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Equivalent Conditions and Persistence for Uniformly Exponential Dichotomy

Sutrima Sutrima * and Ririn Setiyowati

Department of Mathematics, University of Sebelas Maret, Ir. Sutami, no.36 A Kentingan, 57126, Surakarta, Indonesia.

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Abstract: The purpose of this paper is to provide equivalence conditions of existing conditions for the uniformly exponential dichotomy of strongly continuous quasi groups (C_0-quasi groups) on Banach spaces. There are four equivalent conditions for the existence of uniformly exponential dichotomy in the used classes of continuous and integrable function spaces over $\mathbb{R}$. Each condition emphasizes the existence and uniqueness of mild solutions of the corresponding inhomogeneous equation on the corresponding space in the C_0-quasi group term. The results are parallel with the dichotomy for the evolution family. Moreover, a small time-dependent perturbation of the infinitesimal generator of the C_0-quasi groups persists the uniformly exponential dichotomy. The results are also motivated by illustrative examples.

Keywords: *strongly continuous quasi semigroup; uniformly exponential dichotomy; mild solution; time-dependent perturbation; persistence.*

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1 Introduction

As a generalization of exponential stability and dichotomy for the evolution family [1, 2], the Dichotomy Theorem of C_0-quasi groups on Banach spaces has just been developed in [3], see Theorem 4. The theorem implies that a uniformly exponential dichotomy of the C_0-quasi groups on Banach spaces X is equivalent to the spectral property of the corresponding evolution semigroup on $L_p(\mathbb{R}, X)$. Besides, the uniformly exponential dichotomy is also equivalent to the existence and uniqueness of Green's function for the quasi group, Theorem 9 of [3]. The uniformly exponential stability in this paper refers to the term in [4–6].

* Corresponding author: sutrima@mipa.uns.ac.id

Consider a non-autonomous abstract Cauchy problem

$$\dot{u}(t) = A(t)u(t), \quad u(r) = u_r, \quad u_r \in \mathcal{D}, \quad t \geq r, \quad t, r \in \mathbb{R}, \tag{1}$$

where $A(t)$ is a linear closed operator in X with the domain $\mathcal{D}(A(t)) = \mathcal{D}$ being independent of t and dense in X. Assume that (1) is well-posed in the sense that there exists a quasi group $\{R(t, s\}_{t,s\in\mathbb{R}}$ which gives a differential function u [7,8]. In fact, if $u_r \in \mathcal{D}$, then $u(t) = R(r, t-r)u_r$, $t \geq r$, is a solution of (1) and $u(t) \in \mathcal{D}$. This confirms that the uniformly exponential dichotomy is a fundamental asymptotic property of the solutions of (1). The important examples of the finite cases of (1) are given in [9,10].

Next, consider the inhomogeneously non-autonomous abstract Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{2}$$

where f is a locally integrable X-valued function on $\mathbb{R}$. It can be verified that the function u, which satisfies the integral equation

$$u(t) = R(r, t-r)u(r) + \int_r^t R(s, t-s)f(s)ds, \quad t \geq r, \tag{3}$$

is a solution (mild solution) of (2). In particular, this confirms that the uniformly exponential dichotomy of solutions of the non-autonomous abstract Cauchy problem (1) is equivalent to the existence and uniqueness of the mild solution of the inhomogeneous equation (2) for some integrable functions f. In other words, it allows to characterize the uniformly exponential dichotomy for the quasi groups in terms of "Perron-type" theorems of classes of integrable function spaces over $\mathbb{R}$. These questions are the counterpart of the classical theorems of the Perron type for the evolutionary families with varying $A(t)$ and classes of f's are discussed in [11–17].

In [7, 18], a bounded time-dependent perturbation under certain conditions of an infinitesimal generator of C_0-quasi semigroups produces a perturbed C_0-quasi semigroup on the same space. The classical and mild solutions of the new non-autonomous abstract Cauchy problem induced by the perturbed infinitesimal generator retain dependence on the similar solutions of the old problem. A question arises whether this situation applies to the quasi groups. Further, if the old quasi groups have a uniformly exponential dichotomy, whether the perturbed quasi group also persists the uniformly exponential dichotomy. As a comparison, under time-dependent Miyadera-type perturbations, the evolution family persists the uniformly exponential dichotomy [1,2].

This paper focuses on characterizations of the equivalent conditions for the uniformly exponential dichotomy of the C_0-quasi group using classes of integrable function spaces and investigates the persistence of the uniformly exponential dichotomy due to the time-dependent perturbations. The organization of this paper is as follows. In Section 2, re-exposure of the existing results for the uniformly exponential dichotomy of the C_0-quasi groups on a Banach space is considered. Characterizations for the uniformly exponential dichotomy using four spaces of $C_b(\mathbb{R}, X)$, $C_0(\mathbb{R}, X)$, $L_p(\mathbb{R}, X)$, and a scale space of continuous functions $\mathcal{F}_\alpha$ are considered in Section 3. Section 4 investigates the persistence for the uniformly exponential dichotomy under a bounded time-dependent perturbation of the infinitesimal generator.

2 Preliminaries

In this section, we recall the results about the sufficient and necessary conditions for the uniformly exponential dichotomy of the strongly continuous quasi groups on Banach spaces [3]. The quasi group itself is a generalization of the strongly continuous quasi semigroup [19].

