Nonlinear Dynamics and Systems Theory, **22** (3) (2022) 281–290

The Effects of Pesticide as Optimal Control of Agriculture Pest Growth Dynamical Model

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Received: August 18, 2021; Revised: July 3, 2022

Abstract: Indonesia has tropical climate so that many crops can be harvested. One of agricultural problems is the agricultural pest (*Nilaparvata lugens*) in a rice field. This pest can be devastated by the natural predator spider (*Lycosa pseudoannulata*). To reduce the number of pests, we use pesticide as a control which is applied in the pest population. For the problem, we can construct the model as a predator-prey model with the pest as the prey and the spider as the natural predator. This paper discusses stability analysis and optimal control of the agricultural pest growth dynamical model by pesticide. In the agricultural pest dynamical model, there are populations of pests and spiders. From the mathematical model of agricultural pest growth, we obtain three equilibrium points. We will analyze the stability of each equilibrium point by using the eigenvalue. In this paper, for the original mathematical model of agricultural pest growth, we will introduce a control variable, i.e., pesticide. Then we will formulate an optimal control problem. The forward-backward sweep method is employed to solve the optimal control problem and to obtain the numerical solutions. According to simulation results, pesticide usage can minimize the number of pests achieving the minimum performance index.

Keywords: *optimal control; pesticide; pest growth dynamical model.*

Mathematics Subject Classification (2010): 49-06, 49M30, 93D05, 49J15.

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1 Introduction

Indonesia has a tropical climate so that many crops can be harvested. Indonesia has large fertile lands for rice fields and rice is one of the primary foods in Indonesia. One of agricultural problems is the agricultural pest (*Nilaparvata lugens*) in the rice field. This pest can be devastated by using the natural predator such as the spider (*Lycosa pseudoannulata*).

The behavior of the agricultural pest and the natural predator can be constructed as a predator-prey mathematical model. In a predator-prey model, there are pest and predator populations. The pest is the attacking organism damaging crops, while the predator is the eating organism consuming the pest [1]. In this research, the agricultural pest (*Nilaparvata lugens*) and the natural predator spider (*Lycosa pseudoannulata*) will be included.

In the previous research, there were numerous works on modelling of diseases, for example, influenza [2,3], bird flu [4], dengue fever [5,6], cancer [7] and Corona virus [8]. Generally, the mathematical model of spreading diseases divides the population into several subpopulations such as the susceptible population, infected population, and recovered population [9–11]. For the three subpopulations, we can determine the reproduction number and the stability by using the available parameters. Let us mention that the predator-prey model has been used for determining the stability in the case of natural selection [12].

In order to reduce the number of pests, we use pesticide as a control variable which is applied in the pest population. However, the usage of pesticide should be proportional. Using more pesticide causes side effects on crops and high cost of pesticide. On the other hand, using less pesticide causes pest growth. For the problem, we can construct a predator-prey model with the pest as the prey and the spider as the natural predator. This paper focuses on the stability analysis and optimal control of the agricultural pest growth dynamical model by using pesticide as the control variable.

In the agricultural pest dynamical model, there are two subpopulations: pest and spider. From the mathematical model of agricultural pest growth, we obtain three equilibrium points. We will analyze the stability of each equilibrium point by using the eigenvalue. The first equilibrium point is unstable, whereas the second and third equilibriums are stable, which depends on certain conditions. From the preceding mathematical model of agricultural pest growth, we introduce a control variable that represents pesticide. Next, we formulate an optimal control problem: the objective function and the constraints. We use the forward-backward sweep method to obtain the solution of the optimal control problem and to compute the numerical solutions. This method leverages the state variables with certain initial condition and adjoint variables with certain final condition [13]. According to the simulation results, pesticide usage can minimize the number of pests achieving the minimum performance index.

2 Mathematical Model of Agricultural Pest Growth

In the mathematical model of agricultural pest growth, there are two populations used, namely, the agricultural pest (*Nilaparvata lugens*) as the prey and the spider (*Lycosa pseudoannulata*) as the natural predator. This model can be constructed as a predator-prey model.

