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Some Results on Controllability for a Class of Non-Integer Order Differential Equations with Impulses

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Abstract: In this paper, we considered a class of impulsive fractional differential equations of order $1 < \alpha \leq 2$, in a Banach space. An associated integral equation is obtained by using the fractional integral and the cosine or sine family of linear operators. By using the measure of non-compactness and Mönch's condition, we prove that the problem under consideration is controllable. Abstract results are illustrated by an example in the last section.

Keywords: *controllability; non-integer order differential equation; impulsive condition; measure of non-compactness; Mönch's condition.*

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1 Introduction

Controllability is a fundamental concept in the theory of control dynamic systems, which plays an important role in the investigations and design of various kinds of control dynamic processes in finite and infinite dimensional spaces. An extensive study on controllability of various types of differential equations in abstract spaces has been done by many authors [2, 3, 5–8, 11–13]. In papers [2, 6, 9], the authors proved the results on controllability for second order control systems. Controllability of damped second order integrodifferential systems with impulses has been studied by Arthi and Balachandran [6].

The present work has been motivated by the work of Ravichandran and Baleanu [3], in which a control problem involving non-integer order (Caputo) derivatives is studied by using the measure of non-compactness and Mönch's condition. There are only few papers dealing with the study of controllability for a dynamic system with impulses. Impulse conditions describe the dynamics of a process in which discontinuous jumps occur. Such

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processes are generally seen in biology, physics and engineering. For earlier works on impulsive differential equations, we refer the readers to [10, 13, 14] and references cited in these papers.

In this manuscript, we are concerned with the controllability of the following fractional impulsive differential equation in a Banach space $(Y, \|\cdot\|)$:

$$\begin{cases} {}_CD^{\alpha}y(t) = Ay(t) + Ew(t) + g(t, y(t), y(\nu(t))), \quad t \in [0, b_0],\ t \neq t_i, \\ \Delta y(t_i) = \hat{I}_i(y(t_i)), \quad \Delta y'(t_i) = \hat{J}_i(y(t_i)), \quad i \in \mathbb{N},\ 1 \leq i \leq q, \\ y(t) = h(t), \quad t \in [-\tau, 0], \end{cases} \tag{1}$$

where $1 < \alpha \leq 2$, ${}_CD^{\alpha}$, denotes the (Caputo) fractional derivative, A is a densely defined closed linear operator, which generates a strongly continuous cosine family in Y, E denotes the bounded linear operators defined on W, $w \in L^2([0, b_0], W)$ denotes the control function, which takes the values in a Banach space W. The maps $g : [0, b_0] \times Y^2 \to Y$ and the maps $\hat{I}_i$, $\hat{J}_i$ defined on Y satisfy some suitable conditions, and the function $\nu : [0, b_0] \to [0, b_0]$ is continuous such that $0 \leq \nu(t) \leq t$, $t_i \in [0, b_0]$ for all $i \in \mathbb{N},\ 1 \leq i \leq q$ such that $t_1 < t_2 < \cdots < t_q$, and $q \in \mathbb{N},\ b_0 > 0$. $h \in C^2([-\tau, 0], Y)$, i.e., h is twice continuously differentiable on $[-\tau, 0]$. Let $I_0 = [0, b_0]$.

The main aim of this paper is to prove the controllability of the problem (1) by using the measure of non-compactness and Mönch's condition.

2 Preliminaries and Assumptions

It is well known that if A generates a strongly continuous cosine family, then A also generates an analytic semigroup. The fractional power A^{β} of A from $D(A^{\beta}) \subset Y$ into Y is well defined for all $0 \leq \beta \leq 1$ (cf., A. Pazy [1], pp. 69-75). The space $Y_{\beta} = \big(D(A^{\beta}), \|.\|_{\beta}\big)$ is a Banach space, where

