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Existence, Uniqueness of Weak Solution to the Thermoelastic Plates

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Abstract: In this paper, we study a model of dynamic von Karman equation coupled to the thermoelastic equation, with rotational forces and not clamped boundary conditions. Our fundamental goal is to establish the existence as well as the uniqueness of a weak solution for the so-called global energy. In the end, we display a numerical simulation.

Keywords: *von Karman equation; nonlinear plates; rotational inertia; non-coupled method; finite difference method.*

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1 Introduction

In nonlinear oscillation of elastic plates, a dynamic von Karman equation with rotational forces, ($\alpha > 0$) [1], describes the buckling and flexible phenomenon of small nonlinear vibration of vertical displacement to the elastic plates. In nonlinear thermoelastic plate interaction, we study in this paper the case when the plate is coupled with thermal dissipation. From physical point of view, the main peculiarities of the model are the possibility of large deflections of the plate and small changes of the temperature near the reference temperature of the plate. As is well-known, the model with clamped boundary conditions, taking and not taking into account the rotational terms, for displacement u, the Airy stress function ϕ and the thermal function θ, can be formulated by the following system, see for instance [1].

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Find $(u,\phi,\theta) \in \left(L^2([0,T],(H_0^2(\omega))^2\right) \times H_0^1(\omega)$ such that

$$(\mathbb{P}_0)\begin{cases} u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u + \mu\Delta\theta - [\phi + F_0, u] = p(x) & in \quad \omega \times [0,T], \\ k\theta_t - \eta\Delta\theta - \mu\Delta u_t = 0 & in \quad \omega \times [0,T], \\ u_{|t=0} = u_0, \quad (u_t)_{|t=0} = u^1, \quad \theta_{|t=0} = \theta_0 & in \quad \omega, \\ u = \partial_\nu u = 0 & on \quad \Gamma \times [0,T], \\ \theta = 0 & on \quad \Gamma, \end{cases}$$

and

$$(\mathbb{Q})\begin{cases} \Delta^2\phi + [u,u] = 0 & in \quad \omega \times [0,T], \\ \phi = 0, \ \partial_\nu\phi = 0 & on \quad \Gamma \times [0,T], \end{cases}$$

where ω is the surface plate, u_0, u_1 and θ_0 are the initial data and $[.,.]$ is the so-called Monge-Ampère operator defined by [2]

$$[\phi, u] = \partial_{11}\phi\partial_{22}u + \partial_{11}u\partial_{22}\phi - 2\partial_{12}\phi\partial_{12}u. \tag{1}$$

The parameters μ, η are positive and α, k are non negative. The case $\alpha > 0$ corresponds to the equation with rotational term. But the parameter k has the meaning of heat/thermal capacity. Now, in the case $k = 0$ and $\alpha = 0$, the model $(\mathbb{P}_0)$ without rotational inertia can be decoupled, if we substitute $\Delta\theta$ from the second equation, the first equation becomes just a model of dynamic von Karman equations with internal viscous damping [1].

The plate is subjected to the internal force F_0 and external force p_0. In [1], Chueshov and Lasiecka studied the problem of structural interaction coupled with the von Karman evolution and established the theoretical result for a strong, generalized and weak solution by using the theory of nonlinear semi-group, if one chooses $0 \le \alpha \le 1$ and $0 \le k \le 1$. To justify the uniqueness, the authors used the limit definition of a generalized solution along weak continuity of the nonlinear terms involving the Airy stress function and known Lip continuity of the von Karman bracket with the Airy stress function.

The aim of this paper is to give a condition verified by the external, internal loads and the initial datums to have a unique weak solution of the von Karman evolution with rotational terms and not clamped boundary conditions subject to thermal dissipation and for all $\alpha > 0$, $k > 0$ and $0 < \mu \le \eta$. Our approach is based on an iterative problem $(P_n)_{n\ge0}$ whose sequence-solution $(u_n, \phi_n, \theta_n)_{n\ge0}$ converges to the unique solution of the problem under consideration.

This paper is organized as follows. Section 2 is devoted to the description of the mathematical structure of the model. In Section 3, we use the iterative method for establishing the uniqueness of weak solution of the associated dynamical plates with rotational terms, subject to thermal dissipation. In Section 3, we describe the numerical test.

2 Preliminary Results and Needed Tools

Throughout the following consideration, ω denotes a nonempty bounded domain in $\mathbb{R}^2$, with the regular boundary $\Gamma = \partial\omega$ and $\alpha > 0$, $k > 0$, $0 < \mu \le \eta$ are the reals.

Let $p \geq 1$ be a real number and $m \geq 1$ be an integer. We denote by $|.|_{p,\omega}$ the classical norm of $L^p(\omega)$ and by $\|.\|_{m,\omega}$ that of $H^m(\omega)$. For $u \in H^2(\omega)$, we set $\|u\| = |\Delta u|_{2,\omega}$ for the sake of simplicity. We also set

$$W(0,T) = \big\{u, \;\; u \in L^2\big([0,T] , H_0^2(\omega)\big), u_t \in L^2\big([0,T] , L^2(\omega)\big)\big\},$$

which is a Hilbert space with the associated norm

$$\Big(|u|^2_{L^2\big([0,T],H_0^2(\omega)\big)} + |u_t|^2_{L^2\big([0,T],L^2(\omega)\big)} \Big)^{1/2}.$$

In this paper, for the sake of simplicity, we denote

$$\|u\|_\alpha = \|u\|^2 + \alpha \, |\nabla u_t|^2_{2,\omega} + |u_t|^2_{2,\omega} \, . \tag{2}$$

We recall the following result [3, 4].

Theorem 2.1 *Let $f \in L^2(\omega)$. Then the following problem:*

$$\begin{cases} \Delta^2 v = f & in \;\; \omega, \\ v = 0 & on \;\; \Gamma, \\ \partial_\nu v = 0 & on \;\; \Gamma, \end{cases}$$

has one and only one solution $v \in H_0^2(\omega) \cap H^4(\omega)$ satisfying

$$\|v\| \leq c_0 \, |f|_{1,\omega}$$

for some constant $c_0 > 0$ depending only on $mes(\omega)$.

The following remark is of interest.

Remark 2.1 If the function f is in $L^2\big([0,T] , L^2(\omega)\big)$, then the unique solution of the last problem is in $L^2\big([0,T] , H_0^2(\omega) \cap H^4(\omega)\big)$.

We also need to recall the following result [4, 5].

Theorem 2.2 *Let $g \in L^2\big([0,T] , L^2(\omega)\big)$, $u_0 \in L^2(\omega)$ and k, η, μ are non negative reals. Then the following problem :*

$$(\mathbb{D}) \begin{cases} ku_t - \eta \Delta u = \mu g & in \;\; \omega \times [0,T] , \\ u_{|t=0} = u_0 & in \;\; \omega, \\ u = 0 & on \;\; \Gamma \times [0,T] , \end{cases}$$

has one and only one solution $u \in C\big([0,T] ; H^2(\omega) \cap H_0^1(\omega)\big) \cap C^1\big([0,T] ; L^2(\omega)\big)$.

Proposition 2.1 *Under the assumptions of Theorem 2.2 and if we choose $g = -\Delta f$, then the unique solution of the problem $(\mathbb{D})$ satisfies the following inequality:*

$$\forall 0 \leq t \leq T, \quad k \, |u|^2_{2,\omega} + \eta \int_0^t |\nabla u|^2_{2,\omega} \leq k \, |u_0|^2_{2,\omega} + \mu \int_0^t |\nabla f|^2_{2,\omega} \, , \tag{3}$$

where $f \in H^2(\omega)$, $k > 0$ and $0 < \mu \leq \eta$.