2.1 Mathematical model

The mathematical model of agricultural pest growth with the functional response of Holling $\frac{\gamma SP}{a+S_0}$ and the denominator being a constant value, can be constructed as follows [1]:

$$\dot{S} = rS\left(1-\frac{S}{K}\right) - \frac{\gamma SP}{a+S_0}, \tag{1}$$

$$\dot{P} = \frac{\alpha\gamma SP}{a+S_0} - \delta P \tag{2}$$

with the following parameters:

$S(t)$	:	the population of the pest (*Nilaparvata lugens*) as the prey,
$P(t)$	:	the population of the spider (*Lycosa pseudoannulata*) as the natural predator,
r	:	intrinsic rate of growth of the pest as the prey,
K	:	environmental carrying capacity of the pest as the prey population,
γ	:	search rate of the pest as the prey by the predator,
α	:	conversion factors,
δ	:	natural death rate of predators,
a	:	half saturation constant.

From the model, we conclude the following conditions. Without the existence of predators, the pest as the prey grows based on the logistic function, and without the existence of the pest as the prey, the predators go away. Based on the natural selection, the existence of the pest as the prey will increase the predator, and the existence of predators will decrease the pest as the prey.

2.2 Existence of solutions

The solutions of this problem exist if the populations of predators and preys are greater than or equal to zero, i.e., $S(t) \geq 0$, $P(t) \geq 0$. As it will be clear later, the equilibrium points must satisfy these conditions [14].

2.3 Equilibrium points

In order to compute the equilibrium points, we find the solutions of $\dot{S} = 0$, $\dot{P} = 0$ as follows:

$$rS\left(1-\frac{S}{K}\right) - \frac{\gamma SP}{a+S_0} = 0, \tag{3}$$

$$\frac{\alpha\gamma SP}{a+S_0} - \delta P = 0. \tag{4}$$

By using simple algebraic manipulations, from (3) and (4), we obtain the following equilibrium points:

1. Equilibrium point 1 : $S_{e1} = 0$, $P_{e1} = 0$;
2. Equilibrium point 2 : $S_{e2} = K$, $P_{e2} = 0$;
3. Equilibrium point 3 : $S_{e3} = \frac{\delta(a+S_0)}{\alpha\gamma}$, $P_{e3} = \frac{(a+S_0)r(\alpha\gamma K - \delta(a+S_0))}{\alpha\gamma^2 K}$.

Next, we analyze the stability of each equilibrium point by using the eigenvalue method of the Jacobian matrix.

2.4 Stability analysis

First of all, we derive the Jacobian matrix from (1) and (2). In order to simplify the notations in the Jacobian matrix, we introduce

$$f_1 = rS\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a + S_0} - \varepsilon uS,$$
$$f_2 = \frac{\alpha\gamma SP}{a + S_0} - \delta P.$$

Then the Jacobian matrix is

$$Jac = \begin{bmatrix} \frac{\partial f_1}{\partial S} & \frac{\partial f_1}{\partial P} \\ \frac{\partial f_2}{\partial S} & \frac{\partial f_2}{\partial P} \end{bmatrix} = \begin{bmatrix} r - 2(\frac{r}{K})S - \frac{\gamma P}{a+S_0} & -\frac{\gamma S}{a + S_0} \\ \frac{\alpha\gamma P}{a + S_0} & \frac{\alpha\gamma S}{a + S_0} - \delta \end{bmatrix}. \tag{5}$$

In order to analyze the stability, we compute the eigenvalue of the Jacobian matrix by using the formula $\det(\lambda I - Jac) = 0$ after substituting the equilibrium points. The equilibrium point is stable if the real parts of all eigenvalues are negative. Based on these conditions, we can conclude that

1. Equilibrium point 1 : $S_{e1} = 0, P_{e1} = 0$ is always unstable;
2. Equilibrium point 2 : $S_{e2} = 0, P_{e2} = 0$ is stable if $\frac{\alpha\gamma K}{\delta(a + S_0)} < 1$;
3. Equilibrium point 3 : $S_{s3} = \frac{\delta(a + S_0)}{\alpha\gamma}$, $P_{e3} = \frac{(a + S_0)r(\alpha\gamma K - \delta(a + S_0))}{\alpha\gamma^2 K}$ is stable if $\frac{\alpha\gamma K}{\delta(a + S_0)} > 1$.

