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Solvability of Equations with Time-Dependent Potentials

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Abstract: This paper is devoted to the solution of equations with time-dependent potential, at which the heat and wave equations are taken as prototype problems. The method of separating variables failed to be applied to the equations. The well-posedness of the problems is justified by strongly continuous quasi semigroups. The positive solution of the heat equations is conditioned by the maximum principle depending on the potential. For the wave equations, the bounded potentials imply the well-posedness of the problems. Further, firstly approximate solutions can be schemed. The heat and wave equations with specific potentials are considered.

Keywords: *strongly continuous quasi semigroup; heat equation; wave equation; time-dependent potential; well-posedness.*

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1 Introduction

Some phenomena of reaction-diffusion in physical systems have models as equations with time-dependent potentials [1, 2]. In general, they take the forms of nonautonomous Cauchy problems (NCP) on Banach spaces [3–6],

$$\begin{aligned} \dot{u}(t) &= A(t)u(t), \qquad t \geq 0, \\ u(0) &= u_0, \qquad u_0 \in X, \end{aligned} \tag{1}$$

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where u is an unknown function from $[0,\infty)$ into a Banach space X and every $A(t)$ is a densely defined closed linear operator on $\mathcal{D}(A(t)) = \mathcal{D} \subseteq X$, the domain which is independent of t.

A strongly continuous quasi semigroup (C_0-quasi semigroup) is a sophisticated method to characterize the well-posedness of the NCP (1) [7]. Another awesomeness of the C_0-quasi semigroups in analyzing the nonautonomous problems can be found in [8–13]. The method is an alternative to the evolution operator or propagation method in [1–3, 5, 6].

The heat and wave equations with time-dependent potentials are the useful prototypes of the equations of the same types. The variable separation method failed to be applied to solve the heat and wave equations of this type. The heat equation with time-dependent potential on $\mathbb{R}^+ \times \Omega$ has a form

$$u_t = \Delta u - V(t,x)u + f(t,x), \quad (t,x) \in \mathbb{R}^+ \times \Omega, \tag{2}$$

where Δ is the Laplace operator on an open bounded set Ω in $\mathbb{R}^n$. By the evolution operator or propagation, problem (2) have been discussed [14,15]. Moreover, determination of a time-dependent heat transfer known as an inverse time-dependent source problem in various conditions also has been justified [16–21].

The wave equation with time-dependent potential on $\mathbb{R}^+ \times \Omega$ takes a form

$$u_{tt} = \Delta u - V(t,x)u + f(t,x), \quad (t,x) \in \mathbb{R}^+ \times \Omega. \tag{3}$$

Under suitable assumptions on the potential $V(t,x)$, the main goal of (3) is to show the existence of the scattering operator of the propagation [22–25]. In particular, $V(t,x) = V(x)$, the recovery of $V(x)$ and uniqueness results have been investigated [26, 27].

In fact, the procedure used to analyze problems (2) and (3) is very complicated. It seems that it will be simpler if the problems are modeled as NCP (1) and the C_0-quasi semigroup approach is used. Therefore, in this paper, we focus on the solvability of (2) and (3) using the C_0-quasi semigroups. In the preliminaries, we recall the well-posedness of (1) that has been developed in [7]. The main results are the well-posedness of problems (2) and (3).

2 Preliminaries

This work is a continuation of the paper of Sutrima et al. [7]. The paper characterized the well-posedness of the nonautonomous abstract Cauchy problems using a strongly continuous quasi semigroup approach. The characterization centers on the infinitesimal generators of the corresponding quasi-semigroups. Therefore, the materials of this paper involve the well-posedness results in [7].

Definition 2.1 Let $\mathcal{L}(X)$ be the set of all bounded linear operators on a Banach space X. A two-parameter commutative family $\{R(t,s)\}_{s,t\geq 0}$ in $\mathcal{L}(X)$ is called a strongly continuous quasi semigroup (in short C_0-quasi semigroup) on X if:

(a) $R(t,0) = I$, the identity operator on X,

(b) $R(t,s+r) = R(t+r,s)R(t,r)$,

(c) $\lim_{s\to 0^+} \|R(t,s)x - x\| = 0$,

(d) there is a continuous increasing function $M : [0, \infty) \to [1, \infty)$ such that

$$\|R(t, s)\| \leq M(s)$$

for all $r, s, t \geq 0$ and $x \in X$.

