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The Solution of the Second Part of the 16th Hilbert Problem for a Class of Piecewise Linear Hamiltonian Saddles Separated by Conics

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Abstract: In this paper, we study the existence of the maximum number of crossing limit cycles of planar piecewise differential systems formed by linear Hamiltonian saddles. Firstly, we prove that if we separate these systems by either a parabola or hyperbola or an ellipse, they can have at most three crossing limit cycles. Secondly, we provide an example of four crossing limit cycles when these systems have four zones separated by two intersecting straight lines $xy = 0$.

Keywords: *piecewise differential system, limit cycles, linear Hamiltonian saddles, conics.*

Mathematics Subject Classification (2010): 34A36, 34A07, 34C25.

1 Introduction

One of the important and difficult problems in the qualitative study of differential systems is the determination of the existence or non-existence of limit cycles and their position in the plane, the same problem arises for the piecewise linear differential systems separated by an algebraic curve. Planar discontinuous piecewise linear differential systems were firstly studied by Andronov, Vit and Khaikin [1].

Recently, these systems have been of great importance to the mathematical community due to their applicability to modeling and control of the environment, see for example the books [7, 14].

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Many authors studied the upper bound of crossing limit cycles that some families of discontinuous piecewise differential systems can have. In 2010, Han and Zhang [9] conjectured that we can have two crossing limit cycles when we separate planar discontinuous piecewise linear differential systems by a straight line, but in 2012, Huan and Yang [10] proved that the conjecture of Han is wrong by proving the existence of a numerical example with three limit cycles. Afterward, Llibre and Ponce in [12] proved analytically the existence of this example. In 2015, Llibre et al. [11] showed that the discontinuous piecewise linear differential centers sepatated by a straight line can not exhibit any limit cycle, while if we consider that the curve of discontinuity is different from a straight line, we can produce limit cycles, see for example the papers [3–5]. For another kind of discontinuous planar piecewise differential systems, Benterki and Llibre [2, 6] studied the existence of limit cycles of planar piecewise linear Hamiltonian systems without equilibrium points, where they solved the 16th Hilbert problem of these systems when the curve of separation are conics or irreducible cubic curves.

In [8], Damene and Benterki provided the maximum number of crossing limit cycles of two different families of discontinuous piecewise linear differential systems separated by cubic curves.

Our objective in this paper is to study the crossing limit cycles of planar piecewise differential systems with linear Hamiltonian saddles separated by conics.

We recognize that each conic occurs in nine canonical forms, but we omit some of them due to the fact that they do not separate the plane into connected regions such as $x^2 + 1 = 0$, $x^2 + y^2 = 0$, and $x^2 + y^2 + 1 = 0$.

In [13], the authors proved that the maximum number of limit cycles for discontinuous planar piecewise differential systems formed by linear Hamiltonian saddles and separated by two parallel straight lines is at most one.

The main goal of our work is to provide the upper bounds of crossing limit cycles of discontinuous planar piecewise differential linear Hamiltonian saddles (or simply **PHS**) separated by either an ellipse $x^2 + y^2 - 1 = 0$, or a parabola $y - x^2 = 0$, or a hyperbola $x^2 - y^2 = 1$ or by the two intersecting straight lines $xy = 0$. The main tool that we used to prove our results is the first integrals method.

A normal form for an arbitrary linear differential system with Hamiltonian saddles is given in the following proposition. For the proof, see for instance [13].

Proposition 1.1 *Differential systems with a linear Hamiltonian saddle can be written as*

$$\dot{x} = -bx - \delta y + d, \qquad \dot{y} = \alpha x + by + c, \tag{1}$$

where $\alpha \in \{0, 1\}$ and $b, \delta, c, d \in \mathbb{R}$. Moreover, if $\alpha = 0$, then $c = 0$, and if $\alpha = 1$, then $\delta = b^2 - \omega^2$ with $\omega \neq 0$. The corresponding first integral of system (1) is

$$H(x, y) = -(\alpha/2)x^2 - bxy - (\delta/2)y^2 - cx + dy.$$

2 Statements of the Main Results

In this section, and specialy in Theorem 2.1, we prove our results for discontinuous piecewise differential systems formed by linear Hamiltonian saddles intersecting the parabola, or hyperbola or ellipse at two points. While in Theorem 2.2 we are interested in studying the number of crossing limit cycles intersecting the straight lines $xy = 0$ at exactly four points. Our first main result is the following.

Theorem 2.1 *The following statements hold.*

- (i) *The maximum number of crossing limit cycles of* **PHS** *intersecting the parabola at two points, is at most three. This maximum is reached in Figure 1.*
- (ii) *The maximum number of crossing limit cycles of* **PHS** *intersecting the hyperbola at two points, is at most three. This maximum is reached in Figure 2.*
- (iii) *The maximum number of crossing limit cycles of* **PHS** *intersecting the ellipse at two points, is at most three. This maximum is also reached in Figure 3.*

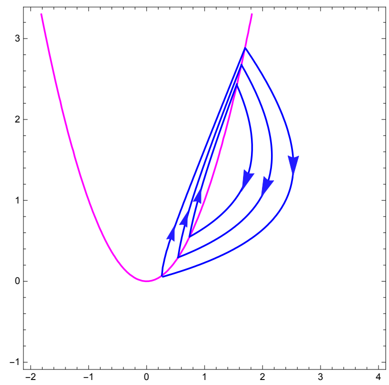

Figure 1: Three crossing limit cycles of piecewise differential system (7)–(8).

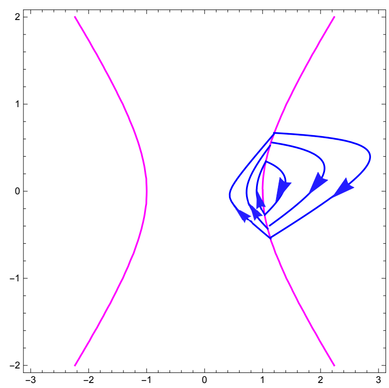

Figure 2: Three crossing limit cycles of piecewise differential system (10)–(11).

Theorem 2.2 *The maximum number of crossing limit cycles of piecewise linear differential systems formed by four linear Hamiltonian saddles and separated by the two intersecting straight lines $xy = 0$, is at most eight. There is an example of these systems with exactly four limit cycles.*

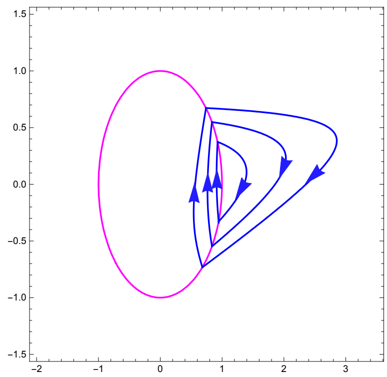

Figure 3: Three crossing limit cycles of piecewise differential system (13)–(14).

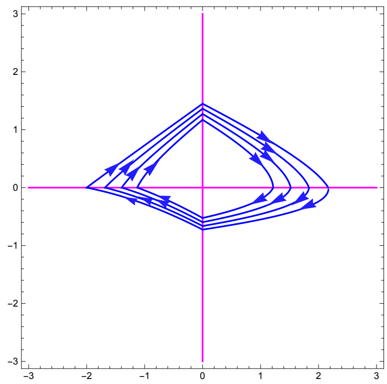

Figure 4: Four crossing limit cycles of piecewise differential system (21)–(24).

3 Proof of Theorem 2.1

Proof. In this part, we are going to prove the statement (i) of Theorem 2.1. Then in the first region $R_1 = \{(x, y) : y - x^2 \geq 0\}$, we consider the planar discontinuous piecewise Hamiltonian saddle

$$\dot{x} = -b_1 x - \delta_1 y + d_1, \quad \dot{y} = \alpha_1 x + b_1 y + c_1, \tag{2}$$

its corresponding Hamiltonian function is

$$H_1(x, y) = -\frac{\alpha_1}{2} x^2 - b_1 xy - \frac{\delta_1}{2} y^2 - c_1 x + d_1 y. \tag{3}$$

In the second region $R_2 = \{(x, y) : y - x^2 \leq 0\}$, we consider the **PHS** system

$$\dot{x} = -b_2 x - \delta_2 y + d_2, \quad \dot{y} = \alpha_2 x + b_2 y + c_2, \tag{4}$$

with its corresponding Hamiltonian function

$$H_2(x, y) = -\frac{\alpha_2}{2} x^2 - b_2 xy - \frac{\delta_2}{2} y^2 - c_2 x + d_2 y. \tag{5}$$

In order to have a crossing limit cycle that intersects the parabola $y - x^2 = 0$ at the points (x_i, y_i) and (x_k, y_k), with $i \neq k$, these points must satisfy the following system:

$$\begin{aligned} &H_1(x_i, y_i) - H_1(x_k, y_k)) = 0,\\ &H_2(x_i, y_i) - H_2(x_k, y_k) = 0,\\ &y_i - x_i^2 = 0,\ y_k - x_k^2 = 0. \end{aligned} \tag{6}$$

We suppose that the system (2)–(4) has four crossing limit cycles. Then, system (6) must have four pairs of points as solutions, namely, p_i and q_i taking the forms $p_i = (r_i, r_i^2)$ and $q_i = (s_i, s_i^2)$, with $i = 1, 2, 3, 4$. Due to the fact that these points satisfy system (6) and if we consider the points $p_1 = (r_1, r_1^2)$ and $q_1 = (s_1, s_1^2)$, then simple calculations give the following expressions of the parameters c_1 and c_2:

$$c_1 = \frac{1}{2}\Big(2d_1(r_1 + s_1) - 2b_1(r_1^2 + r_1 s_1 + s_1^2) - (r_1 + s_1)(\alpha_1 + (r_1^2 + s_1^2)\delta_1)\Big),$$

and c_2 has the same expression as c_1 with the change of $(d_1, \delta_1, b_1, \alpha_1)$ by $(d_2, \delta_2, b_2, \alpha_2)$.

If the two points $p_2 = (r_2, r_2^2)$ and $q_2 = (s_2, s_2^2)$ satisfy system (6), then by solving the two first equations of (6), we obtain the expressions of the two parameters d_1 and d_2

$$\begin{aligned} d_1 = \ & \frac{1}{2(r_1 - r_2 + s_1 - s_2)}\Big(2b_1(r_1^2 - r_2^2 + r_1 s_1 + s_1^2 - r_2 s_2 - s_2^2) - r_2\alpha_1 + s_1\alpha_1\\ & -s_2\alpha_1 + r_1^3\delta_1 - r_2^3\delta_1 + r_1^2 s_1\delta_1 + s_1^3\delta_1 - r_2^2 s_2\delta_1 - r_2 s_2^2\delta_1 - s_2^3\delta_1 + r1(\alpha_1 + s_1^2\delta_1)\Big), \end{aligned}$$

and d_2 has the same expression as d_1 with the change of $(\delta_1, b_1, \alpha_1)$ by $(\delta_2, b_2, \alpha_2)$.

Now let us suppose that the points $p_3 = (r_3, r_3^2)$ and $q_3 = (s_3, s_3^2)$ satisfy system (6), then the parameters δ_1 and δ_2 must be $\delta_1 = A/B$, where

$$\begin{aligned} A = \ & -2b_1\Big((s_1 - s_2)(r_3^2 + (s_1 - s_3)(s_2 - s_3) - r_3(s_1 + s_2 - s_3)) + r_1^2(r_2 - r_3 + s_2\\ & -s_3) + r_2^2(r_3 - s_1 + s_3) + r_1(-r_2^2 + r_3^2 - r_3 s_1 + r_2(s_1 - s_2) + s_1 s_2 - s_2^2 + r_3 s_3\\ & -s_1 s_3 + s_3^2) - r_2(r_3^2 + r_3(-s_2 + s_3) - (s_1 - s_3)(s_1 - s_2 + s_3))\Big), \end{aligned}$$

$$\begin{aligned} B = \ & r_1^3(r_2 - r_3 + s_2 - s_3) + r_1^2 s_1(r_2 - r_3 + s_2 - s_3) + r_2^3(r_3 - s_1 + s_3) + r_2^2 s_2(r_3\\ & -s_1 + s_3) + r_1(-r_2^3 + r_3^3 - r_3 s_1^2 - r_2^2 s_2 + s_1^2 s_2 - s_2^3 + r_2(s_1^2 - s_2^2) + r_3^2 s_3 + s_3^3\\ & +r_3 s_3^2 - s_1^2 s_3) + (s_1 - s_2)(r_3^3 + r_3^2 s_3 + (s_1 - s_3)(s_2 - s_3)(s_1 + s_2 + s_3) - r_3(s_1^2\\ & +s_1 s_2 + s_2^2 - s_3^2)) - r_2(r_3^3 - s_1^3 + s_1 s_2^2 + r_3^2 s3 - s_2^2 s3 + s_3^3 + r_3(-s_2^2 + s_3^2)), \end{aligned}$$

and we get the expression of δ_2 by changing (b_1, α_1) by (b_2, α_2) in the expression of δ_1. Finally, if we suppose that the points $p_4 = (r_4, r_4^2)$ and $q_4 = (s_4, s_4^2)$ satisfy system (6) and if $\alpha_i \in \{0, 1\}$ with $i = 1, 2$, then we obtain $b_1 = 0$ and $b_2 = 0$.

We replace $c_1, d_1, \delta_1, \alpha_1$ and b_1 in the expression of $H_1(x, y)$, and $c_2, d_2, \delta_2, \alpha_2$ and b_2 in the expression of $H_2(x, y)$, we known that the expression of the first integral $H_1(x, y)$ is the same as the expression of the first integral $H_2(x, y)$, i.e., $H_1(x, y) = H_2(x, y)$. Therefore, the piecewise linear differential system (2)–(4) becomes a linear differential system, which does not have limit cycles. Consequently, the maximum number of crossing limit cycles in this case is at most three.

Example with three limit cycles. Consider the planar discontinuous piecewise linear Hamiltonian saddle

$$\dot{x} = 75x + 250y - 550, \quad \dot{y} = -75y - 100, \tag{7}$$

in the region R_1 with its corresponding Hamiltonian function

$$H_1(x,y) = 125y^2 + 75xy + 100x - 550y.$$

In the region R_2, we consider the **PHS** system

$$\dot{x} = 0.07125x + 0.2375y - 0.0225, \quad \dot{y} = x - 0.07125y - 0.095, \tag{8}$$

its corresponding Hamiltonian function is

$$H_2(x,y) = -\frac{x^2}{2} + 0.0712..xy + 0.1187..y^2 - 0.0224..y + 0.095..x.$$

Now system (6) has the three solutions $(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}) = (1.5569.., 2.4239.., 0.7417.., 0.5502..)$, $(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}) = (1.6356.., 2.6753.., 0.543.., 0.2948..)$, and $(x_1^{(3)}, y_1^{(3)}, x_2^{(3)}, y_2^{(3)}) = (1.6977.., 2.8823.., 0.264.., 0.0698..)$, which provide the three limit cycles shown in Figure 1. This completes the proof of statement (i) of Theorem 2.1.

To prove the statement (ii) of Theorem 2.1, we consider in the region $R_1 = \{(x,y) : x^2 - y^2 - 1 \geq 0\}$ the **PHS** given in (2), with its corresponding Hamiltonian function given in (3).

In the region $R_2 = \{(x,y) : x^2 - y^2 - 1 \leq 0\}$, we consider the **PHS** given in (4), with its corresponding Hamiltonian function given in (5). In order to have a crossing limit cycle that intersects the hyperbola $x^2 - y^2 - 1 = 0$ at the points (x_i, y_i) and (x_k, y_k), with $i \neq k$, they must satisfy the system of equations

$$\begin{array}{l} H_1(x_i,y_i) - H_1(x_k,y_k) = 0, \\ H_2(x_i,y_i) - H_2(x_k,y_k) = 0, \\ x_i^2 - y_i^2 - 1 = 0, \ x_k^2 - y_k^2 - 1 = 0, \end{array} \tag{9}$$

we suppose that system (2)–(4) has four crossing limit cycles. So, system (9) must have four pairs of solutions which can be written as $p_i = (\cosh r_i, \sinh r_i)$ and $q_i = (\cosh s_i, \sinh s_i)$, for $i = 1, 2, 3, 4$.

Due to the fact that the two points $p_1 = (\cosh r_1, \sinh r_1)$ and $q_1 = (\cosh s_1, \sinh s_1)$ satisfy system (9), then by solving the two first equations in (9), we obtain the parameters c_1 and c_2 as follows:

$$\begin{aligned} c_1 = \; & \frac{1}{2(\cosh r_1 - \cosh s_1)}\Big(- \alpha_1 \cosh^2 r_1 + \alpha_1 \cosh^2 s_1 + 2d_1 \sinh r_1 - 2b_1 \cosh r_1 \\ & \sinh r_1 - \delta_1 \sinh^2 r_1 - 2d_1 \sinh s_1 + \delta_1 \sinh s_1^2 + b_1 \sinh 2s_1\Big). \end{aligned}$$

By changing $(\alpha_1, \delta_1, b_1, d_1)$ by $(\alpha_2, \delta_2, b_2, d_2)$ in the expression of c_1, we get the expression of c_2. We know that the two points $p_2 = (\cosh r_2, \sinh r_2)$ and $q_2 = (\cosh s_2, \sinh s_2)$ satisfy system (9), then from this system, we get the parameters d_1 and d_2, where

$$\begin{aligned} d_1 = \; & \frac{1}{4\big(\cosh\big(\dfrac{r_1 - 2r_2 + s_1}{2}\big) - \cosh\big(\dfrac{r_1 + s_1 - 2s_2}{2}\big)\big)}\Big(csch(\tfrac{r_1 - s_1}{2})\Big(\alpha_1 \cosh^2 r_2 \cosh s_1 \\ & +\alpha_1 \cosh^2 r_1(\cosh r_2 - \cosh s_2) + \alpha_1 \cosh^2 s_1 \cosh s_2 - \alpha_1 \cosh s_1 \cosh^2 s_2 - \delta_1 \\ & \cosh s_2 \sinh^2 r_1 + \delta_1 \cosh s_1 \sinh^2 r_2 + b_1 \cosh s_1 \sinh(2r_2) + \delta_1 \cosh s_2 \sinh^2 s_1 \\ & +b_1 \cosh s_2 \sinh(2s_1) + \cosh r_2\big(- \alpha_1 \cosh^2 s_1 + \sinh(r_1 - s_1)\big(2b_1 \cosh(r_1 + s_1) \\ & +\delta_1 \sinh(r_1 + s_1)\big)\big) - 2b_1 \cosh s_1 \cosh s_2 \sinh s_2 - \delta_1 \cosh s_1 \sinh^2 s_2 \\ & + \cosh r_1\big(- \alpha_1 \cosh^2 r_2 + \alpha_1 \cosh^2 s_2 - 2b_1 \cosh s_2 \sinh r_1 + 2b_1 \cosh r_2 \sinh r_2 \\ & -\delta_1 \sinh^2 r_2 + \delta_1 \sinh^2 s_2 + b_1 \sinh(2s_2)\big)\Big)\Big), \end{aligned}$$

and by changing $(\alpha_1, \delta_1, b_1)$ by $(\alpha_2, \delta_2, b_2)$ in the expression of d_1, we obtain d_2.

We know that the points $p_3 = (\cosh r_3, \sinh r_3)$ and $q_3 = (\cosh s_3, \sinh s_3)$ satisfy system (9), then we obtain the values of δ_1 and δ_2. The value of δ_1 is given by $\delta_1 = A/B$, where

$$\begin{aligned}
A = \; & -\alpha_1\Big(\cosh(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}) + \cosh(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}) \\
& -\cosh(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}) + \cosh(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2} \\
& +\cosh(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}) + \cosh(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}) \\
& -\cosh(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}) + \cosh(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}) \\
& -\cosh(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}) - \cosh(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}) \\
& +\cosh(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}) - \cosh(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}) \\
& +\cosh(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2})\Big) + 2b_1\Big(\sinh(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}) \\
& -\sinh(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2} + \sinh(\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}) \\
& -\sinh(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}) + \sinh(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}) \\
& -\sinh(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}) + \sinh(\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}) \\
& -\sinh(\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}) - \sinh(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}) \\
& +\sinh(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}) - \sinh(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}) \\
& +\sinh(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2})\Big),
\end{aligned}$$

$$\begin{aligned}
B = \; & \cosh(\frac{r_1-r_2-r_3+s_1-s_2-3s_3}{2}) - \cosh(\frac{r_1-r_2-r_3+s_1-3s_2-s_3}{2}) \\
& +\cosh\frac{r_1-r_2-3r_3+s_1-s_2-s_3}{2}) - \cosh(\frac{r_1-3r_2-r_3+s_1-s_2-s_3}{2}) \\
& -\cosh(\frac{3r_1+r_2-r_3+s_1+s_2-s_3}{2}) + \cosh(\frac{r_1+3r_2-r_3+s_1+s_2-s_3}{2}) \\
& -\cosh\frac{r_1+r_2-r_3+3s_1+s_2-s_3}{2}) + \cosh\frac{r_1+r_2-r_3+s_1+3s_2-s_3}{2}) \\
& +\cosh(\frac{3r_1-r_2+r_3+s_1-s_2+s_3}{2}) - \cosh(\frac{r_1-r_2+3r_3+s_1-s_2+s_3}{2}) \\
& +\cosh(\frac{r_1-r_2+r_3+3s_1-s_2+s_3}{2}) - \cosh(\frac{r_1-r_2+r_3+s_1-s_2+3s_3}{2}).
\end{aligned}$$

We get the expression of δ_2 by changing (α_1, b_1) by (α_2, b_2) in the expression of δ_1.

If $\alpha_1 = \alpha_2 = 1$ or ($\alpha_1 = \alpha_2 = 0$), we assume that the points $p_4 = (\cosh r_4, \sinh r_4)$ and $q_4 = (\cosh s_4, \sinh s_4)$ satisfy system (9), then we obtain $b_1 = 0$ and $b_2 = 0$.

We replace c_1, d_1, δ_1, α_1 and b_1 in the expression of $H_1(x, y)$, and c_2, d_2, δ_2, α_2 and b_2 in the expression of $H_2(x, y)$ and we obtain $H_1(x, y) = H_2(x, y)$. Hence, in these cases, the piecewise linear differential system becomes a linear differential system, which does not have any limit cycle. Therefore, the maximum number of crossing limit cycles in the piecewise linear Hamiltonian saddles separated by a hyperbola is at most three.

Example with three limit cycles. We consider the **PHS** separated by the hyperbola

$$\dot{x} = -18x + 95y + 15, \quad \dot{y} = 18y - 14, \tag{10}$$

in the region $R_1 = \{(x,y) : x^2 - y^2 - 1 \geq 0\}$, which has the Hamiltonian function

$$H_1(x,y) = \frac{95}{2}y^2 - 18xy + 14x + 15y.$$

Now, in the region $R_2 = \{(x,y) : x^2 - y^2 - 1 \leq 0\}$, we consider the second **PHS**

$$\dot{x} = -0.2699..x + 2.425..y + 0.225.., \quad \dot{y} = x + 0.2699..y - 0.21, \tag{11}$$

its corresponding first integral is

$$H_2(x,y) = \frac{x^2}{2} - 0.2699..xy + 1.2124..y^2 + 0.21x + 0.2249..y.$$

The **PHS** (10)–(11) has exactly three crossing limit cycles because the system of equations (9) has three real solutions $(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}) = (1.0571.., 0.3427.., 1.0362.., -0.2715..)$, $(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}) = (1.1283.., 0.5227.., 1.0885.., -0.43..)$, and $(x_1^{(3)}, y_1^{(3)}, x_2^{(3)}, y_2^{(3)}) = (1.1969.., 0.6577.., 1.1385.., -0.5442)$, see Figure 2. This completes the proof of statement (ii).

Finally, to prove the statement (iii), we consider the **PHS** given in (2) in the region $R_1 = \{(x,y) : x^2 + y^2 - 1 \geq 0\}$, with its corresponding Hamiltonian function (3). We consider the **PHS** given in (4) in the region $R_2 = \{(x,y) : x^2 + y^2 - 1 \leq 0\}$, with its corresponding Hamiltonian function (5). In order that system (2)–(4) has crossing limit cycles intersecting the ellipse $y^2 + x^2 - 1 = 0$ at the points (x_i, y_i) and (x_k, y_k), with $i \neq k$, they must satisfy the system

$$\begin{aligned} &H_1(x_i, y_i) - H_1(x_k, y_k)) = 0, \\ &H_2(x_i, y_i) - H_2(x_k, y_k) = 0, \\ &y_i^2 + x_i^2 - 1 = 0, \;\; y_k^2 + x_k^2 - 1 = 0. \end{aligned} \tag{12}$$

Now we assume that system (2)–(4) has four crossing limit cycles. Consequently, system (12) must have four pairs of points $p_i = (\cos r_i, \sin r_i)$ and $q_i = (\cos s_i, \sin s_i)$ with $i = 1, \ldots, 4$ as solutions. So, if we consider the points $p_1 = (\cos r_1, \sin r_1)$ and $q_1 = (\cos s_1, \sin s_1)$ from (12), we obtain that the parameters c_1 and c_2 must be

$$\begin{aligned} c_1 = \; & \frac{1}{2(\cos r_1 - \cos s_1)}\Big(- \alpha_1 \cos^2 r_1 + \alpha_1 \cos^2 s_1 + 2d_1 \sin r_1 - 2b_1 \cos r_1 \sin r_1 - \\ & \delta_1 \sin^2 r_1 - 2d_1 \sin s_1 + \delta_1 \sin^2 s_1 + b_1 \sin(2s_1)\Big). \end{aligned}$$

Changing $(d_1, \delta_1, \alpha_1, b_1)$ by $(d_2, \delta_2, \alpha_2, b_2)$ in the expression of c_1, we get the expression of c_2. Due to the fact that the two points $p_2 = (\cos r_2, \sin r_2)$ and $q_2 = (\cos s_2, \sin s_2)$ satisfy system (12), then the parameters d_1 and d_2 have the expressions

$$\begin{aligned} d_1 = \; & \frac{\csc((r_1 - s_1)/2)}{4\big(\cos(2(r_1 - 2r_2 + s_1)/2) - \cos((r_1 + s_1 - 2s_2)/2)\big)}(\alpha_1 \cos^2 r_2 \cos s_1 \\ & +\alpha_1 \cos^2 r_1 (\cos r_2 - \cos s_2) + \alpha_1 \cos^2 s_1 \cos s_2 - \alpha_1 \cos s_1 \cos^2 s_2 - \delta_1 \cos s_2 \sin^2 r_1 \\ & +\delta_1 \cos s_1 \sin^2 r_2 + b_1 \cos s_1 \sin(2r_2) + \delta_1 \cos s_2 \sin^2 s_1 + b_1 \\ & \cos s_2 \sin(2s_1) + \cos r_2(-\alpha_1 \cos^2 s_1 + \sin(r_1 - s_1)(2b_1 \cos(r_1 + s_1) + \delta_1 \sin(r_1 + s_1))) \\ & -2b_1 \cos s_1 \cos s_2 \sin s_2 - \delta_1 \cos s_1 \sin^2 s_2 + \cos r_1(-\alpha_1 \cos^2 r_2 + \alpha_1 \cos^2 s_2 - 2b_1 \\ & \cos s_2 \sin r_1 - 2b_1 \cos r_2 \sin r_2 - \delta_1 \sin^2 r_2 + \delta_1 \sin^2 s_2 + b_1 \sin(2s_2))). \end{aligned}$$

We get the expression of d_2 by changing $(\delta_1, \alpha_1, b_1)$ by $(\delta_2, \alpha_2, b_2)$ in the expression of d_1.