$$\|\psi\|_{\beta} = \|A^{\beta}\psi\|, \quad \psi \in D(A^{\beta}).$$

Let $PC([0, b_0], Y_{\beta})$ denote the set of all piecewise continuous functions on $[0, b_0]$, and $\Omega_{\beta}^{b_0} = \big\{y \mid y, y' \in PC([0, b_0], Y_{\beta})$ such that $y(t), y'(t)$ are left continuous at $t = t_i$ and the right-hand limit of $y(t), y'(t)$ exists at $t = t_i,\ i \in \mathbb{N},\ 1 \leq i \leq q\big\}$. Eventually, $\big(\Omega_{\beta}^{b_0}, \|.\|_{\beta, b_0}\big)$ is a Banach space, where

$$\|\psi\|_{\beta, b_0} = \sup_{s \in [0, b_0]} \|\psi(s)\|_{\beta}, \quad \psi \in \Omega_{\beta}^{b_0}.$$

For $R_0 > 0$, let

$$B_{R_0}(\Omega_{\beta}^{b_0}, \tilde{h}) = \{y \in \Omega_{\beta}^{b_0} \ : \ \|y - \tilde{h}\|_{\beta, b_0} \leq R_0\},$$

where

$$\tilde{h}(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0), & t \in [0, b_0]. \end{cases}$$

Let $\{C_{\alpha}(t) : t \geq 0\}$ denote the cosine family generated by A. For $t \geq 0$, we define

$$S_{\alpha}(t) = \int_0^t C_{\alpha}(s) ds, \quad \text{and} \quad P_{\alpha}(t) = \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t - s)^{\alpha - 2} C_{\alpha}(s) ds.$$

Definition 2.1 [10] A measure of non-compactness defined on a Banach space Y is a function defined from Y to a positive cone of an ordered Banach space $(F, \leq)$ such that $\phi(\bar{ch}(B)) = \phi(B)$ for all bounded subset B of Y, where $\bar{ch}(B)$ denotes the closure of convex hull of B.

Lemma 2.1 *[10] Let Ω_0 be a closed convex subset of a Banach space Y and f_0 be a continuous map defined on Ω_0. If f_0 satisfies the (Mönch's) condition: $C_0 \subseteq \Omega_0$ is countable, $C_0 \subseteq \bar{ch}(\{0\} \cup f_0(C_0)) \Rightarrow \bar{C}_0$ is compact, then f_0 has a fixed point in Ω_0.*

Lemma 2.2 *[4] There are constants $T_C > 0, T_S > 0$ and $T_P > 0$ such that*

$$\|C_\alpha(s'') - C_\alpha(s')\| \leq T_C|s'' - s'|,$$

$$\|S_\alpha(s'') - S_\alpha(s')\| \leq T_S|s'' - s'|,$$

$$\|P_\alpha(s'') - P_\alpha(s')\| \leq T_P|s'' - s'|,$$

for $s', s'' \in I_0$.

Assumption 2.1 *Consider the following assumptions:*

(H1) There exists an increasing function $L_g : R^+ \to R^+$ such that

$$\|g(t, \hat{\phi}_1, \hat{\psi}_1) - g(s, \hat{\phi}_2, \hat{\psi}_2)\| \leq L_g(r)\left[|t - s| + \|\hat{\phi}_1 - \hat{\phi}_2\|_\beta + \|\hat{\psi}_1 - \hat{\psi}_2\|_\beta\right]$$

for all $\hat{\phi}_1, \hat{\phi}_2, \hat{\psi}_1, \hat{\psi}_2 \in B_{R_0}(\Omega_\beta^{b_0}, \tilde{h})$, and $t, s \in [0, b_0]$.

(H2) There are positive constants C_i, D_i and L_i, N_i, ($i \in \mathbb{N}$, $1 \leq i \leq q$) such that

(i) $\|\hat{I}_i(\hat{z})\|_\beta \leq C_i, \quad \|\hat{I}_i(\hat{z}_1) - \hat{I}_i(\hat{z}_2)\|_\beta \leq L_i\|\hat{z}_1 - \hat{z}_2\|_\beta$,

(ii) $\|\hat{J}_i(\hat{z})\|_\beta \leq D_i, \quad \|\hat{J}_i(\hat{z}_1) - \hat{J}_i(\hat{z}_2)\|_\beta \leq N_i\|\hat{z}_1 - \hat{z}_2\|_\beta$

for all $\hat{z}, \hat{z}_1, \hat{z}_2 \in B_{R_0}(\Omega_\beta^{b_0}, \tilde{h})$.

