} \quad \text{for large } r. \tag{56}$$

By (52), (55) and (56), we get the function $r \to r^{\frac{\alpha}{\beta}-1}u^q(r)$ and the function $r \to r^{\frac{\alpha}{\beta}-2}|u'(r)|^{p-1}$ belong to $L^1(r_0,+\infty)$ for any $r_0 > 0$; therefore $h'(r) \in L^1(r_0,+\infty)$ for any $r_0 > 0$. Hence,

$$\lim_{r\to+\infty} h(r) = h(r_0) + \int_{r_0}^{+\infty} h'(s)\, ds \tag{57}$$

exists and is finite. Then by Step 1, $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r)$ exists and is finite. Let $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \Gamma \geq 0$.

Step 3: $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \Gamma > 0$ and $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}+1}u'(r) = \dfrac{-\alpha}{\beta}\Gamma < 0$.

We argue by contradiction and assume that $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = 0$. Then by the first step, $\lim\limits_{r\to+\infty} h(r) = 0$. Therefore, using L'Hospital's rule, we obtain

$$\lim_{r\to+\infty} \frac{h'(r)}{\left(r^{\frac{\alpha}{\beta}}u(r)\right)'} = \lim_{r\to+\infty} \frac{h(r)}{r^{\frac{\alpha}{\beta}}u(r)} = \beta. \tag{58}$$

On the other hand, we have

$$h'(r) = r^{\frac{\alpha}{\beta}-2}|u'(r)|^{p-1}\left(N - \frac{\alpha}{\beta} - \frac{ru^q}{|u'|^{p-1}}\right). \tag{59}$$

Let $0 < c < \dfrac{\alpha}{\beta}$, then by Proposition 2.5, we have $g_c(r) < 0$ for large r, then

$$|u'(r| > \frac{c\,u(r)}{r} \quad \text{for large } r. \tag{60}$$

This leads to

$$0 < \frac{r\,u^q(r)}{|u'(r)|^{p-1}} < c^{1-p}r^p u^{q+1-p}(r). \tag{61}$$

Since $\dfrac{\alpha}{\beta} = \dfrac{p}{q+1-p}$, then $\lim\limits_{r\to+\infty} r^p u^{q+1-p}(r) = 0$, therefore by (61), $\lim\limits_{r\to+\infty} \dfrac{ru^q}{|u'(r)|^{p-1}} = 0$. Using the fact that $\dfrac{\alpha}{\beta} < N$ and $|u'(r)| > 0$, we obtain by (59), $h'(r) > 0$ for large r. Therefore by (58), we have $\left(r^{\frac{\alpha}{\beta}}u(r)\right)' > 0$ for large r, but this contradicts Proposition 2.7. Consequently, $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \Gamma > 0$. Hence, using L'Hospital's rule (because $\lim\limits_{r\to+\infty} u(r) = 0$), we get

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}+1}u'(r) = \frac{-\alpha}{\beta}\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \frac{-\alpha}{\beta}\Gamma < 0.$$

Step 4: $\Gamma = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}$.

By (28), we have

$$-\beta r\, g_{\alpha/\beta}(r) = |u'|^{p-2}\, u'(r)\Bigg[\left(N - p - \frac{\alpha}{\beta}(p-1)\right) + (p-1)\frac{g'_{\alpha/\beta}(r)}{u'(r)} + \frac{ru^q(r)}{|u'|^{p-2}\, u'(r)}\Bigg]. \tag{62}$$

Since $\lim\limits_{r\to+\infty} u(r) = 0$ and $\lim\limits_{r\to+\infty} ru'(r) = 0$ (by Step 3), one has $\lim\limits_{r\to+\infty} g_{\frac{\alpha}{\beta}}(r) = 0$. Therefore, using again Step 3 and L'Hospital's rule, we obtain

$$\lim_{r\to+\infty} \frac{g'_{\frac{\alpha}{\beta}}(r)}{u'(r)} = \lim_{r\to+\infty} \frac{g_{\frac{\alpha}{\beta}}(r)}{u(r)} = \lim_{r\to+\infty} \left(\frac{\alpha}{\beta} + \frac{ru'(r)}{u(r)}\right) = 0. \tag{63}$$

Moreover, since $\frac{\alpha}{\beta} = \frac{p}{q+1-p}$, we have

$$\lim_{r\to+\infty} \frac{ru^q(r)}{|u'|^{p-2}u'(r)} = \frac{-\Gamma^{q+1-p}}{\left(\frac{\alpha}{\beta}\right)^{p-1}}. \tag{64}$$