Proof. Since u is a solution of the problem $(\mathbb{D})$, we have

$$\frac{k}{2}\frac{d}{dt}|u|^2_{2,\omega}+\eta|\nabla u|^2_{2,\omega}=\int_\omega gu=\int_\omega -\Delta f u=\int_\omega \nabla f\nabla u\leq \frac{1}{2}|\nabla f|^2_{2,\omega}+\frac{1}{2}|\nabla u|^2_{2,\omega}.$$

Now, if we integrate the latter inequality with respect to $t>0$, we then deduce, by using the fact that $(u)_{|t=0}=u_0$ in ω,

$$\frac{k}{2}|u|^2_{2,\omega}+\eta\int_0^t|\nabla u|^2_{2,\omega}\leq \frac{k}{2}|u_0|^2_{2,\omega}+\frac{\mu}{2}\int_0^t|\nabla f|^2_{2,\omega}+\frac{\mu}{2}\int_0^t|\nabla u|^2_{2,\omega},$$

we have that $0\prec\mu\leq\eta$, then we conclude that

$$k|u|^2_{2,\omega}+\eta\int_0^t|\nabla u|^2_{2,\omega}\leq k|u_0|^2_{2,\omega}+\mu\int_0^t|\nabla f|^2_{2,\omega}.$$

The following theorem is of interest, see [1].

Theorem 2.3 *Assume that for $f\in L^2(\omega)$, $\alpha>0$ and $(u_0,u^1)\in H_0^2(\omega)\times H_0^1(\omega)$, the problem*

$$(\mathbb{S})\begin{cases}(1-\alpha\Delta)u_{tt}+\Delta^2 u=f & in\ \ \omega\times[0,T],\\ u=\partial_\nu u=0 & on\ \ \Gamma\times[0,T],\\ u_{|t=0}=u_0,\ \ (u_t)_{|t=0}=u^1 & in\ \ \omega,\end{cases}$$

has a unique solution $(u,u_t)\in C([\,0,T\,],H_0^2(\omega)\times H_0^1(\omega))$, and the energy equality

$$E_0(u,u_t)=E_0(u_0,u^1)+\int_0^t\int_\omega fu_t$$

holds, here

$$E_0(u_0,u^1)=\frac{1}{2}\int_\omega(\|u_0\|^2_{2,\omega}+\left|u^1\right|^2_{2,\omega}+\alpha\left|\nabla u^1\right|^2_{2,\omega}).$$

Now, let us put

$$F_1(u,\phi)=[\phi+F_0,u]. \tag{4}$$

Before giving our main result, we now state the following results.

Proposition 2.2 *Let $(u,v)\in(H_0^2(\omega))^2$ and $F_0\in H^4(\omega)$ be with small norms. Let $\phi,\varphi\in H_0^2(\omega)$ be the solutions of the following two problems:*

$$\Delta^2\phi=-[u,u]\ \ \text{and}\ \ \Delta^2\varphi=-[v,v].$$

Then the following estimations:

$$\Big|[u,\phi]-[v,\varphi]\Big|_{2,\omega}\leq c_1\|u-v\|$$

and

$$\|F_1(u,\phi)-F_1(v,\varphi)\|_{(L^2(\omega))^3}\leq c_1\|u-v\|$$

hold for some $0<c_1<1$.

Proof. Following [1], we have

$$\Big|\,[u,\phi]-[v,\varphi]\,\Big|_{2,\omega}\leq c_0\big(\,\|u\|^2+\|v\|^2\,\big)\,\|u-v\|$$

for some $c_0>0$. Let $c>0$ be small enough such that $\|u\|\leq c$ and $\|v\|\leq c$. We have

$$\Big|\,[u,\phi]-[v,\varphi]\,\Big|_{2,\omega}\leq 2c_0c^2\,\|u-v\|$$

and so

$$\begin{aligned}\|F_1(u,\phi)-F_1(v,\varphi)\|_{(L^2(\omega))^3} &\leq \Big|\,[\phi+F_0,u]-[\varphi+F_0,v]\,\Big|_{2,\omega},\\ &\leq \Big|\,[\phi,u]-[\varphi,v]\,\Big|_{2,\omega}+\Big|\,[F_0,u-v]\,\Big|_{2,\omega},\\ &\leq \big(2c_0c^2+4\,\|F_0\|_{4,\omega}\,\big)\,\|u-v\|\,.\end{aligned}$$

If we choose

$$\|F_0\|_{4,\omega}<\frac{1}{4}\quad and\quad 0<c<\sqrt{\frac{1-4\,\|F_0\|_{4,\omega}}{2c_0}},$$

we have

$$0<c_1=2c_0c^2+4\,\|F_0\|_{4,\omega}<1,$$

we then conclude the proof. The following proposition is of interest.

Proposition 2.3 *Let $f\in L^2([0,T],L^2(\omega))$, $\theta_0\in H_0^1(\omega)$ and $(u_0,u^1)\in H_0^2(\omega)\times H_0^1(\omega)$. The following problem :*

$$(\mathbb{S}_1)\begin{cases}(u)_{tt}-\alpha\Delta(u)_{tt}+\Delta^2u+\mu\Delta\theta=f & in\quad \omega\times[0,T]\,,\\ k\theta_t-\eta\Delta\theta=\mu\Delta u_t & in\quad \omega\times[0,T]\,,\\ u=\partial_\nu u=\theta=0 & on\quad \Gamma\times[0,T]\,,\\ (u)_{|t=0}=u_0,(u_t)_{|t=0}=u^1,(\theta)_{|t=0}=\theta_0 & in\quad \omega,\end{cases}$$

has one and only one solution $(u,\theta)\in L^2([\,0,T\,],H_0^2(\omega)\times H_0^1(\omega))$ and $u_t\in L^2([\,0,T\,],H_0^1(\omega))$ satisfies

$$\|u\|_\alpha+k\,|\theta|_{2,\omega}^2+2\eta\int_0^t|\nabla\theta|_{2,\omega}^2\leq e^T\Big(\,\|u_0\|^2+\alpha\,\big|\nabla u^1\big|_{2,\omega}^2+\big|u^1\big|_{2,\omega}^2+k\,|\theta_0|_{2,\omega}^2+\int_0^T|f|_{2,\omega}^2\Big).\quad(5)$$

Proof. For establishing the existence and uniqueness of solution of the problem under consideration, we will study the problem $(\mathbb{S})_1$ by considering the n-order approximate solution and we use the variational problem.
Let $\{\,e_k,e_k^1\,\}$ be a basis in the space $H_0^2(\omega)\times H_0^1(\omega)$. We define an n-order Galerkin approximate solution to the problem $(\mathbb{S})_1$ with clamped boundary conditions on the interval $[\,0,T\,]$, as a function $(u^n(t),\theta^n(t))$ of the form, see for instance [1,6],

$$u^n=\sum_{k=1}^n h_k(t)e_k\ \ and\ \theta^n=\sum_{k=1}^n l_k(t)e_k^1\quad n=1,2,3,...,$$

where $(h_k(t), l_k(t)) \in W^{2,+\infty}(0,T,IR) \times W^{1,+\infty}(0,T,IR)$ and ϕ^n is determined by u^n according to the problem $(\mathbb{Q})$ and (u_{n0}, θ_{n0}), u_{n1} are chosen such that (u_{n0}, θ_{n0}) converges to (u_0, θ_0) in $L^2([0,T], H_0^2(\omega) \times H_0^1(\omega))$ and u_{n1} converges to u^1 in $L^2([0,T], H_0^1(\omega))$. Let the variational problem of $(\mathbb{S})_1$ be

$$\int_\omega u_{tt}^n u_t^n + \alpha \int_\omega \nabla u_{tt}^n \nabla u_t^n + \int_\omega \Delta u^n \Delta u_t^n + \mu \int_\omega \Delta\theta^n u_t^n = \int_\omega f u_t^n$$

and

$$\int_\omega \theta_t^n \theta^n - \eta \int_\omega (\nabla\theta^n)^2 = \mu \int_\omega \Delta u_t^n \theta^n.$$