(H3) The linear operator $E : L^2(I_0, W) \to: L^1(I_0, W)$ is bounded. Also, the operator $Q : L^2(I_0, W) \to Y$ defined by

$$Qw = \int_0^{b_0} P_\alpha(b_0 - s)Ew(s)ds$$

has bounded inverse, i.e., $\|E\| \leq M_2$ and $\|Q^{-1}\| \leq M_3$, for some $M_2, M_3 > 0$.

3 Main Results

We assume that the families $\{C_\alpha(t)\}$, $\{S_\alpha(t)\}$, $\{P_\alpha(t)\}$ and $\{AP_\alpha(t)\}$ are unifromly bounded, i.e., there are constants r_1, r_2, r_3, r_4 such that

$$\|C_\alpha(t)\| \leq r_1, \quad \|S_\alpha(t)\| \leq r_2, \quad \|P_\alpha(t)\| \leq r_3 \quad \|AP_\alpha(t)\| \leq r_4, \quad t \in [0, b_0].$$

Lemma 3.1 *If $y(t)$ satisfies the control system (1), then $y(t)$ also satisfies the integral equation*

$$y(t) = \begin{cases} \tilde{h}(t), \quad -\tau \le t \le 0, \\ C_\alpha(t)h(0) + S_\alpha(t)h'(0) + \int_0^t P_\alpha(t-s)\big(g(s,y(s),y(\nu(s))) + Ew(s)\big)ds \\ \quad + \sum\limits_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum\limits_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \le b_0. \end{cases}$$

Proof. If $t \in [-\tau, 0]$, then $u(t) = h(t) = \tilde{h}(t)$. If $t \in [0, t_1)$, then

$$\begin{aligned} {}_C D_t^\alpha y(t) &= Ay(t) + Ew(t) + g(t, y(t), y(\nu(t))), \\ y(0) &= h(0), \ y'(0) = h'(0). \end{aligned}$$

Integrating, we get

$$y(t) + c_1 + c_2 t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[Ax(s) + Ew(s) + g(s, y(s), y(\nu(s)))\Big] ds.$$

Using $y(0) = h(0)$, $y'(0) = h'(0)$, we get $c_1 = -h(0)$, $\quad c_2 = -h'(0)$. Thus

$$y(t) = h(0) + h'(0)t + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[Ay(s) + Ew(s) + g(s, y(s), y(\nu(s)))\Big] ds.$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} {}_C D_t^\alpha y(t) &= Ay(t) + Ew(t) + g(t, y(t), y(\nu(t))) \\ y(t_1^+) &= y(t_1^-) + \hat{I}_1(y(t_1)) \\ y'(t_1^+) &= y'(t_1^-) + \hat{J}_1(y(t_1)). \end{aligned}$$

Again, integrating, we get

$$y(t) + c_3 + c_4 t = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[Ay(s) + Ew(s) + g(s, y(s), y(\nu(s)))\Big] ds.$$

Using $y(t_1^+) = y((t_1^-) + \hat{I}_1(y(t_1)))$ and $y'(t_1^+) = y'((t_1^-) + \hat{J}_1(y(t_1)))$, we get $c_3 = -h(0) - \hat{I}_1(y(t_1)) + t_1\hat{J}_1(y(t_1))$, $c_4 = -h'(0) - \hat{J}_1(y(t_1))$. Thus,

$$\begin{aligned} y(t) &= h(0) + h'(0)t + \hat{I}_1(y(t_1)) + (t - t_1)\hat{J}_1(y(t_1)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[Ax(s) + Ew(s) + g(s, y(s), y(\nu(s)))\Big] ds. \end{aligned}$$

Similarly, if $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} x(t) &= h(0) + h'(0)t + \sum_{1\le i\le q} \hat{I}_i(y(t_i)) + \sum_{1\le i\le q} (t - t_i)\hat{J}_i(y(t_i)) \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \Big[Ay(s) + Ew(s) + g(s, y(s), y(\nu(s)))\Big] ds. \end{aligned}$$