Suppose by contradiction that

$$N - p - \frac{\alpha}{\beta}(p-1) - \frac{\Gamma^{q+1-p}}{\left(\frac{\alpha}{\beta}\right)^{p-1}} \neq 0. \tag{65}$$

Then, according to (62), (63) and (64), we have

$$-\beta r g_{\frac{\alpha}{\beta}}(r) \underset{+\infty}{\sim} |u'|^{p-2}u'(r)\left[N - p - \frac{\alpha}{\beta}(p-1) - \frac{\Gamma^{q+1-p}}{\left(\frac{\alpha}{\beta}\right)^{p-1}}\right]. \tag{66}$$

This gives $g_{\frac{\alpha}{\beta}}(r) \neq 0$ for large r, that is, $r^{\frac{\alpha}{\beta}}u(r)$ is strictly monotone for large r, but this contradicts Proposition 2.7. Consequently,

$$\Gamma = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}.$$

The proof of this theorem is complete. □

The following Figures 1 and 2 describe the strictly positive solution and its comparison with the function $r^{-\alpha/\beta}$.

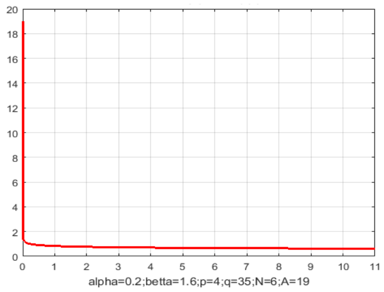

Figure 1: Strictly positive solution u.

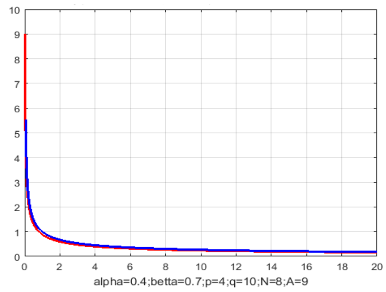

Figure 2: Comparison of solution u with $r^{-\alpha/\beta}$.

4 Structure of Radial Solutions

In this section, we investigate the structure of the solutions of the problem (P). The study depends strongly on the sign of $N\beta-\alpha$ and the comparison between the three determining values $\dfrac{\alpha}{\beta}$, $\dfrac{p}{q+1-p}$ and $\dfrac{N-p}{p-1}$.

Theorem 4.1 *Assume* $\dfrac{\alpha}{\beta} \geq N$. *Then the solution* u *of problem* (P) *changes the sign.*

Proof. We consider the following function:

$$\varphi(r)=r^{N-1}|u^{'}|^{p-2}u^{'}(r)+\beta r^{N}u(r). \tag{67}$$

Therefore by (9), we get

$$\varphi'(r)=(\beta N-\alpha)\, r^{N-1}u(r)-r^{N-1}|u|^{q-1}u(r). \tag{68}$$

Suppose that $u(r)>0$ for all $r\in[0,+\infty)$. As $\alpha \geqslant \beta N$, then $\varphi'(r)<0$. Therefore, as $\varphi(0)=0$, we have $\varphi(r)\leqslant 0 \ \ \forall r\in[0,+\infty)$. Consequently, the function $r\to H(r)=\dfrac{p}{p-2}u^{\frac{p-2}{p-1}}(r)+\beta^{\frac{1}{p-1}}r^{\frac{p}{p-1}}$ is decreasing. Then for any $r\in[0,+\infty)$, we have

$$H(r)\leqslant H(0)=\frac{p}{p-2}A^{\frac{p-2}{p-1}}. \tag{69}$$

When letting $r\to+\infty$, the term on the left-hand part of the inequality converges to $+\infty$, so we reach a contradiction.

Now, let r_0 be the first zero of u, then $\varphi'(r)<0$ for all $r\in(0,r_0)$, thus $\varphi(r_0)<\varphi(0)=0$. Therefore $u'(r_0)<0$, consequently, u changes the sign. □

The solution that changes the sign is illustrated by Figure 3.

Theorem 4.2 *Assume* $\dfrac{\alpha}{\beta}<N$. *Then the solution* u *of problem* (P) *is not strictly positive in the following cases:*

(i) $\dfrac{N-p}{p-1}\leqslant\dfrac{\alpha}{\beta}$.

(ii) $q\leqslant p-1$ *and* $\dfrac{\alpha}{\beta}<\dfrac{N-p}{p-1}$.