Likewise, the points $p_3 = (\cos r_3, \sin r_3)$ and $q_3 = (\cos s_3, \sin s_3)$ satisfy system (12), then we obtain the expressions of δ_1 and δ_2 such that $\delta_1 = A/B$, where

$$\begin{aligned}
A = \quad & \alpha_1\Big(\cos\Big(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\Big) - \cos\Big(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\Big)\\
& + \cos\Big(\frac{r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3}{2}\Big) - \cos\Big(\frac{r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3}{2}\Big)\\
& - \cos\Big(\frac{3r_1 + r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) + \cos\Big(\frac{r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big)\\
& - \cos\Big(\frac{r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) + \cos\Big(\frac{r_1 + r_2 - r_3 + s_1 + 3s_2 - s_3}{2}\Big)\\
& + \cos\Big(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\Big) - \cos\Big(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\Big)\\
& + \cos\Big(\frac{r1 - r_2 + r_3 + 3s_1 - s_2 + s_3}{2}\Big) - \cos\Big(\frac{r1 - r_2 + r_3 + s_1 - s_2 + 3s_3}{2}\Big)\Big)\\
& -2b_1\Big(\sin\Big(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\Big) + \sin\Big(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\Big)\\
& - \sin\Big(\frac{r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3}{2}\Big) + \sin\Big(\frac{r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3}{2}\Big)\\
& - \sin\Big(\frac{3r_1 + r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) + \sin\Big(\frac{r_1 + 3r_2 - r_3 + s_1 + s2 - s3}{2}\Big)\\
& - \sin\Big(\frac{r_1 + r_2 - r_3 + 3s_1 + s2 - s3}{2}\Big) + \sin\Big(\frac{r_1 + r_2 - r_3 + s_1 + 3s2 - s3}{2}\Big)\\
& + \sin\Big(\frac{3r_1 - r_2 + r_3 + s_1 - s2 + s3}{2}\Big) - \sin\Big(\frac{r_1 - r_2 + 3r_3 + s_1 - s2 + s3}{2}\Big)\\
& + \sin\Big(\frac{r_1 - r_2 + r_3 + 3s_1 - s2 + s3}{2}\Big) - \sin\Big(\frac{r_1 - r_2 + r_3 + s_1 - s_2 + 3s_3}{2}\Big)\Big),
\end{aligned}$$

and the expression of B is

$$\begin{aligned}
B = \quad & \cos\Big(\frac{r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3}{2}\Big) - \cos\Big(\frac{r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3}{2}\Big)\\
& + \cos\Big(\frac{r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3}{2}\Big) - \cos\Big(\frac{r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3}{2}\Big)\\
& - \cos\Big(\frac{3r_1 + r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big) + \cos\Big(\frac{r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3}{2}\Big)\\
& - \cos\Big(\frac{r_1 + r_2 - r_3 + 3s_1 + s_2 - s_3}{2}\Big) + \cos\Big(\frac{r_1 + r_2 - r_3 + s_1 + 3s_2 - s_3}{2}\Big)\\
& + \cos\Big(\frac{3r_1 - r_2 + r_3 + s_1 - s_2 + s_3}{2}\Big) - \cos\Big(\frac{r_1 - r_2 + 3r_3 + s_1 - s_2 + s_3}{2}\Big)\\
& + \cos\Big(\frac{r_1 - r_2 + r_3 + 3s_1 - s_2 + s_3}{2}\Big) - \cos\Big(\frac{r_1 - r_2 + r_3 + s_1 - s_2 + 3s_3}{2}\Big).
\end{aligned}$$

A simple change of (α_1, b_1) to (α_2, b_2) in the expression of δ_1, allows us to get the expression of δ_2.

Now, if we suppose that the points $p_4 = (\cos r_4, \sin r_4)$ and $q_4 = (\cos s_4, \sin s_4)$ satisfy equation (12) and if $\alpha_i \in \{0, 1\}$ with $i = 1, 2$, then we obtain $b_1 = 0$ and $b_2 = 0$.
We replace $c_1, d_1, \delta_1, \alpha_1$ and b_1 in the expression of $H_1(x, y)$, and $c_2, d_2, \delta_2, \alpha_2$ and b_2 in the expression of $H_2(x, y)$, we have $H_1(x, y) = H_2(x, y)$. Therefore, the piecewise linear differential system becomes a linear differential system, which does not have limit cycles. Therefore, the maximum number of crossing limit cycles in this case is at most three.

Example with three limit cycles. In the region $R_1 = \{(x, y) : x^2 + y^2 - 1 \geq 0\}$, we consider the linear **PHS**

$$\dot{x} = -18x + 95y + 15, \quad \dot{y} = 18y - 14, \tag{13}$$

with its Hamiltonian function $H_1(x, y) = 14x + 15y - 18xy + \frac{95}{2}y^2$. In the region $R_1 = \{(x, y) : x^2 + y^2 - 1 \leq 0\}$, we consider the linear **PHS**

$$\dot{x} = -0.2442..x + 0.2892..y + 0.203571, \quad \dot{y} = x + 0.2442..y - 0.19, \tag{14}$$

which has the Hamiltonian function

$$H_2(x, y) = -\frac{x^2}{2} - 0.2442..xy + 0.1446..y^2 + 0.19x + 0.20357..y.$$

The linear **PHS** (13)-(14) has exactly three crossing limit cycles because the system of equations (12) has exactly three real solutions $(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}) = (0.9273.., 0.3741.., 0.9445.., -0.3282..)$, $(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}) = (0.8357.., 0.5491..,$ $0.83658.., -0.5478..)$, and $(x_1^{(3)}, y_1^{(3)}, x_2^{(3)}, y_2^{(3)}) = (0.7397.., 0.6729.., 0.6809.., -0.732..)$, see Figure 3.

4 Proof of Theorem 2.2

In the quarter-plane $R_1 = \{(x, y) : x > 0, y < 0\}$, we consider the **PHS** given by (2). Its corresponding Hamiltonian function is given by equation (3).

In the quarter-plane $R_2 = \{(x, y) : x < 0, y < 0\}$, we consider the **PHS** given by (4), with its corresponding Hamiltonian function (5).

In the quarter-plane $R_3 = \{(x, y) : x < 0, y > 0\}$, we consider the **PHS**

$$\dot{x} = -b_3x - \delta_3y + d_3, \quad \dot{y} = \alpha_3x + b_3y + c_3, \tag{15}$$

its corresponding Hamiltonian function is

$$H_3(x, y) = -\frac{\alpha_3}{2}x^2 - b_3xy - \frac{\delta_3}{2}y^2 - c_3x + d_3y. \tag{16}$$

In the quarter-plane $R_4 = \{(x, y) : x > 0, y > 0\}$, we consider the **PHS**

$$\dot{x} = -b_4x - \delta_4y + d_4, \quad \dot{y} = \alpha_4x + b_4y + c_4. \tag{17}$$

Its corresponding Hamiltonian function is

$$H_4(x, y) = -\frac{\alpha_4}{2}x^2 - b_4xy - \frac{\delta_4}{2}y^2 - c_4x + d_4y. \tag{18}$$

In order to have a crossing limit cycle that intersects the two intersecting straight lines $xy = 0$ at the points $(x_1, 0)$, $(x_2, 0)$, $(0, y_1)$ and $(0, y_2)$, we must satisfy the following system:

$$\begin{aligned} P_1(x_1, y_1) &= H_1(x_1, 0) - H_1(0, y_1) = 0, \\ P_2(x_2, y_1) &= H_2(0, y_1) - H_2(x_2, 0) = 0, \\ P_3(x_2, y_2) &= H_3(x_2, 0) - H_3(0, y_2) = 0, \\ P_4(x_1, y_2) &= H_4(0, y_2) - H_4(x_1, 0) = 0, \end{aligned} \tag{19}$$

or, equivalently,

$$\begin{array}{l} P_1(x_1,y_1) = -2c_1x_1 - 2d_1y_1 - x_1^2\alpha_1 + y_1^2\delta_1 = 0, \\ P_2(x_2,y_1) = 2c_2x_2 + 2d_2y_1 + x_2^2\alpha_2 - y_1^2\delta_2 = 0, \\ P_3(x_2,y_2) = -2c_3x_2 - 2d_3y_2 - x_2^2\alpha_3 + y_2^2\delta_3 = 0, \\ P_4(x_1,y_2) = 2c_4x_1 + 2d_4y_2 + x_1^2\alpha_4 - y_2^2\delta_4 = 0. \end{array} \tag{20}$$

As $x_1 \neq x_2$ and $y_1 \neq y_2$, we know that the polynomials $P_1(x_1, y_1)$, $P_2(x_2, y_1)$, $P_3(x_2, y_2)$ and $P_4(x_1, y_2)$ are of degree 2. By using Bézout Theorem, we know that the number of solutions of the system (19) is bounded by the product of the degrees of the four polynomials $P_i(x_k, y_j)$, with $k, j = 1, 2$, which is equal to 16. According to the symmetry of the solutions of this system, we know that the maximum number of solutions satisfying (20) is at most 8. Then the upper bound of limit cycles of system (2)–(17) is eight.

Because of the higher degree of these polynomials and the number of their parameters, we only can give an example with four limit cycles.

Example of four limit cycles for PHS separated by $xy = 0$. In the quarter-plane $R_1 = \{(x, y) : x > 0, y < 0\}$, we consider the **PHS**

$$\dot{x} = -4x - 35y + 8, \quad \dot{y} = -x + 4y + 8, \tag{21}$$

its Hamiltonian function is

$$H_1(x, y) = -\frac{1}{2}x^2 + 4xy + \frac{35}{2}y^2 + 8x - 8y.$$

In the quarter-plane $R_2 = \{(x, y) : x < 0, y < 0\}$, we consider the **PHS**

$$\dot{x} = -3x - 10.32..y + 5.17.., \quad \dot{y} = -x + 3y - 4.23.., \tag{22}$$

its Hamiltonian function is

$$H_2(x, y) = -\frac{1}{2}x^2 + 3xy + 5.16..y^2 - 4.23..x - 5.17..y.$$

In the quarter-plane $R_3 = \{(x, y) : x < 0, y > 0\}$, we consider the **PHS**

$$\dot{x} = +2.5x - 9y + 1, \quad \dot{y} = -x - 2.5y - 5, \tag{23}$$

where its Hamiltonian function is

$$H_3(x, y) = -\frac{1}{2}x^2 - 2.5xy + \frac{9}{2}y^2 - y - 5x.$$

In the quarter-plane $R_4 = \{(x, y) : x > 0, y > 0\}$, we consider the **PHS**

$$\dot{x} = -4x - 22.61..y + 4.46.., \quad \dot{y} = -x + 4y + 9.05.., \tag{24}$$

with the Hamiltonian function

$$H_4(x, y) = -\frac{1}{2}x^2 + 4xy + 11.3..y^2 - 4.46..y + 9.05..x.$$

The **PHS** (21)–(24) has exactly four crossing limit cycles because the system of equations (20) has four real solutions $(x_1^{(1)}, y_1^{(1)}, x_2^{(1)}, y_2^{(1)}) = (-0.524.., 1.2176.., -1.12.., 1.171..)$, $(x_1^{(2)}, y_1^{(2)}, x_2^{(2)}, y_2^{(2)}) = (-0.59.., 1.519.., -1.394.., 1.27..)$, $(x_1^{(3)}, y_1^{(3)}, x_2^{(3)}, y_2^{(3)}) = (-0.66.., 1.83.., -1.68.., 1.36..)$ and $(x_1^{(4)}, y_1^{(4)}, x_2^{(4)}, y_2^{(4)}) = (-0.72.., 2.16.., -2, 0, 1.44..)$, see Figure 4. This completes the proof of Theorem 2.2.

5 Conclusion

We have solved the extension of the second part of the 16th Hilbert problem for a family of discontinuous planar differential systems separated by conics. These piecewise differential systems are formed by planar linear Hamiltonian saddles. By using the fisrt integrals of these systems, we proved that the maximum number of crossing limit cycles of this family of systems is either three or eight depending on the curve of separation.

References

[1] A. Andronov, A. Vitt and S. Khaikin. *Theory of Oscillations.* Pergamon Press, Oxford, 1966.

[2] A. Belfar, R. Benterki and J. Llibre. Limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points and separated by irreducible cubics. *Dynamics of Continuous, Discrete and Impulsive Systems Series B: Applications and Algorithms* **28** (2021) 399–421.

[3] R. Benterki and J. Llibre. The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves I. *Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis* **28** (2021) 153-192.

[4] R. Benterki, L. Damene and J. Llibre. The limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves II. *Differential Equations and Dynamical Systems* (2021).

[5] R. Benterki and J. Llibre. On the limit cycles of discontinuous piecewise linear differential systems formed by centers and separated by irreducible cubic curves III. Submitted (2021).

[6] R. Benterki and J. Llibre. Crossing limit cycles of planar piecewise linear Hamiltonian systems without Equilibrium Points. *Mathematics* **8** (5) 755 (2020).

[7] M. Di Bernardo, C.J. Budd, A.R. Champneys and P. Kowalczyk. *Piecewise-Smooth Dynamical Systems: Theory and Applications.* Appl. Math. Sci. Series, Springer-Verlag, London **163** (2008).

[8] L. Damene and R. Benterki. Limit cycles of discontinuous piecewise linear differential systems formed by centers or Hamiltonian without equilibria separated by irreducible cubics. *Moroccan Journal of Pure and Applied Analysis* **7** (2021) 248–276.

[9] M. Han and W. Zhang. On Hopf bifurcation in non-smooth planar systems. *Journal of Differential Equations* **248** (9) (2010) 2399–2416.

[10] S.M. Huan and X.S. Yang. On the number of limit cycles in general planar piecewise linear systems. *Discrete and Continuous Dynamical Systems* **32** (2012) 2147–2164.

[11] J. Llibre, D.D. Novaes and M.A. Teixeira. Maximum number of limit cycles for certain piecewise linear dynamical systems. *Nonlinear Dynamics* **82** (2015) 1159–1175.

[12] J. Llibre and E. Ponce. Three nested limit cycles in discontinuous piecewise linear differential systems with two zones. *Dyn. Cont. Disc. Impul. Syst. Ser. B* **19** (2012) 325–335.

[13] J. Llibre and C. Valls. Limit cycles of planar piecewise differential systems with linear Hamiltonian saddles. *Symmetry* **13** (7) (2021).

[14] D.J.W. Simpson. Bifurcations in piecewise-Smooth Continuous Systems. *World Scientific series on Nonlinear Science A* **70** (2010).

Nonlinear Dynamics and Systems Theory, **22** (3) (2022) 243–262

Asymptotic Analysis of a Nonlinear Elliptic Equation with a Gradient Term

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Abstract: The main purpose of the present paper is to study the equation

$$div(|\nabla u|^{p-2}\nabla u) + \alpha u + \beta x.\nabla u + |u|^{q-1}u = 0, \quad x \in \mathbb{R}^N,$$

where $p > 2$, $q > 1$, $N \geqslant 1$, $\alpha > 0$ and $\beta > 0$. We investigate the structure of radial solutions and we present the asymptotic behavior of positive solutions near infinity. The study depends strongly on the sign of $N\beta - \alpha$ and the comparison between the three determining values $\frac{\alpha}{\beta}$, $\frac{p}{q+1-p}$ and $\frac{N-p}{p-1}$. More precisely, we prove under some assumptions that there exists a positive solution u which has the following behavior near infinity:

$$u(r) \underset{+\infty}{\sim} \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}} r^{-\alpha/\beta}.$$

Keywords: *nonlinear elliptic equation; radial self-similar solution; global existence; energy function; asymptotic behavior; equilibrium point; nonlinear dynamical systems.*

Mathematics Subject Classification (2010): 35A01, 35J60, 35B08, 35B09, 35B40, 35C06, 70K20, 70K42, 93C10, 93C15.

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1 Introduction

The main purpose of this paper is to study the nonlinear elliptic equation

$$div(|\nabla U|^{p-2}\nabla U) + \alpha U + \beta x.\nabla U + |U|^{q-1}U = 0, \quad x \in \mathbb{R}^N, \tag{1}$$

where $p > 2$, $q > 1$, $N \geqslant 1$, $\alpha > 0$ and $\beta > 0$. The equation (1) is derived from the self-similar solutions of the nonlinear parabolic equation

$$v_t - \Delta_p v - |v|^{q-1}v = 0, \quad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{2}$$

These particular solutions are of the form

$$v(t,x) = t^{-\alpha}U(t^{-\beta}|x|), \tag{3}$$

where

$$\alpha = \frac{1}{q-1} \quad \text{and} \quad \beta = \frac{q+1-p}{p(q-1)}.$$

If $p = 2$, $\alpha = 0$ and $\beta = 0$, the equation (1) is due to Emden-Fowler and plays an important role in astrophysics, this motivates many researchers to be interested in the study of this case, the examples include (but are not limited to) [3, 7–10, 12, 17, 18]. In the case $p = 2$, $\alpha > 0$ and $\beta > 0$, the equation (1) was studied in [6, 14–16, 19, 20, 22–24]. In the case $p > 2$, $\alpha = 0$ and $\beta = 0$, (1) was investigated in [2], [13] and [21]. When $p > 2$, $\alpha > 0$ and $\beta = 1$, equation (1) was studied in [1]. When $p > 2$, $\alpha = \dfrac{1}{q-1}$ and $\beta = \dfrac{q+1-p}{p(q-1)}$, equation (1) was studied in [11]. In the case $p > 2$, $\alpha < 0$ and $\beta < 0$, we have studied an equation similar to (1) but with the term $|U|^{q-1}U$ weakened by its multiplication by the function $|x|^l$ with $l < 0$ that tends to 0 at infinity. This study was carried out in [4] and gave the existence and asymptotic behavior of unbounded solutions near infinity using nonlinear dynamical systems theory. In this paper, we consider the case where $\alpha > 0$, $\beta > 0$ and $l = 0$. It is also a generalization of the study carried out in [11]. We will present a result that improves asymptotic behavior near infinity of positive solutions, we investigate the structure of solutions of problem (P) in the cases $\dfrac{\alpha}{\beta} \geq N$ and $\dfrac{\alpha}{\beta} < N$ and we give an important relation between the solutions of the problem (P) and those of a nonlinear dynamical system obtained by using the logarithmic change.

If we put $U(x) = u(|x|)$, it is easy to see that u satisfies the equation

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u(r) + \beta r u'(r) + |u|^{q-1}u(r) = 0, \quad r > 0. \tag{4}$$

Since we are interested in radial regular solutions, we impose the condition $u'(0) = 0$. Thus we consider the following Cauchy problem.

Problem (P): Find a function u defined on $[0, +\infty[$ such that $|u'|^{p-2}u' \in C^1([0, +\infty[)$ and satisfying

$$(|u'|^{p-2}u')' + \frac{N-1}{r}|u'|^{p-2}u' + \alpha u(r) + \beta r u'(r) + |u|^{q-1}u(r) = 0, \quad r > 0 \tag{5}$$

and

$$u(0) = A > 0, \quad u'(0) = 0, \tag{6}$$

where $p > 2$, $q > 1$, $N \geqslant 1$, $\alpha > 0$ and $\beta > 0$.

By reducing the problem (P) to a fixed point for a suitable integral operator (see for example [5]), we prove that for each $A > 0$, the problem (P) has a unique global solution $u(., A, \alpha, \beta)$.

The main results are the following.

Theorem 1.1 *Problem (P) has a unique solution $u(., A)$. Moreover,*

$$(|u^{'}|^{p-2}u^{'})^{'}(0) = \frac{-A}{N}\left(\alpha + A^{q-1}\right). \tag{7}$$

Theorem 1.2 *Problem (P) has no positive solutions in the following cases:*
(i) $\frac{\alpha}{\beta} \geq N$.
(ii) $\frac{N-p}{p-1} \leqslant \frac{\alpha}{\beta} < N$.
(iii) $q \leqslant p-1$ *and* $\frac{\alpha}{\beta} < \frac{N-p}{p-1}$.
(iv) $q > p-1$ *and* $\frac{\alpha}{\beta} \neq \frac{p}{q+1-p} < \frac{N-p}{p-1}$.

Theorem 1.3 *Assume* $\frac{\alpha}{\beta} < N$. *Then the solution $u(., A)$ of problem (P) is strictly positive in the following cases:*
(i) $0 < A < (\beta N - \alpha)^{\frac{1}{q-1}}$.
(ii) $\frac{\alpha}{\beta} = \frac{p}{q+1-p} < \min\left(\frac{N-p}{p}, \frac{p}{2}\right)$.

Theorem 1.4 *Assume* $\frac{\alpha}{\beta} = \frac{p}{q+1-p} < \frac{N-p}{p-1}$. *Let u be a strictly positive solution of problem (P). Then*

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}} u(r) = \Gamma > 0$$

and

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}+1} u'(r) = \frac{-\alpha}{\beta}\Gamma,$$

where

$$\Gamma = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}.$$

The rest of the paper is organized as follows. In the second section, we present basic tools for the study of the problem (P). The third section concerns asymptotic behavior near infinity of solutions of problem (P); more precisely, we give explicit equivalents of solutions and their derivatives near infinity. The fourth section concerns the structure of solutions of problem (P). The last section, in the form of a conclusion, presents the asymptotic behavior of the solution of a nonlinear dynamical system around its equilibrium point and explains its relation with the asymptotic behavior of the solution of the problem (P).

2 Preliminaries and Basic Tools

In this section, we give existence of global solutions of problem (P) and we present the necessary basic tools that will be useful to us in the rest of the work.

Theorem 2.1 *Problem (P) has a unique solution $u(.,A)$. Moreover,*

$$(|u'|^{p-2}u')'(0) = \frac{-A}{N}\left(\alpha + A^{q-1}\right). \tag{8}$$

Proof. The proof of theorem is divided into three steps.
Step 1: Existence and uniqueness of a local solution.
Multiply equation (5) by r^{N-1}, we obtain

$$\left(r^{N-1}|u'|^{p-2}u'(r) + \beta r^N u(r)\right)' = (\beta N - \alpha)\, r^{N-1}u(r) - r^{N-1}|u|^{q-1}u(r). \tag{9}$$

Integrating (9) twice from 0 to r and taking into account (6), we see that problem (P) is equivalent to the equation

$$u(r) = A - \int_0^r G(F[u](s))\, ds, \tag{10}$$

where

$$G(s) = |s|^{(2-p)/(p-1)}s, \qquad s \in \mathbb{R}, \tag{11}$$

and the nonlinear mapping F is given by the formula

$$F[u](s) = \beta s u(s) + s^{1-N}\int_0^s \sigma^{N-1}u(\sigma)\left((\alpha - \beta N) + |u(\sigma)|^{q-1}\right)\, d\sigma. \tag{12}$$

Now, we consider for $A > M > 0$, the complete metric space

$$E_A = \{\varphi \in C\left([0,R]\right) \;\; \textit{such that} \;\; ||\varphi - A||_0 \leqslant M\}. \tag{13}$$

Next, we define the mapping Ψ on E_A by

$$\Psi[\varphi](r) = A - \int_0^r G(F[\varphi](s))\, ds. \tag{14}$$

Claim 1: Ψ maps E_A into itself for some small M and $R > 0$.
Obviously, $\Psi[\varphi] \in C([0,R])$. From the definition of the space E_A, $\varphi(r) \in [A-M, A+M]$, for any $r \in [0,R]$. It is easy to prove that $F[\varphi]$ has a constant sign in $[0,R]$ for every $\varphi \in E_A$. Moreover, there exists a constant $K > 0$ such that

$$F[\varphi](s) \geq Ks \qquad \textit{for all} \;\; s \in [0,R], \tag{15}$$

where $K = \dfrac{A}{2N}\left(\alpha + A^{q-1}\right)$.
Taking into account that the function $r \to \dfrac{G(r)}{r}$ is decreasing on $(0,+\infty)$, we have

$$|\Psi[\varphi](r) - A| \leq \int_0^r \frac{G(F[\varphi](s))}{F[\varphi](s)}|F[\varphi](s)|\, ds \leq \int_0^r \frac{G(Ks)}{Ks}|F[\varphi](s)|\, ds$$

for $r \in [0, R]$. On the other hand,

$$|F[\varphi](s)| \leq Cs, \; where \quad C = [\beta + |\frac{\alpha}{N} - \beta| + (A+M)^{q-1}](A+M).$$

We thus get

$$|\Psi[\varphi](r) - A| \leq \frac{p-1}{p} C K^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}}$$

for every $r \in [0, R]$. Choose R small enough such that

$$|\Psi[\varphi](r) - A| \leq M, \quad \varphi \in E_A.$$

And thereby $\Psi[\varphi] \in E_A$. The first claim is thus proved.
Claim 2**:** Ψ is a contraction in some interval $[0, r_A]$.
According to Claim 1, if r_A is small enough, the space E_A applies into itself. For any $\varphi, \psi \in E_A$, we have

$$|\Psi[\varphi](r) - \Psi[\psi](r)| \leq \int_0^r |G(F[\varphi](s)) - G(F[\psi](s))| \, ds, \tag{16}$$

where $F[\varphi]$ is given by (12. Next, let

$$\Phi(s) = \min(F[\varphi](s), F[\psi](s)).$$

As a consequence of estimate (15), we have

$$\Phi(s) \geq Ks \qquad for \;\; 0 \leq s \leq r < r_A$$

and then

$$\begin{aligned} |G(F[\varphi](s)) - G(F[\psi](s))| &\leq \frac{G(\Phi(s))}{\Phi(s)} |F[\varphi](s) - F[\psi](s)| \\ &\leq \frac{G(Ks)}{Ks} |F[\varphi](s) - F[\psi](s)|. \end{aligned} \tag{17}$$

Moreover,

$$|F[\varphi](s) - F[\psi](s)| \leq C' ||\varphi - \psi||_0 s, \tag{18}$$

where $C' = [\beta + |\frac{\alpha}{N} - \beta| + (A+M)^{q-1}](A+M)$. Combining (16), (17) and (18), we have

$$|\Psi[\varphi](s) - \Psi[\psi](s)| \leq \frac{p-1}{p} C' K^{\frac{2-p}{p-1}} r^{\frac{p}{p-1}} ||\varphi - \psi||_0 \tag{19}$$

for any $r \in [0, r_A]$. When choosing r_A small enough, Ψ is a contraction. This proves the second claim.

The Banach Fixed Point Theorem then implies the existence of a unique fixed point of Ψ in E_A, which is a solution of (10) and consequently, of problem (P). As usual, this solution can be extended to a maximal interval $[0, r_{max}[, \; 0 < r_{max} \leq +\infty$.