Definition 2.1 (Definition 1 [3]) *Let $\mathcal{L}(X)$ be the set of all bounded linear operators on a Banach space X. A two-parameter commutative family $\{R(t,s)\}_{s,t\in\mathbb{R}}$ in $\mathcal{L}(X)$ is called a strongly continuous quasi group (C_0-quasi group) on X if for each $r,s,t \in \mathbb{R}$ and $x \in X$:*

(a) $R(t,0) = I$, *the identity operator on* X,

(b) $R(t,s+r) = R(t+r,s)R(t,r)$,

(c) $\lim_{s\to 0} \|R(t,s)x - x\| = 0$,

(d) there is a continuous increasing function $M : \mathbb{R} \to [1,\infty)$ such that

$$\|R(t,s)\| \le M(t+s).$$

Let $\mathcal{D}$ be the set of all $x \in X$ such that the following limits exist:

$$\lim_{s\to 0} \frac{R(t,s)x - x}{s}, \quad s,t \in \mathbb{R}.$$

For $t \in \mathbb{R}$, we define an operator $A(t)$ on $\mathcal{D}$ as

$$A(t)x = \lim_{s\to 0} \frac{R(t,s)x - x}{s}.$$

The family of operators $\{A(t)\}_{t\in\mathbb{R}}$ is called an infinitesimal generator of the C_0-quasi group $\{R(t,s)\}_{s,t\in\mathbb{R}}$. In what follows, for simplicity, we denote the quasi group $\{R(t,s)\}_{s,t\in\mathbb{R}}$ and the family $\{A(t)\}_{t\in\mathbb{R}}$ by $R(t,s)$ and $A(t)$, respectively.

We have identified the dichotomy for the C_0-quasi groups using uniformly exponential stability, an extension of the similar term for C_0-quasi semigroups [18].

Definition 2.2 (Definition 2 [3]) *A C_0-quasi group $R(t,s)$ is said to be uniformly exponentially stable on a Banach space X if there exist constants $\gamma > 0$ and $N \ge 1$ such that*

$$\|R(t,s)x\| \le Ne^{-\gamma|s|}\|x\|, \quad t,s \in \mathbb{R}, \quad x \in X. \tag{4}$$

Definition 2.3 *The C_0-quasi group $R(t,s)$ is said to be exponentially bounded on a Banach space X if there exist a constant $\omega \in \mathbb{R}$ and a function $N_\omega : \mathbb{R}^+ \to [1,\infty)$ such that*

$$\|R(t,s)x\| \le N_\omega(t)e^{\omega|s|}\|x\|, \quad t,s \in \mathbb{R}, \quad x \in X.$$

Sometimes, we have to convert a quasi-group to be an evolution semigroup. For example, the uniformly exponential stability for a quasi-group is more easily identified by the spectrum of the infinitesimal generator of the corresponding evolution semigroup. For a Banach space X, $L_p(\mathbb{R},X)$, $1 \le p < \infty$, denotes the space of all functions $f : \mathbb{R} \to X$ with the norm $\|f\|_{L_p(\mathbb{R},X)} = \left(\int_{-\infty}^{\infty} \|f(t)\|_X^p dt\right)^{\frac{1}{p}}$. Henceforth, in this paper we always assume that $L_p(\mathbb{R},X)$ with $1 \le p < \infty$.

Definition 2.4 (Definition 3 [3]) *Let $R(t,s)$ be a C_0-quasi group on a Banach space X. The evolution semigroup associated with $R(t,s)$ on $L_p(\mathbb{R},X)$ is a family of operators $\{E^s\}_{s\geq 0}$ given by*

$$(E^s f)(t) = R(t-s,s)f(t-s), \quad s \geq 0, \quad t \in \mathbb{R}, \quad f \in L_p(\mathbb{R},X). \tag{5}$$

For simplicity, the evolution semigroup $\{E^s\}_{s\geq 0}$ is simply written as E^s. We see that E^s is strongly continuous on $L_p(\mathbb{R},X)$. Moreover, if $A(t)$ is the infinitesimal generator of the C_0-quasi group $R(t,s)$ with domain $\mathcal{D}$, then an operator Γ defined by

$$(\Gamma f)(t) = -\frac{df}{dt} + A(t)f(t), \quad t \in \mathbb{R}, \tag{6}$$

is the infinitesimal generator of E^s with the domain

$$\mathcal{D}(\Gamma) = \{f \in L_p(\mathbb{R},X) : f \text{ is absolutely continuous}, \ f(t) \in \mathcal{D}\}.$$

The uniformly exponential dichotomy for the C_0-quasi groups is an extension of the similar term for the C_0-quasi semigroups introduced by Cuc [4]. Let $P : \mathbb{R} \to \mathcal{L}(X)$ be a projection-valued function, the complementary projection is given by $Q(t) = I - P(t)$ for all $t \in \mathbb{R}$. If $P(t+s)R(t,s) = R(t,s)P(t)$, then

$$R_P(t,s) := P(t+s)R(t,s)P(t) \quad \text{and} \quad R_Q(t,s) := Q(t+s)R(t,s)Q(t)$$

are the restrictions of $R(t,s)$ on $\operatorname{ran} P(t)$ and $\operatorname{ran} Q(t)$, respectively. The $R_P(t,s)$ is the operator from $\operatorname{ran} P(t)$ to $\operatorname{ran} P(t+s)$, while $R_Q(t,s)$ maps $\operatorname{ran} Q(t)$ to $\operatorname{ran} Q(t+s)$.