Taking the Laplace transform, we get

$$\begin{aligned} \tilde{y}(\lambda) &= \frac{h(0)}{\lambda} + \frac{h'(0)}{\lambda^2} + \sum_{1\le i\le q} \frac{e^{-t_i\lambda}}{\lambda}\hat{I}_i(y(t_i)) + \sum_{1\le i\le q} \frac{e^{-t_i\lambda}}{\lambda^2}\hat{J}_i(y(t_i)) \\ &\quad - \frac{1}{\lambda^\alpha}A\tilde{y}(\lambda) + \frac{1}{\lambda^\alpha}\tilde{g}(\lambda) + \frac{1}{\lambda^\alpha}E\tilde{w}(\lambda), \end{aligned}$$

where $\tilde{y}(\lambda) = L[y(t)]$, $\tilde{g}(\lambda) = L[g(t, y(t), y(\nu(t)))]$, and $\tilde{w}(\lambda) = L[w(t)]$.

$$\begin{aligned}\Rightarrow \tilde{y}(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}h(0) + \lambda^{\alpha-2}(\lambda^\alpha I - A)^{-1}h'(0) \\ &\quad +\lambda^{\alpha-1}\sum_{1\le i\le q} e^{-t_i\lambda}(\lambda^\alpha I - A)^{-1}\hat{I}_i(y(t_i)) \\ &\quad +\lambda^{\alpha-2}\sum_{1\le i\le q} e^{-t_i\lambda}(\lambda^\alpha I - A)^{-1}\hat{J}_i(y(t_i)).\end{aligned}$$

Using the properties of resolvent operator [4], we get

$$\begin{aligned}\tilde{y}(\lambda) &= \left\{\int_0^\infty e^{-\lambda t}C_\alpha(t)dt\right\}h(0) + \left\{\int_0^\infty e^{-\lambda t}S_\alpha(t)dt\right\}h'(0) \\ &\quad + \sum_{1\le i\le q} e^{-t_i\lambda}\int_0^\infty e^{-\lambda t}C_\alpha(t)\hat{I}_i(y(t_i))dt \\ &\quad + \sum_{1\le i\le q} e^{-t_i\lambda}\int_0^\infty e^{-\lambda t}S_\alpha(t)\hat{J}_i(y(t_i))dt \\ &\quad + \int_0^\infty e^{-\lambda t}P_\alpha(t)\tilde{g}(\lambda)dt + \int_0^\infty e^{-\lambda t}P_\alpha(t)E\tilde{w}(\lambda)dt. \end{aligned} \tag{2}$$

Consider

$$\begin{aligned}&\sum_{1\le i\le q} e^{-\lambda t_i}\int_0^\infty e^{-\lambda t}C_\alpha(t)\hat{I}_i(y(t_i))dt \\ &\quad = \int_0^\infty e^{-\lambda t}\left\{\sum_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i))\right\}dt. \end{aligned} \tag{3}$$

Similarly,

$$\begin{aligned}&\sum_{1\le i\le q} e^{-\lambda t_i}\int_0^\infty e^{-\lambda t}S_\alpha(t)\hat{J}_i(y(t_i))dt \\ &\quad = \int_0^\infty e^{-\lambda t}\left\{\sum_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i))\right\}dt, \end{aligned} \tag{4}$$

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}P_\alpha(t)\tilde{g}(\lambda)dt \\ &\quad = \int_0^\infty e^{-\lambda t}P_\alpha(t)\int_0^\infty e^{-\lambda s}g(s, y(s), y(\nu(s)))dsdt \\ &\quad = \int_0^\infty e^{-\lambda t}\left\{\int_0^t P_\alpha(t-s)g(s, y(s), y(\nu(s)))ds\right\}dt, \end{aligned} \tag{5}$$

and

$$\int_0^\infty e^{-\lambda t} P_\alpha(t) E\tilde{w}(\lambda) dt$$
$$= \int_0^\infty \int_0^\infty e^{-\lambda(t+s)} P_\alpha(t) E w(s) ds dt$$
$$= \int_0^\infty e^{-\lambda t} \left\{ \int_0^t P_\alpha(t-s) E w(s) ds \right\} dt. \tag{6}$$