Step 2: Existence of a global solution.
Define the energy function

$$E(r) = \frac{p-1}{p}|u'|^p + \frac{\alpha}{2}u^2(r) + \frac{1}{q+1}|u|^{q+1}. \tag{20}$$

Then by equation (5), the energy function satisfies

$$E'(r) = -\left(\frac{N-1}{r}|u'|^p + \beta r u'^2\right). \tag{21}$$

Then E is decreasing, hence it is bounded. Consequently, u and u' are also bounded and the local solution constructed above can be extented to $\mathbb{R}^+$.
Step 3: $(|u^{'}|^{p-2}u^{'})^{'}(0) = \frac{-A}{N}\left(\alpha + A^{q-1}\right)$.
Integrating (9) between 0 and r, we get

$$\frac{|u^{'}|^{p-2}u^{'}}{r} = -\beta u(r) + (\beta N - \alpha)r^{-N}\int_0^r s^{N-1}u(s)\,ds - r^{-N}\int_0^r s^{N-1}|u|^{q-1}u(s)\,ds.$$

Hence, using L'Hospital's rule and letting $r \to 0$, we obtain the desired result. The proof of the theorem is complete. □

Proposition 2.1 *Let u be a solution of problem (P) and let $S_u := \{r > 0,\ u(r) > 0\}$. Then $u'(r) < 0$ for any $r \in S_u$.*

Proof. We argue by contradiction. Let $r_0 > 0$ be the first zero of u'. Since by (8) $u'(r) < 0$ for $r \sim 0$, we have by continuity and the definition of r_0, there exists a left neighborhood $]r_0 - \varepsilon, r_0[$ (for some $\varepsilon > 0$), where u' is strictly increasing and strictly negative, that is, $(|u'|^{p-2}u')'(r) > 0$ for any $r \in]r_0 - \varepsilon, r_0[$, hence, by letting $r \to r_0$, we get $(|u'|^{p-2}u')'(r_0) \geq 0$. But by equation (5), we have $(|u'|^{p-2}u')'(r_0) = -\alpha u(r_0) - |u|^{q-1}u(r_0) < 0$ since $u(r_0) > 0$, $u'(r_0) = 0$ and $\alpha > 0$. This is a contradiction. The proof is complete. □

Proposition 2.2 *Assume $N > 1$. Let u be a solution of problem (P). Then*

$$\lim_{r\to+\infty} u(r) = \lim_{r\to+\infty} u^{'}(r) = 0. \tag{22}$$

Proof. Since $E^{'}(r) \leqslant 0$ and $E(r) \geqslant 0$ for all $r > 0$, there exists a constant $l \geqslant 0$ such that $\lim_{r\to+\infty} E(r) = l$. Suppose $l > 0$. Then there exists $r_1 > 0$ such that

$$E(r) \geqslant \frac{l}{2} \quad \text{for } r \geqslant r_1. \tag{23}$$

Now consider the function

$$D(r) = E(r) + \frac{N-1}{2r}|u'|^{p-2}u'(r)u(r) + \frac{\beta(N-1)}{4}u^2(r) + \beta\int_0^r su'^2(s)\,ds.$$

Then

$$D'(r) = -\frac{N-1}{2r}\left[|u'(r)|^p + \frac{N}{r}|u'|^{p-2}u'u(r) + |u(r)|^{q+1} + \alpha u^2(r)\right].$$

Recall that u and u' are bounded (because E is bounded), then

$$\lim_{r\to+\infty}\frac{|u^{'}|^{p-2}u^{'}u(r)}{r} = 0.$$

Moreover, by (20) and (23), we have for $r \geqslant r_1$,

$$\alpha u^2(r) + |u'(r)|^p + |u(r)|^{q+1} \geqslant \frac{p-1}{p}|u'(r)|^p + \frac{1}{q+1}|u(r)|^{q+1} + \frac{\alpha}{2}u^2(r) = E(r) \geqslant \frac{l}{2}.$$

Consequently, there exist two constants $c > 0$ and $r_2 \geqslant r_1$ such that

$$D'(r) \leqslant -\frac{c}{r} \quad \text{for } r \geqslant r_2.$$

Integrating the last inequality between r_2 and r, we get

$$D(r) \leqslant D(r_2) - c\ln(\frac{r}{r_2}) \quad \text{for } r \geqslant r_2.$$

In particular, we obtain $\lim\limits_{r\to+\infty} D(r) = -\infty$. Since

$$E(r) + \frac{N-1}{2r}|u'|^{p-2}u'(r)u(r) \leqslant D(r),$$

we get $\lim\limits_{r\to+\infty} E(r) = -\infty$. This is impossible, hence the conclusion. □

Proposition 2.3 *Let u be a strictly positive solution of problem (P), then u and u' have the same behavior (22).*

Proof. If $N > 1$, then by Proposition 2.2, $\lim\limits_{r\to+\infty} u(r) = \lim\limits_{r\to+\infty} u'(r) = 0$. If $N = 1$, let

$$J(r) = |u'|^{p-2}u'(r) + \beta r\, u(r). \tag{24}$$

Then by equation (5),

$$J'(r) = (\beta - \alpha)u - |u|^{q-1}u(r). \tag{25}$$

Since u is strictly positive, it is strictly decreasing by Proposition 2.1. Therefore $\lim\limits_{r\to+\infty} u(r) \in [0, +\infty[$. Since the energy function E given by (20) converges (because it is positive and decreasing), u' also necessarily converges and $\lim\limits_{r\to+\infty} u'(r) = 0$. Suppose by contradiction that $\lim\limits_{r\to+\infty} u(r) = L > 0$. Therefore $\lim\limits_{r\to+\infty} J(r) = +\infty$.
Using L'Hospital's rule, we have

$$\lim_{r\to+\infty} J'(r) = \lim_{r\to+\infty}\frac{J(r)}{r}.$$

That is,

$$(\beta - \alpha)L - L^q = \beta\, L.$$

Therefore $-\alpha L - L^q = 0$. But this contradicts the fact that $L > 0$ and $\alpha > 0$. Hence $\lim\limits_{r\to+\infty} u(r) = 0$. □

Proposition 2.4 *Let $0 < c \neq \dfrac{\alpha}{\beta}$. Let u be a strictly positive solution of problem (P). Then the function $r^c u(r)$ is strictly monotone for large r.*

Proof. For any $c > 0$, we consider the function

$$g_c(r) = cu(r) + ru'(r), \quad r > 0. \tag{26}$$

It is clear that

$$(r^c u(r))' = r^{c-1} g_c(r), \quad r > 0. \tag{27}$$

The monotonicity of the function $r^c u(r)$ can be obtained by the sign of the function $g_c(r)$. Using (5), we have for any $r > 0$ such that $u'(r) \neq 0$,

$$\begin{aligned}(p-1)|u'(r)|^{p-2} g_c'(r) = {} & (p-1)(c - \frac{N-p}{p-1})|u'|^{p-2}u'(r) \\ & - \beta r^2 u'(r) - \alpha r\, u(r) - r|u|^{q-1}u(r).\end{aligned} \tag{28}$$

Consequently, if $g_c(r_0) = 0$ for some $r_0 > 0$, we obtain by (26) and (28),

$$\begin{aligned}(p-1)|u'|^{p-2}(r_0) g_c'(r_0) = {} & r_0 u(r_0)\Big[(\beta c - \alpha) - |u(r_0)|^{q-1} \\ & + (p-1)c^{p-1}\left(\frac{N-p}{p-1} - c\right)\frac{|u(r_0)|^{p-2}}{r_0^p}\Big].\end{aligned} \tag{29}$$

Suppose that there exists a large r_0 such that $g_c(r_0) = 0$. Since $\lim\limits_{r \to +\infty} u(r) = 0$ and according to (29), we have for $c > \dfrac{\alpha}{\beta}$ (respectively, $c < \dfrac{\alpha}{\beta}$), $g_c'(r_0) > 0$ (respectively, $g_c'(r_0) < 0$) and thereby $g_c(r) \neq 0$ for large r if $c \neq \dfrac{\alpha}{\beta}$. Consequently, the function $r^c u(r)$ is strictly monotone for large r if $c \neq \dfrac{\alpha}{\beta}$. □

Proposition 2.5 *Let u be a strictly positive solution of problem (P). Then for any $0 < c < \dfrac{\alpha}{\beta}$, we have $g_c(r) < 0$ for large r and $\lim\limits_{r \to +\infty} r^c u(r) = 0$.*

Proof. We know by Proposition 2.4 that if $0 < c < \dfrac{\alpha}{\beta}$, $g_c(r) \neq 0$ for large r. Suppose that $g_c(r) > 0$ for large r. Then, by (26) and the fact that $u'(r) < 0$, we get

$$|u'(r)| < \frac{cu(r)}{r} \qquad for\ large\ r. \tag{30}$$

This gives by equation (5),

$$(|u'|^{p-2}u')'(r) < u(r)\left[(\beta c - \alpha) + (N-1)c^{p-1}\frac{u^{p-2}(r)}{r^p}\right]. \tag{31}$$

As $0 < c < \dfrac{\alpha}{\beta}$, $u(r) > 0$ and $\lim\limits_{r \to +\infty} u(r) = 0$, then $(|u'|^{p-2}u')'(r) < 0$ for large r. Combining with $u' < 0$, we get $\lim\limits_{r \to +\infty} u'(r) \in [-\infty, 0[$, which is impossible. Hence, $g_c(r) < 0$ for large r and by (27), $\lim\limits_{r \to +\infty} r^c u(r) \in [0, +\infty[$. Suppose that $\lim\limits_{r \to +\infty} r^c u(r) = L > 0$. Then $\lim\limits_{r \to +\infty} r^{c+\varepsilon} u(r) = +\infty$ for $0 < c + \varepsilon < \dfrac{\alpha}{\beta}$, but this contradicts the fact that $g_{c+\varepsilon}(r) < 0$ for large r. Consequently, $\lim\limits_{r \to +\infty} r^c u(r) = 0$. □

Proposition 2.6 *Let u be a strictly positive solution of problem (P). Then for any $\frac{\alpha}{\beta} < c \leqslant N$, we have $g_c(r) > 0$ for large r and $\lim_{r\to+\infty} r^c u(r) = +\infty$.*

Proof. Let $\frac{\alpha}{c} < k < \beta$. We introduce the following energy function:

$$\phi(r) = r^{c-1}|u'|^{p-2}u' + kr^c u(r). \tag{32}$$

Using equation (5), we have

$$\begin{aligned}\phi'(r) =&(c-N)r^{c-2}|u'|^{p-2}u' + (k-\beta)r^c u'(r) + \\ &(kc-\alpha)r^{c-1}u(r) - r^{c-1}|u|^{q-1}u(r).\end{aligned} \tag{33}$$

As $u' < 0$, $c \leqslant N$ and $k < \beta$, then

$$\phi'(r) > r^{c-1}u\left[kc - \alpha - |u|^{q-1}\right]. \tag{34}$$

As $kc-\alpha > 0$ and $\lim_{r\to+\infty} u(r) = 0$, then $\phi'(r) > 0$ for large r, therefore $\phi(r) \neq 0$ for large r. Suppose that $\phi(r) < 0$ for large r, then

$$|u'|^{p-2}u' < -k\, r\, u(r) \quad \text{for large } r. \tag{35}$$

Therefore

$$u' u^{\frac{-1}{p-1}} < -k^{\frac{1}{p-1}} r^{\frac{1}{p-1}} \quad \text{for large } r. \tag{36}$$

Integrating this last inequality on (R, r) for large R, we obtain

$$u^{\frac{p-2}{p-1}}(r) < u^{\frac{p-2}{p-1}}(R) - \frac{p-2}{p} k^{\frac{1}{p-1}} r^{\frac{p}{p-1}} + \frac{p-2}{p} k^{\frac{1}{p-1}} R^{\frac{p}{p-1}}.$$

Letting $r \to +\infty$, we obtain $\lim_{r\to+\infty} u(r) = -\infty$, which is a contradiction. Consequently, $\phi(r) > 0$ for large r. Since ϕ is strictly increasing for large r, we have $\lim_{r\to+\infty} \phi(r) \in]0, +\infty]$, so there exists $C_1 > 0$ such that $\phi(r) > C_1$ for large r. This gives by (32) and the fact that $u'(r) < 0$,

$$r^c u(r) > \frac{C_1}{k} \quad \text{for large } r.$$

On the other hand, using (34) and the fact that $\lim_{r\to+\infty} u(r) = 0$, we obtain

$$r\phi'(r) > \frac{kc-\alpha}{2} r^c u(r) \quad \text{for large } r. \tag{37}$$

This implies that

$$r\phi'(r) > C \quad \text{for large } r, \tag{38}$$

where $C = \frac{C_1(kc-\alpha)}{2k} > 0$. Integrating this last inequality on (R, r) for large R, we obtain $\lim_{r\to+\infty} \phi(r) = +\infty$. Consequently, by (32) and the fact that $u'(r) < 0$, we have $\lim_{r\to+\infty} r^c u(r) = +\infty$. Moreover, since $g_c(r) \neq 0$ for large r, using (27), we have necessarily $g_c(r) > 0$ for large r. □

Proposition 2.7 *Assume* $\dfrac{\alpha}{\beta} < N$. *Let* u *be a strictly positive solution of problem* (P). *Then the function* $r^{\alpha/\beta}u(r)$ *is not strictly monotone for large* r.

Proof. Assume by contradiction that $r^{\alpha/\beta}u(r)$ is strictly monotone for large r. Then by (27), $g_{\frac{\alpha}{\beta}}(r) \neq 0$ for large r. We distinguish two cases.
Case 1: $g_{\frac{\alpha}{\beta}}(r) < 0$ for large r.
We set

$$V(r) = u(r) - r^{p-1}|u'|^{p-1}. \tag{39}$$

Then by equation (5),

$$V'(r) = r^{p-1}u\left[-\alpha - u^{q-1}\right] + r^p u'\left[-\beta + r^{-p} + (p-N)r^{-2}|u'|^{p-2}\right]. \tag{40}$$

Using Proposition 2.6, we have $g_N(r) > 0$ for large r. Then

$$0 < r|u'(r)| < Nu(r) \quad \text{for large } r, \tag{41}$$

so $\lim\limits_{r\to+\infty} ru'(r) = 0$ and therefore

$$\lim_{r\to+\infty} V(r) = 0.$$

Using again inequality 41, we have

$$V(r) > u(r)\left(1 - N^{p-1}u^{p-2}(r)\right) \quad \text{for large } r. \tag{42}$$

Since $\lim\limits_{r\to+\infty} u(r) = 0$, one has $V(r) > 0$ for large r.
On the other hand, since $\lim\limits_{r\to+\infty} u(r) = 0$, $\lim\limits_{r\to+\infty} u'(r) = 0$ and $g_{\frac{\alpha}{\beta}}(r) < 0$ for large r, one has by (40),

$$V'(r) \underset{+\infty}{\sim} -\alpha r^{p-1}u(r) - \beta r^p u'(r) = -\beta r^{p-1} g_{\frac{\alpha}{\beta}}(r) > 0 \quad \text{for large } r. \tag{43}$$

But this contradicts the fact that $V(r) > 0$ for large r and $\lim\limits_{r\to+\infty} V(r) = 0$.
Case 2: $g_{\frac{\alpha}{\beta}}(r) > 0$ for large r.
Using equation (5), we obtain

$$(|u'|^{p-2}u')'(r) = -r\,u'(r)\left[\beta + \frac{N-1}{r^2}|u'|^{p-2}\right] - u(r)\left[\alpha + |u|^{q-1}\right]. \tag{44}$$

Since $\lim\limits_{r\to+\infty} u(r) = 0$, $\lim\limits_{r\to+\infty} u'(r) = 0$ and $g_{\frac{\alpha}{\beta}}(r) > 0$ for large r, we have

$$(|u'|^{p-2}u')'(r) \underset{+\infty}{\sim} -\beta r u'(r) - \alpha u(r) = -\beta g_{\frac{\alpha}{\beta}}(r) < 0 \quad \text{for large } r. \tag{45}$$

But this contradicts the fact that $u'(r) < 0$ and $\lim\limits_{r\to+\infty} u'(r) = 0$. $\square$

3 Asymptotic Behavior Near Infinity

In this section, we give explicit equivalents of the strictly positive solutions of the problem (P) and their derivatives near infinity.

Theorem 3.1 *Assume* $\dfrac{\alpha}{\beta} = \dfrac{p}{q+1-p} < \dfrac{N-p}{p-1}$. *Let u be a strictly positive solution of problem (P). Then*

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}} u(r) = \Gamma > 0 \tag{46}$$

and

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}+1} u'(r) = \frac{-\alpha}{\beta}\Gamma, \tag{47}$$

where

$$\Gamma = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}. \tag{48}$$

Proof. We consider the following function:

$$h(r) = r^{\frac{\alpha}{\beta}} u(r) \left[\beta + \frac{|u'|^{p-2}u'(r)}{r\,u}\right]. \tag{49}$$

Using equation (5), we have

$$h'(r) = \left(\frac{\alpha}{\beta} - N\right) r^{\frac{\alpha}{\beta}-2} |u'|^{p-2}u'(r) - r^{\frac{\alpha}{\beta}-1} u^q(r). \tag{50}$$

The proof will be done in four steps.
Step 1: $h(r) \underset{+\infty}{\sim} \beta r^{\alpha/\beta} u(r)$.
We know by Proposition 2.6 that $g_N(r) > 0$ for large r, then using (41), we get

$$0 < \frac{|u'(r)|^{p-1}}{ru(r)} < N^{p-1}\frac{u^{p-2}(r)}{r^p} \quad \text{for large } r. \tag{51}$$

As $p > 2$ and $\lim\limits_{r\to+\infty} u(r) = 0$, we get $\lim\limits_{r\to+\infty} \dfrac{|u'(r)|^{p-1}}{ru(r)} = 0$. Consequently, by (49), we get $h(r) \underset{+\infty}{\sim} \beta r^{\alpha/\beta} u(r)$.
Step 2: $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}} u(r)$ exists and is finite. By Proposition 2.5, we have for any $\sigma > 0$, $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}-\sigma} u(r) = 0$. In particular, for

$$0 < \sigma < \min\left(\frac{\alpha}{\beta}\frac{(q-1)}{q}, \frac{1}{p-1}\left(\frac{\alpha}{\beta}(p-2) + p\right)\right) < \frac{\alpha}{\beta}, \tag{52}$$

there exists a constant $M > 0$ such that

$$u(r) \leqslant M\, r^{\sigma - \frac{\alpha}{\beta}} \quad \text{for large } r. \tag{53}$$

We have also by (41),

$$|u'(r)|^{p-1} < \frac{N^{p-1}u^{p-1}(r)}{r^{p-1}} \quad \text{for large } r. \tag{54}$$

Combining (53) and (54), we obtain

$$r^{\frac{\alpha}{\beta}-1}u^q(r) < M^q r^{q(\sigma-\frac{\alpha}{\beta})+\frac{\alpha}{\beta}-1} \quad \text{for large } r \tag{55}$$

and

$$r^{\frac{\alpha}{\beta}-2}|u'(r)|^{p-1} < (MN)^{p-1} r^{\frac{\alpha}{\beta}(2-p)+\sigma(p-1)-p-1} \quad \text{for large } r. \tag{56}$$

By (52), (55) and (56), we get the function $r \to r^{\frac{\alpha}{\beta}-1}u^q(r)$ and the function $r \to r^{\frac{\alpha}{\beta}-2}|u'(r)|^{p-1}$ belong to $L^1(r_0,+\infty)$ for any $r_0 > 0$; therefore $h'(r) \in L^1(r_0,+\infty)$ for any $r_0 > 0$. Hence,

$$\lim_{r\to+\infty} h(r) = h(r_0) + \int_{r_0}^{+\infty} h'(s)\, ds \tag{57}$$

exists and is finite. Then by Step 1, $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r)$ exists and is finite. Let $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \Gamma \geq 0$.

Step 3: $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \Gamma > 0$ and $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}+1}u'(r) = \dfrac{-\alpha}{\beta}\Gamma < 0$.

We argue by contradiction and assume that $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = 0$. Then by the first step, $\lim\limits_{r\to+\infty} h(r) = 0$. Therefore, using L'Hospital's rule, we obtain

$$\lim_{r\to+\infty} \frac{h'(r)}{\left(r^{\frac{\alpha}{\beta}}u(r)\right)'} = \lim_{r\to+\infty} \frac{h(r)}{r^{\frac{\alpha}{\beta}}u(r)} = \beta. \tag{58}$$

On the other hand, we have

$$h'(r) = r^{\frac{\alpha}{\beta}-2}|u'(r)|^{p-1}\left(N - \frac{\alpha}{\beta} - \frac{ru^q}{|u'|^{p-1}}\right). \tag{59}$$

Let $0 < c < \dfrac{\alpha}{\beta}$, then by Proposition 2.5, we have $g_c(r) < 0$ for large r, then

$$|u'(r| > \frac{c\,u(r)}{r} \quad \text{for large } r. \tag{60}$$

This leads to

$$0 < \frac{r\,u^q(r)}{|u'(r)|^{p-1}} < c^{1-p}r^p u^{q+1-p}(r). \tag{61}$$

Since $\dfrac{\alpha}{\beta} = \dfrac{p}{q+1-p}$, then $\lim\limits_{r\to+\infty} r^p u^{q+1-p}(r) = 0$, therefore by (61), $\lim\limits_{r\to+\infty} \dfrac{ru^q}{|u'(r)|^{p-1}} = 0$. Using the fact that $\dfrac{\alpha}{\beta} < N$ and $|u'(r)| > 0$, we obtain by (59), $h'(r) > 0$ for large r. Therefore by (58), we have $\left(r^{\frac{\alpha}{\beta}}u(r)\right)' > 0$ for large r, but this contradicts Proposition 2.7. Consequently, $\lim\limits_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \Gamma > 0$. Hence, using L'Hospital's rule (because $\lim\limits_{r\to+\infty} u(r) = 0$), we get

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}+1}u'(r) = \frac{-\alpha}{\beta}\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \frac{-\alpha}{\beta}\Gamma < 0.$$

Step 4: $\Gamma = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}$.
By (28), we have

$$-\beta r\, g_{\alpha/\beta}(r) = |u'|^{p-2}\, u'(r)\Bigg[\left(N - p - \frac{\alpha}{\beta}(p-1)\right) + (p-1)\frac{g'_{\alpha/\beta}(r)}{u'(r)} + \frac{ru^q(r)}{|u'|^{p-2}\, u'(r)}\Bigg]. \tag{62}$$

Since $\lim\limits_{r\to+\infty} u(r) = 0$ and $\lim\limits_{r\to+\infty} ru'(r) = 0$ (by Step 3), one has $\lim\limits_{r\to+\infty} g_{\frac{\alpha}{\beta}}(r) = 0$. Therefore, using again Step 3 and L'Hospital's rule, we obtain

$$\lim_{r\to+\infty} \frac{g'_{\frac{\alpha}{\beta}}(r)}{u'(r)} = \lim_{r\to+\infty} \frac{g_{\frac{\alpha}{\beta}}(r)}{u(r)} = \lim_{r\to+\infty} \left(\frac{\alpha}{\beta} + \frac{ru'(r)}{u(r)}\right) = 0. \tag{63}$$

Moreover, since $\frac{\alpha}{\beta} = \frac{p}{q+1-p}$, we have

$$\lim_{r\to+\infty} \frac{ru^q(r)}{|u'|^{p-2}u'(r)} = \frac{-\Gamma^{q+1-p}}{\left(\frac{\alpha}{\beta}\right)^{p-1}}. \tag{64}$$

Suppose by contradiction that

$$N - p - \frac{\alpha}{\beta}(p-1) - \frac{\Gamma^{q+1-p}}{\left(\frac{\alpha}{\beta}\right)^{p-1}} \neq 0. \tag{65}$$

Then, according to (62), (63) and (64), we have

$$-\beta r g_{\frac{\alpha}{\beta}}(r) \underset{+\infty}{\sim} |u'|^{p-2}u'(r)\left[N - p - \frac{\alpha}{\beta}(p-1) - \frac{\Gamma^{q+1-p}}{\left(\frac{\alpha}{\beta}\right)^{p-1}}\right]. \tag{66}$$

This gives $g_{\frac{\alpha}{\beta}}(r) \neq 0$ for large r, that is, $r^{\frac{\alpha}{\beta}}u(r)$ is strictly monotone for large r, but this contradicts Proposition 2.7. Consequently,

$$\Gamma = \left(N - p - \frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}} \left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}.$$

The proof of this theorem is complete. □

The following Figures 1 and 2 describe the strictly positive solution and its comparison with the function $r^{-\alpha/\beta}$.

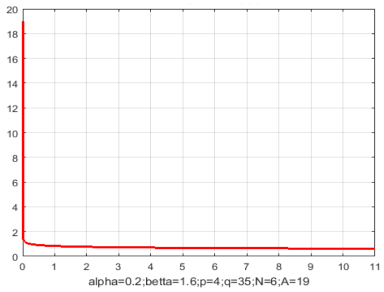

Figure 1: Strictly positive solution u.

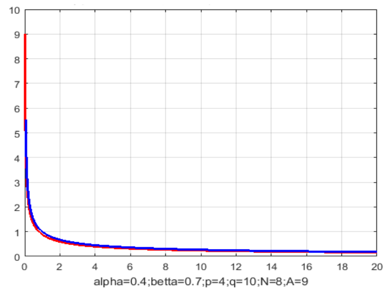

Figure 2: Comparison of solution u with $r^{-\alpha/\beta}$.

4 Structure of Radial Solutions

In this section, we investigate the structure of the solutions of the problem (P). The study depends strongly on the sign of $N\beta - \alpha$ and the comparison between the three determining values $\dfrac{\alpha}{\beta}$, $\dfrac{p}{q+1-p}$ and $\dfrac{N-p}{p-1}$.

Theorem 4.1 *Assume* $\dfrac{\alpha}{\beta} \geq N$. *Then the solution u of problem (P) changes the sign.*

Proof. We consider the following function:

$$\varphi(r) = r^{N-1}|u^{'}|^{p-2}u^{'}(r) + \beta r^N u(r). \tag{67}$$

Therefore by (9), we get

$$\varphi'(r) = (\beta N - \alpha)\, r^{N-1}u(r) - r^{N-1}|u|^{q-1}u(r). \tag{68}$$

Suppose that $u(r) > 0$ for all $r \in [0, +\infty)$. As $\alpha \geqslant \beta N$, then $\varphi'(r) < 0$. Therefore, as $\varphi(0) = 0$, we have $\varphi(r) \leqslant 0 \quad \forall r \in [0, +\infty)$. Consequently, the function $r \to H(r) = \dfrac{p}{p-2}u^{\frac{p-2}{p-1}}(r) + \beta^{\frac{1}{p-1}}r^{\frac{p}{p-1}}$ is decreasing. Then for any $r \in [0, +\infty)$, we have

$$H(r) \leqslant H(0) = \frac{p}{p-2}A^{\frac{p-2}{p-1}}. \tag{69}$$

When letting $r \to +\infty$, the term on the left-hand part of the inequality converges to $+\infty$, so we reach a contradiction.

Now, let r_0 be the first zero of u, then $\varphi'(r) < 0$ for all $r \in (0, r_0)$, thus $\varphi(r_0) < \varphi(0) = 0$. Therefore $u'(r_0) < 0$, consequently, u changes the sign. □

The solution that changes the sign is illustrated by Figure 3.

Theorem 4.2 *Assume* $\dfrac{\alpha}{\beta} < N$. *Then the solution u of problem (P) is not strictly positive in the following cases:*

(i) $\dfrac{N-p}{p-1} \leqslant \dfrac{\alpha}{\beta}$.

(ii) $q \leqslant p-1$ *and* $\dfrac{\alpha}{\beta} < \dfrac{N-p}{p-1}$.

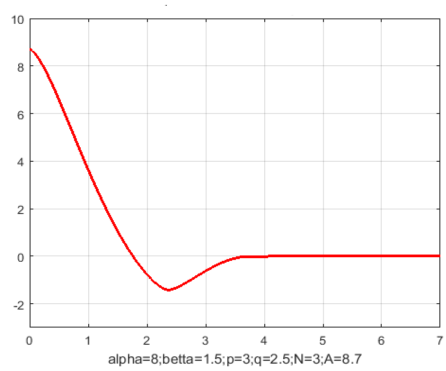

Figure 3: Solution that changes the sign.

(iii) $q > p-1$ *and* $\dfrac{\alpha}{\beta} \neq \dfrac{p}{q+1-p} < \dfrac{N-p}{p-1}$.

Proof. Assume by contradiction that u is strictly positive. The idea is to show that under this assumption, we have $g_{\alpha/\beta}(r) \neq 0$ for large r in these three cases, which is not possible by Proposition 2.7.