Definition 2.5 (Definition 4 [3]) *The C_0-quasi group $R(t,s)$ is said to have a uniformly exponential dichotomy on X if there exist constants $N \geq 1$, $\gamma > 0$ and a projection-valued function $P : \mathbb{R} \to \mathcal{L}(X)$ such that for each $x \in X$, the function $x \mapsto P(t)x$ is continuous and bounded, and, for all $t, s \in \mathbb{R}$, the following conditions hold:*

(a) $P(t+s)R(t,s) = R(t,s)P(t)$,

(b) $R_Q(t,s)$ *is invertible as an operator from* $\operatorname{ran} Q(t)$ *to* $\operatorname{ran} Q(t+s)$,

(c) $\|R_P(t,s)\| \leq Ne^{-\gamma|s|}$,

(d) $\|[R_Q(t,s)]^{-1}\| \leq Ne^{-\gamma|s|}$.

The pair of γ and N in Definition 2.5 is called the dichotomy constants of $R(t,s)$. Definition 2.5 states that if the quasi group $R(t,s)$ has a uniformly exponential dichotomy on X, then $R(t,s)$ and $R^{-1}(t,s)$ are uniformly exponentially stable on $\operatorname{ran} P(t)$ and on $\operatorname{ran} Q(t)$, respectively. The dichotomy bound of $R(t,s)$ is defined as

$$\gamma(R) := \sup\{\gamma > 0 : R(t,s) \text{ has exponential dichotomy with constants } \gamma \text{ and } N = N(\gamma)\}. \tag{7}$$

The sufficient and necessary conditions for the uniformly exponential dichotomy of the C_0-quasi groups are given by the following theorems.

Theorem 2.1 (Dichotomy Theorem, Theorem 4 [3]) *Assume that $R(t,s)$ is a C_0-quasi group on a Banach space X. Let E^s be the corresponding evolution semigroup given by* (5) *on $L_p(\mathbb{R},X)$ and let Γ denote its infinitesimal generator given by* (6)*. The following statements are equivalent:*

(a) The quasi group $R(t,s)$ has a uniformly exponential dichotomy on X.

(b) For each $s>0$, $\sigma(E^s)\cap\mathbb{T}=\emptyset$, where $\mathbb{T}:=\{z\in\mathbb{C}:|z|=1\}$.

(c) $0\in\rho(\Gamma)$.

Let $C_b(\mathbb{R},X)$ be the space of all bounded continuous functions $f:\mathbb{R}\to X$ with the supremum norm. Let $P(\cdot)\in C_b(\mathbb{R},\mathcal{L}_s(X))$ be the projection that satisfies (a) and (b) of Definition 2.5. Green's function for $R(t,s)$ is a map $G_P:\mathbb{R}^2\setminus\{(0,0)\}\to\mathcal{L}_s(X)$ defined by

$$G_P(t,s)=R_P(t,s)P(t),\quad t>s,$$
$$G_P(t,s)=-[R_Q(t,s)]^{-1}Q(t),\quad t<s.$$

Green's operator $\mathbb{G}$ associated with G_P on $L_p(\mathbb{R},X)$ is defined by

$$(\mathbb{G}f)(t)=\int_{-\infty}^{\infty}G_P(s,t-s)f(s)ds,\quad f\in L_p(\mathbb{R},X). \tag{8}$$

Theorem 2.2 (Theorem 9 [3]) *Let Γ be the infinitesimal generator of the evolution semigroup E^s corresponding to a C_0-quasi group $R(t,s)$ defined by* (5) *on $L_p(\mathbb{R},X)$. The quasi group $R(t,s)$ has a uniformly exponential dichotomy on X if and only if there exists a unique Green's function G_P for $R(t,s)$. Moreover, if the associated Green's operator is given by* (8)*, then $\mathbb{G}=-\Gamma^{-1}$ on $L_p(\mathbb{R},X)$.*

We summarize that the sufficient and necessary conditions for a C_0-quasi group to have a uniformly exponential dichotomy are that the corresponding evolution semigroup is hyperbolic. Moreover, the dichotomy is equivalent to the uniqueness of Green's function for the C_0-quasi group.

3 Equivalent Conditions for Uniformly Exponential Dichotomy

In the section, we shall characterize the others equivalent conditions for the uniformly exponential dichotomy of the C_0-quasi groups. The characterizations refer to the method used in [1, 13] for the family of the evolution operators.

We start with defining Green's operator $\mathbb{G}$ for the C_0-quasi group $R(t,s)$ as in (8) on $C_b(\mathbb{R},X)$ by

$$(\mathbb{G}f)(t)=\int_{-\infty}^{\infty}G_P(s,t-s)f(s)ds,\quad f\in C_b(\mathbb{R},X). \tag{9}$$

We see that $\mathbb{G}$ is a bounded operator on $C_b(\mathbb{R},X)$.

Condition (M). For each $g\in C_b(\mathbb{R},X)$, there exists a unique function $u\in C_b(\mathbb{R},X)$ such that

$$u(t)=R(r,t-r)u(r)+\int_r^t R(s,t-s)g(s)ds,\quad t\geq r. \tag{10}$$

Remark 3.1 *Condition (M) states that for each $g\in C_b(\mathbb{R},X)$, there exists a unique mild solution $u\in C_b(\mathbb{R},X)$ of the integral equation* (10)*. Thus, if we define an operator*

$Gg = u$ on $C_b(\mathbb{R}, X)$, then G is closed. In fact, if $g_n \to g$ and $u_n := Gg_n \to u$ in $C_b(\mathbb{R}, X)$, then for each $t \in \mathbb{R}$,

$$\begin{aligned} u(t) = \lim_{n\to\infty} u_n(t) &= \lim_{n\to\infty} \left(R(r, t-r)u_n(r) + \int_r^t R(s, t-s)g_n(s)ds \right) \\ &= R(r, t-r)u(r) + \int_r^t R(s, t-s)g(s)ds. \end{aligned}$$

This gives $u = Gg$.