Putting the values of (3), (4), (5) and (6) in (2), and by taking the inverse Laplace transform, we get

$$\begin{aligned} y(t) &= C_\alpha(t)h(0) + S_\alpha(t)h'(0) + \int_0^t P_\alpha(t-s)(g(s, y(s), y(\nu(s))) + Ew(s))ds \\ &\quad + \sum_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \le b_0. \end{aligned}$$

Definition 3.1 A mild solution of the problem (1) is a fucntion $y \in \Omega_\beta^{b_0}$ satisfying the integral equation

$$y(t) = \begin{cases} \tilde{h}(t), \quad -\tau \le t \le 0, \\ C_\alpha(t)h(0) + S_\alpha(t)h'(0) + \int_0^t P_\alpha(t-s)(g(s, y(s), y(\nu(s))) + Ew(s))ds \\ \quad + \sum\limits_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum\limits_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \le b_0. \end{cases} \tag{7}$$

Definition 3.2 The system (1) is said to be controllable on the interval I_0 if for every $h(t) \in C^2([-\tau, 0], Y)$, $y_1 \in Y$, there is a control function $w \in L^2(I_0, W)$ such that the mild solution $y(t)$ of (1) satisfies $y(b_0) = y_1$.

For any $y \in \Omega_\beta^{b_0}$, we define the control function

$$\begin{aligned} w_y(t) = Q^{-1}\Big\{ & y_1 - C_\alpha(b_0)h(0) - S_\alpha(b_0)h'(0) - \int_0^{b_0} P_\alpha(b_0 - t) g(t, y(t), y(\nu(t)))dt \\ & - \sum_{1\le i\le q} C_\alpha(b_0 - t_i)\hat{I}_i(y(t_i)) - \sum_{1\le i\le q} S_\alpha(b_0 - t_i)\hat{J}_i(y(t_i))\Big\}. \end{aligned}$$

Using (H1)-(H3), we can find a constant $K_w > 0$ s.t. $\|w_y(t)\| \le K_w$.

Theorem 3.1 *If (H1)-(H3) hold, then control system (1) is controllable.*

Proof. Using the control w_y, we show that the operator $F : B_{R_0}(\Omega_\beta^{b_0}, \tilde{h}) \to B_{R_0}(\Omega_\beta^{b_0}, \tilde{h})$, defined by

$$Fy(t) = \begin{cases} \tilde{h}(t), \quad -\tau \le t \le 0, \\ C_\alpha(t)h(0) + S_\alpha(t)h'(0) + \int_0^t P_\alpha(t-s)(g(s, y(s), y(\nu(s))) + Ew_y(s))ds \\ \quad + \sum\limits_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum\limits_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \le b_0, \end{cases}$$

has a fixed point. This fixed point is then a solution of the given system. Clearly, $Fy(b_0) = y_1$, which shows that the given system is controllable on I_0.

We define

$$\hat{h}(t) = \begin{cases} \tilde{h}(t), & -\tau \le t \le 0, \\ C_\alpha(t)h(0) + S_\alpha(t)h'(0), & 0 < t \le b_0, \end{cases}$$

and

$$z(t) = \begin{cases} 0, \quad -\tau \le t \le 0, \\ \int_0^t P_\alpha(t-s)(g(s, y(s), y(\nu(s))) + Ew(s))ds \\ \quad + \sum\limits_{1 \le i \le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum\limits_{1 \le i \le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \le b_0. \end{cases}$$