Assume that there exists a large r_0 such that $g_{\alpha/\beta}(r_0) = 0$, we obtain by (29),

$$(p-1)|u'|^{p-2}(r_0)g'_{\alpha/\beta}(r_0) = r_0 u^q(r_0)\Bigg[-1 + (p-1)\left(\frac{\alpha}{\beta}\right)^{p-1} \times \left(\frac{N-p}{p-1} - \frac{\alpha}{\beta}\right) r_0^{-p} u^{p-1-q}(r_0)\Bigg]. \tag{70}$$

Using the fact that $\lim\limits_{r\to+\infty} u(r) = 0$, we have in the cases (i) and (ii), $g'_{\alpha/\beta}(r_0) < 0$. For the case (iii), we have by Proposition 2.5 and Proposition 2.6 $\lim\limits_{r\to+\infty} r^{\frac{p}{q+1-p}} u(r) = 0$ or $\lim\limits_{r\to+\infty} r^{\frac{p}{q+1-p}} u(r) = +\infty$, then we get $g'_{\frac{\alpha}{\beta}}(r_0) \neq 0$. Therefore, in the three cases, we have $g_{\alpha/\beta}(r) \neq 0$ for large r, that is, $r^{\alpha/\beta}u(r)$ is strictly monotone for large r. But this contradicts Proposition 2.7. Consequently, u is not strictly positive in the three cases. □

Theorem 4.3 *Assume* $\dfrac{\alpha}{\beta} < N$. *Then for any* $0 < A < (\beta N - \alpha)^{\frac{1}{q-1}}$, *the solution* $u(.,A)$ *of problem* (P) *is strictly positive.*

Proof. Let r_0 be the first zero of u, then $u(r_0) = 0$ and $u'(r_0) \leqslant 0$. Integrating (9) on $(0, r_0)$, we obtain

$$r_0^{N-1}|u'|^{p-2}u'(r_0) = \int_0^{r_0} \left[(\beta N - \alpha) - u^{q-1}(s)\right] s^{N-1} u(s)\, ds. \tag{71}$$

As $u(r) > 0$ and $u'(r) < 0$ on $(0, r_0)$, then

$$\beta N - \alpha - u^{q-1}(s) > \beta N - \alpha - A^{q-1} > 0 \quad \text{for any } s \in (0, r_0). \tag{72}$$

Therefore by (71), we get $u'(r_0) > 0$, but this contradicts the fact that $u'(r_0) \leqslant 0$. Hence $u(., A)$ is strictly positive. □

Theorem 4.4 *Assume* $\dfrac{\alpha}{\beta} = \dfrac{p}{q+1-p} < \min\left(\dfrac{N-p}{p}, \dfrac{p}{2}\right)$. *Then the solution* u *of problem* (P) *is strictly positive.*

Before giving the proof of the theorem, we need the following result.

Proposition 4.1 *Let* u *be a solution of problem* (P). *Assume that there exists* $R > 0$, *the first zero of* u. *Then for* $\lambda \geq 1$ *and* $0 < \gamma < \rho$, *we have*

$$\int_0^R u^\lambda |u'|^\gamma s^{\rho-1}\, ds \leq \frac{\lambda+\gamma}{\rho-\gamma} \int_0^R u^{\lambda-1} |u'|^{\gamma+1} s^\rho \, ds. \tag{73}$$

Proof. By Holder's inequality, we have

$$\int_0^R u^\lambda |u'|^\gamma s^{\rho-1}\, ds \leq \left(\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds\right)^{\frac{1}{\gamma+1}} \left(\int_0^R u^{\lambda-1}|u'|^{\gamma+1} s^\rho\, ds\right)^{\frac{\gamma}{\gamma+1}}. \tag{74}$$

On the other hand, using the fact that $u(R) = 0$, we obtain

$$\int_0^R \left(u^{\lambda+\gamma} s^{\rho-1-\gamma}\right)' s\, ds = -\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds. \tag{75}$$

Therefore

$$\begin{aligned}&(\lambda+\gamma)\int_0^R u' u^{\lambda+\gamma-1} s^{\rho-\gamma}\, ds + (\rho-1-\gamma)\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds = \\ &-\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds.\end{aligned} \tag{76}$$

Using the fact that $u' < 0$ in $(0, R)$, we get

$$\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds = \frac{\lambda+\gamma}{\rho-\gamma}\int_0^R |u'| u^{\lambda+\gamma-1} s^{\varrho-\gamma}\, ds. \tag{77}$$

Applying Holder's inequality again, we obtain

$$\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds \leq \frac{\lambda+\gamma}{\rho-\gamma}\left(\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds\right)^{\frac{\gamma}{\gamma+1}} \left(\int_0^R u^{\lambda-1}|u'|^{\gamma+1} s^\rho\, ds\right)^{\frac{1}{\gamma+1}}. \tag{78}$$

Therefore,

$$\left(\int_0^R u^{\lambda+\gamma} s^{\rho-1-\gamma}\, ds\right)^{1-\frac{\gamma}{\gamma+1}} \leq \frac{\lambda+\gamma}{\rho-\gamma}\left(\int_0^R u^{\lambda-1}|u'|^{\gamma+1} s^\rho\, ds\right)^{\frac{1}{\gamma+1}}. \tag{79}$$

Combining (74) and (79), we easily obtain the estimation (73). This completes the proof of this proposition. □

Now we turn to the proof of Theorem 4.4.

Proof. Assume that there exists $r_0 > 0$, the first zero of u. Then $u(r) > 0$ $\forall r \in [0, r_0[$, $u'(r) < 0$ $\forall r \in (0, r_0)$ and $u'(r_0) \leqslant 0$.
Since $\dfrac{p}{q+1-p} < \dfrac{N-p}{p}$, one has $\dfrac{N-p}{p} > \dfrac{N}{q+1}$.
Let $\dfrac{N}{q+1} < \delta < \dfrac{N-p}{p}$ and we consider the following energy function:

$$G(r) = r^N \left(\frac{p-1}{p}|u'|^p + \frac{1}{q+1}|u|^{q+1}\right) + \delta r^{N-1} u|u'|^{p-2}u'. \tag{80}$$

Using equation (5), we get

$$\begin{aligned} G'(r) = \left(\delta - \frac{N-p}{p}\right) r^{N-1}|u'|^p + \left(\frac{N}{q+1} - \delta\right) r^{N-1}|u|^{q+1} + \\ (\alpha + \beta\delta) r^N u|u'| - \alpha\delta r^{N-1}u^2(r) - \beta r^{N+1}u'^2(r). \end{aligned} \tag{81}$$

Integrating the last inequality on $(0, r_0)$, we obtain

$$\begin{aligned} G(r_0) = \left(\delta - \frac{N-p}{p}\right) \int_0^{r_0} s^{N-1}|u'|^p \, ds + \left(\frac{N}{q+1} - \delta\right) \int_0^{r_0} s^{N-1}|u|^{q+1}(s) \, ds \\ + (\alpha + \beta\delta) \int_0^{r_0} s^N u|u'| \, ds - \alpha\delta \int_0^{r_0} s^{N-1}u^2(s) \, ds - \beta \int_0^{r_0} s^{N+1}u'^2(s) \, ds. \end{aligned} \tag{82}$$

With the choice of δ and the fact that $u > 0$ and $u' < 0$ on $(0, r_0)$, we obtain by (82),

$$G(r_0) < (\alpha + \beta\delta) \int_0^{r_0} s^N u|u'| \, ds - \beta \int_0^{r_0} s^{N+1}u'^2(s) \, ds. \tag{83}$$

According to Proposition 4.1, we have

$$\int_0^{r_0} s^N u|u'| \, ds \leqslant \frac{2}{N} \int_0^{r_0} s^{N+1}u'^2(s) \, ds. \tag{84}$$

Then by (83) and (84), we see that

$$G(r_0) < \left(\frac{2}{N}(\alpha + \beta\delta) - \beta\right) \int_0^{r_0} s^{N+1}u'^2(s) \, ds. \tag{85}$$

Since $N > p$ and $\dfrac{p}{q+1-p} < \dfrac{p}{2}$, one has $\dfrac{N-p}{p} < \dfrac{N}{2} - \dfrac{\alpha}{\beta}$. Again, with the choice of δ, we have $\delta < \dfrac{N}{2} - \dfrac{\alpha}{\beta}$, which implies that $\left(\dfrac{2}{N}(\alpha + \beta\delta) - \beta\right) < 0$, that is, $G(r_0) < 0$, but this contradicts the fact that

$$G(r_0) = \frac{p-1}{p} r_0^N |u'(r_0)|^p \geqslant 0.$$

Consequently, u is strictly positive. This completes the proof. □

5 Conclusion

In this work, we studied the Cauchy problem (P). We proved the existence of global solutions, we presented their complete classification in the cases $\dfrac{\alpha}{\beta} \geq N$ and $\dfrac{\alpha}{\beta} < N$, and we gave an explicit behavior near infinity of the positive solutions. More precisely, we have given explicit equivalents to the positive solution u of problem (P) and its negative derivative u'. The study of asymptotic behavior of positive solutions is carried out in the case $\dfrac{\alpha}{\beta} = \dfrac{p}{q+1-p} < \dfrac{N-p}{p-1}$, which recalls the form of radial self-similar solutions of the parabolic problem (2) from which the problem (P) is derived.

Asymptotic behavior of positive solutions is ensured by the study of a nonlinear dynamical system that we obtained by using the logarithmic change

$$v(t) = r^{\alpha/\beta} u(r), \quad r > 0 \text{ and } t = Log(r). \tag{86}$$

This obtained system, which we call (S), is as following:

$$(S) \begin{cases} v'(t) = |w(t)|^{\frac{2-p}{p-1}} w(t) + \dfrac{\alpha}{\beta} v(t), \\ w'(t) = -(N-p-\dfrac{\alpha}{\beta}(p-1))w(t) - \alpha e^{(p+\frac{\alpha}{\beta}(p-2))t} v(t) - \beta e^{(p+\frac{\alpha}{\beta}(p-2))t} z(t) - |v|^{q-1} v(t), \end{cases}$$

where

$$w(t) = |z|^{p-2} z(t) \tag{87}$$

and

$$z(t) = v'(t) - \frac{\alpha}{\beta} v(t) = r^{\frac{\alpha}{\beta}+1} u'(r). \tag{88}$$

The solution (v, w) of the system (S) satisfies $v > 0$ and $w < 0$ (because $u > 0$ and $u' < 0$) and tends near infinity to the equilibrium point $\left(\Gamma, -\left(\dfrac{\alpha}{\beta}\Gamma\right)^{p-1}\right)$, where Γ is explicitly dependent on p, q and N. Indeed, rewriting the second equation of the system (S) by using expression (88), we obtain

$$-\beta e^{(p+\frac{\alpha}{\beta}(p-2))t} v'(t) = w\left(N - p - \frac{\alpha}{\beta}(p-1) + \frac{w'}{w} + \frac{v^q}{w}\right). \tag{89}$$

We have by (63) and (64),

$$\lim_{t\to+\infty} \frac{w'}{w} = \lim_{r\to+\infty} (p-1) \frac{g'_{\frac{\alpha}{\beta}}(r)}{u'(r)} = 0 \tag{90}$$

and

$$\lim_{t\to+\infty} \frac{v^q}{w} = \lim_{r\to+\infty} \frac{r u^q(r)}{|u'|^{p-2} u'(r)} = \frac{-\Gamma^{q+1-p}}{\left(\dfrac{\alpha}{\beta}\right)^{p-1}}. \tag{91}$$

Therefore

$$\lim_{t\to+\infty} -\beta e^{(p+\frac{\alpha}{\beta}(p-2))t} \frac{v'(t)}{w} = N - p - \frac{\alpha}{\beta}(p-1) - \frac{\Gamma^{q+1-p}}{\left(\dfrac{\alpha}{\beta}\right)^{p-1}}. \tag{92}$$

Recall by Proposition 2.7, that $v(t)$ is not strictly monotone for large t, then since w is strictly negative, necessarily we have by (92),

$$\lim_{t\to+\infty} -\beta e^{(p+\frac{\alpha}{\beta}(p-2))t}\frac{v'(t)}{w} = 0.$$

Hence the explicit expression of Γ given by (48).

Finally, using expressions (86), (87) and (88), the convergence of the solution (v, w) of the system (S) to the equilibrium point $\left(\Gamma, -\left(\frac{\alpha}{\beta}\Gamma\right)^{p-1}\right)$ near infinity is expressed in terms of u and u' by

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}}u(r) = \left(N-p-\frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}}\left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}$$

and

$$\lim_{r\to+\infty} r^{\frac{\alpha}{\beta}+1}u'(r) = \frac{-\alpha}{\beta}\left(N-p-\frac{\alpha}{\beta}(p-1)\right)^{\frac{1}{q+1-p}}\left(\frac{\alpha}{\beta}\right)^{\frac{p-1}{q+1-p}}.$$

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References

[1] M. F. Bidaut-Véron. The p-Laplacien heat equation with a source term: self-similar solutions revisited. *Advanced Nonlinear Studies* **6** (2006) 69–108.

[2] M. F. Bidaut-Véron. Local and global behavior of solutions of quasilinear equations of Emden-Fowler type. *Archive for Rational Mechanics and Analysis* **107** (4) (1989) 293–324.

[3] M. Bonforte, F. Gazzola, G. Grillo and J.L.Vazquez. Classification of radial solutions to the Emden-Fowler equation on the hyperbolic space. *Calc. Var.* **46** (2013) 375–401.

[4] A. Bouzelmate and A. Gmira. Existence and Asymptotic Behavior of Unbounded Solutions of a Nonlinear Degenerate Elliptic Equation. *Nonlinear Dynamics and Systems Theory* **21** (1) (2021) 27–55.

[5] A. Bouzelmate, A. Gmira and G. Reyes. Radial selfsimilar solutions of a nonlinear Ornstein-Uhlenbeck equation. *Electronic Journal of Differential Equations* **2007** (67) (2007) 1–21.

[6] C. Dohmen and M. Hirose. Structure of the positive radial solutions to the HarauxWeissler equation. *Nonlinear Anal.* **33** (1998) 51–69.

[7] R. Emden. *Gaskugeln Anwendungen der Mechanischen Warmen-theorie auf Kosmologie und Meteorologische Probleme.* Leipzig, Teubner, 1907.

[8] R.H. Fowler. The form near infinity of real continuous solutions of a certain differential equation of second order. *Quart. J. Math.* **45** (1914) 289–350.

[9] R.H. Fowler. The solutions of Emden's and similar differential equation. *Monthly notices Roy. Astr. Soc.* **91** (1920) 63–91.

[10] R.H. Fowler. Further studies on Emden's and similar differential equation. *Quart. J. Math.* **2** (1931) 259–288.

[11] A. Gmira and B. Bettioui. On the radial solutions of a degenerate quasilinear elliptic equation in $\mathbb{R}^N$. *Annals of the Faculty of Sciences of Toulouse* **8** (3) (1999) 411–438.

[12] B. Gidas and J. Spruck. Global and local behavior of positive solutions of nonlinear elliptic equations. *Comm. Pure Appl. Math.* **34** (1980) 525–598.

[13] M. Guedda and L. Véron. Local and global properties of solutions of quasilinear elliptic equations. *J. Differential Equations* **76** (1988) 159–189.

[14] H. Haraux and F.B. Weissler. Non-uniqueness for a semilinear initial value problem. *indiana Univ. Math. J.* **31** (1982) 167–189.

[15] M. Hirose. Structure of positive radial solutions to a semilinear elliptic pde with a gradient-term. *Funkcialaj Ekvacioj* **39** (1996) 323–345.

[16] M. Hirose and E. Yanagida. Global structure of self-similar solutions in a semilinear parabolic equation. *J. Mathematical Analysis and Applications* **244** (2000) 348–368.

[17] P.L.Lions. Isolated singularities in semilinear problems. *J. Diff. Eq.* **38** (1980) 441–450.

[18] W. M. Ni and J. Serrin. Nonexistence theorems for singular solutions of quasilinear partial differential equations. *Comm. Pure Appl. Math.* **38** (1986) 379–399.

[19] Y. Naito and T. Suzuki. Radial symmetry self similar solutions for semilinear heat equations. *J. Diff. Equ.* **163** (2000) 407–428.

[20] L. A. Peletier, D. Terman and F.B. Weissler. On the equation $\Delta u + \frac{1}{2}x.\nabla u + f(u) = 0$. *Arch. Rational Mech. Anal.* **94** (1986) 83–99.

[21] Y-W-Qi. The global existence and non uniqueness of a non linear degenerate equation. *Nonlinear Analysis* **31** (12) (1998) 117–136.

[22] F. B. Weissler. Asymptotic analysis of an ordinary differential equation and nonuniqueness for a semilinear partial differential equation. *Arch. Rat. Mech. Anal.* **91** (1986) 231–245.

[23] F. B. Weissler. Rapidly decaying solutions of an ordinary differential equation with applications to semilinear elliptic and parabolic partial differential equations. *Arch. Rat. Mech. Anal.* **91** (1986) 247–266.

[24] E. Yanagida. Uniqueness of rapidly decaying solutions to the Haraux-Weissler Equation. *J. Diff. Equ.* **127** (1996) 561–570.

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Existence, Uniqueness of Weak Solution to the Thermoelastic Plates

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Abstract: In this paper, we study a model of dynamic von Karman equation coupled to the thermoelastic equation, with rotational forces and not clamped boundary conditions. Our fundamental goal is to establish the existence as well as the uniqueness of a weak solution for the so-called global energy. In the end, we display a numerical simulation.

Keywords: *von Karman equation; nonlinear plates; rotational inertia; non-coupled method; finite difference method.*

Mathematics Subject Classification (2010): 74F10; 74B20; 74K25; 65N06.

1 Introduction

In nonlinear oscillation of elastic plates, a dynamic von Karman equation with rotational forces, $(\alpha > 0)$ [1], describes the buckling and flexible phenomenon of small nonlinear vibration of vertical displacement to the elastic plates. In nonlinear thermoelastic plate interaction, we study in this paper the case when the plate is coupled with thermal dissipation. From physical point of view, the main peculiarities of the model are the possibility of large deflections of the plate and small changes of the temperature near the reference temperature of the plate. As is well-known, the model with clamped boundary conditions, taking and not taking into account the rotational terms, for displacement u, the Airy stress function ϕ and the thermal function θ, can be formulated by the following system, see for instance [1].

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Find $(u, \phi, \theta) \in \left(L^2([0,T], (H_0^2(\omega))^2\right) \times H_0^1(\omega)$ such that

$$(\mathbb{P}_0)\begin{cases} u_{tt} - \alpha\Delta u_{tt} + \Delta^2 u + \mu\Delta\theta - [\phi + F_0, u] = p(x) & in \quad \omega \times [0,T], \\ k\theta_t - \eta\Delta\theta - \mu\Delta u_t = 0 & in \quad \omega \times [0,T], \\ u_{|t=0} = u_0, \quad (u_t)_{|t=0} = u^1, \quad \theta_{|t=0} = \theta_0 & in \quad \omega, \\ u = \partial_\nu u = 0 & on \quad \Gamma \times [0,T], \\ \theta = 0 & on \quad \Gamma, \end{cases}$$

and

$$(\mathbb{Q})\begin{cases} \Delta^2\phi + [u,u] = 0 & in \quad \omega \times [0,T], \\ \phi = 0, \ \partial_\nu\phi = 0 & on \quad \Gamma \times [0,T], \end{cases}$$

where ω is the surface plate, u_0, u_1 and θ_0 are the initial data and $[.,.]$ is the so-called Monge-Ampère operator defined by [2]

$$[\phi, u] = \partial_{11}\phi\partial_{22}u + \partial_{11}u\partial_{22}\phi - 2\partial_{12}\phi\partial_{12}u. \tag{1}$$

The parameters μ, η are positive and α, k are non negative. The case $\alpha > 0$ corresponds to the equation with rotational term. But the parameter k has the meaning of heat/thermal capacity. Now, in the case $k = 0$ and $\alpha = 0$, the model $(\mathbb{P}_0)$ without rotational inertia can be decoupled, if we substitute $\Delta\theta$ from the second equation, the first equation becomes just a model of dynamic von Karman equations with internal viscous damping [1].

The plate is subjected to the internal force F_0 and external force p_0. In [1], Chueshov and Lasiecka studied the problem of structural interaction coupled with the von Karman evolution and established the theoretical result for a strong, generalized and weak solution by using the theory of nonlinear semi-group, if one chooses $0 \le \alpha \le 1$ and $0 \le k \le 1$. To justify the uniqueness, the authors used the limit definition of a generalized solution along weak continuity of the nonlinear terms involving the Airy stress function and known Lip continuity of the von Karman bracket with the Airy stress function.

The aim of this paper is to give a condition verified by the external, internal loads and the initial datums to have a unique weak solution of the von Karman evolution with rotational terms and not clamped boundary conditions subject to thermal dissipation and for all $\alpha > 0$, $k > 0$ and $0 < \mu \le \eta$. Our approach is based on an iterative problem $(P_n)_{n\ge 0}$ whose sequence-solution $(u_n, \phi_n, \theta_n)_{n\ge 0}$ converges to the unique solution of the problem under consideration.

This paper is organized as follows. Section 2 is devoted to the description of the mathematical structure of the model. In Section 3, we use the iterative method for establishing the uniqueness of weak solution of the associated dynamical plates with rotational terms, subject to thermal dissipation. In Section 3, we describe the numerical test.

2 Preliminary Results and Needed Tools

Throughout the following consideration, ω denotes a nonempty bounded domain in $\mathbb{R}^2$, with the regular boundary $\Gamma = \partial\omega$ and $\alpha > 0$, $k > 0$, $0 < \mu \le \eta$ are the reals.

Let $p \geq 1$ be a real number and $m \geq 1$ be an integer. We denote by $|.|_{p,\omega}$ the classical norm of $L^p(\omega)$ and by $\|.\|_{m,\omega}$ that of $H^m(\omega)$. For $u \in H^2(\omega)$, we set $\|u\| = |\Delta u|_{2,\omega}$ for the sake of simplicity. We also set

$$W(0,T) = \{u, \quad u \in L^2\big([0,T], H_0^2(\omega)\big), u_t \in L^2\big([0,T], L^2(\omega)\big)\},$$

which is a Hilbert space with the associated norm

$$\Big(|u|^2_{L^2\left([0,T],H_0^2(\omega)\right)} + |u_t|^2_{L^2\left([0,T],L^2(\omega)\right)}\Big)^{1/2}.$$

In this paper, for the sake of simplicity, we denote

$$\|u\|_\alpha = \|u\|^2 + \alpha\,|\nabla u_t|^2_{2,\omega} + |u_t|^2_{2,\omega}\,. \tag{2}$$

We recall the following result [3, 4].

Theorem 2.1 *Let $f \in L^2(\omega)$. Then the following problem:*

$$\begin{cases} \Delta^2 v = f & in \quad \omega, \\ v = 0 & on \quad \Gamma, \\ \partial_\nu v = 0 & on \quad \Gamma, \end{cases}$$

has one and only one solution $v \in H_0^2(\omega) \cap H^4(\omega)$ satisfying

$$\|v\| \leq c_0\,|f|_{1,\omega}$$

for some constant $c_0 > 0$ depending only on $mes(\omega)$.

The following remark is of interest.

Remark 2.1 If the function f is in $L^2\big([0,T], L^2(\omega)\big)$, then the unique solution of the last problem is in $L^2\big([0,T], H_0^2(\omega) \cap H^4(\omega)\big)$.

We also need to recall the following result [4, 5].

Theorem 2.2 *Let $g \in L^2\big([0,T], L^2(\omega)\big)$, $u_0 \in L^2(\omega)$ and k, η, μ are non negative reals. Then the following problem :*

$$(\mathbb{D}) \begin{cases} ku_t - \eta\Delta u = \mu g & in \quad \omega \times [0,T], \\ u_{|t=0} = u_0 & in \quad \omega, \\ u = 0 & on \quad \Gamma \times [0,T], \end{cases}$$

has one and only one solution $u \in C\big([0,T]; H^2(\omega) \cap H_0^1(\omega)\big) \cap C^1\big([0,T]; L^2(\omega)\big)$.

Proposition 2.1 *Under the assumptions of Theorem 2.2 and if we choose $g = -\Delta f$, then the unique solution of the problem $(\mathbb{D})$ satisfies the following inequality:*

$$\forall 0 \leq t \leq T, \quad k\,|u|^2_{2,\omega} + \eta \int_0^t |\nabla u|^2_{2,\omega} \leq k\,|u_0|^2_{2,\omega} + \mu \int_0^t |\nabla f|^2_{2,\omega}\,, \tag{3}$$

where $f \in H^2(\omega)$, $k > 0$ and $0 < \mu \leq \eta$.

Proof. Since u is a solution of the problem $(\mathbb{D})$, we have

$$\frac{k}{2}\frac{d}{dt}\left|u\right|_{2,\omega}^{2}+\eta\left|\nabla u\right|_{2,\omega}^{2}=\int_{\omega}gu=\int_{\omega}-\Delta fu=\int_{\omega}\nabla f\nabla u\leq\frac{1}{2}\left|\nabla f\right|_{2,\omega}^{2}+\frac{1}{2}\left|\nabla u\right|_{2,\omega}^{2}.$$

Now, if we integrate the latter inequality with respect to $t>0$, we then deduce, by using the fact that $(u)_{|t=0}=u_0$ in ω,

$$\frac{k}{2}\left|u\right|_{2,\omega}^{2}+\eta\int_{0}^{t}\left|\nabla u\right|_{2,\omega}^{2}\leq\frac{k}{2}\left|u_0\right|_{2,\omega}^{2}+\frac{\mu}{2}\int_{0}^{t}\left|\nabla f\right|_{2,\omega}^{2}+\frac{\mu}{2}\int_{0}^{t}\left|\nabla u\right|_{2,\omega}^{2},$$

we have that $0\prec\mu\leq\eta$, then we conclude that

$$k\left|u\right|_{2,\omega}^{2}+\eta\int_{0}^{t}\left|\nabla u\right|_{2,\omega}^{2}\leq k\left|u_0\right|_{2,\omega}^{2}+\mu\int_{0}^{t}\left|\nabla f\right|_{2,\omega}^{2}.$$

The following theorem is of interest, see [1].

Theorem 2.3 *Assume that for $f\in L^2(\omega)$, $\alpha>0$ and $(u_0,u^1)\in H_0^2(\omega)\times H_0^1(\omega)$, the problem*

$$(\mathbb{S})\begin{cases}(1-\alpha\Delta)u_{tt}+\Delta^2u=f & in\quad \omega\times[0,T],\\ u=\partial_\nu u=0 & on\quad \Gamma\times[0,T],\\ u_{|t=0}=u_0,\ \ (u_t)_{|t=0}=u^1 & in\quad \omega,\end{cases}$$

has a unique solution $(u,u_t)\in C([\,0,T\,],H_0^2(\omega)\times H_0^1(\omega))$, and the energy equality

$$E_0(u,u_t)=E_0(u_0,u^1)+\int_0^t\int_\omega fu_t$$

holds, here

$$E_0(u_0,u^1)=\frac{1}{2}\int_\omega(\|u_0\|_{2,\omega}^2+\left|u^1\right|_{2,\omega}^2+\alpha\left|\nabla u^1\right|_{2,\omega}^2).$$

Now, let us put

$$F_1(u,\phi)=[\phi+F_0,u]. \tag{4}$$

Before giving our main result, we now state the following results.

Proposition 2.2 *Let $(u,v)\in(H_0^2(\omega))^2$ and $F_0\in H^4(\omega)$ be with small norms. Let $\phi,\varphi\in H_0^2(\omega)$ be the solutions of the following two problems:*

$$\Delta^2\phi=-\left[u,u\right]\quad and\quad \Delta^2\varphi=-\left[v,v\right].$$

Then the following estimations:

$$\Big|\,[u,\phi]-[v,\varphi]\,\Big|_{2,\omega}\leq c_1\|u-v\|$$

and

$$\|F_1(u,\phi)-F_1(v,\varphi)\|_{(L^2(\omega))^3}\leq c_1\left\|u-v\right\|$$

hold for some $0<c_1<1$.