3 Optimal Control of Agriculture Pest Growth

In the optimal control of agricultural pest growth, we introduce a control variable u. The control variable is used to reduce the number of pests. The effectiveness range of the control variable u lies in the interval $[0, 1]$, where the value of 0 represents the failure of control functions or the control functions are not to be applied, and the value of 1 represents the successful control functions or the control functions are applied to the entire population. Therefore, after introducing the control variable u, the mathematical model in (1) and (2) becomes (6) and (7), respectively.

$$\dot{S} = rS\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a + S_0} - \varepsilon uS, \tag{6}$$
$$\dot{P} = \frac{\alpha\gamma SP}{a + S_0} - \delta P \tag{7}$$

with ε being the rate of reducing the pest as the prey due to pesticide.

Now, we formulate an optimal control problem. First, we define an objective function. The objective function is minimizing the number of pests and the cost of pesticides. As such, the objective function is defined as follows:

$$J = \int_T^0 (A_1 S + A_2 u^2)\, dt, \tag{8}$$

where the weights $A_1 > 0$, $A_2 > 0$ are associated with the number of pests as the prey and the cost of pesticides, respectively. The solution is an optimal control u^*.

3.1 Pontryagin's maximum principle

If u is an optimal control associated with the state of the system, then there exist adjoint variables $\begin{pmatrix}\lambda_S & \lambda_P\end{pmatrix}$ that satisfy the following conditions [10]:

$$\begin{aligned}\dot{\lambda}_S = -\frac{\partial H}{\partial S} &= -A_1 - \lambda_S\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a+S_0}\left(r - \frac{2r}{K}S - \frac{\gamma P}{a+S_0} - \varepsilon u\right) \\ &\quad - \lambda_P\left(\frac{\alpha\gamma P}{a+S_0}\right)\end{aligned} \tag{9}$$

$$\dot{\lambda}_P = -\frac{\partial H}{\partial P} = -\lambda_S\left(-\frac{\gamma S}{a+S_0}\right) - \lambda_P\left(\frac{\alpha\gamma S}{a+S_0} - \delta\right) \tag{10}$$

$$\lambda_S(T) = \lambda_P(T) = 0, \tag{11}$$

where the Hamiltonian is

$$H = A_1 S + A_2 u^2 + \begin{pmatrix}\lambda_S & \lambda_P\end{pmatrix}\begin{pmatrix} rS\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a+S_0} - \varepsilon uS \\ \frac{\alpha\gamma SP}{a+S_0} - \delta P\end{pmatrix}. \tag{12}$$

An optimal control u^* is obtained as follows:

$$\frac{\partial H}{\partial u} = 0, \tag{13}$$

$$2A_2 u + \lambda_S(-\varepsilon S) = 0, \tag{14}$$

$$u = \min\left(1, \max\left(0, \frac{\lambda_S\, \varepsilon S}{2A_2}\right)\right). \tag{15}$$

3.2 Forward-backward sweep method

In order to compute the optimal control, we use the forward-backward sweep method. When we apply the method to the optimal control problem of agricultural pest growth, the steps are as follows [15]. Notice that the state and adjoint variables are

$$\begin{aligned} f_1 &= rS\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a+S_0} - \varepsilon uS, \\ f_2 &= \frac{\alpha\gamma SP}{a+S_0} - \delta P, \\ g_1 &= -A_1 - \lambda_S\left(r - \frac{2r}{K}S - \frac{\gamma P}{a+S_0} - \varepsilon u\right) - \lambda_P\left(\frac{\alpha\gamma P}{a+S_0}\right), \\ g_2 &= -\lambda_S\left(-\frac{\gamma S}{a+S_0}\right) - \lambda_P\left(\frac{\alpha\gamma S}{a+S_0} - \delta\right). \end{aligned}$$

Next, we describe the forward-backward sweep method. The method is written as an algorithm so that we can implement the method easily. Here is the complete algorithm:

$$u_{old} = u.$$

1. Calculate the solution of state variables, where the initial conditions are S_0, P_0, by using the Runge-Kutta fourth-order method. For the agricultural pest growth model, the steps are