Let $y(t) = \hat{h}(t) + z(t)$, then $y(t)$ satisfies (7). Define

$$\Omega_{\beta,0}^{b_0} = \left\{ y \in \Omega_\beta^{b_0} \mid y(t) = 0, \quad -\tau \le t \le 0 \right\},$$

and the operator $\tilde{F} : \Omega_{\beta,0}^{b_0} \to \Omega_{\beta,0}^{b_0}$ is

$$\tilde{F}z(t) = \begin{cases} 0, \quad -\tau \le t \le 0, \\ \int_0^t P_\alpha(t-s) \Big(g\big(s, \hat{h}(s) + z(s), \hat{h}(\nu(s)) + z(\nu(s))\big) + Ew_{\hat{h}+z}(s) \Big) ds \\ \quad + \sum\limits_{1 \le i \le q} C_\alpha(t-t_i)\hat{I}_i\big(\hat{h}(t_i) + z(t_i)\big) \\ \quad + \sum\limits_{1 \le i \le q} S_\alpha(t-t_i)\hat{J}_i\big(\hat{h}(t_i) + z(t_i)\big), \quad 0 < t \le b_0. \end{cases}$$

Obviously, to show that F has a fixed point, it is sufficient to show that $\tilde{F}$ has a fixed point. For this, we use Lemma 2.1. Let

$$B_R = \left\{ x \in \Omega_{\beta,0}^{b_0} \mid \|x\|_{\beta,b_0} \le R \right\}.$$

We prove this result in four steps.

Step 1: There is a number $R > 0$ such that

$$\tilde{F}(B_R) \subseteq B_R.$$

Let $z \in B_R$, $t \in (0, b_0]$, we have

$$\begin{aligned} \|(\tilde{F}z)(t)\|_\beta \quad \le \quad & r_4 \|A^{\beta-1}\| \int_0^t \Big[\|g\big(s, \hat{h}(s) + z(s), \hat{h}(\nu(s)) + z(\nu(s))\big)\| + \|Ew_{\hat{h}+z}(s)\| \Big] ds \\ & + r_1 \sum_{1 \le i \le q} \|\hat{I}_i\big(\hat{h}(t_i) + z(t_i)\big)\|_\beta + r_2 \sum_{1 \le i \le q} \|\hat{J}_i\big(\hat{h}(t_i) + z(t_i)\big)\|_\beta. \end{aligned}$$

Using (H1)-(H3), and the inequality $\|w_y(t)\| \le K_w$, and then taking R sufficiently large, we have

$$\|(\tilde{F}z)\|_{\beta,b_0} \le R.$$

Thus there is a $R > 0$ such that

$$\tilde{F}(B_R) \subseteq B_R.$$

Step 2: $\tilde{F}$ is continuous on B_R. We consider a sequence $\{z_n\}$ in B_R such that $z_n \to z \in B_R$. Then we have

$$\begin{aligned} \|(\tilde{F}z_n)(t) - (\tilde{F}z)(t)\|_\beta \leq \; & r_4\|A^{\beta-1}\| \int_0^t \|g\big(s, \hat{h}(s) + z_n(s), \hat{h}(\nu(s)) + z_n(\nu(s))\big) \\ & - g\big(s, \hat{h}(s) + z(s), \hat{h}(\nu(s)) + z(\nu(s))\big)\|ds \\ & + r_4\|A^{\beta-1}\| \int_0^t \|E\|\|w_{\hat{h}+z_n}(s) - w_{\hat{h}+z}(s)\|ds \\ & + r_1 \sum_{1\leq i\leq q} \|\hat{I}_i\big(\hat{h}(t_i) + z_n(t_i)\big) - \hat{I}_i\big(\hat{h}(t_i) + z(t_i)\big)\|_\beta \\ & + r_2 \sum_{1\leq i\leq q} \|\hat{J}_i\big(\hat{h}(t_i) + z_n(t_i)\big) - \hat{J}_i\big(\hat{h}(t_i) + z(t_i)\big)\|_\beta. \end{aligned} \tag{8}$$

Using (H1)-(H3), and taking supremum over $[0, b_0]$, we have

$$\|\tilde{F}z_n - \tilde{F}z\|_{\beta,b_0} \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that $\tilde{F}$ is continuous on B_R.