Proof. Following [1], we have

$$\Big| [u,\phi]-[v,\varphi]\Big|_{2,\omega} \le c_0\big(\|u\|^2+\|v\|^2\big)\|u-v\|$$

for some $c_0>0$. Let $c>0$ be small enough such that $\|u\|\le c$ and $\|v\|\le c$. We have

$$\Big| [u,\phi]-[v,\varphi]\Big|_{2,\omega} \le 2c_0c^2\|u-v\|$$

and so

$$\begin{aligned}\|F_1(u,\phi)-F_1(v,\varphi)\|_{(L^2(\omega))^3} &\le \Big|\,[\phi+F_0,u]-[\varphi+F_0,v]\,\Big|_{2,\omega},\\ &\le \Big|\,[\phi,u]-[\varphi,v]\,\Big|_{2,\omega}+\Big|\,[F_0,u-v]\,\Big|_{2,\omega},\\ &\le \big(2c_0c^2+4\|F_0\|_{4,\omega}\big)\|u-v\|.\end{aligned}$$

If we choose

$$\|F_0\|_{4,\omega}<\frac{1}{4}\quad \textit{and}\quad 0<c<\sqrt{\frac{1-4\|F_0\|_{4,\omega}}{2c_0}},$$

we have

$$0<c_1=2c_0c^2+4\|F_0\|_{4,\omega}<1,$$

we then conclude the proof. The following proposition is of interest.

Proposition 2.3 *Let $f\in L^2([0,T],L^2(\omega))$, $\theta_0\in H_0^1(\omega)$ and $(u_0,u^1)\in H_0^2(\omega)\times H_0^1(\omega)$. The following problem :*

$$(\mathbb{S}_1)\begin{cases}(u)_{tt}-\alpha\Delta(u)_{tt}+\Delta^2u+\mu\Delta\theta=f & in\ \ \omega\times[0,T],\\ k\theta_t-\eta\Delta\theta=\mu\Delta u_t & in\ \ \omega\times[0,T],\\ u=\partial_\nu u=\theta=0 & on\ \ \Gamma\times[0,T],\\ (u)_{|t=0}=u_0,(u_t)_{|t=0}=u^1,(\theta)_{|t=0}=\theta_0 & in\ \ \omega,\end{cases}$$

has one and only one solution $(u,\theta)\in L^2([\,0,T\,],H_0^2(\omega)\times H_0^1(\omega))$ and $u_t\in L^2([\,0,T\,],H_0^1(\omega))$ satisfies

$$\|u\|_\alpha+k\,|\theta|_{2,\omega}^2+2\eta\int_0^t|\nabla\theta|_{2,\omega}^2\le e^T\Big(\|u_0\|^2+\alpha\left|\nabla u^1\right|_{2,\omega}^2+\left|u^1\right|_{2,\omega}^2+k\,|\theta_0|_{2,\omega}^2+\int_0^T|f|_{2,\omega}^2\Big). \quad (5)$$

Proof. For establishing the existence and uniqueness of solution of the problem under consideration, we will study the problem $(\mathbb{S})_1$ by considering the n-order approximate solution and we use the variational problem.
Let $\{\,e_k,e_k^1\,\}$ be a basis in the space $H_0^2(\omega)\times H_0^1(\omega)$. We define an n-order Galerkin approximate solution to the problem $(\mathbb{S})_1$ with clamped boundary conditions on the interval $[\,0,T\,]$, as a function $(u^n(t),\theta^n(t))$ of the form, see for instance [1, 6],

$$u^n=\sum_{k=1}^n h_k(t)e_k\ \ \textit{and}\ \theta^n=\sum_{k=1}^n l_k(t)e_k^1\quad n=1,2,3,...,$$

where $(h_k(t), l_k(t)) \in W^{2,+\infty}(0,T,IR) \times W^{1,+\infty}(0,T,IR)$ and ϕ^n is determined by u^n according to the problem $(\mathbb{Q})$ and (u_{n0}, θ_{n0}), u_{n1} are chosen such that (u_{n0}, θ_{n0}) converges to (u_0, θ_0) in $L^2([0,T], H_0^2(\omega) \times H_0^1(\omega))$ and u_{n1} converges to u^1 in $L^2([0,T], H_0^1(\omega))$.
Let the variational problem of $(\mathbb{S})_1$ be

$$\int_\omega u_{tt}^n u_t^n + \alpha \int_\omega \nabla u_{tt}^n \nabla u_t^n + \int_\omega \Delta u^n \Delta u_t^n + \mu \int_\omega \Delta\theta^n u_t^n = \int_\omega f u_t^n$$

and

$$\int_\omega \theta_t^n \theta^n - \eta \int_\omega (\nabla\theta^n)^2 = \mu \int_\omega \Delta u_t^n \theta^n.$$

Since $(u_t^n, \theta^n) \in H_0^2(\omega) \times H_0^1(\omega)$ and $\int_\omega \Delta\theta^n u_t^n = \int_\omega \theta^n \Delta u_t^n$, we have

$$\frac{1d}{2dt}(|u_t^n|_{2,\omega}^2 + \|u^n\|^2 + \alpha\,|\nabla u_t^n|_{2,\omega}^2) + \mu \int_\omega \theta^n \Delta u_t^n = \int_\omega f u_t^n,$$

and

$$\frac{kd}{2dt}\,|\theta^n|_{2,\omega}^2 + \eta\,|\nabla\theta^n|_{2,\omega}^2 = \int_\omega \Delta\theta^n u_t^n.$$

Hence

$$\frac{1d}{2dt}(|u_t^n|_{2,\omega}^2 + \|u^n\|^2 + \alpha\,|\nabla u_t^n|_{2,\omega}^2) + \frac{kd}{2dt}\,|\theta^n|_{2,\omega}^2 + \eta\,|\nabla\theta^n|_{2,\omega}^2 = \int_\omega f u_t^n.$$

Now, if we integrate the latter inequality with respect to $t > 0$, with (2) and by using the fact that $u^n_{|t=0} = u_{n0}, (u_t^n)_{|t=0} = u_{n1}$ and $\theta^n_{|t=0} = \theta_{n0}$, we deduce that

$$\frac{1}{2}(\|u^n\|_\alpha + k\,|\theta^n|_{2,\omega}^2) + \eta \int_0^t |\nabla\theta^n|_{2,\omega}^2 = \frac{1}{2}(|u_{n1}|_{2,\omega}^2 + \alpha\,|\nabla u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2) + \int_0^t \int_\omega f u_t^n.$$

And for all $0 \le s \le t$,

$$\|u^n\|_\alpha + k\,|\theta^n|_{2,\omega}^2 + 2\eta \int_0^t |\nabla\theta^n|_{2,\omega}^2 \le |u_{n1}|_{2,\omega}^2 + \alpha\,|\nabla u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2$$
$$+ \int_0^t \Big(\|u^n\|_\alpha + k\,|\theta^n|_{2,\omega}^2 + 2\eta \int_0^s |\nabla\theta^n|_{2,\omega}^2 \Big). \qquad (6)$$

For any $0 \le s \le t$, we put

$$I(s) = \|u^n\|_\alpha + k\,|\theta^n|_{2,\omega}^2 + 2\eta \int_0^s |\nabla\theta^n|_{2,\omega}^2.$$

The inequality (6) yields

$$e^{-s}\Big(I(s) - \int_0^s I(\sigma)d\sigma\Big) \le e^{-s}\Big(|u_{n1}|_{2,\omega}^2 + \alpha\,|\nabla u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \int_0^T |f|_{2,\omega}^2 \Big).$$

Now, we have

$$\begin{aligned} \tfrac{d}{ds}\Big(e^{-s}\textstyle\int_0^s I(\sigma)d\sigma\Big) &= e^{-s}I(s) - e^{-s}\textstyle\int_0^s I(\sigma)d\sigma = e^{-s}\Big(I(s) - \textstyle\int_0^s I(\sigma)d\sigma\Big), \\ &\le e^{-s}\Big(|u_{n1}|_{2,\omega}^2 + \alpha\,|\nabla u_{n1}|_{2,\omega}^2 + \|u_{n0}\|^2 + k\,|\theta_{n0}|_{2,\omega}^2 + \textstyle\int_0^T |f|_{2,\omega}^2 \Big), \end{aligned}$$

and

$$|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega} + \int_0^T |f|^2_{2,\omega} = I(0) + \int_0^T |f|^2_{2,\omega}$$

does not depend on s, then

$$\int_0^t \frac{d}{ds}\Big(e^{-s}\int_0^s I(\sigma)d\sigma\Big)ds \le (\int_0^t e^{-s}ds)\Big(\,|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega}$$
$$+\int_0^T |f|^2_{2,\omega}\Big),$$

from which we deduce

$$e^{-t}\int_0^t I(\sigma)d\sigma \le (1-e^{-t})\Big(\,|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega} + \int_0^T |f|^2_{2,\omega}\Big).$$

Since

$$\int_0^t (\|u^n\|_\alpha + k\,|\theta^n|^2_{2,\omega} + 2\eta\int_0^s |\nabla\theta^n|^2_{2,\omega}) = \int_0^t I(\sigma)d\sigma,$$

it follows that

$$\begin{aligned}
\textstyle\int_0^t I(\sigma)d\sigma \;&\le \tfrac{(1-e^{-t})}{e^{-t}}\Big(\,|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big),\\
&\le (e^t-1)\Big(\,|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big),\\
&\le (e^T-1)\Big(\,|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big).
\end{aligned}$$

This, with (6), yields

$$\begin{aligned}
\|u^n\|_\alpha + k\,|\theta^n|^2_{2,\omega} + 2\eta\textstyle\int_0^t |\nabla\theta^n|^2_{2,\omega} &\le \Big(\,|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big)\\
&\quad+(e^T-1)\Big(\,|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big),\\
&\le e^T\Big(\,|u_{n1}|^2_{2,\omega} + \alpha\,|\nabla u_{n1}|^2_{2,\omega} + \|u_{n0}\|^2 + k\,|\theta_{n0}|^2_{2,\omega} + \textstyle\int_0^T |f|^2_{2,\omega}\Big).
\end{aligned}$$

This estimate implies that there exists a subsequence (u^{n_l},θ^{n_l}) such that $(u^{n_l},\theta^{n_l}) \rightharpoonup (u,\theta)$ weakly in $H_0^2(\omega)\times L^2(\omega))$ and $((u^{n_l})_t,\nabla\theta^{n_l}) \rightharpoonup ((u)_t,\nabla\theta)$ weakly in $H_0^1(\omega)\times L^2(\omega)$.

For showing that (u,θ) is a weak solution of the problem $(\mathbb{S})_1$, we use the same method as in [6]. Let $\varphi_j \in C^1(0,T)$, $1\le j\le j_0$, such that $\varphi_j(T)=0$ and

$$\psi = \sum_{j=1}^{j_0}\varphi_j\otimes e_j, \quad \varphi = \sum_{j=1}^{j_0}\varphi_j\otimes e_j^1.$$

After the variational problem, we have

$$-\int_0^T\int_\omega u_t^{nl}\psi_t + \alpha\int_0^T\int_\omega \nabla u_t^{nl}\nabla\psi_t + \mu\int_0^T\int_\omega \nabla\theta^{nl}\nabla\psi + \int_0^T\int_\omega \Delta u^{nl}\Delta\psi$$

$$= \int_0^T \int_\omega f\psi - \int_\omega u_{nl1}\psi(0) - \alpha \int_\omega \nabla u_{nl1} \nabla\psi(0) \tag{7}$$

and

$$-\int_0^T \Big(\int_\omega \theta^{nl} \varphi_t + \eta \int_\omega \nabla\theta^{nl} \nabla\varphi - \mu \int_\omega \nabla u^{nl} \nabla\varphi_t \Big) = -\int_\omega \theta_{nl0}\varphi(0) + \mu \int_\omega \nabla u_{nl1} \nabla\varphi(0). \tag{8}$$

Now, we can pass to the limit $nl \to +\infty$, in (7) and (8), we find that for all $\psi \in L^2([0,T], H_0^2(\omega))$, $\psi_t \in L^2([0,T], H^1(\omega))$, $\varphi \in L^2([0,T], H_0^1(\omega))$ and $\varphi_t \in L^2([0,T], L^2(\omega))$ such that $\psi(T) = \varphi(T) = 0$. We deduce that

$$-\int_0^T \int_\omega u_t \psi_t + \alpha \int_0^T \int_\omega \nabla u_t \nabla\psi_t + \mu \int_0^T \int_\omega \nabla\theta \nabla\psi + \int_0^T \int_\omega \Delta u \Delta\psi$$

$$= \int_0^T \int_\omega f\psi - \int_\omega u^1 \psi(0) - \alpha \int_\omega \nabla u^1 \nabla\psi(0)$$

and

$$-\int_0^T \Big(\int_\omega \theta \varphi_t + \eta \int_\omega \nabla\theta \nabla\varphi - \mu \int_\omega \nabla u \nabla\varphi_t \Big) = -\int_\omega \theta_0 \varphi(0) + \mu \int_\omega \nabla u^1 \nabla\varphi(0).$$

This shows that (u, θ) is a weak solution of the problem $(\mathbb{S})_1$, by the some method as in the last proof, we deduce the following inequality:

$$\|u\|_\alpha + k\,|\theta|^2_{2,\omega} + 2\eta \int_0^t |\nabla\theta|^2_{2,\omega} \leq e^T (\left|u^1\right|^2_{2,\omega} + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \|u_0\|^2 + k\,|\theta_0|^2_{2,\omega} + \int_0^T |f|^2_{2,\omega}).$$

For the uniqueness, let (u_1, θ_1) and (u_2, θ_2) be two solutions. We use a similar proof as that of inequality (5), for the solution $(u_1 - u_2, \theta_1 - \theta_2)$ of the following problem:

$$\begin{cases} (1 - \alpha\Delta)(u_1 - u_2)_{tt} + \Delta^2(u_1 - u_2) + \mu\Delta(\theta_1 - \theta_2) = 0 & in \quad \omega \times [0,T]\,, \\ k(\theta_1 - \theta_2)_t - \eta\Delta(\theta_1 - \theta_2) = \mu\Delta(u_1 - u_2)_t & in \quad \omega \times [0,T]\,, \\ \theta_1 - \theta_2 = u_1 - u_2 = \partial_\nu(u_1 - u_2) = 0 & on \quad \Gamma \times [0,T]\,, \\ (u_1 - u_2)_{|t=0} = 0, ((u_1 - u_2)_t)_{|t=0} = 0, (\theta_1 - \theta_2)_{|t=0} = 0 & in \quad \omega, \end{cases}$$

it follows that

$$\|u_1 - u_2\|_\alpha + k\,|\theta_1 - \theta_2|^2_{2,\omega} + 2\eta \int_0^t |\nabla(\theta_1 - \theta_2)|^2_{2,\omega} \leq e^T (\left|(u_1)^1 - (u_2)^1\right|^2_{2,\omega}$$

$$+ \|(u_1)_0 - (u_2)_0\|^2 + \alpha \left|\nabla((u_1)^1 - (u_2)^1)\right|^2_{2,\omega} + k\,|(\theta_1)_0 - (\theta_2)_0|^2_{2,\omega}).$$

Then $u_1 = u_2$ and $\theta_1 = \theta_2$. The proof of the proposition is completed.

3 Iterative Approach: The Main Results

For establishing the existence and uniqueness of solution of the problem $(\mathbb{P}_0)$ in the case of rotational terms $\alpha > 0$, we use the following iterative approach.

Let $n \geq 2$ and let $0 \neq u_1 \in H_0^2(\omega)$ be given. We first find $\phi_{n-1} \in H_0^2(\omega)$ as the solution of the equation $\Delta^2\phi_{n-1} = -[u_{n-1}, u_{n-1}]$ and (u_n, θ_n) as the solution of the following problem:

$$(\mathbb{P}_n)\begin{cases} (u_n)_{tt} - \alpha\Delta(u_n)_{tt} + \Delta^2 u_n = F(u_{n-1}, \phi_{n-1}, \theta_n) & in \quad \omega \times [0,T], \\ k(\theta_n)_t - \eta\Delta\theta_n = \mu\Delta(u_n)_t & in \quad \omega \times [0,T], \\ u_n = \partial_\nu u_n = \theta_n = 0 & on \quad \Gamma \times [0,T], \\ (u_n)_{|t=0} = u_0, ((u_n)_t)_{|t=0} = u^1, (\theta_n)_{|t=0} = \theta_0 & in \quad \omega, \end{cases}$$

where

$$F(u, \phi, \theta) = F_1(u, \phi) - \mu\Delta\theta + p,$$

and F_1 is defined by (4).

We are now in a position to state our main result of this section.

Theorem 3.1 *Let $p \in L^2(\omega)$, $(u_0, u^1) \in H_0^2(\omega) \times H_0^1(\omega)$ and $\theta_0 \in H_0^1(\omega)$. Assume that all the following quantities:*

$$\|F_0\|_{4,\omega}, \ |p|_{2,\omega}, \ \|u_0\|^2 + \left|u^1\right|_{2,\omega}^2 + \alpha\left|\nabla u^1\right|_{2,\omega}^2 \quad and \quad \|\theta_0\|_{1,\omega}^2$$

are small with $0 < \mu \leq \eta$. Then the problem $(\mathbb{P}_0)$ with rotational forces has one and only one weak solution (u, ϕ, θ) in $L^2\big([0,T], H_0^2(\omega) \times H_0^2(\omega) \times H_0^1(\omega)\big)$ such that $u_t \in L^2\big([0,T], H_0^1(\omega)\big)$ and $u_{tt} \in L^2\big([0,T], L^2(\omega)\big)$.

Proof. We divide the proof into four steps.
Step 1: Let us consider the problem $(\mathbb{P}_n)$, where $0 \neq u_1$ does not depend on t.
Throughout this proof, we use the notation

$$\|(u,\theta)\|_* = \|u\|_\alpha + k\,|\theta|_{2,\omega}^2 + 2\eta\int_0^t |\nabla\theta|_{2,\omega}^2,$$

where $\|.\|_\alpha$ is defined by (2). According to Proposition 2.2 and Theorem 2.1, there exists a constant $c_0 > 0$. Now, for $\|F_0\|_{4,\omega} < \frac{1}{4}$, we can choose $c := c(\|F_0\|_{4,\omega}, c_0, T) > 0$ such that

$$0 < 4c_0c < 1, \ 0 < c < \sqrt{\frac{1 - 4\|F_0\|_{4,\omega}}{2c_0}} \quad \text{and} \quad \|u_1\|_{2,\omega} < c < 1.$$

By a mathematical induction on $n \geq 1$, we will prove that the following two inequalities:

$$\|u\|_\alpha = \|u_n\|^2 + \alpha\,|\nabla(u_n)_t|_{2,\omega}^2 + |(u_n)_t|_{2,\omega}^2 \leq \|u_1\|_{2,\omega}^2 \quad \text{and} \quad \|\phi_n\|_{2,\omega} \leq \|u_1\|_{2,\omega}$$

hold for all $n \geq 1$ and any $0 \leq t \leq T$. For $n = 1$, we have

$$\|u_1\|_\alpha = \|u_1\|^2 + |(u_1)_t|_{2,\omega}^2 = \|u_1\|_{2,\omega}^2$$

since u_1 does not depend on t. Otherwise, for ϕ_1 being the solution of the problem $\Delta^2\phi_1 = -[\, u_1, u_1\,]$, Theorem 2.1 ensures that there exists $c_0 > 0$ such that

$$\|\phi_1\|_{2,\omega} \leq c_0\,|[\, u_1, u_1\,]|_{1,\omega},$$

using the proof of Proposition 2.2 with $\|u_1\|_{2,\omega} < c$ and $0 < 4c_0c < 1$, we can deduce that

$$\|\phi_1\|_{2,\omega} \leq 4c_0 \|u_1\|^2_{2,\omega} \leq 4c_0c \|u_1\|_{2,\omega} \leq \|u_1\|_{2,\omega}.$$

The desired inequalities are true for $n = 1$.

Suppose that for $k = 2, ..., n$ and $0 \leq t \leq T$, we have

$$\|u_k\|_\alpha \leq \|u_1\|^2_{2,\omega} \quad \text{and} \quad \|\phi_k\|_{2,\omega} \leq \|u_1\|_{2,\omega}.$$

According to Proposition 2.2 and Theorem 2.1, we have

$$\|\phi_n\|_{2,\omega} \leq c_0 \,|[\, u_n, u_n \,]|_{1,\omega} \leq 4c_0 \|u_n\|^2 \leq 4c_0c \|u_n\| \leq c_1 \|u_n\|.$$

Since u_{n+1} is a solution of $(\mathbb{P}_{n+1})$, Proposition 2.3, Proposition 2.2 and Theorem 2.1 imply that there exists $0 < c_1 = 2c_0c^2 + 4\|F_0\|_{4,\omega} < 1$ such that

$$\|(u_{n+1}, \theta_{n+1})\|_* \leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + \textstyle\int_0^T (\|F_1(u_n, \phi_n)\|_{(L^2(\omega))^2}$$

$$+p)^2 \leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + 2\textstyle\int_0^T \Big(\, \|F_1(u_n, \phi_n)\|^2_{(L^2(\omega))^2}\,,$$

$$+\,|p|^2_{2,\omega}\Big) \leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + 2\textstyle\int_0^T c_1^2 \|u_n\|^2 + 2T\,|p|^2_{2,\omega}),$$

$$\leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} + 2\textstyle\int_0^T c_1 \|u_n\|^2 + 2T\,|p|^2_{2,\omega}),$$

$$\leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} + 2\textstyle\int_0^T c_1(\|u_n\|^2 + 2T\,|p|^2_{2,\omega}),$$

$$\leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} + 2Tc_1 \left\|u^1\right\|^2_{2,\omega} + 2T\,|p|^2_{2,\omega}).$$

If we choose $c > 0$ sufficiently small, then $0 < c_1 < 1$, $0 < c_2 := 2e^Tc_1 < 1$, and we have

$$\|(u_{n+1}, \theta_{n+1})\|_* \leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} + 2T\,|p|^2_{2,\omega}) + c_2 \|u_1\|^2_{2,\omega},$$

and we can choose

$$\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + 2T\,|p|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} \leq \frac{(1-c_2)}{e^T} \|u_1\|^2_{2,\omega}.$$

We have

$$\|u_{n+1}\|_\alpha = \|u_{n+1}\|^2 + \alpha\,|\nabla(u_{n+1})_t|^2_{2,\omega} + |(u_{n+1})_t|^2_{2,\omega} \leq \|(u_{n+1}, \theta_{n+1})\|_*$$

and

$$\|\phi_n\|_{2,\omega} \leq c_1 \|u_n\|_{2,\omega} \leq \|u_1\|_{2,\omega}.$$

It follows that

$$\begin{aligned} \|u_{n+1}\|_\alpha &\leq e^T(\|u_0\|^2 + \alpha \left|\nabla u^1\right|^2_{2,\omega} + \left|u^1\right|^2_{2,\omega} + k\,|\theta_0|^2_{2,\omega} + 2T\,|p|^2_{2,\omega}) + c_2 \|u_1\|^2_{2,\omega}, \\ &\leq e^T \tfrac{(1-c_2)}{e^T} \|u_1\|^2_{2,\omega} + c_2 \|u_1\|^2_{2,\omega} = \|u_1\|^2_{2,\omega}. \end{aligned}$$

Further, we have

$$\|\phi_{n+1}\|_{2,\omega} \leq c_0 \,|[\, u_{n+1}, u_{n+1} \,]|_{1,\omega}\,,$$

which, with $\|u_1\|_{2,\omega} < c$ and $0 < 4c_0c < 1$, immediately yields

$$\|\phi_{n+1}\|_{2,\omega} \leq 4c_0 \|u_{n+1}\|^2 \leq 4c_0 \|u_1\|^2_{2,\omega} \leq 4c_0c \|u_1\|_{2,\omega} \leq \|u_1\|_{2,\omega}\,.$$

Summarizing, we have proved that, for all $n \geq 1$ and any $\forall 0 \leq t \leq T$, we have

$$\|u_n\|_{\alpha} \leq \|u_1\|^2_{2,\omega} \quad \text{and} \quad \|\phi_n\|_{2,\omega} \leq \|u_1\|_{2,\omega}\,.$$

Moreover, we have

$$k\,|\theta_n|^2_{2,\omega} + 2\eta \int_0^t |\nabla\theta_n|^2_{2,\omega} \leq \|(u_n,\theta_n)\|_* \leq \|u_1\|^2_{2,\omega}\,.$$

Step 2: For $n \geq 2$, let u_n, θ_n be the solution of $(\mathbb{P}_n)$.

Let $2 \leq m \leq n$, then it is easy to see that $\theta_n - \theta_m$ and $u_n - u_m$ are solutions of the following problem:

$$\begin{cases} (1-\alpha\Delta)(u_n-u_m)_{tt} + \Delta^2(u_n-u_m) + \mu\Delta(\theta_n-\theta_m) = F_1(u_{n-1},\phi_{n-1}) & \\ \qquad\qquad -F_1(u_{m-1},\phi_{m-1}) & in \quad \omega\times[0,T]\,, \\ k((\theta_n)_t-(\theta_m)_t) - \eta\Delta(\theta_n-\theta_m) = \mu\Delta((u_n)_t-(u_n)_t) & in \quad \omega\times[0,T]\,, \\ u_n-u_m = \theta_n-\theta_m = \partial_\nu(u_n-u_m) = 0 & on \quad \Gamma\times[0,T]\,, \\ (u_n-u_m)_{|t=0} = ((u_n)_t-(u_m)_t)_{|t=0} = ((\theta_n)_t-(\theta_m)_t)_{|t=0} = 0 & in \quad \omega. \end{cases}$$

According to Proposition 2.2 and Theorem 2.1 we deduce, for all $0 \leq t \leq T$,

$$\|(\phi_{n-1}-\phi_{m-1})\|_{2,\omega} \leq 4c_0c\,\|u_{n-1}-u_{m-1}\|\,.$$

Using Proposition 2.3 and Proposition 2.2, again we have, with $0 < c_3 = Te^Tc_1 < 1$,

$$\begin{aligned} \|(u_n-u_m,\theta_n-\theta_m)\|_* &\leq e^T \textstyle\int_0^T |F_1(u_{n-1},\phi_{n-1}) - F_1(u_{m-1},\phi_{m-1})|^2_{(L^2(\omega))^2}\,, \\ &\leq e^T \textstyle\int_0^t c_1\,\|u_{n-1}-u_{m-1}\|^2\,. \end{aligned}$$

It follows that

$$\begin{aligned} \|(u_n-u_m,\theta_n-\theta_m)\|_* &\leq c_3 \textstyle\int_0^t \|(u_{n-1}-u_{m-1},\theta_{n-1}-\theta_{m-1})\|_* \\ &\leq (c_3)^{m-2} \textstyle\int_0^t \cdots \int_0^t \Big(\|(u_{n-m+2}-u_1,\theta_{n-m-2}-\theta_1\|_* \\ &\leq (c_3)^{m-2} \textstyle\int_0^t \cdots \int_0^t \sum_{k=0}^{n-m+1} (c_3)^k \int_0^t \cdots \int_0^t \|(u_2-u_1\theta_2-\theta_1)\|_* \\ &\leq (c_3)^{m-2} \textstyle\int_0^t \cdots \int_0^t \sum_{k=0}^{n-m+1} (c_3)^k \int_0^t \cdots \int_0^t \Big(\|(u_2,\theta_2)\|_* \\ &\quad + \|(u_1,\theta_1)\|_* \Big) \leq (c_3T)^{m-2} \textstyle\sum_{k=0}^{n-m+1} (c_3T)^k \big(2\,\|u_1\|^2_{2,\omega}\big) \end{aligned}$$

and

$$\int_0^T \|(u_n - u_m, \theta_n - \theta_m)\|_*^2 \leq T(c_3T)^{m-2} \sum_{k=0}^{n-m+1}(c_3T)^k(2\|u_1\|_{2,\omega}^2).$$

And so we have

$$\|\phi_n - \phi_m\|_{2,\omega} \leq 4c_0c\|u_n - u_m\|.$$

The sequence $(u_n, \phi_{n-1})_{n\geq 2}$ is a Cauchy sequence in $H_0^2(\omega) \times H_0^2(\omega)$ and $(u_n)_{n\geq 2}$ is also a Cauchy sequence in $W(0,T)$. It follows that (u_n, ϕ_{n-1}) converges to (u, ϕ) in $H_0^2(\omega) \times H_0^2(\omega)$, $(u_n)_t$ converges to $(u)_t$ in $L^2(\omega)$ and $\nabla(u_n)_t$ converges to ∇u_t in $L^2(\omega)$. We then have $\Delta^2(u_n, \phi_{n-1})$ weakly converges to $\Delta^2(u, \phi)$ in $L^2(\omega)\times L^2(\omega)$.