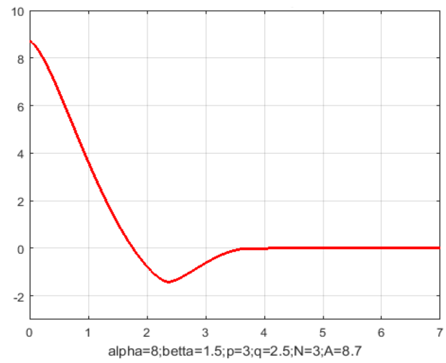

Figure 3: Solution that changes the sign.

(iii) $q > p-1$ *and* $\dfrac{\alpha}{\beta} \neq \dfrac{p}{q+1-p} < \dfrac{N-p}{p-1}$.

Proof. Assume by contradiction that u is strictly positive. The idea is to show that under this assumption, we have $g_{\alpha/\beta}(r) \neq 0$ for large r in these three cases, which is not possible by Proposition 2.7.

Assume that there exists a large r_0 such that $g_{\alpha/\beta}(r_0) = 0$, we obtain by (29),

$$(p-1)|u'|^{p-2}(r_0)g'_{\alpha/\beta}(r_0) = r_0 u^q(r_0)\Bigg[-1+(p-1)\left(\frac{\alpha}{\beta}\right)^{p-1} \times \left(\frac{N-p}{p-1} - \frac{\alpha}{\beta}\right) r_0^{-p} u^{p-1-q}(r_0)\Bigg]. \tag{70}$$

Using the fact that $\lim\limits_{r\to+\infty} u(r) = 0$, we have in the cases (i) and (ii), $g'_{\alpha/\beta}(r_0) < 0$. For the case (iii), we have by Proposition 2.5 and Proposition 2.6 $\lim\limits_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) = 0$ or $\lim\limits_{r\to+\infty} r^{\frac{p}{q+1-p}}u(r) = +\infty$, then we get $g'_{\frac{\alpha}{\beta}}(r_0) \neq 0$. Therefore, in the three cases, we have $g_{\alpha/\beta}(r) \neq 0$ for large r, that is, $r^{\alpha/\beta}u(r)$ is strictly monotone for large r. But this contradicts Proposition 2.7. Consequently, u is not strictly positive in the three cases. □

Theorem 4.3 *Assume* $\dfrac{\alpha}{\beta} < N$. *Then for any* $0 < A < (\beta N - \alpha)^{\frac{1}{q-1}}$, *the solution* $u(.,A)$ *of problem* (P) *is strictly positive.*

Proof. Let r_0 be the first zero of u, then $u(r_0) = 0$ and $u'(r_0) \leqslant 0$. Integrating (9) on $(0, r_0)$, we obtain

$$r_0^{N-1}|u'|^{p-2}u'(r_0) = \int_0^{r_0} \left[(\beta N - \alpha) - u^{q-1}(s)\right] s^{N-1}u(s)\, ds. \tag{71}$$

As $u(r) > 0$ and $u'(r) < 0$ on $(0, r_0)$, then

$$\beta N - \alpha - u^{q-1}(s) > \beta N - \alpha - A^{q-1} > 0 \quad \text{for any } s \in (0, r_0). \tag{72}$$

Therefore by (71), we get $u'(r_0) > 0$, but this contradicts the fact that $u'(r_0) \leqslant 0$. Hence $u(.,A)$ is strictly positive. □

Theorem 4.4 *Assume* $\dfrac{\alpha}{\beta} = \dfrac{p}{q+1-p} < \min\left(\dfrac{N-p}{p}, \dfrac{p}{2}\right)$. *Then the solution* u *of problem* (P) *is strictly positive.*

Before giving the proof of the theorem, we need the following result.

Proposition 4.1 *Let* u *be a solution of problem* (P). *Assume that there exists* $R > 0$, *the first zero of* u. *Then for* $\lambda \geq 1$ *and* $0 < \gamma < \rho$, *we have*

$$\int_0^R u^{\lambda}|u'|^{\gamma} s^{\rho-1}\, ds \leq \frac{\lambda+\gamma}{\rho-\gamma}\int_0^R u^{\lambda-1}|u'|^{\gamma+1} s^{\rho}\, ds. \tag{73}$$

Proof. By Holder's inequality, we have

$$\int_0^R u^{\lambda}|u'|^{\gamma} s^{\rho-1}\, ds \leq \left(\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds\right)^{\frac{1}{\gamma+1}} \left(\int_0^R u^{\lambda-1}|u'|^{\gamma+1} s^{\rho}\, ds\right)^{\frac{\gamma}{\gamma+1}}. \tag{74}$$