In particular, if $R(t, s)$ is uniformly exponentially dichotomic, then G is equal to Green's operator $\mathbb{G}$ in (9).

Lemma 3.1 *If Green's operator $\mathbb{G}$ defined in* (9) *is bounded on $C_b(\mathbb{R}, X)$, then for each $g \in C_b(\mathbb{R}, X)$, there exists a solution $u \in C_b(\mathbb{R}, X)$ of* (10).

Proof. For $g \in C_b(\mathbb{R}, X)$, we set $u := \mathbb{G}g$. For $t \geq r$, we show that u satisfies (10). In this proof, we use the fact that $R^{-1}(k, l-k) = R(l, k-l)$. For $t \geq r$,

$$\begin{aligned} u(t) - R(r, t-r)u(r) &= (\mathbb{G}g)(t) - R(r, t-r)(\mathbb{G}g)(r) \\ &= \int_r^t P(t)R(s, t-s)P(s)g(s)ds - \int_t^\infty R_Q^{-1}(s, t-s)Q(s)g(s)ds \\ &\quad + \int_r^t R(s, t-s)R(r, s-r)R_Q^{-1}(s, r-s)Q(s)g(s)ds \\ &\quad + \int_t^\infty R(r, t-r)[R_Q(t, r-t)R_Q(s, t-s)]^{-1}Q(s)g(s)ds \\ &= \int_r^t P(t)R(s, t-s)P(s)g(s)ds + \int_r^t R(s, t-s)Q(s)g(s)ds \\ &= \int_r^t R(s, t-s)g(s)ds. \end{aligned}$$

As a generalization of Theorem 10 from [18], we have the following lemma which implies that the infinitesimal generator Γ is invertible on $L_p(\mathbb{R}, X)$.

Lemma 3.2 *Let E^s be the evolution semigroup defined in* (5) *on $L_p(\mathbb{R}, X)$ with its infinitesimal generator Γ in* (6). *If $u, g \in L_p(\mathbb{R}, X)$, then the following statements are equivalent.*

(a) $u \in \mathcal{D}(\Gamma)$ dan $\Gamma u = -g$.

(b) u is a solution of the integral equation (10) *that corresponds to g.*

Proof. (a) $\Rightarrow$ (b). Assume that (a) holds. By an elementary property of C_0-semigroup, we have

$$E^s u - u = \int_0^s E^r \Gamma u dr = -\int_0^s E^r g\, dr, \quad s \geq 0. \tag{11}$$

Substituting $(E^s u)(t) = R(t-s, s)u(t-s)$ (definition of E^s) into (11) gives

$$R(t-s, s)u(t-s) - u(t) = -\int_0^s R(t-v, v)g(t-v)\, dv.$$

The transformation of variable $r = t - s$ gives statement (b).

(b)⇒ (a). Assume that (b) holds. If $s \geq 0$, $t - s \geq r$, and u is a solution of (10), then

$$\begin{aligned}(E^s u)(t) &= R(t-s,s)\left[R(r,t-s-r)u(r) + \int_r^{t-s} R(v,t-s-v)g(v)\,dv\right] \\ &= R(r,t-r)u(r) + \int_r^{t-s} R(v,t-v)g(v)\,dv.\end{aligned}$$

Consequently, for $s > 0$, we obtain

$$\begin{aligned}s^{-1}\left[(E^s u)(t) - u(t)\right] &= s^{-1}\Big[R(r,t-r)u(r) + \int_r^{t-s} R(v,t-v)g(v)\,dv \\ &\qquad - \left(R(r,t-r)u(r) + \int_r^{t} R(v,t-v)g(v)\,dv\right)\Big] \\ &= -s^{-1}\int_{t-s}^{t} R(v,t-v)g(v)\,dv = -s^{-1}\int_0^{s} R(t-v,v)g(t-v)\,dv.\end{aligned}$$

Therefore,

$$s^{-1}(E^s u - u) = -s^{-1}\int_0^s E^v g\,dv.$$

Passing to the limit as $s \to 0^+$ proves that $u \in \mathcal{D}(\Gamma)$ and $\Gamma u = -g$.

Remark 3.2 *Lemma 3.2 remains valid if $L_p(\mathbb{R},X)$ is replaced by $C_0(\mathbb{R},X)$, the space of all continuous functions $f : \mathbb{R} \to X$ such that $\lim_{t\to\pm\infty} f(t) = 0$ with the supremum norm. Moreover, Condition (M) holds for some $g, u \in L_p(\mathbb{R},X)$.*

Theorem 3.1 *An exponentially bounded C_0-quasi group $R(t,s)$ on a Banach space X has a uniformly exponential dichotomy if and only if Condition (M) is satisfied.*

Proof. (⇒). Let $R(t,s)$ be uniformly exponentially dichotomic. By Theorem 9 of [3], there exists Green's operator $\mathbb{G}$ as defined in (9) corresponding to Green's function G_P and dichotomy projection P. Lemma 3.1 guarantees the existence of a solution $u \in C_b(\mathbb{R},X)$ of (10) for each $g \in C_b(\mathbb{R},X)$.