Since $(u_t^n, \theta^n) \in H_0^2(\omega) \times H_0^1(\omega)$ and $\int_\omega \Delta\theta^n u_t^n = \int_\omega \theta^n \Delta u_t^n$, we have

$$\frac{1d}{2dt}(|u_t^n|_{2,\omega}^2 + \|u^n\|^2 + \alpha |\nabla u_t^n|_{2,\omega}^2) + \mu \int_\omega \theta^n \Delta u_t^n = \int_\omega f u_t^n,$$

and

$$\frac{kd}{2dt} |\theta^n|_{2,\omega}^2 + \eta |\nabla\theta^n|_{2,\omega}^2 = \int_\omega \Delta\theta^n u_t^n.$$

Hence

$$\frac{1d}{2dt}(|u_t^n|_{2,\omega}^2 + \|u^n\|^2 + \alpha |\nabla u_t^n|_{2,\omega}^2) + \frac{kd}{2dt} |\theta^n|_{2,\omega}^2 + \eta |\nabla\theta^n|_{2,\omega}^2 = \int_\omega f u_t^n.$$

Now, if we integrate the latter inequality with respect to $t > 0$, with (2) and by using the fact that $u^n_{|t=0} = u_{n0}, (u_t^n)_{|t=0} = u_{n1}$ and $\theta^n_{|t=0} = \theta_{n0}$, we deduce that

$$\frac{1}{2}(\|u^n\|_\alpha + k |\theta^n|_{2,\omega}^2) + \eta \int_0^t |\nabla\theta^n|_{2,\omega}^2 = \frac{1}{2}(|u_{n1}|_{2,\omega}^2 + \alpha |\nabla u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k |\theta_{n0}|_{2,\omega}^2) + \int_0^t \int_\omega f u_t^n.$$

And for all $0 \le s \le t$,

$$\|u^n\|_\alpha + k |\theta^n|_{2,\omega}^2 + 2\eta \int_0^t |\nabla\theta^n|_{2,\omega}^2 \le |u_{n1}|_{2,\omega}^2 + \alpha |\nabla u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k |\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2$$
$$+ \int_0^t \Big(\|u^n\|_\alpha + k |\theta^n|_{2,\omega}^2 + 2\eta \int_0^s |\nabla\theta^n|_{2,\omega}^2 \Big). \qquad (6)$$

For any $0 \le s \le t$, we put

$$I(s) = \|u^n\|_\alpha + k |\theta^n|_{2,\omega}^2 + 2\eta \int_0^s |\nabla\theta^n|_{2,\omega}^2 .$$

The inequality (6) yields

$$e^{-s}\Big(I(s) - \int_0^s I(\sigma)d\sigma\Big) \le e^{-s}\Big(|u_{n1}|_{2,\omega}^2 + \alpha |\nabla u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k |\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big).$$

Now, we have

$$\begin{aligned} \tfrac{d}{ds}\Big(e^{-s} \textstyle\int_0^s I(\sigma)d\sigma\Big) &= e^{-s} I(s) - e^{-s} \textstyle\int_0^s I(\sigma)d\sigma = e^{-s}\Big(I(s) - \textstyle\int_0^s I(\sigma)d\sigma\Big), \\ &\le e^{-s}\Big(|u_{n1}|_{2,\omega}^2 + \alpha |\nabla u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k |\theta_{n0}|_{2,\omega}^2 + \textstyle\int_0^T |f|_{2,\omega}^2 \Big), \end{aligned}$$

and

$$|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \int_0^T |f|^2_{2,\omega} = I(0) + \int_0^T |f|^2_{2,\omega}$$

does not depend on s, then

$$\int_0^t \frac{d}{ds}\Big(e^{-s}\int_0^s I(\sigma)d\sigma\Big)ds \leq (\int_0^t e^{-s}ds)\Big(|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \int_0^T |f|^2_{2,\omega}\Big),$$

from which we deduce

$$e^{-t}\int_0^t I(\sigma)d\sigma \leq (1-e^{-t})\Big(|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \int_0^T |f|^2_{2,\omega}\Big).$$

Since

$$\int_0^t (\|u^n\|_\alpha + k\, |\theta^n|^2_{2,\omega} + 2\eta \int_0^s |\nabla\theta^n|^2_{2,\omega}) = \int_0^t I(\sigma)d\sigma,$$

it follows that

$$\begin{aligned} \textstyle\int_0^t I(\sigma)d\sigma &\leq \tfrac{(1-e^{-t})}{e^{-t}}\Big(|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big),\\ &\leq (e^{t}-1)\Big(|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big),\\ &\leq (e^{T}-1)\Big(|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big). \end{aligned}$$

This, with (6), yields

$$\begin{aligned} \|u^n\|_\alpha + k\, |\theta^n|^2_{2,\omega} + 2\eta \textstyle\int_0^t |\nabla\theta^n|^2_{2,\omega} &\leq \Big(|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big)\\ &+(e^T-1)\Big(|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big),\\ &\leq e^T\Big(|u_{n1}|^2_{2,\omega} + \alpha\, |\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\, |\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big). \end{aligned}$$

This estimate implies that there exists a subsequence (u^{n_l}, θ^{n_l}) such that $(u^{n_l}, \theta^{n_l}) \rightharpoonup (u, \theta)$ weakly in $H^2_0(\omega) \times L^2(\omega))$ and $((u^{n_l})_t, \nabla\theta^{n_l}) \rightharpoonup ((u)_t, \nabla\theta)$ weakly in $H^1_0(\omega) \times L^2(\omega)$.

For showing that (u, θ) is a weak solution of the problem $(\mathbb{S})_1$, we use the same method as in [6]. Let $\varphi_j \in C^1(0,T)$, $1 \leq j \leq j_0$, such that $\varphi_j(T) = 0$ and

$$\psi = \sum_{j=1}^{j_0} \varphi_j \otimes e_j, \quad \varphi = \sum_{j=1}^{j_0} \varphi_j \otimes e^1_j.$$

After the variational problem, we have

$$-\int_0^T\int_\omega u^{nl}_t \psi_t + \alpha \int_0^T\int_\omega \nabla u^{nl}_t \nabla\psi_t + \mu \int_0^T\int_\omega \nabla\theta^{nl}\nabla\psi + \int_0^T\int_\omega \Delta u^{nl}\Delta\psi$$

$$= \int_0^T \int_\omega f\psi - \int_\omega u_{nl1}\psi(0) - \alpha \int_\omega \nabla u_{nl1}\nabla\psi(0) \tag{7}$$

and

$$-\int_0^T \Big(\int_\omega \theta^{nl}\varphi_t + \eta \int_\omega \nabla\theta^{nl}\nabla\varphi - \mu \int_\omega \nabla u^{nl}\nabla\varphi_t \Big) = -\int_\omega \theta_{nl0}\varphi(0) + \mu \int_\omega \nabla u_{nl1}\nabla\varphi(0). \tag{8}$$

Now, we can pass to the limit $nl \to +\infty$, in (7) and (8), we find that for all $\psi \in L^2([0,T], H_0^2(\omega))$, $\psi_t \in L^2([0,T], H^1(\omega))$, $\varphi \in L^2([0,T], H_0^1(\omega))$ and $\varphi_t \in L^2([0,T], L^2(\omega))$ such that $\psi(T) = \varphi(T) = 0$. We deduce that

$$-\int_0^T \int_\omega u_t\psi_t + \alpha \int_0^T \int_\omega \nabla u_t \nabla\psi_t + \mu \int_0^T \int_\omega \nabla\theta\nabla\psi + \int_0^T \int_\omega \Delta u \Delta\psi$$

$$= \int_0^T \int_\omega f\psi - \int_\omega u^1\psi(0) - \alpha \int_\omega \nabla u^1 \nabla\psi(0)$$

and

$$-\int_0^T \Big(\int_\omega \theta\varphi_t + \eta \int_\omega \nabla\theta\nabla\varphi - \mu \int_\omega \nabla u\nabla\varphi_t \Big) = -\int_\omega \theta_0\varphi(0) + \mu \int_\omega \nabla u^1\nabla\varphi(0).$$