Step 3: $\tilde{F}(B_R)$ is equicontinuous on I_0. For this, we assume $z \in \tilde{F}(B_R)$ and $0 \leq s' < s'' \leq b_0$. Then there is a $y \in B_R$ such that

$$\begin{aligned} \|z(s'') - z(s')\| \leq \; & \int_0^{s'} \|P_\alpha(s'' - s) - P_\alpha(s' - s)\|_\beta \Big[\|g\big(s, \hat{h}(s) + y(s), \\ & \hat{h}(\nu(s)) + y(\nu(s))\big)\| + \|E\|\|w_{\hat{h}+y}(s)\|ds\Big] \\ & + \int_{s'}^{s''} \|P_\alpha(s'' - s)\|_\beta \Big[\|g\big(s, \hat{h}(s) + y(s), \\ & \hat{h}(\nu(s)) + y(\nu(s))\big)\| + \|E\|\|w_{\hat{h}+y}(s)\|\Big]ds \\ & + \sum_{1\leq i\leq q} \|C_\alpha(s'' - t_i) - C_\alpha(s' - t_i)\|\|\hat{I}_i\big(\hat{h}(t_i) + y(t_i)\big)\|_\beta \\ & + \sum_{1\leq i\leq q} \|S_\alpha(s'' - t_i) - S_\alpha(s' - t_i)\|\|\hat{J}_i\big(\hat{h}(t_i) + y(t_i)\big)\|_\beta. \end{aligned}$$

Using (H1), (H2), (H3), Lemma 2.2 and $\|w_y(t)\| \leq K_w$, we can find a constant $G_1 > 0$ such that

$$\|z(s'') - z(s')\| \leq G_1|s'' - s'|.$$

From the above inequality, it is clear that $\|z(t'') - z(t')\| \to 0$ as $t'' \to t'$. Therefore $\tilde{F}(B_R)$ is equicontinuous on I_0.

Step 4: Next, we show that Mönch's condition is satisfied, i.e., if $V \subseteq B_R$ is countable and $V \subseteq \bar{ch}\Big(\{0\} \cup \tilde{F}(V)\Big)$, then $\bar{V}$ is compact. According to the idea used in [13], we can show that $\tilde{F}(V)$ is relatively compact, i.e., if ϕ is a monotone, nonsingular measure of non-compactness, then $\phi\big(\tilde{F}(V)\big) = 0$.

Since $V \subseteq \bar{ch}(\{0\} \cup \tilde{F}(V))$, by using the definition of ϕ, we have

$$\phi(V) \quad \leq \quad \phi\Big(\bar{ch}(\{0\} \cup \tilde{F}(V))\Big) = \phi\big(\tilde{F}(V)\big) = 0.$$

This implies that V is relatively compact, i.e., $\bar{V}$ is compact. Thus Mönch's condition is satisfied. Therefore, by Lemma 2.1, $\tilde{F}$ has a fixed point. This completes the proof.

4 Application

Consider the problem

$$\begin{cases} {}_C D^{\frac{4}{3}} w(y,s) = \dfrac{\partial^2 w}{\partial x^2} + \mu_0(y,s) + \mathcal{H}(y,s,w(y,s),w(y,s-\tau)), \\ \qquad\qquad\qquad\qquad y \in (0,\pi), \quad s \in [0,b_0],\ s \neq s_i, \\ w(0,s) = w(\pi,s) = 0, \quad s \in (0,b_0], \\ \Delta w(y,s_i) = \dfrac{3w(y,s_i)}{4 + w(y,s_i)}, \quad y \in (0,\pi), \\ \Delta w'(y,s_i) = \dfrac{5u_1(x,s_i)}{6 + w(y,s_i)}, \quad y \in (0,\pi), \\ w(y,s) = \chi(y,s), \quad y \in [0,\pi], \quad s \in [-\tau,0], \\ \qquad\qquad i \in \mathbb{N},\ 1 \leq i \leq q, \end{cases} \tag{9}$$

where $\mathcal{H}, \chi$ are sufficiently smooth real-valued functions. $\Delta w(y,s_i) = w(y,s_i^+) - w(y,s_i^-)$, $\Delta w'(y,s_i) = w'(y,s_i^+) - w'(y,s_i^-)$, where $w(y,s_i^-)(w(y,s_i^+))$ are the left-(right-) hand limits of w and w' at $(y,s) = (y,s_i)$, respectively. Let the control function $\mu_0 : [0,b_0] \times (0,\pi) :\to R$ be continuous on $[0,b_0]$.