To prove the uniqueness of the solution of (10), let $g = 0$ and suppose there exists $u \in C_b(\mathbb{R},X)$ such that $u(t) = R(r,t-r)u(r), t \geq r$. It suffices to prove that $u = 0$. The uniformly exponential dichotomy of $R(t,s)$ implies

$$P(t)u(t) = R_P(r,t-r)P(r)u(r) \quad \text{and} \quad Q(t)u(t) = R_Q(r,t-r)Q(r)u(r), \quad t \geq r.$$

The boundedness of $\|u(\cdot)\|$ and condition (c) of Definition 2.5 give

$$\|P(t)u(t)\| \leq Ne^{-\gamma(t-r)}\|u(r)\|.$$

Passing to the limit as $r \to -\infty$ provides that $P(t)u(t) = 0$ for all $t \in \mathbb{R}$. On the other hand, condition (d) of Definition 2.5 forces

$$\|Q(r)u(r)\| = \|[R_Q(r,t-r)]^{-1}Q(t)u(t)\| \leq Ne^{-\gamma(t-r)}\|u(t)\|.$$

Passing to the limit as $t \to \infty$ implies that $Q(r)u(r) = 0$ for all $r \in \mathbb{R}$. Therefore, $u = 0$.

($\Leftarrow$). Let Condition (M) be satisfied. We define an operator G on $C_b(\mathbb{R}, X)$ by $Gg = u$. By Theorem 2.1, it suffices to show that Γ is invertible on $C_b(\mathbb{R}, X)$. Since $u = Gg$ and $g \in L_p(\mathbb{R}, X)$, Lemma 3.2 implies that $u \in \mathcal{D}(\Gamma)$ and $\Gamma(-G)g = \Gamma(-u) = g$. Thus, Γ is right invertible. On the other hand, the linearity of G implies that $(-G)\Gamma u = (-G)(-g) = u$. This proves the left invertibility of Γ. Thus, Γ is invertible with $\Gamma^{-1} = -G$.

We shall characterize the other conditions for the uniformly exponential dichotomy of the quasi groups. We start with defining the scale of function space $\mathcal{F}_\alpha$, $\alpha > 0$, by

$$\mathcal{F}_\alpha := \{f \in C(\mathbb{R}, X) : e^{-\alpha|\cdot|} f(\cdot) \in C_b(\mathbb{R}, X)\}.$$

Thus, $\mathcal{F}_\alpha$ is the space of continuous, exponentially bounded functions with exponent α. These spaces provide three conditions formulated as follows.

Condition ($\mathbf{M}_{C_0}$). For each $g \in C_0(\mathbb{R}, X)$, the integral equation (10) has a unique solution $u \in C_0(\mathbb{R}, X)$.

Condition ($\mathbf{M}_{L_p}$). For each $g \in L_p(\mathbb{R}, X)$, $1 \leq p \leq \infty$, the integral equation (10) has a unique solution $u \in L_p(\mathbb{R}, X)$.

Condition ($\mathbf{M}_{\mathcal{F}_\alpha}$). For each $g \in \mathcal{F}_\alpha$, the integral equation (10) has a unique solution $u \in \mathcal{F}_\alpha$.

Theorem 3.2 *Let $R(t, s)$ be an exponentially bounded C_0-quasi group on X.*

(a) The following statements are equivalent:

(i) $R(t, s)$ has uniformly exponential dichotomy.

(ii) Condition (M) holds.

(iii) Condition (M_{C_0}) holds.

(iv) Condition (M_{L_p}) holds.

(b) The operator G defined by Conditions (M), (M_{C_0}), or (M_{L_p}) as in Remark 3.1, is equal to Green's operator $\mathbb{G}$ as in (9). *Further, if E^s is the evolution semigroup on the space $C_0(\mathbb{R}, X)$ or $L_p(\mathbb{R}, X)$ with the infinitesimal generator Γ, then $G = -\Gamma^{-1}$.*

Proof. Theorem 3.1 guarantees that Condition (M) is equivalent to (i).

Let G be an operator defined using Condition (M_{C_0}) (resp. (M_{L_p})) as in Remark 3.1. Lemma 3.2 together with Dichotomy Theorem 2.1 implies the uniformly exponential dichotomy for $R(t, s)$. These show that (iii)(resp. (iv)) is equivalent to (i).

If $R(t, s)$ has a uniformly exponential dichotomy, then by Theorem 2.2, Green's operator $\mathbb{G}$ is defined on $L_p(\mathbb{R}, X)$ or $C_0(\mathbb{R}, X)$ satisfies $\mathbb{G} = -\Gamma^{-1}$. Moreover, using the same argument as in the proof of the necessity of Theorem 3.1, we conclude that (M_{C_0}) and (M_{L_p}) hold, and $G = \mathbb{G}$.

Lemma 3.3 *Condition ($M_{\mathcal{F}_\alpha}$) holds for $R(t, s)$ if and only if Condition (M) holds for $R_\alpha(t, s)$, where $R_\alpha(t, s) = e^{-\alpha(|t+s|-|t|)} R(t, s)$ and $\alpha \in [0, \beta)$ for some $\beta > 0$.*

Proof. If Condition (M) holds for $R_\alpha(t, s)$, there exists a bounded operator G_α on $C_b(\mathbb{R}, X)$ defined by $G_\alpha g = u$. We define an operator $J_\alpha : \mathcal{F}_\alpha \to C_b(\mathbb{R}, X)$ by $(J_\alpha f)(t) = e^{-\alpha|t|} f(t), t \in \mathbb{R}$. Similarly, if Condition ($\mathrm{M}_{\mathcal{F}_\alpha}$) holds for $R(t, s)$, then there exists a bounded operator $G \in \mathcal{L}(\mathcal{F}_\alpha)$ defined by $Gg = u$. We see that $G_\alpha = J_\alpha G J_\alpha^{-1}$. Thus, Condition (M) holds for $R_\alpha(t,s)$ if and only if $G_\alpha \in \mathcal{L}(\mathbb{R}, X)$. However, $G \in \mathcal{L}(\mathcal{F}_\alpha)$ if and only if Condition ($M_{\mathcal{F}_\alpha}$) holds for $R(t,s)$.