This shows that (u, θ) is a weak solution of the problem $(\mathbb{S})_1$, by the some method as in the last proof, we deduce the following inequality:

$$\|u\|_\alpha + k\,|\theta|^2_{2,\omega} + 2\eta \int_0^t |\nabla\theta|^2_{2,\omega} \le e^T(\left|u^1\right|^2_{2,\omega} + \alpha\left|\nabla u^1\right|^2_{2,\omega} + \|u_0\|^2 + k\,|\theta_0|^2_{2,\omega} + \int_0^T |f|^2_{2,\omega}).$$

For the uniqueness, let (u_1, θ_1) and (u_2, θ_2) be two solutions. We use a similar proof as that of inequality (5), for the solution $(u_1 - u_2, \theta_1 - \theta_2)$ of the following problem:

$$\begin{cases} (1-\alpha\Delta)(u_1-u_2)_{tt} + \Delta^2(u_1-u_2) + \mu\Delta(\theta_1-\theta_2) = 0 & in \quad \omega\times[0,T], \\ k(\theta_1-\theta_2)_t - \eta\Delta(\theta_1-\theta_2) = \mu\Delta(u_1-u_2)_t & in \quad \omega\times[0,T], \\ \theta_1-\theta_2 = u_1-u_2 = \partial_\nu(u_1-u_2) = 0 & on \quad \Gamma\times[0,T], \\ (u_1-u_2)_{|t=0} = 0, ((u_1-u_2)_t)_{|t=0} = 0, (\theta_1-\theta_2)_{|t=0} = 0 & in \quad \omega, \end{cases}$$

it follows that

$$\|u_1-u_2\|_\alpha + k\,|\theta_1-\theta_2|^2_{2,\omega} + 2\eta\int_0^t |\nabla(\theta_1-\theta_2)|^2_{2,\omega} \le e^T(\left|(u_1)^1-(u_2)^1\right|^2_{2,\omega}$$

$$+ \|(u_1)_0-(u_2)_0\|^2 + \alpha\left|\nabla((u_1)^1-(u_2)^1)\right|^2_{2,\omega} + k\,|(\theta_1)_0-(\theta_2)_0|^2_{2,\omega}).$$

Then $u_1 = u_2$ and $\theta_1 = \theta_2$. The proof of the proposition is completed.

3 Iterative Approach: The Main Results

For establishing the existence and uniqueness of solution of the problem $(\mathbb{P}_0)$ in the case of rotational terms $\alpha > 0$, we use the following iterative approach.

Let $n \geq 2$ and let $0 \neq u_1 \in H_0^2(\omega)$ be given. We first find $\phi_{n-1} \in H_0^2(\omega)$ as the solution of the equation $\Delta^2\phi_{n-1} = -[u_{n-1}, u_{n-1}]$ and (u_n, θ_n) as the solution of the following problem:

$$(\mathbb{P}_n)\begin{cases} (u_n)_{tt} - \alpha\Delta(u_n)_{tt} + \Delta^2 u_n = F(u_{n-1}, \phi_{n-1}, \theta_n) & in \quad \omega \times [0,T]\,, \\ k(\theta_n)_t - \eta\Delta\theta_n = \mu\Delta(u_n)_t & in \quad \omega \times [0,T]\,, \\ u_n = \partial_\nu u_n = \theta_n = 0 & on \quad \Gamma \times [0,T]\,, \\ (u_n)_{|t=0} = u_0, ((u_n)_t)_{|t=0} = u^1, (\theta_n)_{|t=0} = \theta_0 & in \quad \omega, \end{cases}$$

where

$$F(u, \phi, \theta) = F_1(u, \phi) - \mu\Delta\theta + p,$$

and F_1 is defined by (4).

We are now in a position to state our main result of this section.

Theorem 3.1 *Let $p \in L^2(\omega)$, $(u_0, u^1) \in H_0^2(\omega) \times H_0^1(\omega)$ and $\theta_0 \in H_0^1(\omega)$. Assume that all the following quantities:*

$$\|F_0\|_{4,\omega}\,,\ |p|_{2,\omega}\,,\ \|u_0\|^2 + \left|u^1\right|_{2,\omega}^2 + \alpha\left|\nabla u^1\right|_{2,\omega}^2 \quad and \quad \|\theta_0\|_{1,\omega}^2$$

are small with $0 < \mu \leq \eta$. Then the problem $(\mathbb{P}_0)$ with rotational forces has one and only one weak solution (u, ϕ, θ) in $L^2\big([0,T]\,, H_0^2(\omega) \times H_0^2(\omega) \times H_0^1(\omega)\big)$ such that $u_t \in L^2\big([0,T]\,, H_0^1(\omega)\big)$ and $u_{tt} \in L^2\big([0,T]\,, L^2(\omega)\big)$.

Proof. We divide the proof into four steps.

Step 1: Let us consider the problem $(\mathbb{P}_n)$, where $0 \neq u_1$ does not depend on t.

Throughout this proof, we use the notation

$$\|(u, \theta)\|_* = \|u\|_\alpha + k\,|\theta|_{2,\omega}^2 + 2\eta\int_0^t |\nabla\theta|_{2,\omega}^2\,,$$

where $\|.\|_\alpha$ is defined by (2). According to Proposition 2.2 and Theorem 2.1, there exists a constant $c_0 > 0$. Now, for $\|F_0\|_{4,\omega} < \frac{1}{4}$, we can choose $c := c(\|F_0\|_{4,\omega}\,, c_0, T) > 0$ such that

$$0 < 4c_0c < 1,\ \ 0 < c < \sqrt{\frac{1 - 4\,\|F_0\|_{4,\omega}}{2c_0}} \quad \text{and} \quad \|u_1\|_{2,\omega} < c < 1.$$

By a mathematical induction on $n \geq 1$, we will prove that the following two inequalities:

$$\|u\|_\alpha = \|u_n\|^2 + \alpha\,|\nabla(u_n)_t|_{2,\omega}^2 + |(u_n)_t|_{2,\omega}^2 \leq \|u_1\|_{2,\omega}^2 \quad \text{and} \quad \|\phi_n\|_{2,\omega} \leq \|u_1\|_{2,\omega}$$

hold for all $n \geq 1$ and any $0 \leq t \leq T$. For $n = 1$, we have

$$\|u_1\|_\alpha = \|u_1\|^2 + |(u_1)_t|_{2,\omega}^2 = \|u_1\|_{2,\omega}^2$$

since u_1 does not depend on t. Otherwise, for ϕ_1 being the solution of the problem $\Delta^2\phi_1 = -[\,u_1, u_1\,]$, Theorem 2.1 ensures that there exists $c_0 > 0$ such that

$$\|\phi_1\|_{2,\omega} \leq c_0\,|[\,u_1, u_1\,]|_{1,\omega}\,,$$

using the proof of Proposition 2.2 with $\|u_1\|_{2,\omega} < c$ and $0 < 4c_0c < 1$, we can deduce that

$$\|\phi_1\|_{2,\omega} \leq 4c_0 \|u_1\|^2_{2,\omega} \leq 4c_0c \|u_1\|_{2,\omega} \leq \|u_1\|_{2,\omega} .$$

The desired inequalities are true for $n = 1$.