The infinitesimal generator of the C_0-quasi semigroup $\{R(t, s)\}_{s,t\geq 0}$ is a family of operators $\{A(t)\}_{t\geq 0}$ on $\mathcal{D}$, where

$$A(t)x = \lim_{s\to 0^+} \frac{R(t, s)x - x}{s}$$

and $\mathcal{D}$ is the set of all $x \in X$ such that the right-hand limits exist.

For simplicity, we denote by $R(t, s)$ and $A(t)$ the quasi semigroup $\{R(t, s)\}_{s,t\geq 0}$ and the infinitesimal generator $\{A(t)\}_{t\geq 0}$, respectively. Further, we always consider the C_0-quasi semigroups whose infinitesimal generator has a dense domain in the Banach spaces.

Let $\mathcal{R}(\lambda, A(t)) := (\lambda - A(t))^{-1}$ be the resolvent operator of $A(t)$ for $\lambda \in \rho(A(t))$, where $\rho(A(t))$ is the resolvent set of $A(t)$. The following result is the version of the Hille-Yosida Theorem for a C_0-quasi semigroup.

Theorem 2.1 (Theorem 2.3 of [7]) *For each $t \geq 0$, let $A(t)$ be a closed and densely defined operator on $\mathcal{D}$ and the map $t \mapsto A(t)y$ is a continuous function from $\mathbb{R}^+$ to X for all $y \in \mathcal{D}$. If $\mathcal{R}(\lambda, A(\cdot))$ is locally integrable and there exist constants $N, \omega \geq 0$ such that $[\omega, \infty) \subseteq \rho(A(t))$ and*

$$\|\mathcal{R}(\lambda, A(t))^r\| \leq \frac{N}{(\lambda - \omega)^r}, \quad \lambda > \omega, \quad r \in \mathbb{N},$$

then $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup.

We recall the well-posedness of the Cauchy problem (1) that has been discussed in [7]. First, we consider the inhomogeneous form of the Cauchy problem (1)

$$\begin{aligned} \dot{u}(t) &= A(t)u(t) + f(t), & t &\geq 0, \\ u(0) &= u_0, & u_0 &\in X, \end{aligned} \tag{4}$$

where f is a continuous function from $[0, \infty)$ to a Banach space X. Let $\mathcal{C}(\Omega, X)$ and $\mathcal{C}^1(\Omega, X)$ denote the set of all continuous functions on Ω and the set of all functions with continuous derivative on Ω, respectively.

Definition 2.2 A function u is called a classical solution of (4) on $[0, \tau]$ if $u \in \mathcal{C}^1([0, \tau], X)$, $u(t) \in \mathcal{D}$ for all $t \in [0, \tau]$ and $u(t)$ satisfies (4) for all $t \in [0, \tau]$. The function u is called a classical solution on $[0, \infty)$ if u is a classical solution on $[0, \tau]$ for each $\tau > 0$.

Therefore, the classical solution of the nonautonomous abstract Cauchy problem (1) is the classical solution of (4) when $f = 0$.

Lemma 2.1 (Lemma 3.2 of [7]) *Let $A(t)$ be the infinitesimal generator of a C_0-quasi semigroup $R(t, s)$ on a Banach space X and $u_0 \in \mathcal{D}$. If $f \in \mathcal{C}([0, \tau], X)$ and u is a classical solution of (4), then $A(\cdot)u(\cdot) \in \mathcal{C}([0, \tau], X)$ and*

$$u(t) = R(0, t)u_0 + \int_0^t R(s, t - s)f(s)ds. \tag{5}$$

Definition 2.3 The nonautonomous abstract Cauchy problem (1) is said to be *well-posed* if it satisfies the following conditions:

(WP1) *Existence.* For each $u_0 \in \mathcal{D}$, there exists a classical solution u of (1) on $[0, \infty)$.

(WP2) *Uniqueness.* If $u, v : [0, \tau] \to X$ are the classical solutions of (1), then $u(t) = v(t)$ for all $t \in [0, \tau]$, $\tau > 0$.

(WP3) *Continuous dependence.* The classical solution x depends continuously on $t \in [0, \infty)$ and $u_0 \in \mathcal{D}$, i.e., the map $\phi : [0, \infty) \times \mathcal{D} \to X$ with $\phi(t, u_0) = u(t)$ is continuous.

The well-posedness of the nonautonomous Cauchy problem (1) is characterized by the existence and uniqueness of the infinitesimal generator of the related C_0-quasi semigroup.