$$\begin{aligned}
k_{11} &= f_1\left(S_i, P_i, u_i\right), \\
k_{12} &= f_2\left(S_i, P_i, u_i\right), \\
k_{21} &= f_1\left(S_i + \frac{h}{2}k_{11}, P_i + \frac{h}{2}k_{12}, \frac{u_i + u_{i+1}}{2}\right), \\
k_{22} &= f_2\left(S_i + \frac{h}{2}k_{11}, P_i + \frac{h}{2}k_{12}, \frac{u_i + u_{i+1}}{2}\right), \\
k_{31} &= f_1\left(S_i + \frac{h}{2}k_{21}, P_i + \frac{h}{2}k_{22}, \frac{u_i + u_{i+1}}{2}\right), \\
k_{32} &= f_2\left(S_i + \frac{h}{2}k_{21}, P_i + \frac{h}{2}k_{22}, \frac{u_i + u_{i+1}}{2}\right), \\
k_{41} &= f_1\left(S_i + hk_{31}, P_i + hk_{32}, u_{i+1}\right), \\
k_{42} &= f_2\left(S_i + hk_{31}, P_i + hk_{32}, u_{i+1}\right), \\
S_{i+1} &= S_i + \frac{h}{6}\left(k_{11} + 2k_{21} + 2k_{31} + k_{41}\right), \\
P_{i+1} &= P_i + \frac{h}{6}\left(k_{12} + 2k_{22} + 2k_{32} + k_{42}\right).
\end{aligned}$$

2. Calculate the solution of adjoint variables, where the final conditions are $\lambda_{N(T)}, \lambda_{P(T)}$, by using the Runge-Kutta fourth-order method as follows:

$$\begin{aligned}
l_{11} &= g_1\left(\lambda_{S(i)}, \lambda_{P(i)}, S_i, P_i, u_i\right), \\
l_{12} &= g_2\left(\lambda_{S(i)}, \lambda_{P(i)}, S_i, P_i, u_i\right), \\
l_{21} &= g_1\left(\lambda_{S(i)} - \frac{h}{2}l_{11}, \lambda_{P(i)} - \frac{h}{2}l_{12}, \frac{S_i + S_{i-1}}{2}.\frac{P_i + Pi - 1}{2}, \frac{u_i + u_{i-1}}{2}\right), \\
l_{22} &= g_2\left(\lambda_{S(i)} - \frac{h}{2}l_{11}, \lambda_{P(i)} - \frac{h}{2}l_{12}, \frac{S_i + S_{i-1}}{2}.\frac{P_i + Pi - 1}{2}, \frac{u_i + u_{i-1}}{2}\right), \\
l_{31} &= g_1\left(\lambda_{S(i)} - \frac{h}{2}l_{21}, \lambda_{P(i)} - \frac{h}{2}l_{22}, \frac{S_i + S_{i-1}}{2}.\frac{P_i + Pi - 1}{2}, \frac{u_i + u_{i-1}}{2}\right), \\
l_{32} &= g_2\left(\lambda_{S(i)} - \frac{h}{2}l_{21}, \lambda_{P(i)} - \frac{h}{2}l_{22}, \frac{S_i + S_{i-1}}{2}.\frac{P_i + Pi - 1}{2}, \frac{u_i + u_{i-1}}{2}\right), \\
l_{41} &= g_1\left(\lambda_{S(i)} - hl_{31}, \lambda_{P(i)} - hl_{32}, S_{i-1}, P_{i-1}, u_{i-1}\right), \\
l_{42} &= g_2\left(\lambda_{S(i)} - hl_{31}, \lambda_{P(i)} - hl_{32}, S_{i-1}, P_{i-1}, u_{i-1}\right), \\
\lambda_{S(i-1)} &= \lambda_{S(i)} - \frac{h}{6}\left(l_{11} + 2l_{21} + 2l_{31} + l_{41}\right), \\
\lambda_{P(i-1)} &= \lambda_{S(i)} - \frac{h}{6}\left(l_{12} + 2l_{22} + 2l_{32} + l_{42}\right).
\end{aligned}$$

3. Calculate the optimal control u^* using (15).
4. Update the optimal control

$$u \leftarrow \frac{u + u_{old}}{2}. \tag{16}$$

5. Calculate the performance index as the value of objective function

$$J(u) = \sum_{k=0}^{T-1} \left(A_1 S(k)^2 + A_2 u(k)^2 \right). \tag{17}$$

4 Simulation Results

To simulate the closed-loop system, we need to define the parameters. Table 1 describes the parameters used in the simulation.

Table 1: Parameters of optimal control of agricultural pest growth.