Step 3: Using the inequality (3), we have

$$k\,|\theta_{n-1} - \theta_{m-1}|_{2,\omega}^2 \;+\; \eta\int_0^t |\nabla(\theta_{n-1} - \theta_{m-1})|_{2,\omega}^2 \;\;\leq\;\; \mu\int_0^t (|\nabla(u_{n-1} - u_{m-1})_t|_{2,\omega})^2.$$

We deduce that θ_n is a Cauchy sequence in $L^2([0,T], H_0^1(\omega))$, then θ_n converges to θ in $L^2([0,T], H_0^1(\omega))$. By Proposition 2.2, we have $F_1(u_{n-1}, \phi_{n-1})$ converges to $F_1(u, \phi)$ in $(L^2(\omega))^2$.

Since the operator "trace" is continuous, for all $n \geq 2$, we have $(u_n, \phi_{n-1})_\Gamma = (\partial_\nu u_n, \partial_\nu \phi_{n-1}) = (0,0)$ and so $(u, \phi)_\Gamma = (\partial_\nu u, \partial_\nu \phi) = (0,0)$.

Thanks to Theorem 2.3, we have $(u_n, (u_n)_t) \in C([0,T], H_0^2(\omega) \times H_0^1(\omega))$ with $(u_n)_{|t=0} = u_0$, $((u_n)_t)_{|t=0} = u_1$, which implies that $(u)_{|t=0} = u_0$, $((u)_t)_{|t=0} = u^1$. By the assumption $(u_0, u^1) \in H_0^2(\omega) \times H_0^1(\omega)$, we have $u_n \in C^0([0,T], H_0^2(\omega))$ and $(u_n)_{n\geq 2}$ converges to u in $W(0,T)$.

Let $v \in L^2\big([0,T], H_0^2(\omega)\big)$ be such that $v_t \in L^2\big([0,T], L^2(\omega)\big)$, $(1-\alpha\Delta)v_{tt} + \Delta^2 v \in L^2\big([0,T], H^{-2}(\omega)\big)$, $v(x_1, x_2, T) = 0$ and $v_t(x_1, x_2, T) = 0$. Since u_n is a solution of (P_n), by virtue of the transposition theorem, see [4], we deduce that

$$\int_0^T\int_\omega u_n((1-\alpha\Delta)v_{tt}+\Delta^2 v) = \int_0^T\int_\omega F(u_{n-1}, \phi_{n-1}, \theta_{n-1})v + \int_\omega u^1 v(0) - \int_\omega u_0 v_t(0) +$$
$$\alpha\int_\omega (-\nabla(u_t)_n(T)\nabla v(T) + \nabla u^1 \nabla v(0)) + \alpha\int_\omega (\nabla u_n(T)\nabla v_t(T) - \nabla u_0 \nabla v_t(0)).$$

We have u_n converges to u in $H_0^2(\omega)$, then

$$\int_0^T\int_\omega u_n((1-\alpha\Delta)v_{tt} + \Delta^2 v) \text{ converges to } \int_0^T\int_\omega u((1-\alpha\Delta)v_{tt} + \Delta^2 v),$$

and using Proposition 2.2, with

$$\int_0^T\int_\omega F(u, \phi, \theta) = \int_0^T\int_\omega F_1(u, \phi, \theta) + \mu\int_0^T\int_\omega \nabla\theta\nabla u + p,$$

we deduce that

$$\int_0^T\int_\omega F(u_{n-1}, \phi_{n-1}, \theta_{n-1})v \text{ converges to } \int_0^T\int_\omega F(u, \phi, \theta)v,$$

and so we have

$$\int_0^T \int_\omega u((1-\alpha\Delta)v_{tt} + \Delta^2 v) = \int_0^T \int_\omega F(u,\phi,\theta)v + \int_\omega u^1 v(0) - \int_\omega u_0 v_t(0) +$$
$$\alpha \int_\omega (-\nabla u_t(T)\nabla v(T) + \nabla u^1 \nabla v(0)) + \alpha \int_\omega (\nabla u(T)\nabla v_t(T) - \nabla u_0 \nabla v_t(0)).$$

By the transposition theorem, we obtained that u is a solution of the problem $(\mathbb{S})_1$.

In summary, we have proved that (u,ϕ,θ) is a solution of the thermoelastic von Karman evolution.

Step 4: We now prove the uniqueness. Assume that there exist two solutions (u^1,ϕ^1,θ^1) and (u^2,ϕ^2,θ^2) in $L^2([0,T], H_0^2(\omega)\times H_0^2(\omega)\times H_0^1(\omega))$ such that, for some $c>0$ being sufficiently small, we have $\|u^1\|_{W(0,s_0)} \le c$ and $\|u^2\|_{W(0,s_0)} \le c$.
This implies that u^1-u^2 and $(\theta^1-\theta^2)$ satisfies the following problem:

$$(\mathbb{P}_3)\begin{cases} (1-\alpha\Delta)(u^1-u^2)_{tt} + \Delta^2(u^1-u^2) = F(u^1,\phi^1,\theta^1) & \\ \qquad\qquad -F(u^2,\phi^2,\theta^2) & in \ \ \omega\times[0,T], \\ k(\theta^1-\theta^2)_t - \eta\Delta(\theta^1-\theta^2) = \mu\Delta(u^1-u^2)_t & in \ \ \omega\times[0,T], \\ u^1-u^2 = \partial_\nu(u^1-u^2) = \theta^1-\theta^2 = 0 & on \ \ \Gamma\times[0,T], \\ u^1(x_1,x_2,0) - u^2(x_1,x_2,0) = 0 & in \ \ \omega, \\ (u^1)_t(x_1,x_2,0) - (u^2)_t(x_1,x_2,0) = 0 & in \ \ \omega, \\ (\theta^1)_t(x_1,x_2,0) - (\theta^2)_t(x_1,x_2,0) = 0 & in \ \ \omega, \end{cases}$$

which means that $(u^1-u^2,\theta^1-\theta^2)$ is a solution of the problem $(\mathbb{P}_3)$. Proposition 2.2, Proposition 2.3 and Theorem 2.1 ensure that there exists $c_0>0$ such that

$$\|(u^1-u^2,\theta^1-\theta^2)\|_* \le e^T \int_0^T \left|F_1(u^1,\phi^1) - F_1(u^2,\phi^2)\right|^2_{(L^2(\omega))^2}$$
$$\le e^T \int_0^T c_1 \|u^1-u^2\|^2 \le e^T c_1 \int_0^T \|(u^1-u^2,\theta^1-\theta^2)\|_* .$$

Since c is small and thus $0 < c_3 = Te^T c_1 < 1$, it follows that

$$\int_0^T \|(u^1-u^2,\theta^1-\theta^2)\|_* \le c_3 \int_0^T \|(u^1-u^2,\theta^1-\theta^2)\|_* ,$$

which, with $0<c_3<1$, immediately yields $\forall 0 \prec t \prec T$, $u^1=u^2$ in ω, $\phi^1=\phi^2$ in ω and $\theta^1=\theta^2$ in ω.

We conclude that the dynamic von Karman equation coupled with thermal dissipation, without rotational inertia, has one and only one weak solution (u,ϕ,θ) in $L^2\Big([0,T], H_0^2(\omega)\times H_0^2(\omega)\times H_0^1(\omega)\Big)$. The proof of the theorem is completed.

Proposition 3.1 *Let* $(u,\phi,\theta) \in L^2\Big([0,T], H_0^2(\omega)\times H_0^2(\omega)\times H_0^1(\omega)\Big)$ *be the unique solution of* $(\mathbb{P}_0)$*. Then the following equalities:*

$$\widetilde{E}\big(u(t),u_t(t),\phi\big) + \frac{k}{2}\,|\theta|_{2,\omega}^2 - \eta\int_0^t |\nabla\theta_t|_{2,\omega}^2 = \widetilde{E}_1(u_0,u^1,\phi_0) + \frac{k}{2}\,|\theta_0|_{2,\omega}^2\,,$$

with

$$\widetilde{E}\big(u(t),u_t(t),\phi\big) = \frac{1}{2}\big(\,|u_t|_{2,\omega}^2 + \|u\|_{2,\omega}^2 + \alpha\,|\nabla u_t|_{2,\omega}^2\,\big) + \frac{1}{4}\int_\omega \big(\,|\Delta\phi|^2 - 2\,[u,F_0]\,u - 4pu\big)$$

and

$$\widetilde{E}_1(u_0,u^1,\phi_0) = \frac{1}{2}\big(\,\big|u^1\big|_{2,\omega}^2 + \alpha\,\big|\nabla u^1\big|_{2,\omega}^2 + \|u_0\|_{2,\omega}^2\big) + \frac{1}{4}\int_\omega (|\Delta\phi_0|^2 - 2\,[u_0,F_0]\,u_0 - 4pu_0)$$

hold for any $0\le t\le T$*. Here* $\phi_0\in H_0^2(\omega)$ *is the unique solution of the equation* $\Delta^2\phi_0 = -\,[u_0,u_0]$*.*

Proof. According to Theorem 2.3, for any $\forall 0\le t\le T$, u satisfies the following energy equality:

$$\begin{aligned} E_0(u(t),u_t(t)) &= E_0(u_0,u^1) + \int_0^t\int_\omega F(u,\phi,\theta)u_t \\ &= E_0(u_0,u^1) + \int_0^t\int_\omega [u,\phi+F_0]\,u_t - \mu\int_0^t\int_\omega \Delta\theta u_t + \int_0^t\int_\omega p(x_1,x_2)u_t. \end{aligned}$$

First we have

$$\int_0^t\int_\omega p(x_1,x_2)u_t = \int_\omega p(x_1,x_2)u(t) - \int_\omega p(x_1,x_2)u_0.$$

Otherwise, see [1], one has, with $\Delta^2\phi = [u,u]$,

$$\begin{aligned} \int_0^t\int_\omega [u,\phi+F_0]\,u_t &= \int_0^t\int_\omega [u,\phi]\,u_t + \int_0^t\int_\omega [u,F_0]\,u_t, \\ &= \frac{1}{2}\int_0^t\int_\omega \frac{d}{dt}([u,u]\,\phi) + \frac{1}{2}\int_0^t\int_\omega \frac{d}{dt}([u,F_0]\,u), \\ &= -\frac{1}{4}\int_\omega |\Delta\phi|^2 + \frac{1}{4}\int_\omega |\Delta\phi_0|^2 + \frac{1}{2}\int_\omega [u,u]\,F_0 - \frac{1}{2}\int_\omega [u_0,u_0]\,F_0 \end{aligned}$$

and

$$\begin{aligned} \mu\int_0^t\int_\omega \Delta\theta u_t = \mu\int_0^t\int_\omega \theta\Delta u_t &= \frac{k}{2}\int_0^t \frac{d}{dt}\,|\theta|_{2,\omega}^2 - \eta\int_0^t |\nabla\theta|_{2,\omega}^2 \\ &= \frac{k}{2}\,|\theta|_{2,\omega}^2 - \frac{k}{2}\,|\theta_0|_{2,\omega}^2 - \eta\int_0^t |\nabla\theta|_{2,\omega}^2\,. \end{aligned}$$

Finally, we conclude that

$$\widetilde{E}\big(u(t), u_t(t), \phi\big) + \frac{k}{2}\,|\theta|^2_{2,\omega} - \eta \int_0^t |\nabla\theta_t|^2_{2,\omega} = \widetilde{E}_1(u_0, u^1, \phi_0) + \frac{k}{2}\,|\theta_0|^2_{2,\omega}\,.$$

Remark 3.1 In this section, we described an iterative method for constructing a unique weak solution, this method is a very good tool to illustrate this solution from a numerical point of view.

4 Numerical Application

This section displays a numerical resolution in terms of the previous theoretical study.

4.1 Preliminaries

Let ω be defined by

$$\omega =]0,1[\times]0,1[\subset \mathbb{R}^2$$

and $T > 0$. In order to solve numerically the problem $(\mathbb{P}_0)$, we introduce a uniform mesh of width h. Let ω_h be the set of all mesh points inside ω with the internal points

$$x_i = ih,\ \ y_j = jh,\quad i,j = 1,...N-1,\ \ h = \frac{1}{N+1},\ \ \Delta t = \frac{1}{T}.$$

Let $\overline{\omega}_h$ be the set of boundary mesh points and u_h be the finite-difference approximation of u. In [7], Bilbao presented a numerical study of the convergence and stability of the conservative finite difference schemes for the dynamic von Karman plate equations via energy conserving methods.

For approaching the weak unique solution of the dynamic nonlinear plate coupled with structural acoustic model, we will utilize the following discrete model of the von Karman evolution developed by Bilbao and Pereira in [7,8]:

$$(*)\begin{cases} (1-\alpha(\delta_x^2+\delta_y^2))\delta_t^2 u_{ij}^n + \mu(\delta_x^2+\delta_y^2)\theta_{ij}^n + \Delta_h^2 u_{ij}^n = [\ u_{ij}^n\ v_{ij}^n + F_{ij}\ ] + p_{ij} & in \quad \omega_h, \\ k\delta_t\theta_{ij}^n - \eta(\delta_x^2+\delta_y^2)\theta_{ij}^n - \mu\delta_t(\delta_x^2+\delta_y^2)u_{ij}^n = 0 & in \quad \omega_h, \\ \Delta_h^2 v_{ij}^n = -[\ u_{ij}^n\ u_{ij}^n\ ] & in \quad \omega_h, \\ u_{ij}^0 = (\varphi_0)_{ij},\ \delta_t u_{ij}^0 = (\varphi_1)_{ij},\ \theta_{ij}^0 = (\theta_0)_{ij} & in \quad \omega_h, \\ u_{ij}^n = v_{ij}^n = \theta_{ij}^n = 0 & on \quad \overline{\omega_h}, \\ \partial_\nu u_{ij}^n = \partial_\nu v_{ij}^n = 0 & on \quad \overline{\omega_h}, \end{cases}$$

with the following discrete differential operators:

$$\delta_t^2 u_{ij}^n = \frac{u_{ij}^{n+1} - 2u_{ij}^n + u_{ij}^{n-1}}{(\Delta t)^2},$$

$$\delta_t u_{ij}^n = \frac{u_{ij}^{n+1} - u_{ij}^n}{\Delta t},$$

$$\Delta_h^2 u_{ij}^n = h^{-4}\,[\, u_{ij-2} + u_{ij+2} + u_{i-2j} + u_{i+2j} - 8(u_{ij-1} + u_{ij+1} + u_{i-1j} + u_{i+1j}) + 2(u_{i-1j-1} + u_{i-1j+1} + u_{i+1j-1} + u_{i+1j+1}) - 20u_{ij}\,]\,,$$

$$\delta_x^2 u_{ij}^n = \frac{u_{i+1j}^n - 2u_{ij}^n + u_{i-1j}^n}{(h)^2},$$

$$\delta_y^2 u_{ij}^n = \frac{u_{ij+1}^n - 2u_{ij}^n + u_{ij-1}^n}{(h)^2},$$

$$\delta_{xy}^2 u_{ij}^n = \frac{u_{i+1j+1}^n - u_{i+1j-1}^n - u_{i-1j+1}^n + u_{i-1j-1}^n}{(2h)^2},$$

$$[\, u_{ij}^n,\ v_{ij}^n \,] = \delta_x^2 u_{ij}^n \delta_y^2 v_{ij}^n - 2\delta_{xy}^2 u_{ij}^n \delta_{xy}^2 v_{ij}^n + \delta_y^2 u_{ij}^n \delta_x^2 v_{ij}^n.$$

We have transformed the above problem to the numerical resolution in two steps itemized as follows.
First step: We use the numerical procedure of 13-point formula of finite difference developed by Gubta in [9] for illustrating the weak solution of the following biharmonic problem:

$$\begin{cases} \Delta^2 v = f_1 & in \quad \omega, \\ v = g_1 & on \quad \Gamma, \\ \partial_\nu v = g_2 & on \quad \Gamma. \end{cases}$$

Second step: According to the first and second steps, we use the discrete model of the von Karman evolution $(*)$ for illustrating the unique solution of the structural interaction model coupled with the dynamic von Karman evolution.

4.2 Non-coupled approach

In [9], Gubta presented a numerical analysis of the finite-difference method for solving the biharmonic equation. Such method is known as the non-coupled method of 13-point formula of finite difference.

Proposition 4.1 *[9] The 13-point approximation of the biharmonic equation for approaching the unique solution v of the problem (P) is defined by*

$$(1)\begin{cases} L_h v_{ij} = & h^{-4}\,[v_{ij-2} + v_{ij+2} + v_{i-2j} + v_{i+2j} - 8(v_{ij-1} + v_{ij+1} + v_{i-1j} + v_{i+1j}) \\ & +2(v_{i-1j-1} + v_{i-1j+1} + v_{i+1j-1} + v_{i+1j+1}) - 20v_{ij}] = f_1(x_i, y_j) \end{cases}$$

for $i, j = 1, 2, ..., N-1$, where we set $v_{ij} = v(x_i, y_j)$.

When the mesh point (x_i, y_j) is adjacent to the boundary $\overline{\omega}_h$, then the undefined values of v_h are conventionally calculated by the following approximation of $\partial_\nu v$:

$$v_{i-2,j} = \tfrac{1}{2} v_{i+1,j} - v_{ij} + \tfrac{3}{2} v_{i-1,j} - h(\partial_x v)_{i-1,j},$$

$$v_{i,j-2} = \tfrac{1}{2} v_{i,j+1} - v_{ij} + \tfrac{3}{2} v_{i,j-1} - h(\partial_y v)_{i,j-1},$$

$$v_{i+2,j} = \tfrac{1}{2} v_{i+1,j} - v_{ij} + \tfrac{3}{2} v_{i-1,j} - h(\partial_x v)_{i+1,j},$$

$$v_{i,j+2} = \tfrac{1}{2} v_{i,j+1} - v_{ij} + \tfrac{3}{2} v_{i,j-1} - h(\partial_y v)_{i,j+1}.$$

4.3 Numerical test

We consider the following analytical body force and lateral forces:

$$F_0(x,y) = ye^{-x^2-y^2}, \quad p(x,y) = 0.01x(x-y)e^{-x^2-y^2},$$

$$\varphi_0 = 1510^{-6}x^2y^2(x-y-1)^2(y-1)^2e^{-x^2-y^2}, \quad \varphi_1 = 1510^{-6}(\sin(\pi x)\sin(\pi y))^2,$$

$$\theta_0 = 10^{-13}x^2y^3(x-1)^2(y-1)^2(e^{-x^2} - e^{-y^2}).$$

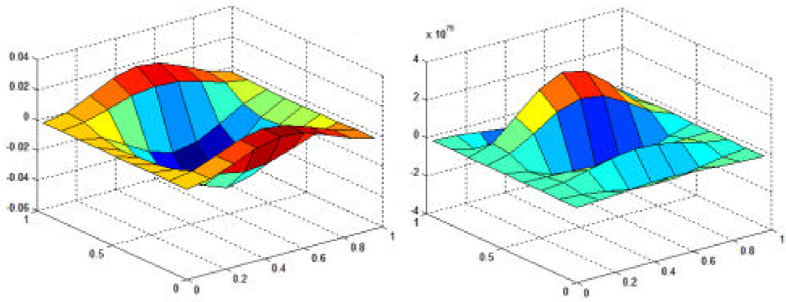

Figure 1: The thermal function θ, $t_1 = 0.2s$ and $t_7 = 60s$.

5 Conclusion

In this paper, we described an iterative method for constructing a unique weak solution to the model of dynamic von Karman equations with a flexible phenomenon of small nonlinear vibration of displacement in nonlinear oscillation of elastic plate, with rotational terms and not clamped boundary conditions subject to thermal dissipation. Our approach is in fact a good tool for justifying the theoretical results. We then use the method of finite difference for approaching the unique solution of the theoretical problem. These results have potential for application in the fields of physics. Similar study for the models of dynamic von Karman equations with thermal dissipation and for free boundary conditions of the shell could be the purpose for future research.

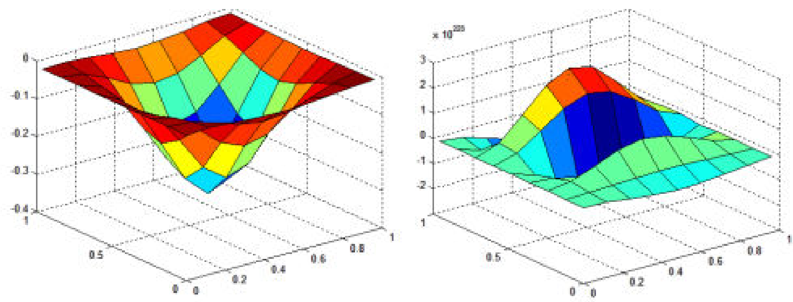

Figure 2: Displacement of plate, $t_1 = 0.2s$ and $t_7 = 60s$.

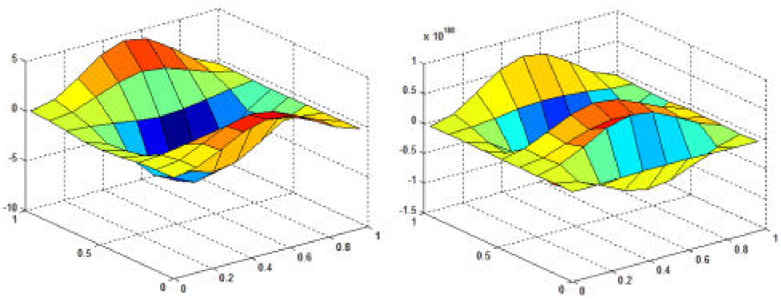

Figure 3: The Airy stress function, $t_1 = 0.2s$ and $t_6 = 32s$.

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References

[1] I. Chueshov and I. Lasiecka. *Von Karman Evolution, Well-posedness and Long Time Dynamics.* New York, Springer, 2010.

[2] S. S. Antman and T. Von-Karman. A Panorama of Hungarian Mathematics in the Twentieth Centuray I, Vol. 14 of the series Bolyai Society Mathematical Studies, 373–382, Springer Berlin Heidelberg (2006).

[3] P. G. Ciarlet and R. Rabier. *Les Equations de von Karman*, Lecture Notes in Mathematics, Vol. 826, New York, Springer, 1980.

[4] J. L. Lions and E. Magenes. *Problèmes aux Limites non Homogènes et Applications*, Vol. 1, Gauthier-Villars, Paris, Dunod, 1968.

[5] H. Bresis. *Analyse Fonctionnelle, Théorie et Application.* Masson, Paris (1983).

[6] J.L. Lions. *Quelques Methodes de Resolution des Problemes aux Limites non Linéaires.* Paris: Dunod (2002).

[7] S. Bilbao. A family of conservative finite difference schemes for the dynamical von Karman plate equations. *Numerical Methods for Partial Differential Equations* **24/1** (2007) 193–218.

[8] D. C. Pereira, C.A. Raposo and A. J. Avila. Numerical solution and exponential decay to von Karman system with frictional damping *Int. J. Math. Information Sci.* **8/4** (2014) 1575–1582.

[9] M. M. Gubta and R. P. Manohar. Direct solution of Biharmonic equation using non coupled approach. *J. Computational Physics* **33/2** (1979) 236–248.

[10] J. Oudaani. Numerical approach to the uniqueness solution of Von-Karman evolution. *IJMOR, International Journal Mathematics in Operational Research* **13** (4) (2018) 450–470.

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The Effects of Pesticide as Optimal Control of Agriculture Pest Growth Dynamical Model

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Abstract: Indonesia has tropical climate so that many crops can be harvested. One of agricultural problems is the agricultural pest (*Nilaparvata lugens*) in a rice field. This pest can be devastated by the natural predator spider (*Lycosa pseudoannulata*). To reduce the number of pests, we use pesticide as a control which is applied in the pest population. For the problem, we can construct the model as a predator-prey model with the pest as the prey and the spider as the natural predator. This paper discusses stability analysis and optimal control of the agricultural pest growth dynamical model by pesticide. In the agricultural pest dynamical model, there are populations of pests and spiders. From the mathematical model of agricultural pest growth, we obtain three equilibrium points. We will analyze the stability of each equilibrium point by using the eigenvalue. In this paper, for the original mathematical model of agricultural pest growth, we will introduce a control variable, i.e., pesticide. Then we will formulate an optimal control problem. The forward-backward sweep method is employed to solve the optimal control problem and to obtain the numerical solutions. According to simulation results, pesticide usage can minimize the number of pests achieving the minimum performance index.

Keywords: *optimal control; pesticide; pest growth dynamical model.*

Mathematics Subject Classification (2010): 49-06, 49M30, 93D05, 49J15.

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1 Introduction

Indonesia has a tropical climate so that many crops can be harvested. Indonesia has large fertile lands for rice fields and rice is one of the primary foods in Indonesia. One of agricultural problems is the agricultural pest (*Nilaparvata lugens*) in the rice field. This pest can be devastated by using the natural predator such as the spider (*Lycosa pseudoannulata*).

The behavior of the agricultural pest and the natural predator can be constructed as a predator-prey mathematical model. In a predator-prey model, there are pest and predator populations. The pest is the attacking organism damaging crops, while the predator is the eating organism consuming the pest [1]. In this research, the agricultural pest (*Nilaparvata lugens*) and the natural predator spider (*Lycosa pseudoannulata*) will be included.

In the previous research, there were numerous works on modelling of diseases, for example, influenza [2, 3], bird flu [4], dengue fever [5, 6], cancer [7] and Corona virus [8]. Generally, the mathematical model of spreading diseases divides the population into several subpopulations such as the susceptible population, infected population, and recovered population [9–11]. For the three subpopulations, we can determine the reproduction number and the stability by using the available parameters. Let us mention that the predator-prey model has been used for determining the stability in the case of natural selection [12].

In order to reduce the number of pests, we use pesticide as a control variable which is applied in the pest population. However, the usage of pesticide should be proportional. Using more pesticide causes side effects on crops and high cost of pesticide. On the other hand, using less pesticide causes pest growth. For the problem, we can construct a predator-prey model with the pest as the prey and the spider as the natural predator. This paper focuses on the stability analysis and optimal control of the agricultural pest growth dynamical model by using pesticide as the control variable.

In the agricultural pest dynamical model, there are two subpopulations: pest and spider. From the mathematical model of agricultural pest growth, we obtain three equilibrium points. We will analyze the stability of each equilibrium point by using the eigenvalue. The first equilibrium point is unstable, whereas the second and third equilibriums are stable, which depends on certain conditions. From the preceding mathematical model of agricultural pest growth, we introduce a control variable that represents pesticide. Next, we formulate an optimal control problem: the objective function and the constraints. We use the forward-backward sweep method to obtain the solution of the optimal control problem and to compute the numerical solutions. This method leverages the state variables with certain initial condition and adjoint variables with certain final condition [13]. According to the simulation results, pesticide usage can minimize the number of pests achieving the minimum performance index.

2 Mathematical Model of Agricultural Pest Growth

In the mathematical model of agricultural pest growth, there are two populations used, namely, the agricultural pest (*Nilaparvata lugens*) as the prey and the spider (*Lycosa pseudoannulata*) as the natural predator. This model can be constructed as a predator-prey model.

2.1 Mathematical model

The mathematical model of agricultural pest growth with the functional response of Holling $\dfrac{\gamma SP}{a+S_0}$ and the denominator being a constant value, can be constructed as follows [1]:

$$\dot{S} = rS\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a+S_0}, \tag{1}$$

$$\dot{P} = \frac{\alpha\gamma SP}{a+S_0} - \delta P \tag{2}$$

with the following parameters:

$S(t)$	:	the population of the pest (*Nilaparvata lugens*) as the prey,
$P(t)$	:	the population of the spider (*Lycosa pseudoannulata*) as the natural predator,
r	:	intrinsic rate of growth of the pest as the prey,
K	:	environmental carrying capacity of the pest as the prey population,
γ	:	search rate of the pest as the prey by the predator,
α	:	conversion factors,
δ	:	natural death rate of predators,
a	:	half saturation constant.