Suppose that for $k = 2, ..., n$ and $0 \leq t \leq T$, we have

$$\|u_k\|_\alpha \leq \|u_1\|^2_{2,\omega} \quad \text{and} \quad \|\phi_k\|_{2,\omega} \leq \|u_1\|_{2,\omega} .$$

According to Proposition 2.2 and Theorem 2.1, we have

$$\|\phi_n\|_{2,\omega} \leq c_0 \,|[\, u_n, u_n \,]|_{1,\omega} \leq 4c_0 \|u_n\|^2 \leq 4c_0c \|u_n\| \leq c_1 \|u_n\| .$$

Since u_{n+1} is a solution of $(\mathbb{P}_{n+1})$, Proposition 2.3, Proposition 2.2 and Theorem 2.1 imply that there exists $0 < c_1 = 2c_0c^2 + 4 \|F_0\|_{4,\omega} < 1$ such that

$$\|(u_{n+1}, \theta_{n+1})\|_* \leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + \textstyle\int_0^T (\|F_1(u_n, \phi_n)\|_{(L^2(\omega))^2}$$

$$+p)^2 \leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + 2 \textstyle\int_0^T \Big(\|F_1(u_n, \phi_n)\|^2_{(L^2(\omega))^2} ,$$

$$+ \left|p\right|^2_{2,\omega} \Big) \leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + 2 \textstyle\int_0^T c_1^2 \|u_n\|^2 + 2T \left|p\right|^2_{2,\omega}),$$

$$\leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} + 2 \textstyle\int_0^T c_1 \|u_n\|^2 + 2T \left|p\right|^2_{2,\omega}),$$

$$\leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} + 2 \textstyle\int_0^T c_1(\|u_n\|^2 + 2T \left|p\right|^2_{2,\omega}),$$

$$\leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} + 2Tc_1 \left\|u^1\right\|^2_{2,\omega} + 2T \left|p\right|^2_{2,\omega}).$$

If we choose $c > 0$ sufficiently small, then $0 < c_1 < 1,\ 0 < c_2 := 2e^Tc_1 < 1$, and we have

$$\|(u_{n+1}, \theta_{n+1})\|_* \leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} + 2T \left|p\right|^2_{2,\omega}) + c_2 \|u_1\|^2_{2,\omega} ,$$

and we can choose

$$\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + 2T \left|p\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} \leq \frac{(1 - c_2)}{e^T} \|u_1\|^2_{2,\omega} .$$

We have

$$\|u_{n+1}\|_\alpha = \|u_{n+1}\|^2 + \alpha \left|\nabla (u_{n+1})_t\right|^2_{2,\omega} + \left|(u_{n+1})_t\right|^2_{2,\omega} \leq \|(u_{n+1}, \theta_{n+1})\|_*$$

and

$$\|\phi_n\|_{2,\omega} \leq c_1 \|u_n\|_{2,\omega} \leq \|u_1\|_{2,\omega} .$$

It follows that

$$\begin{aligned} \|u_{n+1}\|_\alpha &\leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k \left|\theta_0\right|^2_{2,\omega} + 2T \left|p\right|^2_{2,\omega}) + c_2 \|u_1\|^2_{2,\omega} , \\ &\leq e^T \tfrac{(1-c_2)}{e^T} \|u_1\|^2_{2,\omega} + c_2 \|u_1\|^2_{2,\omega} = \|u_1\|^2_{2,\omega} . \end{aligned}$$

Further, we have

$$\|\phi_{n+1}\|_{2,\omega} \leq c_0 \, |[\, u_{n+1}, u_{n+1} \,]|_{1,\omega} \,,$$

which, with $\|u_1\|_{2,\omega} < c$ and $0 < 4c_0c < 1$, immediately yields

$$\|\phi_{n+1}\|_{2,\omega} \leq 4c_0 \, \|u_{n+1}\|^2 \leq 4c_0 \, \|u_1\|^2_{2,\omega} \leq 4c_0 c \, \|u_1\|_{2,\omega} \leq \|u_1\|_{2,\omega} \,.$$

Summarizing, we have proved that, for all $n \geq 1$ and any $\forall 0 \leq t \leq T$, we have

$$\|u_n\|_{\alpha} \leq \|u_1\|^2_{2,\omega} \quad \text{and} \quad \|\phi_n\|_{2,\omega} \leq \|u_1\|_{2,\omega} \,.$$

Moreover, we have

$$k \, |\theta_n|^2_{2,\omega} + 2\eta \int_0^t |\nabla \theta_n|^2_{2,\omega} \leq \|(u_n, \theta_n)\|_* \leq \|u_1\|^2_{2,\omega} \,.$$

Step 2: For $n \geq 2$, let u_n, θ_n be the solution of $(\mathbb{P}_n)$.

Let $2 \leq m \leq n$, then it is easy to see that $\theta_n - \theta_m$ and $u_n - u_m$ are solutions of the following problem:

$$\begin{cases} (1-\alpha\Delta)(u_n - u_m)_{tt} + \Delta^2(u_n - u_m) + \mu\Delta(\theta_n - \theta_m) = F_1(u_{n-1}, \phi_{n-1}) & \\ \qquad\qquad -F_1(u_{m-1}, \phi_{m-1}) & in \quad \omega \times [0,T]\,, \\ k((\theta_n)_t - (\theta_m)_t) - \eta\Delta(\theta_n - \theta_m) = \mu\Delta((u_n)_t - (u_n)_t) & in \quad \omega \times [0,T]\,, \\ u_n - u_m = \theta_n - \theta_m = \partial_\nu(u_n - u_m) = 0 & on \quad \Gamma \times [0,T]\,, \\ (u_n - u_m)_{|t=0} = ((u_n)_t - (u_m)_t)_{|t=0} = ((\theta_n)_t - (\theta_m)_t)_{|t=0} = 0 & in \quad \omega. \end{cases}$$

According to Proposition 2.2 and Theorem 2.1 we deduce, for all $0 \leq t \leq T$,

$$\|(\phi_{n-1} - \phi_{m-1})\|_{2,\omega} \leq 4c_0 c \, \|u_{n-1} - u_{m-1}\| \,.$$

Using Proposition 2.3 and Proposition 2.2, again we have, with $0 < c_3 = Te^T c_1 < 1$,

$$\begin{aligned} \|(u_n - u_m, \theta_n - \theta_m)\|_* \quad &\leq e^T \textstyle\int_0^T |F_1(u_{n-1}, \phi_{n-1}) - F_1(u_{m-1}, \phi_{m-1})|^2_{(L^2(\omega))^2} \,, \\ &\leq e^T \textstyle\int_0^t c_1 \, \|u_{n-1} - u_{m-1}\|^2 \,. \end{aligned}$$

It follows that

$$\begin{aligned} \|(u_n - u_m, \theta_n - \theta_m)\|_* \; &\leq c_3 \textstyle\int_0^t \|(u_{n-1} - u_{m-1}, \theta_{n-1} - \theta_{m-1})\|_* \\ &\leq (c_3)^{m-2} \textstyle\int_0^t \cdots \int_0^t \Big(\|(u_{n-m+2} - u_1, \theta_{n-m-2} - \theta_1\|_* \\ &\leq (c_3)^{m-2} \textstyle\int_0^t \cdots \int_0^t \sum_{k=0}^{n-m+1} (c_3)^k \int_0^t \cdots \int_0^t \|(u_2 - u_1 \theta_2 - \theta_1)\|_* \\ &\leq (c_3)^{m-2} \textstyle\int_0^t \cdots \int_0^t \sum_{k=0}^{n-m+1} (c_3)^k \int_0^t \cdots \int_0^t \Big(\|(u_2, \theta_2)\|_* \\ &+ \|(u_1, \theta_1)\|_* \Big) \leq (c_3 T)^{m-2} \textstyle\sum_{k=0}^{n-m+1} (c_3 T)^k \big(2 \, \|u_1\|^2_{2,\omega} \big) \end{aligned}$$

and

$$\int_0^T \|(u_n - u_m, \theta_n - \theta_m)\|_*^2 \leq T(c_3T)^{m-2} \textstyle\sum_{k=0}^{n-m+1} (c_3T)^k (2\|u_1\|_{2,\omega}^2).$$