On the other hand, using the fact that $u(R) = 0$, we obtain

$$\int_0^R \left(u^{\lambda+\gamma} s^{\rho-1-\gamma}\right)' s\, ds = -\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds. \tag{75}$$

Therefore

$$\begin{aligned}&(\lambda+\gamma)\int_0^R u' u^{\lambda+\gamma-1} s^{\rho-\gamma}\, ds + (\rho-1-\gamma)\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds = \\ &-\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds.\end{aligned} \tag{76}$$

Using the fact that $u' < 0$ in $(0,R)$, we get

$$\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds = \frac{\lambda+\gamma}{\rho-\gamma}\int_0^R |u'| u^{\lambda+\gamma-1} s^{\varrho-\gamma}\, ds. \tag{77}$$

Applying Holder's inequality again, we obtain

$$\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds \leq \frac{\lambda+\gamma}{\rho-\gamma}\left(\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds\right)^{\frac{\gamma}{\gamma+1}} \left(\int_0^R u^{\lambda-1}|u'|^{\gamma+1} s^{\rho}\, ds\right)^{\frac{1}{\gamma+1}}. \tag{78}$$

Therefore,

$$\left(\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds\right)^{1-\frac{\gamma}{\gamma+1}} \leq \frac{\lambda+\gamma}{\rho-\gamma}\left(\int_0^R u^{\lambda-1}|u'|^{\gamma+1} s^{\rho}\, ds\right)^{\frac{1}{\gamma+1}}. \tag{79}$$

Combining (74) and (79), we easily obtain the estimation (73). This completes the proof of this proposition. □

Now we turn to the proof of Theorem 4.4.

Proof. Assume that there exists $r_0 > 0$, the first zero of u. Then $u(r) > 0$ $\forall r \in [0, r_0[,\ u'(r) < 0\ \forall r \in (0, r_0)$ and $u'(r_0) \leqslant 0$.
Since $\dfrac{p}{q+1-p} < \dfrac{N-p}{p}$, one has $\dfrac{N-p}{p} > \dfrac{N}{q+1}$.
Let $\dfrac{N}{q+1} < \delta < \dfrac{N-p}{p}$ and we consider the following energy function:

$$G(r) = r^N \left(\frac{p-1}{p} |u'|^p + \frac{1}{q+1} |u|^{q+1} \right) + \delta r^{N-1} u |u'|^{p-2} u'. \tag{80}$$

Using equation (5), we get

$$\begin{aligned} G'(r) = \left(\delta - \frac{N-p}{p} \right) r^{N-1} |u'|^p + \left(\frac{N}{q+1} - \delta \right) r^{N-1} |u|^{q+1} + \\ (\alpha + \beta\delta) r^N u |u'| - \alpha\delta r^{N-1} u^2(r) - \beta r^{N+1} u'^2(r). \end{aligned} \tag{81}$$

Integrating the last inequality on $(0, r_0)$, we obtain

$$\begin{aligned} G(r_0) = \left(\delta - \frac{N-p}{p} \right) \int_0^{r_0} s^{N-1} |u'|^p \, ds + \left(\frac{N}{q+1} - \delta \right) \int_0^{r_0} s^{N-1} |u|^{q+1}(s) \, ds \\ + (\alpha + \beta\delta) \int_0^{r_0} s^N u |u'| \, ds - \alpha\delta \int_0^{r_0} s^{N-1} u^2(s) \, ds - \beta \int_0^{r_0} s^{N+1} u'^2(s) \, ds. \end{aligned} \tag{82}$$

With the choice of δ and the fact that $u > 0$ and $u' < 0$ on $(0, r_0)$, we obtain by (82),

$$G(r_0) < (\alpha + \beta\delta) \int_0^{r_0} s^N u |u'| \, ds - \beta \int_0^{r_0} s^{N+1} u'^2(s) \, ds. \tag{83}$$

According to Proposition 4.1, we have

$$\int_0^{r_0} s^N u |u'| \, ds \leqslant \frac{2}{N} \int_0^{r_0} s^{N+1} u'^2(s) \, ds. \tag{84}$$

Then by (83) and (84), we see that

$$G(r_0) < \left(\frac{2}{N} (\alpha + \beta\delta) - \beta \right) \int_0^{r_0} s^{N+1} u'^2(s) \, ds. \tag{85}$$