From the model, we conclude the following conditions. Without the existence of predators, the pest as the prey grows based on the logistic function, and without the existence of the pest as the prey, the predators go away. Based on the natural selection, the existence of the pest as the prey will increase the predator, and the existence of predators will decrease the pest as the prey.

2.2 Existence of solutions

The solutions of this problem exist if the populations of predators and preys are greater than or equal to zero, i.e., $S(t) \geq 0$, $P(t) \geq 0$. As it will be clear later, the equilibrium points must satisfy these conditions [14].

2.3 Equilibrium points

In order to compute the equilibrium points, we find the solutions of $\dot{S} = 0$, $\dot{P} = 0$ as follows:

$$rS\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a+S_0} = 0, \tag{3}$$

$$\frac{\alpha\gamma SP}{a+S_0} - \delta P = 0. \tag{4}$$

By using simple algebraic manipulations, from (3) and (4), we obtain the following equilibrium points:

1. Equilibrium point 1 : $S_{e1} = 0$, $P_{e1} = 0$;
2. Equilibrium point 2 : $S_{e2} = K$, $P_{e2} = 0$;
3. Equilibrium point 3 : $S_{e3} = \dfrac{\delta(a+S_0)}{\alpha\gamma}$, $P_{e3} = \dfrac{(a+S_0)r(\alpha\gamma K - \delta(a+S_0))}{\alpha\gamma^2 K}$.

Next, we analyze the stability of each equilibrium point by using the eigenvalue method of the Jacobian matrix.

2.4 Stability analysis

First of all, we derive the Jacobian matrix from (1) and (2). In order to simplify the notations in the Jacobian matrix, we introduce

$$\begin{aligned} f_1 &= rS\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a + S_0} - \varepsilon uS, \\ f_2 &= \frac{\alpha\gamma SP}{a + S_0} - \delta P. \end{aligned}$$

Then the Jacobian matrix is

$$Jac = \begin{bmatrix} \dfrac{\partial f_1}{\partial S} & \dfrac{\partial f_1}{\partial P} \\ \dfrac{\partial f_2}{\partial S} & \dfrac{\partial f_2}{\partial P} \end{bmatrix} = \begin{bmatrix} r - 2(\dfrac{r}{K})S - \frac{\gamma P}{a+S_0} & -\dfrac{\gamma S}{a + S_0} \\ \dfrac{\alpha\gamma P}{a + S_0} & \dfrac{\alpha\gamma S}{a + S_0} - \delta \end{bmatrix}. \tag{5}$$

In order to analyze the stability, we compute the eigenvalue of the Jacobian matrix by using the formula $\det(\lambda I - Jac) = 0$ after substituting the equilibrium points. The equilibrium point is stable if the real parts of all eigenvalues are negative. Based on these conditions, we can conclude that

1. Equilibrium point 1 : $S_{e1} = 0, P_{e1} = 0$ is always unstable;
2. Equilibrium point 2 : $S_{e2} = 0, P_{e2} = 0$ is stable if $\dfrac{\alpha\gamma K}{\delta(a + S_0)} < 1$;
3. Equilibrium point 3 : $S_{s3} = \dfrac{\delta(a + S_0)}{\alpha\gamma}$, $P_{e3} = \dfrac{(a + S_0)r(\alpha\gamma K - \delta(a + S_0))}{\alpha\gamma^2 K}$ is stable if $\dfrac{\alpha\gamma K}{\delta(a + S_0)} > 1$.

3 Optimal Control of Agriculture Pest Growth

In the optimal control of agricultural pest growth, we introduce a control variable u. The control variable is used to reduce the number of pests. The effectiveness range of the control variable u lies in the interval $[0, 1]$, where the value of 0 represents the failure of control functions or the control functions are not to be applied, and the value of 1 represents the successful control functions or the control functions are applied to the entire population. Therefore, after introducing the control variable u, the mathematical model in (1) and (2) becomes (6) and (7), respectively.

$$\dot{S} = rS\left(1 - \frac{S}{K}\right) - \frac{\gamma SP}{a + S_0} - \varepsilon uS, \tag{6}$$

$$\dot{P} = \frac{\alpha\gamma SP}{a + S_0} - \delta P \tag{7}$$

with ε being the rate of reducing the pest as the prey due to pesticide.

Now, we formulate an optimal control problem. First, we define an objective function. The objective function is minimizing the number of pests and the cost of pesticides. As such, the objective function is defined as follows:

$$J = \int_T^0 (A_1 S + A_2 u^2)\, dt, \tag{8}$$

where the weights $A_1 > 0$, $A_2 > 0$ are associated with the number of pests as the prey and the cost of pesticides, respectively. The solution is an optimal control u^*.

3.1 Pontryagin's maximum principle

If u is an optimal control associated with the state of the system, then there exist adjoint variables $\begin{pmatrix}\lambda_S & \lambda_P\end{pmatrix}$ that satisfy the following conditions [10]:

$$\begin{aligned}
\dot{\lambda}_S &= -\frac{\partial H}{\partial S} = -A_1 - \lambda_S \left(1 - \frac{S}{K}\right) - \frac{\gamma S P}{a + S_0}\left(r - \frac{2r}{K}S - \frac{\gamma P}{a + S_0} - \varepsilon u\right) \\
&\quad - \lambda_P \left(\frac{\alpha \gamma P}{a + S_0}\right) && (9) \\
\dot{\lambda}_P &= -\frac{\partial H}{\partial P} = -\lambda_S \left(-\frac{\gamma S}{a + S_0}\right) - \lambda_P \left(\frac{\alpha \gamma S}{a + S_0} - \delta\right) && (10) \\
\lambda_S(T) &= \lambda_P(T) = 0, && (11)
\end{aligned}$$

where the Hamiltonian is

$$H = A_1 S + A_2 u^2 + \begin{pmatrix}\lambda_S & \lambda_P\end{pmatrix} \begin{pmatrix} rS\left(1 - \dfrac{S}{K}\right) - \dfrac{\gamma S P}{a + S_0} - \varepsilon u S \\ \dfrac{\alpha \gamma S P}{a + S_0} - \delta P \end{pmatrix}. \tag{12}$$

An optimal control u^* is obtained as follows:

$$\frac{\partial H}{\partial u} = 0, \tag{13}$$

$$2A_2 u + \lambda_S(-\varepsilon S) = 0, \tag{14}$$

$$u = \min\left(1, \max\left(0, \frac{\lambda_S\, \varepsilon S}{2A_2}\right)\right). \tag{15}$$

3.2 Forward-backward sweep method

In order to compute the optimal control, we use the forward-backward sweep method. When we apply the method to the optimal control problem of agricultural pest growth, the steps are as follows [15]. Notice that the state and adjoint variables are

$$\begin{aligned}
f_1 &= rS\left(1 - \frac{S}{K}\right) - \frac{\gamma S P}{a + S_0} - \varepsilon u S, \\
f_2 &= \frac{\alpha \gamma S P}{a + S_0} - \delta P, \\
g_1 &= -A_1 - \lambda_S \left(r - \frac{2r}{K}S - \frac{\gamma P}{a + S_0} - \varepsilon u\right) - \lambda_P \left(\frac{\alpha \gamma P}{a + S_0}\right), \\
g_2 &= -\lambda_S \left(-\frac{\gamma S}{a + S_0}\right) - \lambda_P \left(\frac{\alpha \gamma S}{a + S_0} - \delta\right).
\end{aligned}$$

Next, we describe the forward-backward sweep method. The method is written as an algorithm so that we can implement the method easily. Here is the complete algorithm:

$$u_{old} = u.$$

1. Calculate the solution of state variables, where the initial conditions are S_0, P_0, by using the Runge-Kutta fourth-order method. For the agricultural pest growth model, the steps are

$$\begin{aligned}
k_{11} &= f_1\left(S_i, P_i, u_i\right),\\
k_{12} &= f_2\left(S_i, P_i, u_i\right),\\
k_{21} &= f_1\left(S_i + \frac{h}{2}k_{11}, P_i + \frac{h}{2}k_{12}, \frac{u_i + u_{i+1}}{2}\right),\\
k_{22} &= f_2\left(S_i + \frac{h}{2}k_{11}, P_i + \frac{h}{2}k_{12}, \frac{u_i + u_{i+1}}{2}\right),\\
k_{31} &= f_1\left(S_i + \frac{h}{2}k_{21}, P_i + \frac{h}{2}k_{22}, \frac{u_i + u_{i+1}}{2}\right),\\
k_{32} &= f_2\left(S_i + \frac{h}{2}k_{21}, P_i + \frac{h}{2}k_{22}, \frac{u_i + u_{i+1}}{2}\right),\\
k_{41} &= f_1\left(S_i + hk_{31}, P_i + hk_{32}, u_{i+1}\right),\\
k_{42} &= f_2\left(S_i + hk_{31}, P_i + hk_{32}, u_{i+1}\right),\\
S_{i+1} &= S_i + \frac{h}{6}\left(k_{11} + 2k_{21} + 2k_{31} + k_{41}\right),\\
P_{i+1} &= P_i + \frac{h}{6}\left(k_{12} + 2k_{22} + 2k_{32} + k_{42}\right).
\end{aligned}$$

2. Calculate the solution of adjoint variables, where the final conditions are $\lambda_{N(T)}, \lambda_{P(T)}$, by using the Runge-Kutta fourth-order method as follows:

$$\begin{aligned}
l_{11} &= g_1\left(\lambda_{S(i)}, \lambda_{P(i)}, S_i, P_i, u_i\right),\\
l_{12} &= g_2\left(\lambda_{S(i)}, \lambda_{P(i)}, S_i, P_i, u_i\right),\\
l_{21} &= g_1\left(\lambda_{S(i)} - \frac{h}{2}l_{11}, \lambda_{P(i)} - \frac{h}{2}l_{12}, \frac{S_i + S_{i-1}}{2}. \frac{P_i + Pi - 1}{2}, \frac{u_i + u_{i-1}}{2}\right),\\
l_{22} &= g_2\left(\lambda_{S(i)} - \frac{h}{2}l_{11}, \lambda_{P(i)} - \frac{h}{2}l_{12}, \frac{S_i + S_{i-1}}{2}. \frac{P_i + Pi - 1}{2}, \frac{u_i + u_{i-1}}{2}\right),\\
l_{31} &= g_1\left(\lambda_{S(i)} - \frac{h}{2}l_{21}, \lambda_{P(i)} - \frac{h}{2}l_{22}, \frac{S_i + S_{i-1}}{2}. \frac{P_i + Pi - 1}{2}, \frac{u_i + u_{i-1}}{2}\right),\\
l_{32} &= g_2\left(\lambda_{S(i)} - \frac{h}{2}l_{21}, \lambda_{P(i)} - \frac{h}{2}l_{22}, \frac{S_i + S_{i-1}}{2}. \frac{P_i + Pi - 1}{2}, \frac{u_i + u_{i-1}}{2}\right),\\
l_{41} &= g_1\left(\lambda_{S(i)} - hl_{31}, \lambda_{P(i)} - hl_{32}, S_{i-1}, P_{i-1}, u_{i-1}\right),\\
l_{42} &= g_2\left(\lambda_{S(i)} - hl_{31}, \lambda_{P(i)} - hl_{32}, S_{i-1}, P_{i-1}, u_{i-1}\right),\\
\lambda_{S(i-1)} &= \lambda_{S(i)} - \frac{h}{6}\left(l_{11} + 2l_{21} + 2l_{31} + l_{41}\right),\\
\lambda_{P(i-1)} &= \lambda_{S(i)} - \frac{h}{6}\left(l_{12} + 2l_{22} + 2l_{32} + l_{42}\right).
\end{aligned}$$

3. Calculate the optimal control u^* using (15).
4. Update the optimal control
$$u \leftarrow \frac{u + u_{old}}{2}. \tag{16}$$
5. Calculate the performance index as the value of objective function
$$J(u) = \sum_{k=0}^{T-1} \left(A_1 S(k)^2 + A_2 u(k)^2 \right). \tag{17}$$

4 Simulation Results

To simulate the closed-loop system, we need to define the parameters. Table 1 describes the parameters used in the simulation.

Table 1: Parameters of optimal control of agricultural pest growth.

Parameters	**Value**
The population of the pest (*Nilaparvata lugens*) as the prey $S(0)$	20
The population of the spider (*Lycosa pseudoannulata*) as the natural predator $P(0)$	10
Intrinsic rate of growth of the pest as the prey r	1
Environmental carrying capacity of the pest as the prey population K	30
Search rate of the pest as the prey by the predator γ	1
Natural death rate of the predator δ	0.6
Half saturation constant a	10
Rate of reducing the pest as the prey due to pesticide ε	5
Weight related to the number of the pest as the prey A_1	1
Weight related to the cost of pesticide A_2	2

The simulation results are applied with two parts because there are three equilibrium points, but an equilibrium (equilibrium of type 1) is unstable. In the first simulation, the conversion factor $\alpha = 0.1$, and in the second simulation, the conversion factor $\alpha = 8$.

4.1 Simulation with equilibrium point of type 2

In this simulation, conversion factors $\alpha = 1$ will be applied. From the results, we obtain
$$\frac{\alpha\gamma K}{\delta(a + S_0)} = \frac{(0.1)(1)(30)}{0.6(10 + 20)} = \frac{3}{18} = 0.167 < 1.$$
At the equilibrium point of type 2, the equilibrium point is stable if $\dfrac{\alpha\gamma K}{\delta(a + S_0)} < 1$. The numerical simulation with the equilibrium point of type 2 can be seen in Figure 1 and Figure 2 (left).

Figure 1 (left) displays the number of agricultural pests (*Nilaparvata lugens*) as preys while Figure 1 (right) displays the number of spiders (*Lycosa pseudoannulata*) as natural predators. In Figure 1 (left), the number of pests as preys with pesticide control is smaller than the number of pests as the prey without control. In Figure 1 (right), the

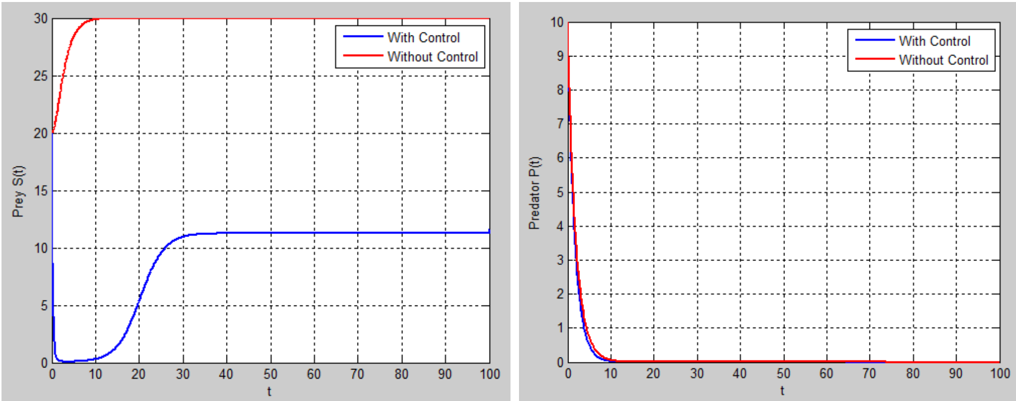

Figure 1: The left panel represents the numerical solution of the number of agricultural pests (*Nilaparvata lugens*) as preys. The right panel denotes the numerical simulation of the number of spiders (*Lycosa pseudoannulata*) as natural predators.

number of predators with pesticide control tends to 0 and is almost similar to the number of predators without control because pesticide is only applied in the pest population.

Figure 2 (left) shows the optimal control of pesticide used. Initially, the pesticides are given to around 95% of the population. Then the number of individuals receiving the pesticides is decreasing. When $t \geq 10$, the pesticides are given to around 55% of the population.

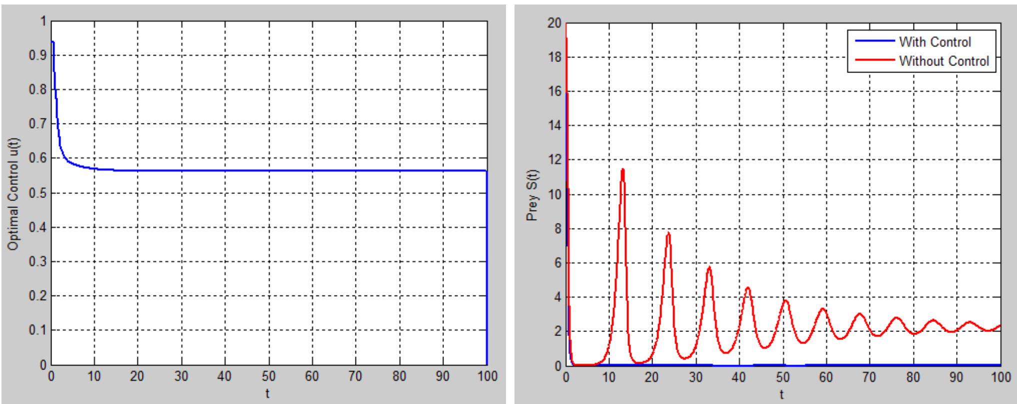

Figure 2: The left panel denotes the optimal control of pesticides. The right panel represents the numerical solution of the number of agricultural pests (*Nilaparvata lugens*) as preys.

4.2 Simulation with equilibrium of type 3

In this simulation, conversion factors $\alpha = 8$ will be applied. From the results, we obtain

$$\frac{\alpha\gamma K}{\delta(a+S_0)} = \frac{(8)(1)(30)}{0.6(10+20)} = \frac{240}{18} = 13.33 > 1.$$

At the equilibrium point of type 3, the equilibrium point is stable if $\dfrac{\alpha\gamma K}{\delta(a+S_0)} > 1$. Numerical simulation with the equilibrium point of type 3 can be seen in Figure 2 (right) and Figure 3.

Figure 2 (right) shows the number of agricultural pests (*Nilaparvata lugens*) as preys, while Figure 3 (left) displays the number of spiders (*Lycosa pseudoannulata*) as natural predators. Both Figure 2 (right) and Figure 3 (left) show fluctuative graphs. When pests as the prey increase, then predators follow the increase, and if pests as the prey decrease, then predators follow the decrease. The pesticide as the control can cause the number of pests as the prey to decrease. However, it also affects predators so that predators also decrease. Figure 3 (right) shows the optimal control of pesticide used. Initially,

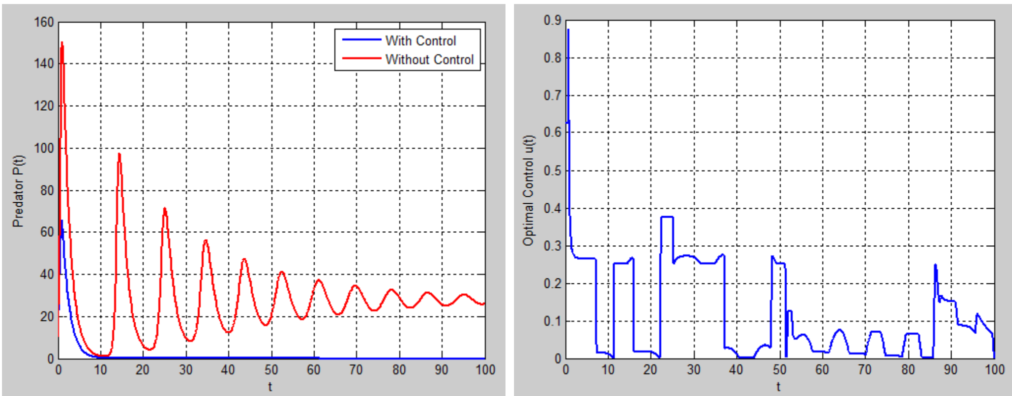

Figure 3: The left panel represents the numerical simulation of the number of spiders (*Lycosa pseudoannulata*) as natural predators. The right panel denotes the optimal control of pesticides.

the pesticides are given to around 88% of the population. After that, the number of individuals receiving the pesticides is decreasing. Starting from $t = 4$, the number of individuals receiving the pesticides is fluctuating between 0 and 3.8.

5 Conclusions

In the agricultural pest dynamical model, there are populations of the pest and the spider. From the mathematical model of agriculture pest growth, we obtain three equilibrium points. We analyze the stability of each equilibrium point by using the eigenvalue. The first equilibrium point is unstable, whereas the second and third equilibriums are stable if certain conditions are satisfied. Furthermore, we have introduced a control variable, which represents pesticide, in the agricultural pest growth model. We have formulated an optimal control problem and solved it numerically. Moreover, we have conducted several simulations to show the effectiveness of the proposed method.

References

[1] T. K. Kar, A. Ghorai and S. Jana. Dynamics of pest and its predator model with disease in the pest and optimal use of pesticide. *Journal of Theoretical Biology* **310** (2012) 187–198.

[2] D. Rahmalia and T. Herlambang. Weight optimization of optimal control influenza model using artificial bee colony. *IJCSAM (International Journal of Computing Science and Applied Mathematics)* **4** (1) (2018) 27–31.

[3] T. Herlambang and D. Rahmalia. Application ant colony optimization on weight selection of optimal control seir epidemic model. In: *Proc. Book 7th Annual Basic Science International Conf.* Malang, Indonesia, 2017, 196–199.

[4] D. Rahmalia. Pemodelan Matematika dan Analisis Stabilitas dari Penyebaran Penyakit Flu Burung. *Unisda Journal of Mathematics and Computer Science (UJMC)* **1** (1) (2015) 11–19.

[5] D. Rahmalia and A. Rohmatullah. Analisis Kestabilan dan Kontrol Optimal Model Penyebaran Penyakit Zika dari Gejala Dengue Menggunakan Forward Backward Sweep Method. *Jurnal Ilmiah Soulmath: Jurnal Edukasi Pendidikan Matematika* **7** (2) (2019) 159–173.

[6] D. Rahmalia, T. Herlambang, A. M. Rohmah and A. Muhith. Weights optimization using Firefly Algorithm on optimal control of zika disease from dengue symptoms by vaccination. *Journal of Physics: Conference Series* **1594** (1) (2020) 012040.

[7] A. Muhith, D. Rahmalia and T. Herlambang. The effects of drugs in chemotherapy as optimal control of tumor growth dynamical model. *Journal of Physics: Conference Series* **1663** (1) (2020) 012006.

[8] A. Zaghdani. Mathematical Study of a Modified SEIR Model for the Novel SARS-Cov-2 Coronavirus. *Nonlinear Dynamics and Systems Theory* **21** (3) (2021) 326–336.

[9] J. D. Murray. *Mathematical biology: I. An introduction.* Springer Science & Business Media, 2007.

[10] M. El Hia, O. Balatif, M. Rachik and J. Bouyaghroumni. Application of optimal control theory to an seir model with immigration of infectives. *International Journal of Computer Science Issues (IJCSI)* **10** (2 Part 2) (2013) 230.

[11] M. H. DarAssi and M. A. Safi. Analysis of an SIRS Epidemic Model for a Disease Geographic Spread. *Nonlinear Dynamics and Systems Theory* **21** (1) (2021) 56–67.

[12] K. Mu'tamar, D. Rahmalia and S. Sutimin. Vaksinasi dan Treatment pada Predator-Prey dengan Dua Jenis Pemangsa yang Salah Satunya Terinfeksi. *EKSAKTA: Journal of Sciences and Data Analysis* **19** (2) (2019) 128–142.

[13] J. B. Burl. *Linear optimal control: H (2) and H (Infinity) methods.* Addison-Wesley Longman Publishing Co., Inc, 1998.

[14] F. Brauer and C. Castillo-Chavez. *Mathematical Models in Population Biology and Epidemiology.* Springer, 2012.

[15] S. Lenhart, and J. T. Workman. *Optimal Control Applied to Biological Models.* CRC press, 2007.

Nonlinear Dynamics and Systems Theory, **22** (3) (2022) 291–302

Solvability of Equations with Time-Dependent Potentials

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Abstract: This paper is devoted to the solution of equations with time-dependent potential, at which the heat and wave equations are taken as prototype problems. The method of separating variables failed to be applied to the equations. The well-posedness of the problems is justified by strongly continuous quasi semigroups. The positive solution of the heat equations is conditioned by the maximum principle depending on the potential. For the wave equations, the bounded potentials imply the well-posedness of the problems. Further, firstly approximate solutions can be schemed. The heat and wave equations with specific potentials are considered.

Keywords: *strongly continuous quasi semigroup; heat equation; wave equation; time-dependent potential; well-posedness.*

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1 Introduction

Some phenomena of reaction-diffusion in physical systems have models as equations with time-dependent potentials [1, 2]. In general, they take the forms of nonautonomous Cauchy problems (NCP) on Banach spaces [3–6],

$$\begin{aligned} \dot{u}(t) &= A(t)u(t), \qquad t \geq 0, \\ u(0) &= u_0, \qquad\quad u_0 \in X, \end{aligned} \tag{1}$$

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where u is an unknown function from $[0, \infty)$ into a Banach space X and every $A(t)$ is a densely defined closed linear operator on $\mathcal{D}(A(t)) = \mathcal{D} \subseteq X$, the domain which is independent of t.

A strongly continuous quasi semigroup (C_0-quasi semigroup) is a sophisticated method to characterize the well-posedness of the NCP (1) [7]. Another awesomeness of the C_0-quasi semigroups in analyzing the nonautonomous problems can be found in [8–13]. The method is an alternative to the evolution operator or propagation method in [1–3, 5, 6].

The heat and wave equations with time-dependent potentials are the useful prototypes of the equations of the same types. The variable separation method failed to be applied to solve the heat and wave equations of this type. The heat equation with time-dependent potential on $\mathbb{R}^+ \times \Omega$ has a form

$$u_t = \Delta u - V(t,x)u + f(t,x), \quad (t,x) \in \mathbb{R}^+ \times \Omega, \tag{2}$$

where Δ is the Laplace operator on an open bounded set Ω in $\mathbb{R}^n$. By the evolution operator or propagation, problem (2) have been discussed [14,15]. Moreover, determination of a time-dependent heat transfer known as an inverse time-dependent source problem in various conditions also has been justified [16–21].

The wave equation with time-dependent potential on $\mathbb{R}^+ \times \Omega$ takes a form

$$u_{tt} = \Delta u - V(t,x)u + f(t,x), \quad (t,x) \in \mathbb{R}^+ \times \Omega. \tag{3}$$

Under suitable assumptions on the potential $V(t,x)$, the main goal of (3) is to show the existence of the scattering operator of the propagation [22–25]. In particular, $V(t,x) = V(x)$, the recovery of $V(x)$ and uniqueness results have been investigated [26, 27].

In fact, the procedure used to analyze problems (2) and (3) is very complicated. It seems that it will be simpler if the problems are modeled as NCP (1) and the C_0-quasi semigroup approach is used. Therefore, in this paper, we focus on the solvability of (2) and (3) using the C_0-quasi semigroups. In the preliminaries, we recall the well-posedness of (1) that has been developed in [7]. The main results are the well-posedness of problems (2) and (3).

2 Preliminaries

This work is a continuation of the paper of Sutrima et al. [7]. The paper characterized the well-posedness of the nonautonomous abstract Cauchy problems using a strongly continuous quasi semigroup approach. The characterization centers on the infinitesimal generators of the corresponding quasi-semigroups. Therefore, the materials of this paper involve the well-posedness results in [7].