Theorem 2.2 (Theorem 3.6 of [7]) *For each $t \geq 0$, let $A(t) : \mathcal{D} \to X$ be a closed and densely defined operator in a Banach space X. The family $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup on X if and only if the nonautonomous abstract Cauchy problem* (1) *is well-posed.*

We have a similar result on the well-posedness of the inhomogeneous nonautonomous Cauchy problem (4).

Theorem 2.3 (Theorem 3.11 of [7]) *If $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup on X, then the inhomogeneous nonautonomous abstract Cauchy problem* (4) *is well-posed.*

We note that Theorem 2.3 remains valid when f belongs to the Sobolev space $W^{1,p}([0, \infty), X)$, $1 \leq p < \infty$.

3 Results and Discussion

As an auxiliary result, we have a perturbation of the infinitesimal generator of the C_0-quasi semigroups. The following one is a special case of the perturbation.

Theorem 3.1 *If $A(t)$ and B are the infinitesimal generators of a C_0-quasi semigroup $R(t, s)$ and C_0-semigroup $T(s)$ on a Banach space X, respectively, such that $R(t, s)$ and $T(s)$ are commutative, then $A(t) + B$ is the infinitesimal generator of a C_0-quasi semigroup $K(t, s)$ given by*

$$K(t, s) = T(s)R(t, s), \quad s, t \geq 0.$$

Proof. It is easy to show that $K(t, s)$ verifies the definition of a C_0-quasi semigroup. By the Hille-Yosida Theorem for $T(s)$ and the fact that $R(t, s)$ is a C_0-quasi semigroup, there exist constants $N, \omega > 0$ and an increasing function M such that

$$\|K(t, s)\| \leq M_K(s), \quad t, s \geq 0,$$

where $M_K(s) = Ne^{\omega s} M(s)$. For $t \geq 0$ and $x \in \mathcal{D} \cap \mathcal{D}(B)$, the continuity of $T(s)$ gives

$$\lim_{s \to 0^+} \frac{K(t, s)x - x}{s} = \lim_{s \to 0^+} T(s) \frac{R(t, s)x - x}{s} + \lim_{s \to 0^+} \frac{T(s)x - x}{s} = [A(t) + B]x.$$

This shows that $A(t) + B$ generates $K(t, s)$.

Theorem 3.1 is to be central in the discussion of the time-dependent potential problems. Theorem 3 of [10] gives the general perturbation of the infinitesimal generator of C_0-quasi semigroups. Also in [10], we see the applications of the perturbation in the linear nonautonomous control systems.

3.1 Heat equation with time-dependent potential

In this subsection, we shall apply the well-posedness to heat equations with time-dependent potentials. Let Ω be an open bounded set of $\mathbb{R}^n$ with a regular boundary $\partial\Omega$. We consider the non-autonomous heat equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \Delta u(t,x) - V(t,x)u(t,x) + f(t,x) \quad \text{in } (0,\tau)\times\Omega, \\ u(t,x) &= 0 \quad \text{on } (0,\tau]\times\partial\Omega, \\ u(0,x) &= u_0(x) \quad \text{in } \Omega, \end{aligned} \tag{6}$$

where Δ is the Laplace operator, V is a Lebesgue measurable potential on $[0,\tau]\times\Omega$ and $f \in L_2([0,\tau]\times\Omega)$. Set $X = L_2(\Omega)$, the Laplace operator Δ in (6) is densely defined in X with

$$\mathcal{D}(\Delta) = H^2(\Omega) \cap H_0^1(\Omega),$$

where H_0^1 denotes a space of functions in $H^1(\Omega)$ that vanish at the boundary. In this case, $H^m(\Omega)$ is the Sobolev space given by

$$H^m(\Omega) := \{u \in L_2(\Omega) : u, \dots, \tfrac{d^{m-1}u}{dx^{m-1}} \text{ are absolutely continuous on } \Omega,\ \tfrac{d^m u}{dx^m} \in L_2(\Omega)\}.$$

Moreover, Δ is a self-adjoint operator such that

$$\langle \Delta u, u\rangle \le \lambda_0 \|u\|^2, \quad u \in \mathcal{D}(\Delta),$$

where λ_0 is the first eigenvalue of Δ. Lumer-Phillips' theorem implies that Δ is the infinitesimal generator of a C_0-semigroup T in X such that

$$\|T(t)\| \le e^{\lambda_0 t}, \quad t \ge 0. \tag{7}$$

Setting $u(t) = u(t,\cdot)$, $V(t) = V(t,\cdot)$, and $A(t) = \Delta - V(t)$, we can rewrite the problem (6) as

$$\dot{u}(t) = A(t)u(t) + f(t), \quad u(0) = u_0, \quad t \in [0,\tau]. \tag{8}$$