Theorem 3.3 *Let $R(t,s)$ be an exponentially bounded C_0-quasi group on X. The quasi group $R(t,s)$ has a uniformly exponential dichotomy if and only if there exists $\beta > 0$ such that if $\alpha \in [0,\beta)$, then Condition ($M_{\mathcal{F}_\alpha}$) holds for $R(t,s)$. Moreover, for each $\alpha > 0$ and $g \in \mathcal{F}_\alpha$, the solution of the integral equation* (10) *is given by $u = Gg$, where $G \in \mathcal{L}(\mathcal{F}_\alpha)$ is equal to Green's operator $\mathbb{G}$ on $\mathcal{F}_\alpha$ as defined in* (9).

Proof. ($\Leftarrow$). If $\alpha = 0$, then Condition ($M_{\mathcal{F}_\alpha}$) and Condition (M) are identical.

($\Rightarrow$). Assume that $R(t,s)$ has a uniformly exponential dichotomy with the dichotomy bound $\gamma > 0$. If $\beta \in (0,\gamma)$, then $R(t,s)$ has a uniformly exponential dichotomy with constants β and $N = N(\beta)$, see (7). Consequently, if $\alpha \in [0,\beta)$, then the quasi group $R_\alpha(t,s)$ defined in Lemma 3.3 has a uniformly exponential dichotomy with constants $N(\beta)$ and $\beta - \alpha$. Theorem 3.1 provides that Condition (M) holds for $R_\alpha(t,s)$. Let $G \in \mathcal{L}(\mathcal{F}_\alpha)$ be the operator defined by $Gg = u$. Since $G_\alpha = \mathbb{G}_\alpha$, where $\mathbb{G}_\alpha$ is Green's operator for the dichotomic quasi group $R_\alpha(t,s)$ and G_α is as in the proof of Lemma 3.3, the assertions follow.

Remark 3.3 *We note that conditions (M), (M_{C_0}), (M_{L_p}), and ($M_{\mathcal{F}_\alpha}$) for the uniformly exponential dichotomy of the C_0-quasi groups are parallel with the similar conditions for exponential dichotomy of the evolution family, see [1, 13].*

Example 3.1 Let $X = \mathbb{R}^2$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous increasing function such that $\lim_{t\to\pm\infty} \varphi(t) < \infty$. Define a C_0-quasi group on X by

$$R(t,s)x = \left(e^{-(v(t+s)-v(t))}x_1, e^{-s\varphi(0)+v(t+s)-v(t)}x_2\right), \quad t,s \in \mathbb{R},$$

where $v(t) = \int_0^t \varphi(s)ds$ and $x = (x_1, x_2)$. The quasi group $R(t,s)$ has a uniformly exponential dichotomy on X.

Similar to Example 3 of [3], we have the evolution semigroup E^s in (5) on the space $L_p(\mathbb{R}, X)$ given by

$$(E^s f)(t) = \left(e^{-(v(t)-v(t-s))}f_1(t-s), e^{-s\varphi(0)+v(t)-v(t-s)}f_2(t-s)\right),$$

where $f(t) = (f_1(t), f_2(t))$, $s \geq 0$, and $t \in \mathbb{R}$ with the infinitesimal generator

$$(\Gamma f)(t) = (-f_1'(t) - \varphi(t)f_1(t), -f_2'(t) + [-\varphi(0) + \varphi(t)]f_2(t)).$$

Moreover,

$$(\Gamma^{-1}f)(t) = -(h_1(t), h_2(t)),$$

where

$$h_1(t) = e^{-\phi(t)}\int f_1(t)e^{\phi(t)}dt, \qquad h_2(t) = e^{-\varphi(0)t+\phi(t)}\int f_2(t)e^{\varphi(0)t-\phi(t)}dt,$$

$$\phi(t) = \int \varphi(t)\,dt.$$

By Condition (M), for each $g \in C_b(\mathbb{R}, X)$, there exists a unique solution $u \in C_b(\mathbb{R}, X)$ satisfying the integral equation (10). In fact, we have $u = -\Gamma^{-1}g$. Therefore, $R(t,s)$ has a uniformly exponential dichotomy on X.

Remark 3.4 *We can easily verify that Example 3.1 fulfills Conditions (M_{C_0}), (M_{L_p}) and ($M_{\mathcal{F}_\alpha}$). It is possible that Condition ($M_{\mathcal{F}_\alpha}$) holds for some $\alpha \in (0, \gamma)$, but the quasi group $R(t,s)$ has no uniformly exponential dichotomy, as shown by the following example.*

Example 3.2 Let X be a Banach space of $\mathbb{R}^2$ with the norm $\|x\| = |x_1| + |x_2|$, where $x = (x_1, x_2)$. The quasi group $R(t,s)$ defined on X by

$$R(t,s)x = \left(e^{(t+s)\cos(t+s) - t\cos t - s} x_1, e^s x_2\right), \qquad t, s \in \mathbb{R},$$

has no uniformly exponential dichotomy, but it satisfies Condition ($\mathrm{M}_{\mathcal{F}_\alpha}$) for all $g \in M_{\mathcal{F}_\alpha}$ and $0 < \alpha < 2$.