And so we have

$$\|\phi_n - \phi_m\|_{2,\omega} \leq 4c_0c \|u_n - u_m\|.$$

The sequence $(u_n, \phi_{n-1})_{n\geq 2}$ is a Cauchy sequence in $H_0^2(\omega) \times H_0^2(\omega)$ and $(u_n)_{n\geq 2}$ is also a Cauchy sequence in $W(0,T)$. It follows that (u_n, ϕ_{n-1}) converges to (u, ϕ) in $H_0^2(\omega) \times H_0^2(\omega)$, $(u_n)_t$ converges to $(u)_t$ in $L^2(\omega)$ and $\nabla(u_n)_t$ converges to ∇u_t in $L^2(\omega)$. We then have $\Delta^2(u_n, \phi_{n-1})$ weakly converges to $\Delta^2(u, \phi)$ in $L^2(\omega) \times L^2(\omega)$.

Step 3: Using the inequality (3), we have

$$k\,|\theta_{n-1} - \theta_{m-1}|_{2,\omega}^2 \;+\; \eta \int_0^t |\nabla(\theta_{n-1} - \theta_{m-1})|_{2,\omega}^2 \;\;\leq\;\; \mu \int_0^t (|\nabla(u_{n-1} - u_{m-1})_t|_{2,\omega})^2.$$

We deduce that θ_n is a Cauchy sequence in $L^2([0,T], H_0^1(\omega))$, then θ_n converges to θ in $L^2([0,T], H_0^1(\omega))$. By Proposition 2.2, we have $F_1(u_{n-1}, \phi_{n-1})$ converges to $F_1(u, \phi)$ in $(L^2(\omega))^2$.

Since the operator "trace" is continuous, for all $n \geq 2$, we have $(u_n, \phi_{n-1})_\Gamma = (\partial_\nu u_n, \partial_\nu \phi_{n-1}) = (0,0)$ and so $(u, \phi)_\Gamma = (\partial_\nu u, \partial_\nu \phi) = (0,0)$.

Thanks to Theorem 2.3, we have $(u_n, (u_n)_t) \in C([0,T], H_0^2(\omega) \times H_0^1(\omega))$ with $(u_n)_{|t=0} = u_0$, $((u_n)_t)_{|t=0} = u_1$, which implies that $(u)_{|t=0} = u_0$, $((u)_t)_{|t=0} = u^1$. By the assumption $(u_0, u^1) \in H_0^2(\omega) \times H_0^1(\omega)$, we have $u_n \in C^0([0,T], H_0^2(\omega))$ and $(u_n)_{n\geq 2}$ converges to u in $W(0,T)$.

Let $v \in L^2\big([0,T], H_0^2(\omega)\big)$ be such that $v_t \in L^2\big([0,T], L^2(\omega)\big)$, $(1-\alpha\Delta)v_{tt} + \Delta^2 v \in L^2\big([0,T], H^{-2}(\omega)\big)$, $v(x_1, x_2, T) = 0$ and $v_t(x_1, x_2, T) = 0$. Since u_n is a solution of (P_n), by virtue of the transposition theorem, see [4], we deduce that

$$\int_0^T \int_\omega u_n((1-\alpha\Delta)v_{tt} + \Delta^2 v) = \int_0^T \int_\omega F(u_{n-1}, \phi_{n-1}, \theta_{n-1})v + \int_\omega u^1 v(0) - \int_\omega u_0 v_t(0) +$$
$$\alpha \int_\omega (-\nabla(u_t)_n(T)\nabla v(T) + \nabla u^1 \nabla v(0)) + \alpha \int_\omega (\nabla u_n(T)\nabla v_t(T) - \nabla u_0 \nabla v_t(0)).$$

We have u_n converges to u in $H_0^2(\omega)$, then

$$\int_0^T \int_\omega u_n((1-\alpha\Delta)v_{tt} + \Delta^2 v) \;\text{ converges to }\; \int_0^T \int_\omega u((1-\alpha\Delta)v_{tt} + \Delta^2 v),$$

and using Proposition 2.2, with

$$\int_0^T \int_\omega F(u, \phi, \theta) = \int_0^T \int_\omega F_1(u, \phi, \theta) + \mu \int_0^T \int_\omega \nabla\theta\nabla u + p,$$

we deduce that

$$\int_0^T \int_\omega F(u_{n-1}, \phi_{n-1}, \theta_{n-1})v \;\text{ converges to }\; \int_0^T \int_\omega F(u, \phi, \theta)v,$$

and so we have

$$\int_0^T \int_\omega u((1-\alpha\Delta)v_{tt} + \Delta^2 v) = \int_0^T \int_\omega F(u,\phi,\theta)v + \int_\omega u^1 v(0) - \int_\omega u_0 v_t(0) +$$
$$\alpha \int_\omega (-\nabla u_t(T)\nabla v(T) + \nabla u^1 \nabla v(0)) + \alpha \int_\omega (\nabla u(T)\nabla v_t(T) - \nabla u_0 \nabla v_t(0)).$$

By the transposition theorem, we obtained that u is a solution of the problem $(\mathbb{S})_1$.

In summary, we have proved that (u, ϕ, θ) is a solution of the thermoelastic von Karman evolution.

Step 4: We now prove the uniqueness. Assume that there exist two solutions (u^1, ϕ^1, θ^1) and (u^2, ϕ^2, θ^2) in $L^2([0,T], H_0^2(\omega) \times H_0^2(\omega) \times H_0^1(\omega))$ such that, for some $c > 0$ being sufficiently small, we have $\left\|u^1\right\|_{W(0,s_0)} \leq c$ and $\left\|u^2\right\|_{W(0,s_0)} \leq c$.
This implies that $u^1 - u^2$ and $(\theta^1 - \theta^2)$ satisfies the following problem:

$$(\mathbb{P}_3)\begin{cases} (1-\alpha\Delta)(u^1-u^2)_{tt} + \Delta^2(u^1-u^2) = F(u^1,\phi^1,\theta^1) & \\ \qquad\qquad -F(u^2,\phi^2,\theta^2) & in \ \ \omega \times [0,T], \\ k(\theta^1-\theta^2)_t - \eta\Delta(\theta^1-\theta^2) = \mu\Delta(u^1-u^2)_t & in \ \ \omega \times [0,T], \\ u^1-u^2 = \partial_\nu(u^1-u^2) = \theta^1-\theta^2 = 0 & on \ \ \Gamma \times [0,T], \\ u^1(x_1,x_2,0) - u^2(x_1,x_2,0) = 0 & in \ \ \omega, \\ (u^1)_t(x_1,x_2,0) - (u^2)_t(x_1,x_2,0) = 0 & in \ \ \omega, \\ (\theta^1)_t(x_1,x_2,0) - (\theta^2)_t(x_1,x_2,0) = 0 & in \ \ \omega, \end{cases}$$

which means that $(u^1 - u^2, \theta^1 - \theta^2)$ is a solution of the problem $(\mathbb{P}_3)$. Proposition 2.2, Proposition 2.3 and Theorem 2.1 ensure that there exists $c_0 > 0$ such that

$$\left\|(u^1-u^2,\theta^1-\theta^2)\right\|_* \leq e^T \int_0^T \left|F_1(u^1,\phi^1) - F_1(u^2,\phi^2)\right|^2_{(L^2(\omega))^2}$$
$$\leq e^T \int_0^T c_1 \left\|u^1-u^2\right\|^2 \leq e^T c_1 \int_0^T \left\|(u^1-u^2,\theta^1-\theta^2)\right\|_*.$$

Since c is small and thus $0 < c_3 = Te^T c_1 < 1$, it follows that

$$\int_0^T \left\|(u^1-u^2,\theta^1-\theta^2)\right\|_* \leq c_3 \int_0^T \left\|(u^1-u^2,\theta^1-\theta^2)\right\|_*,$$

which, with $0 < c_3 < 1$, immediately yields $\forall 0 \prec t \prec T$, $u^1 = u^2$ in ω, $\phi^1 = \phi^2$ in ω and $\theta^1 = \theta^2$ in ω.