Function $\mathcal{H}$, satisfies the condition

$$|\mathcal{H}(y,s_1,\phi_1,\psi_1) - \mathcal{H}(y,s_2,\phi_2,\psi_2)| \leq L[|s_1 - s_2| + |\phi_1 - \phi_2| + |\psi_1 - \psi_2|],$$

where $L > 0$ is a constant

System (9) is a generalization of the wave equation with impulsive conditions. This system represents the acoustic wave propagation through human tissues, sediments, rock layers etc.

To write the problem (9) in abstract form, we define an operator $\mathcal{A}$ by

$$\mathcal{A}w = w''.$$

The domain of $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$ is given as follows. If $w \in \mathcal{D}(\mathcal{A})$, then $w \in L^2(0,\pi), w'' \in L^2(0,\pi)$, and $w(0) = w(\pi) = 0$. $\mathcal{A}$ generates a strongly continuous cosine family on $L^2(0,\pi)$ (see [4]). Therefore, $\mathcal{A}$ also generates an analytic semigroup (see [1]). If we take $\beta = \frac{1}{3}$, then the fractional power $\mathcal{A}^{\frac{1}{3}}$ is well defined (see [1]). $\Big(\mathcal{D}(\mathcal{A}^{\frac{1}{3}}), \|.\|_{\frac{1}{3}}\Big)$ is a Banach space, where for $w \in \mathcal{D}(\mathcal{A})$,

$$\|w\|_{\frac{1}{3}} = \|\mathcal{A}^{\frac{1}{3}} w\|.$$

We denote this Banach space by $Y_{\frac{1}{3}}$.

Let $PC\Big([0,b_0], Y_{\frac{1}{3}}\Big)$ denote the set of all piecewise continuous functions on $[0,b_0]$, and $\Omega_{\frac{1}{3}}^{b_0} = \Big\{y \mid y, y' \in PC\Big([0,b_0], Y_{\frac{1}{3}}\Big)$ such that $y(s), y'(s)$ are left-continuous at $s = s_i$

and the right-hand limit of $y(s), y'(s)$ exists at $s = s_i,\ i \in \mathbb{N},\ 1 \le i \le q\Big\}$. Eventually, $\big(\Omega^{b_0}_{\frac{1}{3}}, \|.\|_{\beta,b_0}\big)$ is a Banach space, where

$$\|\psi\|_{\frac{1}{3},b_0} = \sup_{s\in[0,b_0]} \|\psi(s)\|_{\frac{1}{3}}, \quad \psi \in \Omega^{b_0}_{\frac{1}{3}}.$$

If we also define $w(s)(y) = w(y,s),\ \chi(s)(y) = \chi(y,s),\ \nu(t) = t-\tau,\ h(s, w(s), w(s-\tau))(t) = \mathcal{H}(y, s, w(y,s), w(y, s-\tau))$, and $Ev : [0, b_0] \to L^2(0,\pi)$, by $(Ev)(t)(y) = \mu_0(s,y)$, then the abstract formulation of the problem (9) is

$$\begin{cases} {}_CD^{\frac{4}{3}} w(s) = Aw(s) + Ev(s) + h(s, w(s), w(\nu(s))), & t \in [0, b_0],\ s \ne s_i, \\ \Delta w(s_i) = \hat{I}_i(w(s_i)), \quad \Delta w'(s_i) = \hat{J}_i(w(s_i)), & i \in N,\ 1 \le i \le q, \\ u(s) = \chi(s), \quad s \in [-\tau, 0]. \end{cases}$$

It can be easily shown that all the assumptions of Theorem 3.1 are satisfied. Therefore, we conclude that the control system (9) is controllable.

5 Conclusion

In this paper, we proved the controllability for a class of fractional impulsive differential equations in a Banach space X. An associated integral equation is obtained by using the fractional integral and the family of cosines of linear operators, and then by using the measure of non-compactness and Mönch's fixed point theorem, we proved the existence of mild solution and controllability of the problem. In the last section, we presented an example to illustrate the abstract results.

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