Theorem 3.2 *If $u_0 \in X$ and the operator $-V(t)$ verifies the hypothesis of Theorem 2.1, then the non-autonomous problem* (6) *has a unique solution u which belongs to $\mathcal{C}([0,\tau], X)$.*

Proof. By Theorem 2.1, there exists a unique C_0-quasi semigroup $R(t, s)$ generated by $-V(t)$. Theorem 3 of [10] implies that $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup $K(t, s)$ in X with $K(t, s) = T(s)R(t, s)$. Theorem 2.3 gives that the non-autonomous Cauchy problem (8) is well-posed with a unique solution

$$u(t) = K(0,t)u_0 + \int_0^t K(s, t-s)f(s)ds. \tag{9}$$

Together with Theorem 3.10 of [7], the assertion follows.

A physical interpretation requires the positive solution of the Cauchy problem (6). By virtue of Theorem 10 on page 44 of [28], we obtain the following result.

Proposition 3.1 *If $u_0(x), f(t,x), V(t,x) \geq 0$ for all $(t,x) \in [0,\tau] \times \Omega$, then the solution* (9) *is positive.*

The following example gives an illustration how the heat equation with the time-dependent potential in $\mathbb{R}^3$ is solved.

Example 3.1 Consider the problem (6) in $\mathbb{R}^3$. Let V be a potential on $[0,\tau] \times \Omega$ given by $V(t,x) = \alpha(t)Z(x)$, where $\Omega = (0,1)^3$ and $Z(x) = -|x|^{-1}$ is the Coulomb potential on Ω and α is a measurable function on $[0,\tau]$.

We see that $-V(t)$ verifies Theorem 2.1. In addition, if $\alpha(t) = -\dfrac{1}{t+1}$, $-V(t)$ generates a C_0-quasi semigroup $R(t,s)$ on X given by

$$R(t,s)u(x) = \left(\frac{t+1}{t+s+1}\right)^{\frac{1}{|x|}} u(x), \quad u \in X, \quad x \in \Omega.$$

The C_0-semigroup $T(t)$ in (7) is given by

$$T(t)u = \sum_{l,m,n=1}^{\infty} e^{-(l^2+m^2+n^2)\pi^2 t} \langle u, \phi_{lmn} \rangle \phi_{lmn}, \quad u \in X,$$

where $\phi_{lmn}(x) = 2\sqrt{2}\sin(l\pi x_1)\sin(m\pi x_2)\sin(n\pi x_3)$ and $\langle \cdot, \cdot \rangle$ is the inner product in X. Therefore, for arbitrary $f \in L_2([0,\tau] \times \Omega)$, the solution of (6) is given by

$$u(t,x) = K(0,t)u_0(x) + \int_0^t K(s,t-s)f(s,x)ds, \tag{10}$$

where $K(t,s) = T(s)R(t,s)$. Moreover, we see if $u_0(x), f(t,x) \geq 0$ for all $(t,x) \in [0,\tau] \times \Omega$, then the solution (10) is positive.

Remark 3.1 (1) The quasi semigroup is an alternative approach to solve the non-autonomous heat equation in $[0,\tau] \times \mathbb{R}^n$. In some way, the approach is simpler than the approach used by Gulisashvili [14]. Moreover, if the potential V verifies Theorem 2.1, then it belongs to the Kato class K_n or the uniform Kato class $\mathcal{A}_{n,\tau}$.

(2) The quasi semigroup is also immediately applicable to solve the Schrodinger equation with a time-dependent potential

$$i\frac{\partial \psi}{\partial t}(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V(t,x)\psi(t,x).$$

Refer to Evans [2], the quasi semigroups can eliminate the scattering method.