We conclude that the dynamic von Karman equation coupled with thermal dissipation, without rotational inertia, has one and only one weak solution (u, ϕ, θ) in $L^2\Big([0,T], H_0^2(\omega) \times H_0^2(\omega) \times H_0^1(\omega)\Big)$. The proof of the theorem is completed.

Proposition 3.1 *Let* $(u,\phi,\theta)\in L^2\big([0,T],H_0^2(\omega)\times H_0^2(\omega)\times H_0^1(\omega)\big)$ *be the unique solution of* $(\mathbb{P}_0)$*. Then the following equalities:*

$$\widetilde{E}\big(u(t),u_t(t),\phi\big)+\frac{k}{2}\,|\theta|^2_{2,\omega}-\eta\int_0^t|\nabla\theta_t|^2_{2,\omega}=\widetilde{E}_1(u_0,u^1,\phi_0)+\frac{k}{2}\,|\theta_0|^2_{2,\omega}\,,$$

with

$$\widetilde{E}\big(u(t),u_t(t),\phi\big)=\frac{1}{2}\big(\,|u_t|^2_{2,\omega}+\|u\|^2_{2,\omega}+\alpha\,|\nabla u_t|^2_{2,\omega}\,\big)+\frac{1}{4}\int_\omega\big(\,|\Delta\phi|^2-2\,[u,F_0]\,u-4pu\big)$$

and

$$\widetilde{E}_1(u_0,u^1,\phi_0)=\frac{1}{2}\big(\,\left|u^1\right|^2_{2,\omega}+\alpha\left|\nabla u^1\right|^2_{2,\omega}+\|u_0\|^2_{2,\omega}\big)+\frac{1}{4}\int_\omega(|\Delta\phi_0|^2-2\,[u_0,F_0]\,u_0-4pu_0)$$

hold for any $0\le t\le T$*. Here* $\phi_0\in H_0^2(\omega)$ *is the unique solution of the equation* $\Delta^2\phi_0=-\,[u_0,u_0]$*.*

Proof. According to Theorem 2.3, for any $\forall 0\le t\le T$, u satisfies the following energy equality:

$$\begin{aligned}E_0(u(t),u_t(t))&=E_0(u_0,u^1)+\int_0^t\int_\omega F(u,\phi,\theta)u_t\\&=E_0(u_0,u^1)+\int_0^t\int_\omega[u,\phi+F_0]\,u_t-\mu\int_0^t\int_\omega\Delta\theta u_t+\int_0^t\int_\omega p(x_1,x_2)u_t.\end{aligned}$$

First we have

$$\int_0^t\int_\omega p(x_1,x_2)u_t=\int_\omega p(x_1,x_2)u(t)-\int_\omega p(x_1,x_2)u_0.$$

Otherwise, see [1], one has, with $\Delta^2\phi=[u,u]$,

$$\begin{aligned}\int_0^t\int_\omega[u,\phi+F_0]\,u_t&=\int_0^t\int_\omega[u,\phi]\,u_t+\int_0^t\int_\omega[u,F_0]\,u_t,\\&=\frac{1}{2}\int_0^t\int_\omega\frac{d}{dt}([u,u]\,\phi)+\frac{1}{2}\int_0^t\int_\omega\frac{d}{dt}([u,F_0]\,u),\\&=-\frac{1}{4}\int_\omega|\Delta\phi|^2+\frac{1}{4}\int_\omega|\Delta\phi_0|^2+\frac{1}{2}\int_\omega[u,u]\,F_0-\frac{1}{2}\int_\omega[u_0,u_0]\,F_0\end{aligned}$$

and

$$\begin{aligned}\mu\int_0^t\int_\omega\Delta\theta u_t=\mu\int_0^t\int_\omega\theta\Delta u_t&=\frac{k}{2}\int_0^t\frac{d}{dt}\,|\theta|^2_{2,\omega}-\eta\int_0^t|\nabla\theta|^2_{2,\omega}\\&=\frac{k}{2}\,|\theta|^2_{2,\omega}-\frac{k}{2}\,|\theta_0|^2_{2,\omega}-\eta\int_0^t|\nabla\theta|^2_{2,\omega}\,.\end{aligned}$$

Finally, we conclude that

$$\widetilde{E}\big(u(t), u_t(t), \phi\big) + \frac{k}{2}\,|\theta|^2_{2,\omega} - \eta\int_0^t |\nabla\theta_t|^2_{2,\omega} = \widetilde{E}_1(u_0, u^1, \phi_0) + \frac{k}{2}\,|\theta_0|^2_{2,\omega}\,.$$

Remark 3.1 In this section, we described an iterative method for constructing a unique weak solution, this method is a very good tool to illustrate this solution from a numerical point of view.

4 Numerical Application

This section displays a numerical resolution in terms of the previous theoretical study.

4.1 Preliminaries

Let ω be defined by

$$\omega =]0,1[\times]0,1[\subset \mathbb{R}^2$$

and $T > 0$. In order to solve numerically the problem $(\mathbb{P}_0)$, we introduce a uniform mesh of width h. Let ω_h be the set of all mesh points inside ω with the internal points

$$x_i = ih,\;\; y_j = jh,\quad i,j = 1,...N-1,\;\; h = \frac{1}{N+1},\;\; \Delta t = \frac{1}{T}.$$

Let $\overline{\omega}_h$ be the set of boundary mesh points and u_h be the finite-difference approximation of u. In [7], Bilbao presented a numerical study of the convergence and stability of the conservative finite difference schemes for the dynamic von Karman plate equations via energy conserving methods.

For approaching the weak unique solution of the dynamic nonlinear plate coupled with structural acoustic model, we will utilize the following discrete model of the von Karman evolution developed by Bilbao and Pereira in [7, 8]:

$$(*)\begin{cases} (1-\alpha(\delta_x^2+\delta_y^2))\delta_t^2 u_{ij}^n + \mu(\delta_x^2+\delta_y^2)\theta_{ij}^n + \Delta_h^2 u_{ij}^n = [\; u_{ij}^n\;\, v_{ij}^n + F_{ij}\;] + p_{ij} & in \quad \omega_h,\\ k\delta_t\theta_{ij}^n - \eta(\delta_x^2+\delta_y^2)\theta_{ij}^n - \mu\delta_t(\delta_x^2+\delta_y^2)u_{ij}^n = 0 & in \quad \omega_h,\\ \Delta_h^2 v_{ij}^n = -[\; u_{ij}^n\;\, u_{ij}^n\;] & in \quad \omega_h,\\ u_{ij}^0 = (\varphi_0)_{ij},\; \delta_t u_{ij}^0 = (\varphi_1)_{ij},\; \theta_{ij}^0 = (\theta_0)_{ij} & in \quad \omega_h,\\ u_{ij}^n = v_{ij}^n = \theta_{ij}^n = 0 & on \quad \overline{\omega_h},\\ \partial_\nu u_{ij}^n = \partial_\nu v_{ij}^n = 0 & on \quad \overline{\omega_h}, \end{cases}$$