From Lemma 3.3, it suffices to show that $R_\alpha(t,s)$ satisfies Condition (M) for all $g \in C_b(\mathbb{R}, X)$. In fact, for $g = (g_1, g_2) \in C_b(\mathbb{R}, X)$ and $P(t)x = (x_1, 0)$, we can set $u = \mathbb{G}g$, where $\mathbb{G}$ is Green's operator defined in (9) with respect to $R_\alpha(t,s)$. For $0 < \alpha < 2$, we verify that

$$u(t) = (\mathbb{G}g)(t) = (u_1(t), u_2(t)) \in C_b(\mathbb{R}, X),$$

where

$$\begin{aligned} u_1(t) &= e^{-\alpha|t| - t + t\cos t} \int_{-\infty}^{t} e^{\alpha|s| + s - s\cos s} g_1(s) ds, \\ u_2(t) &= -e^{-\alpha|t|} \int_{t}^{\infty} e^{-\alpha|s| - s} g_2(s) ds. \end{aligned}$$

Suppose that $R(t,s)$ has uniformly exponential dichotomy with respect to the family of projections $P(t)$ above. If N, $\gamma > 0$ are the constants satisfying Definition 2.5, i.e., $\|R_P(t,s)\| \le N e^{-\gamma|s|}$, then

$$e^{(t+s)\cos(t+s) - t\cos t - s} \le N e^{-\gamma|s|}$$

for all $t, s \in \mathbb{R}$. But for $t = (2n-1)\pi$ and $s = \pi$, we have $e^{2(2n-1)\pi} \le N e^{-\gamma\pi}$, which is absurd for large enough n.

4 Persistence under Perturbation

Theorem 2.1 implies that the existence of a dichotomy for a strongly continuous quasi group $R(t,s)$ is a spectral property. It persists under small perturbations. We shall first consider the bounded perturbation.

Theorem 4.1 *Let $R(t,s)$ and $R_1(t,s)$ be the C_0-quasi groups on a Banach space X. If $R(t,s)$ has a uniformly exponential dichotomy on X, then for each $r > 0$, there exists an $\epsilon > 0$ such that $R_1(t,s)$ has a uniformly exponential dichotomy and*

$$\sup_{t \in \mathbb{R}} \|R_1(t,r) - R(t,r)\|_{\mathcal{L}(X)} \le \epsilon.$$

Proof. From (5), for $f \in L_p(\mathbb{R}, X)$, we have

$$(E^r f)(t) = R(t-r, r) f(t-r) \quad \text{and} \quad (E_1^r f)(t) = R_1(t-r, r) f(t-r).$$

We obtain the estimate

$$\|E_1^r f - E^r f\|_{L_p}^p = \int_{\mathbb{R}} \|R_1(t-r,r)f(t-r) - R_1(t-r,r)f(t-r)\|^p dt$$
$$= \int_{\mathbb{R}} \|[R_1(t,r) - R(t,r)]f(t)\|^p dt \le \epsilon^p \|f\|_{L_p}^p.$$

This implies that $\|E_1^r - E^r\|_{\mathcal{L}(L_p(\mathbb{R},X))} \le \epsilon$.

The equivalence of (a) and (b) in the Dichotomy Theorem 2.1 gives $\sigma(E^r) \cap \mathbb{T} = \emptyset$. The semicontinuity of the spectrum implies that $\sigma(E_1^r) \cap \mathbb{T} = \emptyset$ for a sufficiently small ϵ. Therefore, $R_1(t,s)$ has a uniformly exponential dichotomy.

Theorem 4.1 describes that a dichotomy persists under small perturbation of the C_0-quasi groups. The similar result of the additive perturbation is given by the following theorem. The theorem refers to the perturbed generator of the C_0-quasi groups given below.

Theorem 4.2 *Let $A(t)$ be the infinitesimal generator of a C_0-quasi group $R(t,s)$ on a Banach space X. If $B \in C_b(\mathbb{R}, \mathcal{L}(X))$, then there exists a unique C_0-quasi group $R_B(t,s)$ with the infinitesimal generator $A(t) + B(t)$ such that*

$$R_B(t,r)x = R(t,r)x + \int_0^r R(t+s, r-s)B(t+s)R_B(t,s)x ds \tag{12}$$

for all $t \in \mathbb{R}$, $r > 0$, and $x \in X$. Moreover, if $\|R(t,r)\| \le M(r)$, then

$$\|R_B(t,r)\| \le M(r)e^{\|B\|M(r)r}.$$

Proof. The proof is similar to the proof of Theorem 3 of [18].

Theorem 4.3 *Let $R(t,s)$ be the C_0-quasi group with the infinitesimal generator $A(t)$ which has a uniformly exponential dichotomy on a Banach space X. Then, there exists $\epsilon > 0$ such that for each $B \in C_b(\mathbb{R}, \mathcal{L}(X))$ with $\|B\|_\infty := \sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \le \epsilon$, there exists a C_0-quasi group $R_B(t,s)$ with the infinitesimal generator $A(t) + B(t)$ which has a uniformly exponential dichotomy on X.*

Proof. From Theorem 4.2, there exists a C_0-quasi group $R_B(t,s)$ with the infinitesimal generator $A(t) + B(t)$. Further, by (12), for $t > r$ and $x \in X$, we have

$$R_B(r, t-r)x = R(r, t-r)x + \int_0^{t-r} R(r+s, t-r-s)B(r+s)R_B(r,s)x ds. \tag{13}$$

Let Γ and Γ_B be the infinitesimal generators of the evolution semigroups corresponding to the C_0-quasi groups $R(t,s)$ and $R_B(t,s)$, respectively.