Since $N > p$ and $\dfrac{p}{q+1-p} < \dfrac{p}{2}$, one has $\dfrac{N-p}{p} < \dfrac{N}{2} - \dfrac{\alpha}{\beta}$. Again, with the choice of δ, we have $\delta < \dfrac{N}{2} - \dfrac{\alpha}{\beta}$, which implies that $\left(\dfrac{2}{N}(\alpha + \beta\delta) - \beta \right) < 0$, that is, $G(r_0) < 0$, but this contradicts the fact that

$$G(r_0) = \frac{p-1}{p} r_0^N |u'(r_0)|^p \geqslant 0.$$

Consequently, u is strictly positive. This completes the proof. $\square$

5 Conclusion

In this work, we studied the Cauchy problem (P). We proved the existence of global solutions, we presented their complete classification in the cases $\frac{\alpha}{\beta} \geq N$ and $\frac{\alpha}{\beta} < N$, and we gave an explicit behavior near infinity of the positive solutions. More precisely, we have given explicit equivalents to the positive solution u of problem (P) and its negative derivative u'. The study of asymptotic behavior of positive solutions is carried out in the case $\frac{\alpha}{\beta} = \frac{p}{q+1-p} < \frac{N-p}{p-1}$, which recalls the form of radial self-similar solutions of the parabolic problem (2) from which the problem (P) is derived.

Asymptotic behavior of positive solutions is ensured by the study of a nonlinear dynamical system that we obtained by using the logarithmic change

$$v(t) = r^{\alpha/\beta}u(r), \quad r > 0 \text{ and } t = Log(r). \tag{86}$$

This obtained system, which we call (S), is as following:

$$(S)\begin{cases} v'(t) = |w(t)|^{\frac{2-p}{p-1}}w(t) + \dfrac{\alpha}{\beta}v(t), \\ w'(t) = -(N-p-\dfrac{\alpha}{\beta}(p-1))w(t) - \alpha e^{(p+\frac{\alpha}{\beta}(p-2))t}v(t) - \beta e^{(p+\frac{\alpha}{\beta}(p-2))t}z(t) - |v|^{q-1}v(t), \end{cases}$$

where

$$w(t) = |z|^{p-2}z(t) \tag{87}$$

and

$$z(t) = v'(t) - \frac{\alpha}{\beta}v(t) = r^{\frac{\alpha}{\beta}+1}u'(r). \tag{88}$$

The solution (v, w) of the system (S) satisfies $v > 0$ and $w < 0$ (because $u > 0$ and $u' < 0$) and tends near infinity to the equilibrium point $\left(\Gamma, -\left(\frac{\alpha}{\beta}\Gamma\right)^{p-1}\right)$, where Γ is explicitly dependent on p, q and N. Indeed, rewriting the second equation of the system (S) by using expression (88), we obtain

$$-\beta e^{(p+\frac{\alpha}{\beta}(p-2))t}v'(t) = w\left(N - p - \frac{\alpha}{\beta}(p-1) + \frac{w'}{w} + \frac{v^q}{w}\right). \tag{89}$$

We have by (63) and (64),

$$\lim_{t\to+\infty} \frac{w'}{w} = \lim_{r\to+\infty} (p-1)\frac{g'_{\frac{\alpha}{\beta}}(r)}{u'(r)} = 0 \tag{90}$$

and

$$\lim_{t\to+\infty} \frac{v^q}{w} = \lim_{r\to+\infty} \frac{ru^q(r)}{|u'|^{p-2}u'(r)} = \frac{-\Gamma^{q+1-p}}{\left(\frac{\alpha}{\beta}\right)^{p-1}}. \tag{91}$$

Therefore

$$\lim_{t\to+\infty} -\beta e^{(p+\frac{\alpha}{\beta}(p-2))t}\frac{v'(t)}{w} = N - p - \frac{\alpha}{\beta}(p-1) - \frac{\Gamma^{q+1-p}}{\left(\frac{\alpha}{\beta}\right)^{p-1}}. \tag{92}$$

Recall by Proposition 2.7, that $v(t)$ is not strictly monotone for large t, then since w is strictly negative, necessarily we have by (92),

$$\lim_{t\to+\infty} -\beta e^{(p+\frac{\alpha}{\beta}(p-2))t}\frac{v'(t)}{w} = 0.$$

Hence the explicit expression of Γ given by (48).

Finally, using expressions (86), (87) and (88), the convergence of the solution (v, w) of the system (S) to the equilibrium point $\left(\Gamma, -\left(\frac{\alpha}{\beta}\Gamma\right)^{p-1}\right)$ near infinity is expressed in terms of u and u' by

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}} u(r) = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}$$

and

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}+1} u'(r) = \frac{-\alpha}{\beta}\left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}.$$

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