3.2 Wave equation with time-dependent potential

Let Ω be an open bounded set of $\mathbb{R}^n$ with a regular boundary $\partial\Omega$. We consider the wave equation with a time-dependent potential

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) - V(t,x)u(t,x) + f(t,x) \quad \text{in } (0,\tau) \times \Omega, \\ &u(t,x) = 0 \quad \text{on } (0,\tau] \times \partial\Omega, \\ &u(0,x) = u_0(x), \ u_t(0,x) = u_1(x) \quad \text{in } \Omega, \end{aligned} \tag{11}$$

where V is a Lebesgue measurable function on $[0,\tau]\times\Omega$ and $f \in L_2([0,\tau]\times\Omega)$. We also assume that $\|V\|_\infty := \sup_{(t,x)\in[0,\tau]\times\Omega} |V(t,x)| < \infty$.

Using the notations as in Section 3.1, we see that $\Lambda = -\Delta$ is strictly positive self-adjoint operator in the Hilbert space X. We consider a Hilbert space $Z = \mathcal{D}(\sqrt{\Lambda}) \oplus X$ with the generic element

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and the inner product in Z is given by

$$\langle z, w\rangle_Z = \langle \sqrt{\Lambda} z_1, \sqrt{\Lambda} w_1\rangle_X + \langle z_2, w_2\rangle_X.$$

Define a linear operator A in Z by

$$A_0 z = \begin{bmatrix} 0 & 1 \\ -\Lambda & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

with $\mathcal{D}(A_0) = \mathcal{D}(\Lambda) \oplus \mathcal{D}(\sqrt{\Lambda})$.

Theorem 3.3 *If $u_0 \in H_0^1(\Omega)$ and $u_1 \in L_2(\Omega)$, then the non-autonomous problem* (11) *has a unique mild solution u belonging to $\mathcal{C}([0,\tau], H_0^1(\Omega)) \cap \mathcal{C}^1([0,\tau], X)$.*

Proof. When setting

$$Y(t) = \begin{bmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{bmatrix}, \quad Q(t) = \begin{bmatrix} 0 & 0 \\ -V(t,\cdot) & 0 \end{bmatrix}, \quad F(t) = \begin{bmatrix} 0 \\ f(t,\cdot) \end{bmatrix}, \quad Y_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix},$$

the problem (11) is equivalent to

$$\dot{Y}(t) = A(t)Y(t) + F(t), \quad Y(0) = Y_0, \quad t \in [0,\tau], \tag{12}$$

where $A(t) = A_0 + Q(t)$. By Proposition 2.12 of [29], we see that A_0 is the infinitesimal generator of a C_0-quasi semigroup $S(t,s)$ on Z given by

$$S(t,s) = \begin{bmatrix} \cos(\sqrt{\Lambda}s) & \Lambda^{-1/2}\sin(\sqrt{\Lambda}s) \\ -\Lambda^{1/2}\sin(\sqrt{\Lambda}s) & \cos(\sqrt{\Lambda}s) \end{bmatrix}.$$

Since $\|Q(t)\| \le \|V\| < \infty$ for $t \in [0,\tau]$, Theorem 3 of [10] gives that $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup $R(t,s)$ on Z such that

$$R(r,t) = \sum_{n=0}^{\infty} S_n(r,t), \tag{13}$$

where $S_0(r,t)y = S(r,t)y$ and $S_n(r,t)y = \int_0^t S(r+s, t-s)Q(r+s)S_{n-1}(r,s)y ds$ for $r, t \ge s \ge 0$, $y \in Z$ and $n \in \mathbb{N}$. Therefore, Theorem 2.3 implies that the problem (12) has a unique mild solution given by

$$Y(t) = R(0,t)Y_0 + \int_0^t R(s, t-s)F(s)ds. \tag{14}$$

By Theorem 3.10 of [7], the first component in (14) provides the required solution, $u(t,x)$, of the problem (11).

Remark 3.2 (1) Theorem 3.1 is not applicable in the determination of a quasi semigroup $R(t,s)$ since $S(t,s)$ is not commutative with any nontrivial quasi semigroups.

(2) We stress again that the function Y in (14) is the exact solution to (12) whose first component is the solution $u(t,x)$ of the problem (11). However, it is not easy to find the explicit form of $R(t,s)$ in (13). The approximation $Y^{(n)}$, $n = 0,1,2,\ldots$, to the solution Y of (14) gives the properties of $u(t,x)$.