We consider the operator $\Gamma + \mathcal{B}$, where $(\mathcal{B}f)(t) = B(t)f(t)$, $t \in \mathbb{R}$. Since $\mathcal{B}$ is a bounded operator, the operator $\Gamma + \mathcal{B}$ generates a unique C_0-semigroup $T(s)$ satisfying the equation

$$T(s)f = E^s f + \int_0^s E^{s-w}\mathcal{B}T(w)f\, dw, \quad E^s = e^{s\Gamma}, \quad s \ge 0. \tag{14}$$

The implication $(a) \Rightarrow (c)$ of Theorem 2.1 gives $0 \in \rho(\Gamma)$. Consequently, if $\|\mathcal{B}\| = \|B\|_\infty \le \epsilon$, then $0 \in \rho(\Gamma + \mathcal{B}) = \rho(\Gamma_B)$. The implication $(c) \Rightarrow (a)$ of Theorem 2.1 concludes that $R_B(t,s)$ has an exponential dichotomy.

From (13), with $s = t - r$ and $x = f(t - r)$, we have

$$(e^{s\Gamma_B} f)(t) = (E^s f)(t) + \int_0^s (E^{s-w} B e^{w\Gamma_B} f)(t)\, dw, \quad t \in \mathbb{R}.$$

In this case, we have proved that $e^{s\Gamma_B} = T(s)$ satisfies (14) and $\Gamma_B = \Gamma + \mathcal{B}$.

Next, we shall prove the persistence of a uniformly exponential dichotomy for a C_0-quasi group $R(t, s)$ with the infinitesimal generator $A(t)$ relative to the class of perturbations that satisfy the Miyadera condition. Theorem 2.1 implies that if Γ is the infinitesimal generator of the evolution semigroup E^s associated with a uniformly exponentially dichotomic C_0-quasi group $R(t, s)$, then Γ is invertible on $L_p(\mathbb{R}, X)$. Dichotomy Theorem 2.1 implies the following result.

Theorem 4.4 *Let $R(t, s)$ be a uniformly exponentially dichotomic C_0-quasi group with the infinitesimal generator $A(t)$ and $R_1(t, s)$ be a C_0-quasi group with the infinitesimal generator $A(t) + B(t)$. Assume that $\mathcal{B}$ is an operator on the domain $\mathcal{D}(\Gamma) \cap \mathcal{D}(\mathcal{B})$, which has an extension $\hat{\mathcal{B}}$ on $\mathcal{D}(\Gamma)$ such that the operator $\Gamma_1 := \Gamma + \mathcal{B}$ on $\mathcal{D}(\Gamma_1) = \mathcal{D}(\Gamma)$ generates the evolution semigroup associated with $R_1(t, s)$. If there exist constants a and b such that*

$$\|\hat{\mathcal{B}} f\| \le a\|f\| + b\|\Gamma f\| \quad \text{for} \quad f \in \mathcal{D}(\Gamma) \quad \text{and} \quad a\|\Gamma^{-1}\| + b < 1,$$

then the perturbed quasi group $R_1(t, s)$ has a uniformly exponential dichotomy.

Proof. Theorem IV.1.16 [20] implies that Γ_1 is invertible on $L_p(\mathbb{R}, X)$. Since Γ_1 is the infinitesimal generator of the evolution semigroup associated with $R_1(t, s)$, the assertion follows from the implication (c) $\Rightarrow$ (a) of Dichotomy Theorem 2.1.

Example 4.1 Consider the quasi group $R(t, s)$ in Example 3.1, which has a uniformly exponential dichotomy on $X = \mathbb{R}^2$ with the norm $\|x\| = |x_1| + |x_2|$ and $\varphi(0) < -1$. Under a perturbation

$$B(t) = \begin{cases} 0, & t < 0, \\ -t, & 0 \le t \le 1, \\ -1, & t > 1, \end{cases}$$

$R(t, s)$ persists the uniformly exponential dichotomy on X.

We notice that $R(t, s)$ has the infinitesimal generator

$$A(t)x = (-\varphi(t)x_1, [-\varphi(0) + \varphi(t)]x_2), \quad x \in X.$$

Given $\epsilon = 1$, we verify that $B \in C_b(\mathbb{R}, \mathcal{L}(X))$ with $\|B\|_\infty = \epsilon$. By Theorem 4.3, there exists a uniformly exponentially dichotomic quasi group $R_B(t, s)$ on X generated by $A(t) + B(t)$. Indeed, we have $R_B(t, s) = \mathcal{B}(t, s)R(t, s)$, where

$$\mathcal{B}(t, s) = \begin{cases} 1, & t, s < 0, \\ e^{-\frac{1}{2}(s^2 + 2st)}, & 0 \le t, s \le 1, \\ e^{-s}, & t, s > 1. \end{cases}$$

Moreover, by the mean value theorem for the integral with respect to φ, we obtain the dichotomy constants $N = \max\left\{1, e^{\frac{3}{2} + \varphi(0)}\right\}$ and $\gamma = \inf_{t \in \mathbb{R}} \varphi(t)$ in Definition 2.5 for $R_B(t, s)$, where $\beta = \sup_{t \in \mathbb{R}} \varphi(t)$.

5 Conclusions

In this paper, we provide four equivalent conditions for uniformly exponential dichotomy of C_0-quasi groups on Banach spaces. They base on the existence and uniqueness of mild solutions of the inhomogeneous equations on $C_b(\mathbb{R}, X)$, $C_0(\mathbb{R}, X)$, $L_p(\mathbb{R}, X)$, $1 \leq p < \infty$, and $\mathcal{F}_\alpha$, respectively. The equivalent conditions are parallel with the exponential dichotomy for the evolution family. A small time-dependent perturbation of the infinitesimal generator of the C_0-quasi groups persists the uniformly exponential dichotomy.

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