Parameters	**Value**
The population of the pest (*Nilaparvata lugens*) as the prey $S(0)$	20
The population of the spider (*Lycosa pseudoannulata*) as the natural predator $P(0)$	10
Intrinsic rate of growth of the pest as the prey r	1
Environmental carrying capacity of the pest as the prey population K	30
Search rate of the pest as the prey by the predator γ	1
Natural death rate of the predator δ	0.6
Half saturation constant a	10
Rate of reducing the pest as the prey due to pesticide ε	5
Weight related to the number of the pest as the prey A_1	1
Weight related to the cost of pesticide A_2	2

The simulation results are applied with two parts because there are three equilibrium points, but an equilibrium (equilibrium of type 1) is unstable. In the first simulation, the conversion factor $\alpha = 0.1$, and in the second simulation, the conversion factor $\alpha = 8$.

4.1 Simulation with equilibrium point of type 2

In this simulation, conversion factors $\alpha = 1$ will be applied. From the results, we obtain

$$\frac{\alpha\gamma K}{\delta(a + S_0)} = \frac{(0.1)(1)(30)}{0.6(10 + 20)} = \frac{3}{18} = 0.167 < 1.$$

At the equilibrium point of type 2, the equilibrium point is stable if $\dfrac{\alpha\gamma K}{\delta(a + S_0)} < 1$. The numerical simulation with the equilibrium point of type 2 can be seen in Figure 1 and Figure 2 (left).

Figure 1 (left) displays the number of agricultural pests (*Nilaparvata lugens*) as preys while Figure 1 (right) displays the number of spiders (*Lycosa pseudoannulata*) as natural predators. In Figure 1 (left), the number of pests as preys with pesticide control is smaller than the number of pests as the prey without control. In Figure 1 (right), the

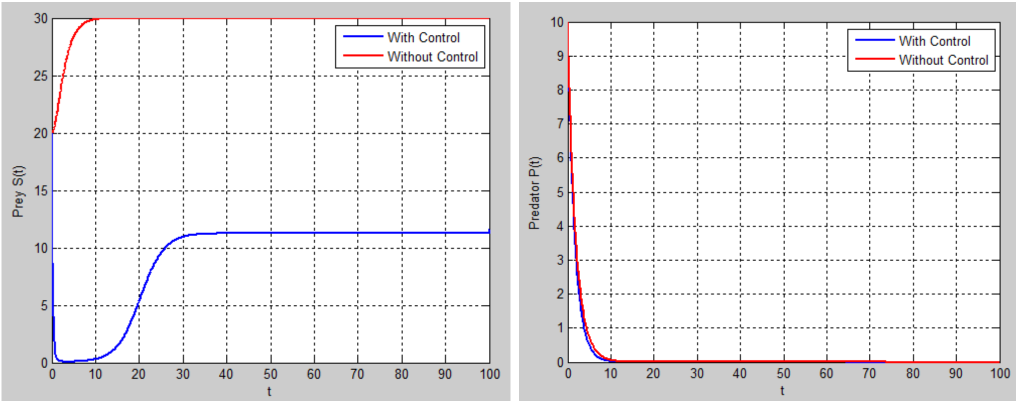

Figure 1: The left panel represents the numerical solution of the number of agricultural pests (*Nilaparvata lugens*) as preys. The right panel denotes the numerical simulation of the number of spiders (*Lycosa pseudoannulata*) as natural predators.

number of predators with pesticide control tends to 0 and is almost similar to the number of predators without control because pesticide is only applied in the pest population.

Figure 2 (left) shows the optimal control of pesticide used. Initially, the pesticides are given to around 95% of the population. Then the number of individuals receiving the pesticides is decreasing. When $t \geq 10$, the pesticides are given to around 55% of the population.