with the following discrete differential operators:

$$\delta_t^2 u_{ij}^n = \frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{(\Delta t)^2},$$

$$\delta_t u_{ij}^n = \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t},$$

$$\Delta_h^2 u_{ij}^n = h^{-4} [\, u_{ij-2} + u_{ij+2} + u_{i-2j} + u_{i+2j} - 8(u_{ij-1} + u_{ij+1} + u_{i-1j} + u_{i+1j}) + 2(u_{i-1j-1} + u_{i-1j+1} + u_{i+1j-1} + u_{i+1j+1}) - 20u_{ij} \,],$$

$$\delta_x^2 u_{ij}^n = \frac{u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n}{(h)^2},$$

$$\delta_y^2 u_{ij}^n = \frac{u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n}{(h)^2},$$

$$\delta_{xy}^2 u_{ij}^n = \frac{u_{i+1j+1}^n - u_{i+1j-1}^n - u_{i-1j+1}^n + u_{i-1j-1}^n}{(2h)^2},$$

$$[\, u_{ij}^n,\ v_{ij}^n \,] = \delta_x^2 u_{ij}^n \delta_y^2 v_{ij}^n - 2\delta_{xy}^2 u_{ij}^n \delta_{xy}^2 v_{ij}^n + \delta_y^2 u_{ij}^n \delta_x^2 v_{ij}^n.$$

We have transformed the above problem to the numerical resolution in two steps itemized as follows.
First step: We use the numerical procedure of 13-point formula of finite difference developed by Gubta in [9] for illustrating the weak solution of the following biharmonic problem:

$$\begin{cases} \Delta^2 v = f_1 & in \quad \omega, \\ v = g_1 & on \quad \Gamma, \\ \partial_\nu v = g_2 & on \quad \Gamma. \end{cases}$$

Second step: According to the first and second steps, we use the discrete model of the von Karman evolution $(*)$ for illustrating the unique solution of the structural interaction model coupled with the dynamic von Karman evolution.

4.2 Non-coupled approach

In [9], Gubta presented a numerical analysis of the finite-difference method for solving the biharmonic equation. Such method is known as the non-coupled method of 13-point formula of finite difference.

Proposition 4.1 *[9] The 13-point approximation of the biharmonic equation for approaching the unique solution v of the problem (P) is defined by*

$$(1) \begin{cases} L_h v_{ij} = & h^{-4} [v_{ij-2} + v_{ij+2} + v_{i-2j} + v_{i+2j} - 8(v_{ij-1} + v_{ij+1} + v_{i-1j} + v_{i+1j}) \\ & +2(v_{i-1j-1} + v_{i-1j+1} + v_{i+1j-1} + v_{i+1j+1}) - 20v_{ij}] = f_1(x_i, y_j) \end{cases}$$

for $i, j = 1, 2, ..., N-1$, where we set $v_{ij} = v(x_i, y_j)$.

When the mesh point (x_i, y_j) is adjacent to the boundary $\overline{\omega}_h$, then the undefined values of v_h are conventionally calculated by the following approximation of $\partial_\nu v$:

$$v_{i-2,j} = \tfrac{1}{2} v_{i+1,j} - v_{ij} + \tfrac{3}{2} v_{i-1,j} - h(\partial_x v)_{i-1,j},$$

$$v_{i,j-2} = \tfrac{1}{2} v_{i,j+1} - v_{ij} + \tfrac{3}{2} v_{i,j-1} - h(\partial_y v)_{i,j-1},$$

$$v_{i+2,j} = \tfrac{1}{2} v_{i+1,j} - v_{ij} + \tfrac{3}{2} v_{i-1,j} - h(\partial_x v)_{i+1,j},$$

$$v_{i,j+2} = \tfrac{1}{2} v_{i,j+1} - v_{ij} + \tfrac{3}{2} v_{i,j-1} - h(\partial_y v)_{i,j+1}.$$

4.3 Numerical test

We consider the following analytical body force and lateral forces:

$$F_0(x,y) = ye^{-x^2-y^2}, \quad p(x,y) = 0.01x(x-y)e^{-x^2-y^2},$$

$$\varphi_0 = 1510^{-6}x^2y^2(x-y-1)^2(y-1)^2e^{-x^2-y^2}, \quad \varphi_1 = 1510^{-6}(\sin(\pi x)\sin(\pi y))^2,$$

$$\theta_0 = 10^{-13}x^2y^3(x-1)^2(y-1)^2(e^{-x^2} - e^{-y^2}).$$

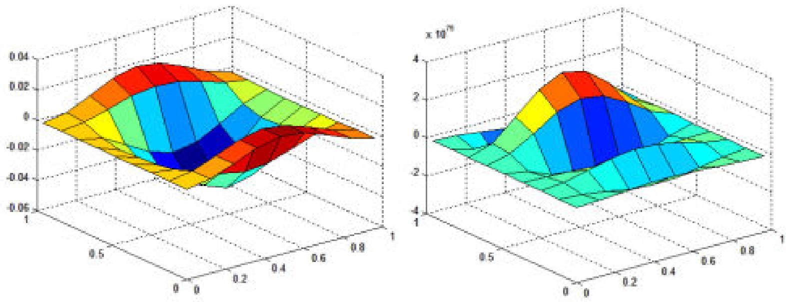

Figure 1: The thermal function θ, $t_1 = 0.2s$ and $t_7 = 60s$.

5 Conclusion

In this paper, we described an iterative method for constructing a unique weak solution to the model of dynamic von Karman equations with a flexible phenomenon of small nonlinear vibration of displacement in nonlinear oscillation of elastic plate, with rotational terms and not clamped boundary conditions subject to thermal dissipation. Our approach is in fact a good tool for justifying the theoretical results. We then use the method of finite difference for approaching the unique solution of the theoretical problem. These results have potential for application in the fields of physics. Similar study for the models of dynamic von Karman equations with thermal dissipation and for free boundary conditions of the shell could be the purpose for future research.

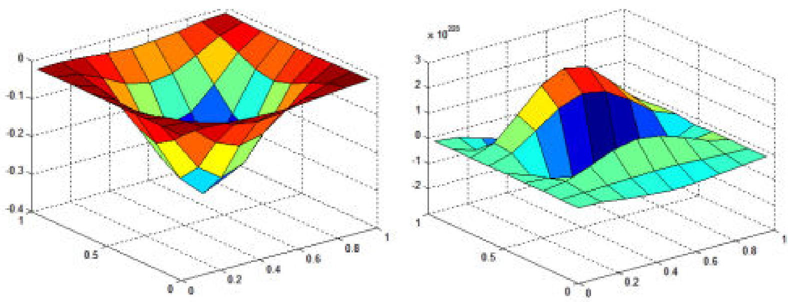

Figure 2: Displacement of plate, $t_1 = 0.2s$ and $t_7 = 60s$.

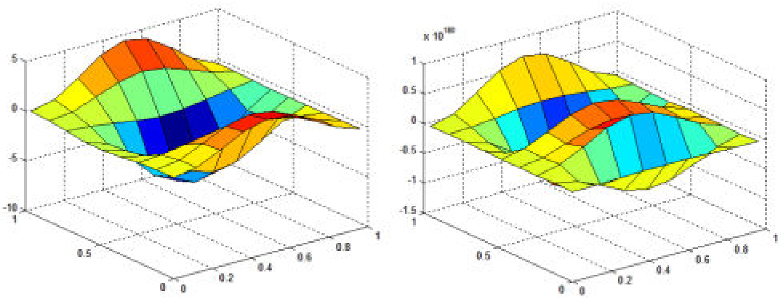

Figure 3: The Airy stress function, $t_1 = 0.2s$ and $t_6 = 32s$.

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