Zero approximation. This is a function $Y^{(0)}$ in the form (14) for $R(t,s) = S(t,s)$. The first component of $Y^{(0)}$ gives $u^{(0)}$, the zero approximation to the solution of the problem (11) given by

$$u^{(0)}(t,\cdot) = [\cos\sqrt{\Lambda}t]u_0 + [\Lambda^{-1/2}\sin\sqrt{\Lambda}t]u_1 + \int_0^t \Lambda^{-1/2}\sin\sqrt{\Lambda}(t-s)f(s,\cdot)ds. \tag{15}$$

By the spectral theorem, this solution can be written in the form

$$\begin{aligned} u^{(0)}(t,\cdot) = \int_0^\infty \cos\sqrt{\lambda}t\, d(E(\lambda)u_0) + \int_0^\infty \frac{\sin\sqrt{\Lambda}t}{\sqrt{\lambda}} d(E(\lambda)u_1) \\ + \int_0^\infty \int_0^t \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} f(s,\cdot)ds\, d(E(\lambda)), \end{aligned} \tag{16}$$

where $E(\lambda) \in \{E(\lambda)\} \equiv \{E(\lambda)\}_{\lambda\in\sigma(\Lambda)}$ is the spectral family for Λ.

The formula (16) indicates that the smoothness of u depends on the smoothness of the data functions. Moreover, the representation implies that the asymptotic behaviour of $u(t,x)$ as $t \to \infty$ is closely related to the properties of the spectrum, $\sigma(\Lambda)$, and the spectral family, $\{E(\lambda)\}$.

Taking into account (15) gives the initial value problem

$$\frac{\partial^2 u^{(0)}}{\partial t^2}(t,\cdot) = \Delta u^{(0)}(t,\cdot) + f(t,\cdot), \quad u^{(0)}(0,\cdot) = u_0,\ u_t^{(0)}(0,\cdot) = u_1.$$

This shows that $u^{(0)}$ is a solution to the problem (11) without the potential or unperturbed problem. The form (14) implies that all subsequent approximations obtained by taking more terms in (13) are the solutions of a perturbed problem.

First approximation. The first two terms of (13) give

$$Y^{(1)}(t) = R(0,t)Y_0 + \int_0^t R(s,t-s)F(s)ds,$$

where $R(r,t) = S(r,t) + S_1(r,t)$. The first component of $Y^{(1)}$ gives $u^{(1)}$ by

$$\begin{aligned} u^{(1)}(t,\cdot) = u^{(0)}(t,\cdot) + \int_0^t [\{-\Lambda^{-1/2}\sin\sqrt{\Lambda}(t-s)\cos\sqrt{\Lambda}t\}V(s)u_0 \\ - \{\Lambda^{-1}\sin\sqrt{\Lambda}(t-s)\sin\sqrt{\Lambda}t\}V(s)u_1]\, ds + \\ \int_0^t \int_0^{t-s} \{-\Lambda^{-1}\sin\sqrt{\Lambda}(t-s-\eta)\sin\sqrt{\Lambda}\eta\}V(s+\eta)f(s,\cdot)d\eta\, ds. \end{aligned} \tag{17}$$

Taking into account (17) gives the initial value problem

$$\frac{\partial^2 u^{(1)}}{\partial t^2}(t,\cdot) = \Delta u^{(1)}(t,\cdot) - V(t,\cdot)u^{(0)}(t,\cdot) + f(t,\cdot),\ u^{(1)}(0,\cdot) = u_0,\ u_t^{(1)}(0,\cdot) = u_1.$$

We see that $u^{(1)}$ is a solution to the wave equation with the nontrivial forcing term on the right-hand side. In particular, the forcing term depends on the potential V and the previously approximate solution $u^{(0)}$.

For a realization, we consider problem (11) in one dimensional space with $\Omega = (0, 1)$, $\tau = 1$, the potential $V(t, x) = h(x)e^{-i\omega t}$, $(t, x) \in [0, \tau] \times \Omega$, and $f = 0$. From (17), the first approximation solution to the problem (11) in the spectral form is

$$\begin{aligned} u^{(1)}(t, \cdot) = &\int_0^\infty [a_0(\lambda) \sin(\sqrt{\lambda} t) + b_0(\lambda) \cos \sqrt{\lambda} t]\, d(E(\lambda)u_0) \\ &+ \int_0^\infty [a_1(\lambda) \sin \sqrt{\Lambda} t + b_1(\lambda) \cos \sqrt{\Lambda} t]\, d(E(\lambda)u_1), \end{aligned} \tag{18}$$