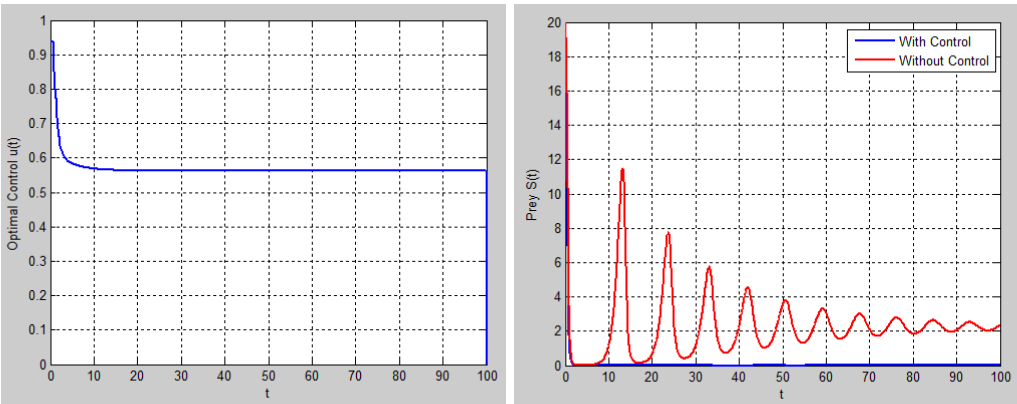

Figure 2: The left panel denotes the optimal control of pesticides. The right panel represents the numerical solution of the number of agricultural pests (*Nilaparvata lugens*) as preys.

4.2 Simulation with equilibrium of type 3

In this simulation, conversion factors $\alpha = 8$ will be applied. From the results, we obtain

$$\frac{\alpha\gamma K}{\delta(a+S_0)} = \frac{(8)(1)(30)}{0.6(10+20)} = \frac{240}{18} = 13.33 > 1.$$

At the equilibrium point of type 3, the equilibrium point is stable if $\dfrac{\alpha\gamma K}{\delta(a+S_0)} > 1$. Numerical simulation with the equilibrium point of type 3 can be seen in Figure 2 (right) and Figure 3.

Figure 2 (right) shows the number of agricultural pests (*Nilaparvata lugens*) as preys, while Figure 3 (left) displays the number of spiders (*Lycosa pseudoannulata*) as natural predators. Both Figure 2 (right) and Figure 3 (left) show fluctuative graphs. When pests as the prey increase, then predators follow the increase, and if pests as the prey decrease, then predators follow the decrease. The pesticide as the control can cause the number of pests as the prey to decrease. However, it also affects predators so that predators also decrease. Figure 3 (right) shows the optimal control of pesticide used. Initially,

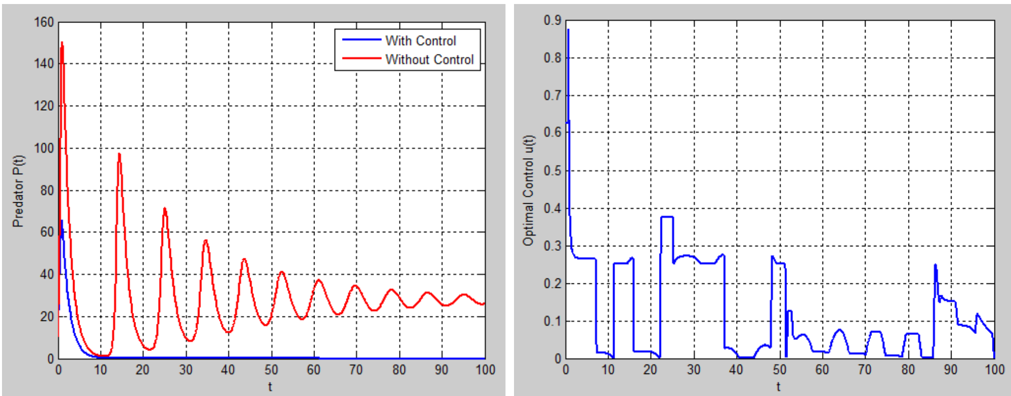

Figure 3: The left panel represents the numerical simulation of the number of spiders (*Lycosa pseudoannulata*) as natural predators. The right panel denotes the optimal control of pesticides.

the pesticides are given to around 88% of the population. After that, the number of individuals receiving the pesticides is decreasing. Starting from $t = 4$, the number of individuals receiving the pesticides is fluctuating between 0 and 3.8.

5 Conclusions

In the agricultural pest dynamical model, there are populations of the pest and the spider. From the mathematical model of agriculture pest growth, we obtain three equilibrium points. We analyze the stability of each equilibrium point by using the eigenvalue. The first equilibrium point is unstable, whereas the second and third equilibriums are stable if certain conditions are satisfied. Furthermore, we have introduced a control variable, which represents pesticide, in the agricultural pest growth model. We have formulated an optimal control problem and solved it numerically. Moreover, we have conducted several simulations to show the effectiveness of the proposed method.

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