where

$$a_0(\lambda) = -\frac{4\lambda e^{-i\omega t} - 4\lambda + 2\omega^2}{2\sqrt{\lambda}\omega(4\lambda - \omega^2)} h(\cdot) i, \qquad b_0(\lambda) = \frac{2\sqrt{\lambda}(4\lambda - \omega^2) - (e^{-i\omega t} - 1)h(\cdot)}{2\sqrt{\lambda}(4\lambda - \omega^2)}$$

$$a_1(\lambda) = \frac{4\lambda - \omega^2 - 2(e^{-i\omega t} + 1)h(\cdot)}{\sqrt{\lambda}(4\lambda - \omega^2)}, \qquad b_1(\lambda) = \frac{4(e^{-i\omega t} - 1)}{\omega(4\lambda - \omega^2)} h(\cdot) i.$$

The eigenvalues and eigenvectors of Λ are $\lambda_n = n^2\pi^2$ and $\phi_n(x) = \sqrt{2} \sin(n\pi x)$, $n = \pm 1, \pm 2, \ldots$, respectively. Hence, by the Cauchy theorem, the solution (18) can be counted as

$$\begin{aligned} u^{(1)}(t, x) = \sum_{n=1}^{\infty} 2 \Big[&\left\{ a_0(n^2\pi^2) \sin(n\pi t) + b_0(n^2\pi^2) \cos(n\pi t) \right\} \langle u_0, \phi_n \rangle \phi_n(x) \\ &+ \left\{ a_1(n^2\pi^2) \sin(n\pi t) + b_1(n^2\pi^2) \cos(n\pi t) \right\} \langle u_1, \phi_n \rangle \phi_n(x) \Big], \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in X.

Remark 3.3 (1) We see that if $\omega^2/4$ in the resolvent set $\rho(\Lambda)$ and h is bounded on Ω, then $u^{(1)}(t, x)$ is bounded as $t \to \infty$. This implies that the solution $u(t, x)$ is also bounded as $t \to \infty$.

(2) If $\omega = 0$, then the problem (11) is the wave problem with the time-independent potential and the first approximation to the solution of the problem (18) can be easily obtained. Again, the solution is bounded as $t \to \infty$.

3.3 Nonlinear equations

We shall show that the quasi semigroup approach is applicable to solve the nonlinear equations. To begin with, we denote by $C_0(\Omega)$ the Banach space of continuous functions on Ω that vanish on $\partial\Omega$ with the sup norm. Given an initial value $u_0 \in C_0(\Omega)$, we consider the nonlinear version of the non-autonomous heat equation (6)

$$\begin{aligned} u_t &= \Delta u - V u + f(u), \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0. \end{aligned} \tag{19}$$

Theorem 2.3 guarantees that the initial value problem (19) is locally well-posed. More precisely, there exists a maximal time $0 < \tau_0 \leq \infty$ and a function $u \in C\left([0, \tau_0], C_0(\Omega)\right) \cap$

$C\left((0,\tau_0), C^2(\overline{\Omega})\right) \cap C^1\left((0,\tau_0), C_0(\Omega)\right)$ which is a classical solution of (19), see [30]. Further, by Theorem 3.2, u is the unique solution of (19) in $L^\infty\left((0,\tau)\times\Omega\right)$ for any $0<\tau<\tau_0$ given by

$$u(t) = K(0,t)u_0 + \int_0^t K(s,t-s)f(u(s))ds. \tag{20}$$

It is clear that the global solution u in (20) depends on u_0 and the nonlinearity of f. Recall that the solution u is global if $\tau_0 = \infty$, meanwhile u is blowing up in finite time if $\tau_0 < \infty$ and $\lim_{t\to\tau_0} \|u(t)\|_{L^\infty} = \infty$. In particular, for $V \equiv 0$, the conditions for the global and blowing up solutions of (19) were discussed in [30]. Analogously, for f and V are positive, we can verify the conditions of the solution u in (20) to be globally positive.

Remark 3.4 We can also apply the quasi semigroups to solve the nonlinear version of the Schrodinger equation in Remark 3.1 and the wave equation (11). Further, we can also classify the nonlinearity of f such that the solutions are global.

4 Conclusions

We have solved the heat and wave equations as prototype problems of the equations with time-dependent potentials, even for the nonlinear ones. The well-posedness is justified by $C_)$-quasi semigroups. By the maximum principle, the positivity of the solution of the heat equations depends on the potential. In the wave equations, the well-posedness is guaranteed by the bounded potentials. Moreover, although general approximations cannot be constructed yet, the first two approximations can be constructed.

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