Definition 2.1 Let $\mathcal{L}(X)$ be the set of all bounded linear operators on a Banach space X. A two-parameter commutative family $\{R(t,s)\}_{s,t\geq 0}$ in $\mathcal{L}(X)$ is called a strongly continuous quasi semigroup (in short C_0-quasi semigroup) on X if:

(a) $R(t, 0) = I$, the identity operator on X,

(b) $R(t, s + r) = R(t + r, s)R(t, r)$,

(c) $\lim_{s\to 0^+} \|R(t,s)x - x\| = 0$,

(d) there is a continuous increasing function $M : [0, \infty) \to [1, \infty)$ such that

$$\|R(t, s)\| \leq M(s)$$

for all $r, s, t \geq 0$ and $x \in X$.

The infinitesimal generator of the C_0-quasi semigroup $\{R(t, s)\}_{s,t\geq 0}$ is a family of operators $\{A(t)\}_{t\geq 0}$ on $\mathcal{D}$, where

$$A(t)x = \lim_{s\to 0^+} \frac{R(t, s)x - x}{s}$$

and $\mathcal{D}$ is the set of all $x \in X$ such that the right-hand limits exist.

For simplicity, we denote by $R(t, s)$ and $A(t)$ the quasi semigroup $\{R(t, s)\}_{s,t\geq 0}$ and the infinitesimal generator $\{A(t)\}_{t\geq 0}$, respectively. Further, we always consider the C_0-quasi semigroups whose infinitesimal generator has a dense domain in the Banach spaces.

Let $\mathcal{R}(\lambda, A(t)) := (\lambda - A(t))^{-1}$ be the resolvent operator of $A(t)$ for $\lambda \in \rho(A(t))$, where $\rho(A(t))$ is the resolvent set of $A(t)$. The following result is the version of the Hille-Yosida Theorem for a C_0-quasi semigroup.

Theorem 2.1 (Theorem 2.3 of [7]) *For each $t \geq 0$, let $A(t)$ be a closed and densely defined operator on $\mathcal{D}$ and the map $t \mapsto A(t)y$ is a continuous function from $\mathbb{R}^+$ to X for all $y \in \mathcal{D}$. If $\mathcal{R}(\lambda, A(\cdot))$ is locally integrable and there exist constants $N, \omega \geq 0$ such that $[\omega, \infty) \subseteq \rho(A(t))$ and*

$$\|\mathcal{R}(\lambda, A(t))^r\| \leq \frac{N}{(\lambda - \omega)^r}, \quad \lambda > \omega, \quad r \in \mathbb{N},$$

then $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup.

We recall the well-posedness of the Cauchy problem (1) that has been discussed in [7]. First, we consider the inhomogeneous form of the Cauchy problem (1)

$$\begin{aligned} \dot{u}(t) &= A(t)u(t) + f(t), \qquad && t \geq 0, \\ u(0) &= u_0, && u_0 \in X, \end{aligned} \tag{4}$$

where f is a continuous function from $[0, \infty)$ to a Banach space X. Let $\mathcal{C}(\Omega, X)$ and $\mathcal{C}^1(\Omega, X)$ denote the set of all continuous functions on Ω and the set of all functions with continuous derivative on Ω, respectively.

Definition 2.2 A function u is called a classical solution of (4) on $[0, \tau]$ if $u \in \mathcal{C}^1([0, \tau], X)$, $u(t) \in \mathcal{D}$ for all $t \in [0, \tau]$ and $u(t)$ satisfies (4) for all $t \in [0, \tau]$. The function u is called a classical solution on $[0, \infty)$ if u is a classical solution on $[0, \tau]$ for each $\tau > 0$.

Therefore, the classical solution of the nonautonomous abstract Cauchy problem (1) is the classical solution of (4) when $f = 0$.

Lemma 2.1 (Lemma 3.2 of [7]) *Let $A(t)$ be the infinitesimal generator of a C_0-quasi semigroup $R(t, s)$ on a Banach space X and $u_0 \in \mathcal{D}$. If $f \in \mathcal{C}([0, \tau], X)$ and u is a classical solution of (4), then $A(\cdot)u(\cdot) \in \mathcal{C}([0, \tau], X)$ and*

$$u(t) = R(0, t)u_0 + \int_0^t R(s, t - s)f(s)ds. \tag{5}$$

Definition 2.3 The nonautonomous abstract Cauchy problem (1) is said to be *well-posed* if it satisfies the following conditions:

(WP1) *Existence.* For each $u_0 \in \mathcal{D}$, there exists a classical solution u of (1) on $[0, \infty)$.

(WP2) *Uniqueness.* If $u, v : [0, \tau] \to X$ are the classical solutions of (1), then $u(t) = v(t)$ for all $t \in [0, \tau]$, $\tau > 0$.

(WP3) *Continuous dependence.* The classical solution x depends continuously on $t \in [0, \infty)$ and $u_0 \in \mathcal{D}$, i.e., the map $\phi : [0, \infty) \times \mathcal{D} \to X$ with $\phi(t, u_0) = u(t)$ is continuous.

The well-posedness of the nonautonomous Cauchy problem (1) is characterized by the existence and uniqueness of the infinitesimal generator of the related C_0-quasi semigroup.

Theorem 2.2 (Theorem 3.6 of [7]) *For each $t \geq 0$, let $A(t) : \mathcal{D} \to X$ be a closed and densely defined operator in a Banach space X. The family $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup on X if and only if the nonautonomous abstract Cauchy problem* (1) *is well-posed.*

We have a similar result on the well-posedness of the inhomogeneous nonautonomous Cauchy problem (4).

Theorem 2.3 (Theorem 3.11 of [7]) *If $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup on X, then the inhomogeneous nonautonomous abstract Cauchy problem* (4) *is well-posed.*

We note that Theorem 2.3 remains valid when f belongs to the Sobolev space $W^{1,p}([0, \infty), X)$, $1 \leq p < \infty$.

3 Results and Discussion

As an auxiliary result, we have a perturbation of the infinitesimal generator of the C_0-quasi semigroups. The following one is a special case of the perturbation.

Theorem 3.1 *If $A(t)$ and B are the infinitesimal generators of a C_0-quasi semigroup $R(t, s)$ and C_0-semigroup $T(s)$ on a Banach space X, respectively, such that $R(t, s)$ and $T(s)$ are commutative, then $A(t) + B$ is the infinitesimal generator of a C_0-quasi semigroup $K(t, s)$ given by*

$$K(t, s) = T(s)R(t, s), \quad s, t \geq 0.$$

Proof. It is easy to show that $K(t, s)$ verifies the definition of a C_0-quasi semigroup. By the Hille-Yosida Theorem for $T(s)$ and the fact that $R(t, s)$ is a C_0-quasi semigroup, there exist constants $N, \omega > 0$ and an increasing function M such that

$$\|K(t, s)\| \leq M_K(s), \quad t, s \geq 0,$$

where $M_K(s) = Ne^{\omega s}M(s)$. For $t \geq 0$ and $x \in \mathcal{D} \cap \mathcal{D}(B)$, the continuity of $T(s)$ gives

$$\lim_{s \to 0^+} \frac{K(t, s)x - x}{s} = \lim_{s \to 0^+} T(s)\frac{R(t, s)x - x}{s} + \lim_{s \to 0^+} \frac{T(s)x - x}{s} = [A(t) + B]x.$$

This shows that $A(t) + B$ generates $K(t, s)$.

Theorem 3.1 is to be central in the discussion of the time-dependent potential problems. Theorem 3 of [10] gives the general perturbation of the infinitesimal generator of C_0-quasi semigroups. Also in [10], we see the applications of the perturbation in the linear nonautonomous control systems.

3.1 Heat equation with time-dependent potential

In this subsection, we shall apply the well-posedness to heat equations with time-dependent potentials. Let Ω be an open bounded set of $\mathbb{R}^n$ with a regular boundary $\partial\Omega$. We consider the non-autonomous heat equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \Delta u(t,x) - V(t,x)u(t,x) + f(t,x) \quad \text{in } (0,\tau)\times\Omega, \\ u(t,x) &= 0 \quad \text{on } (0,\tau]\times\partial\Omega, \\ u(0,x) &= u_0(x) \quad \text{in } \Omega, \end{aligned} \tag{6}$$

where Δ is the Laplace operator, V is a Lebesgue measurable potential on $[0,\tau]\times\Omega$ and $f \in L_2([0,\tau]\times\Omega)$. Set $X = L_2(\Omega)$, the Laplace operator Δ in (6) is densely defined in X with

$$\mathcal{D}(\Delta) = H^2(\Omega) \cap H_0^1(\Omega),$$

where H_0^1 denotes a space of functions in $H^1(\Omega)$ that vanish at the boundary. In this case, $H^m(\Omega)$ is the Sobolev space given by

$$H^m(\Omega) := \{u \in L_2(\Omega) : u, \ldots, \tfrac{d^{m-1}u}{dx^{m-1}} \text{ are absolutely continuous on } \Omega,\ \tfrac{d^m u}{dx^m} \in L_2(\Omega)\}.$$

Moreover, Δ is a self-adjoint operator such that

$$\langle \Delta u, u\rangle \le \lambda_0 \|u\|^2, \quad u \in \mathcal{D}(\Delta),$$

where λ_0 is the first eigenvalue of Δ. Lumer-Phillips' theorem implies that Δ is the infinitesimal generator of a C_0-semigroup T in X such that

$$\|T(t)\| \le e^{\lambda_0 t}, \quad t \ge 0. \tag{7}$$

Setting $u(t) = u(t,\cdot)$, $V(t) = V(t,\cdot)$, and $A(t) = \Delta - V(t)$, we can rewrite the problem (6) as

$$\dot{u}(t) = A(t)u(t) + f(t), \quad u(0) = u_0, \quad t \in [0,\tau]. \tag{8}$$

Theorem 3.2 *If $u_0 \in X$ and the operator $-V(t)$ verifies the hypothesis of Theorem 2.1, then the non-autonomous problem* (6) *has a unique solution u which belongs to* $\mathcal{C}([0,\tau], X)$.

Proof. By Theorem 2.1, there exists a unique C_0-quasi semigroup $R(t,s)$ generated by $-V(t)$. Theorem 3 of [10] implies that $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup $K(t,s)$ in X with $K(t,s) = T(s)R(t,s)$. Theorem 2.3 gives that the non-autonomous Cauchy problem (8) is well-posed with a unique solution

$$u(t) = K(0,t)u_0 + \int_0^t K(s, t-s)f(s)ds. \tag{9}$$

Together with Theorem 3.10 of [7], the assertion follows.

A physical interpretation requires the positive solution of the Cauchy problem (6). By virtue of Theorem 10 on page 44 of [28], we obtain the following result.

Proposition 3.1 *If* $u_0(x), f(t,x), V(t,x) \geq 0$ *for all* $(t,x) \in [0,\tau] \times \Omega$*, then the solution* (9) *is positive.*

The following example gives an illustration how the heat equation with the time-dependent potential in $\mathbb{R}^3$ is solved.

Example 3.1 Consider the problem (6) in $\mathbb{R}^3$. Let V be a potential on $[0,\tau] \times \Omega$ given by $V(t,x) = \alpha(t)Z(x)$, where $\Omega = (0,1)^3$ and $Z(x) = -|x|^{-1}$ is the Coulomb potential on Ω and α is a measurable function on $[0,\tau]$.

We see that $-V(t)$ verifies Theorem 2.1. In addition, if $\alpha(t) = -\dfrac{1}{t+1}$, $-V(t)$ generates a C_0-quasi semigroup $R(t,s)$ on X given by

$$R(t,s)u(x) = \left(\frac{t+1}{t+s+1}\right)^{\frac{1}{|x|}} u(x), \quad u \in X, \quad x \in \Omega.$$

The C_0-semigroup $T(t)$ in (7) is given by

$$T(t)u = \sum_{l,m,n=1}^{\infty} e^{-(l^2+m^2+n^2)\pi^2 t} \langle u, \phi_{lmn} \rangle \phi_{lmn}, \quad u \in X,$$

where $\phi_{lmn}(x) = 2\sqrt{2}\sin(l\pi x_1)\sin(m\pi x_2)\sin(n\pi x_3)$ and $\langle \cdot, \cdot \rangle$ is the inner product in X. Therefore, for arbitrary $f \in L_2([0,\tau] \times \Omega)$, the solution of (6) is given by

$$u(t,x) = K(0,t)u_0(x) + \int_0^t K(s,t-s)f(s,x)ds, \tag{10}$$

where $K(t,s) = T(s)R(t,s)$. Moreover, we see if $u_0(x), f(t,x) \geq 0$ for all $(t,x) \in [0,\tau] \times \Omega$, then the solution (10) is positive.

Remark 3.1 (1) The quasi semigroup is an alternative approach to solve the non-autonomous heat equation in $[0,\tau] \times \mathbb{R}^n$. In some way, the approach is simpler than the approach used by Gulisashvili [14]. Moreover, if the potential V verifies Theorem 2.1, then it belongs to the Kato class K_n or the uniform Kato class $\mathcal{A}_{n,\tau}$.

(2) The quasi semigroup is also immediately applicable to solve the Schrodinger equation with a time-dependent potential

$$i\frac{\partial \psi}{\partial t}(t,x) = -\frac{1}{2}\Delta\psi(t,x) + V(t,x)\psi(t,x).$$

Refer to Evans [2], the quasi semigroups can eliminate the scattering method.

3.2 Wave equation with time-dependent potential

Let Ω be an open bounded set of $\mathbb{R}^n$ with a regular boundary $\partial\Omega$. We consider the wave equation with a time-dependent potential

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) - V(t,x)u(t,x) + f(t,x) \quad \text{in } (0,\tau) \times \Omega, \\ &u(t,x) = 0 \quad \text{on } (0,\tau] \times \partial\Omega, \\ &u(0,x) = u_0(x), \; u_t(0,x) = u_1(x) \quad \text{in } \Omega, \end{aligned} \tag{11}$$

where V is a Lebesgue measurable function on $[0,\tau]\times\Omega$ and $f\in L_2([0,\tau]\times\Omega)$. We also assume that $\|V\|_\infty := \sup_{(t,x)\in[0,\tau]\times\Omega}|V(t,x)| < \infty$.

Using the notations as in Section 3.1, we see that $\Lambda = -\Delta$ is strictly positive self-adjoint operator in the Hilbert space X. We consider a Hilbert space $Z = \mathcal{D}(\sqrt{\Lambda})\oplus X$ with the generic element

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

and the inner product in Z is given by

$$\langle z,w\rangle_Z = \langle\sqrt{\Lambda}z_1,\sqrt{\Lambda}w_1\rangle_X + \langle z_2,w_2\rangle_X.$$

Define a linear operator A in Z by

$$A_0 z = \begin{bmatrix} 0 & 1 \\ -\Lambda & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix},$$

with $\mathcal{D}(A_0) = \mathcal{D}(\Lambda)\oplus\mathcal{D}(\sqrt{\Lambda})$.

Theorem 3.3 *If $u_0\in H_0^1(\Omega)$ and $u_1\in L_2(\Omega)$, then the non-autonomous problem* (11) *has a unique mild solution u belonging to $\mathcal{C}([0,\tau],H_0^1(\Omega))\cap\mathcal{C}^1([0,\tau],X)$.*

Proof. When setting

$$Y(t) = \begin{bmatrix} u(t,\cdot) \\ u_t(t,\cdot) \end{bmatrix},\quad Q(t) = \begin{bmatrix} 0 & 0 \\ -V(t,\cdot) & 0 \end{bmatrix},\quad F(t) = \begin{bmatrix} 0 \\ f(t,\cdot) \end{bmatrix},\quad Y_0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix},$$

the problem (11) is equivalent to

$$\dot{Y}(t) = A(t)Y(t) + F(t),\quad Y(0) = Y_0,\quad t\in[0,\tau], \tag{12}$$

where $A(t) = A_0 + Q(t)$. By Proposition 2.12 of [29], we see that A_0 is the infinitesimal generator of a C_0-quasi semigroup $S(t,s)$ on Z given by

$$S(t,s) = \begin{bmatrix} \cos(\sqrt{\Lambda}s) & \Lambda^{-1/2}\sin(\sqrt{\Lambda}s) \\ -\Lambda^{1/2}\sin(\sqrt{\Lambda}s) & \cos(\sqrt{\Lambda}s) \end{bmatrix}.$$

Since $\|Q(t)\|\le\|V\|<\infty$ for $t\in[0,\tau]$, Theorem 3 of [10] gives that $A(t)$ is the infinitesimal generator of a C_0-quasi semigroup $R(t,s)$ on Z such that

$$R(r,t) = \sum_{n=0}^{\infty} S_n(r,t), \tag{13}$$

where $S_0(r,t)y = S(r,t)y$ and $S_n(r,t)y = \int_0^t S(r+s,t-s)Q(r+s)S_{n-1}(r,s)yds$ for $r,t\ge s\ge 0$, $y\in Z$ and $n\in\mathbb{N}$. Therefore, Theorem 2.3 implies that the problem (12) has a unique mild solution given by

$$Y(t) = R(0,t)Y_0 + \int_0^t R(s,t-s)F(s)ds. \tag{14}$$

By Theorem 3.10 of [7], the first component in (14) provides the required solution, $u(t,x)$, of the problem (11).

Remark 3.2 (1) Theorem 3.1 is not applicable in the determination of a quasi semigroup $R(t,s)$ since $S(t,s)$ is not commutative with any nontrivial quasi semigroups.

(2) We stress again that the function Y in (14) is the exact solution to (12) whose first component is the solution $u(t,x)$ of the problem (11). However, it is not easy to find the explicit form of $R(t,s)$ in (13). The approximation $Y^{(n)}$, $n = 0,1,2,\ldots$, to the solution Y of (14) gives the properties of $u(t,x)$.

Zero approximation. This is a function $Y^{(0)}$ in the form (14) for $R(t,s) = S(t,s)$. The first component of $Y^{(0)}$ gives $u^{(0)}$, the zero approximation to the solution of the problem (11) given by

$$u^{(0)}(t,\cdot) = [\cos\sqrt{\Lambda}t]u_0 + [\Lambda^{-1/2}\sin\sqrt{\Lambda}t]u_1 + \int_0^t \Lambda^{-1/2}\sin\sqrt{\Lambda}(t-s)f(s,\cdot)ds. \tag{15}$$

By the spectral theorem, this solution can be written in the form

$$\begin{aligned} u^{(0)}(t,\cdot) = \int_0^\infty \cos\sqrt{\lambda}t\, d(E(\lambda)u_0) + \int_0^\infty \frac{\sin\sqrt{\Lambda}t}{\sqrt{\lambda}} d(E(\lambda)u_1) \\ + \int_0^\infty \int_0^t \frac{\sin\sqrt{\lambda}(t-s)}{\sqrt{\lambda}} f(s,\cdot)ds\, d(E(\lambda)), \end{aligned} \tag{16}$$

where $E(\lambda) \in \{E(\lambda)\} \equiv \{E(\lambda)\}_{\lambda\in\sigma(\Lambda)}$ is the spectral family for Λ.

The formula (16) indicates that the smoothness of u depends on the smoothness of the data functions. Moreover, the representation implies that the asymptotic behaviour of $u(t,x)$ as $t \to \infty$ is closely related to the properties of the spectrum, $\sigma(\Lambda)$, and the spectral family, $\{E(\lambda)\}$.

Taking into account (15) gives the initial value problem

$$\frac{\partial^2 u^{(0)}}{\partial t^2}(t,\cdot) = \Delta u^{(0)}(t,\cdot) + f(t,\cdot), \quad u^{(0)}(0,\cdot) = u_0,\ u_t^{(0)}(0,\cdot) = u_1.$$

This shows that $u^{(0)}$ is a solution to the problem (11) without the potential or unperturbed problem. The form (14) implies that all subsequent approximations obtained by taking more terms in (13) are the solutions of a perturbed problem.

First approximation. The first two terms of (13) give

$$Y^{(1)}(t) = R(0,t)Y_0 + \int_0^t R(s,t-s)F(s)ds,$$

where $R(r,t) = S(r,t) + S_1(r,t)$. The first component of $Y^{(1)}$ gives $u^{(1)}$ by

$$\begin{aligned} u^{(1)}(t,\cdot) = u^{(0)}(t,\cdot) + \int_0^t [\{-\Lambda^{-1/2}\sin\sqrt{\Lambda}(t-s)\cos\sqrt{\Lambda}t\}V(s)u_0 \\ - \{\Lambda^{-1}\sin\sqrt{\Lambda}(t-s)\sin\sqrt{\Lambda}t\}V(s)u_1]\, ds + \\ \int_0^t \int_0^{t-s} \{-\Lambda^{-1}\sin\sqrt{\Lambda}(t-s-\eta)\sin\sqrt{\Lambda}\eta\}V(s+\eta)f(s,\cdot)d\eta\, ds. \end{aligned} \tag{17}$$

Taking into account (17) gives the initial value problem

$$\frac{\partial^2 u^{(1)}}{\partial t^2}(t,\cdot) = \Delta u^{(1)}(t,\cdot) - V(t,\cdot)u^{(0)}(t,\cdot) + f(t,\cdot),\ u^{(1)}(0,\cdot) = u_0,\ u_t^{(1)}(0,\cdot) = u_1.$$

We see that $u^{(1)}$ is a solution to the wave equation with the nontrivial forcing term on the right-hand side. In particular, the forcing term depends on the potential V and the previously approximate solution $u^{(0)}$.

For a realization, we consider problem (11) in one dimensional space with $\Omega = (0,1)$, $\tau = 1$, the potential $V(t,x) = h(x)e^{-i\omega t}$, $(t,x) \in [0,\tau] \times \Omega$, and $f = 0$. From (17), the first approximation solution to the problem (11) in the spectral form is

$$\begin{aligned} u^{(1)}(t,\cdot) = &\int_0^\infty [a_0(\lambda)\sin(\sqrt{\lambda}t) + b_0(\lambda)\cos\sqrt{\lambda}t]\, d(E(\lambda)u_0) \\ &+ \int_0^\infty [a_1(\lambda)\sin\sqrt{\Lambda}t + b_1(\lambda)\cos\sqrt{\Lambda}t]\, d(E(\lambda)u_1), \end{aligned} \tag{18}$$

where

$$\begin{aligned} a_0(\lambda) &= -\frac{4\lambda e^{-i\omega t} - 4\lambda + 2\omega^2}{2\sqrt{\lambda}\omega(4\lambda - \omega^2)} h(\cdot)i, \qquad & b_0(\lambda) &= \frac{2\sqrt{\lambda}(4\lambda - \omega^2) - (e^{-i\omega t} - 1)h(\cdot)}{2\sqrt{\lambda}(4\lambda - \omega^2)} \\ a_1(\lambda) &= \frac{4\lambda - \omega^2 - 2(e^{-i\omega t} + 1)h(\cdot)}{\sqrt{\lambda}(4\lambda - \omega^2)}, \qquad & b_1(\lambda) &= \frac{4(e^{-i\omega t} - 1)}{\omega(4\lambda - \omega^2)} h(\cdot)i. \end{aligned}$$

The eigenvalues and eigenvectors of Λ are $\lambda_n = n^2\pi^2$ and $\phi_n(x) = \sqrt{2}\sin(n\pi x)$, $n = \pm 1, \pm 2, \ldots$, respectively. Hence, by the Cauchy theorem, the solution (18) can be counted as

$$\begin{aligned} u^{(1)}(t,x) = \sum_{n=1}^{\infty} 2\Big[&\left\{a_0(n^2\pi^2)\sin(n\pi t) + b_0(n^2\pi^2)\cos(n\pi t)\right\} \langle u_0, \phi_n\rangle \phi_n(x) \\ &+ \left\{a_1(n^2\pi^2)\sin(n\pi t) + b_1(n^2\pi^2)\cos(n\pi t)\right\} \langle u_1, \phi_n\rangle \phi_n(x)\Big], \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the inner product in X.

Remark 3.3 (1) We see that if $\omega^2/4$ in the resolvent set $\rho(\Lambda)$ and h is bounded on Ω, then $u^{(1)}(t,x)$ is bounded as $t \to \infty$. This implies that the solution $u(t,x)$ is also bounded as $t \to \infty$.

(2) If $\omega = 0$, then the problem (11) is the wave problem with the time-independent potential and the first approximation to the solution of the problem (18) can be easily obtained. Again, the solution is bounded as $t \to \infty$.

3.3 Nonlinear equations

We shall show that the quasi semigroup approach is applicable to solve the nonlinear equations. To begin with, we denote by $C_0(\Omega)$ the Banach space of continuous functions on Ω that vanish on $\partial\Omega$ with the sup norm. Given an initial value $u_0 \in C_0(\Omega)$, we consider the nonlinear version of the non-autonomous heat equation (6)

$$\begin{aligned} u_t &= \Delta u - Vu + f(u), \\ u|_{\partial\Omega} &= 0, \\ u(0) &= u_0. \end{aligned} \tag{19}$$

Theorem 2.3 guarantees that the initial value problem (19) is locally well-posed. More precisely, there exists a maximal time $0 < \tau_0 \leq \infty$ and a function $u \in C([0,\tau_0], C_0(\Omega)) \cap$

$C\left((0,\tau_0), C^2(\overline{\Omega})\right) \cap C^1\left((0,\tau_0), C_0(\Omega)\right)$ which is a classical solution of (19), see [30]. Further, by Theorem 3.2, u is the unique solution of (19) in $L^\infty\left((0,\tau)\times\Omega\right)$ for any $0<\tau<\tau_0$ given by

$$u(t) = K(0,t)u_0 + \int_0^t K(s,t-s)f(u(s))ds. \tag{20}$$

It is clear that the global solution u in (20) depends on u_0 and the nonlinearity of f. Recall that the solution u is global if $\tau_0=\infty$, meanwhile u is blowing up in finite time if $\tau_0<\infty$ and $\lim_{t\to\tau_0}\|u(t)\|_{L^\infty}=\infty$. In particular, for $V\equiv 0$, the conditions for the global and blowing up solutions of (19) were discussed in [30]. Analogously, for f and V are positive, we can verify the conditions of the solution u in (20) to be globally positive.

Remark 3.4 We can also apply the quasi semigroups to solve the nonlinear version of the Schrodinger equation in Remark 3.1 and the wave equation (11). Further, we can also classify the nonlinearity of f such that the solutions are global.

4 Conclusions

We have solved the heat and wave equations as prototype problems of the equations with time-dependent potentials, even for the nonlinear ones. The well-posedness is justified by $C_)$-quasi semigroups. By the maximum principle, the positivity of the solution of the heat equations depends on the potential. In the wave equations, the well-posedness is guaranteed by the bounded potentials. Moreover, although general approximations cannot be constructed yet, the first two approximations can be constructed.

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References

[1] J.S. Howland. Stationary scattering theory for time-dependent Hamiltonians. *Math. Ann.* **207** (1974) 315–335.

[2] D.E. Evans. Time-dependent perturbations and scattering of strongly continuous groups on Banach spaces. *Math. Ann.* **221** (1976) 275–290.

[3] A.G. Nickel. Evolution semigroups and product formulas for nonautonomous Cauchy problems. *Math. Nachr.* **212** (2000) 101–116.

[4] D. Barcenas and H. Leiva. Characterization of extremal controls for infinite dimensional time-varying systems. *SIAM J. Control Optim.* **40** (2) (2001) 333–347.

[5] J. Schmid. Well-posedness of non-autonomous linear evolution equations for generators whose commutators are scalar. *J. Evol. Equ.* **16** (2016) 21–50.

[6] J. Schmid and M. Griesemer. Well-posedness of non-Autonomous linear evolution equations in uniformly convex spaces. *Math. Nachr.* **290** (2-3) (2017) 435–441.

[7] S. Sutrima, M. Mardiyana, R. Setiyowati and R. Respatiwulan. A new approach on well-posedness of nonautonomous Cauchy problems: An application in population growth. *J. Nonlinear Funct. Anal.* Article ID 29 (2021) 1–15.

[8] Sutrima, C.R. Indrati and L. Aryati. Complete stabilizability and stability radius of time-varying Sturm-Liouville systems. *J. Phys. Conf.* 1088(012021), 2018, 1–10.

[9] S. Sutrima, C.R. Indrati and L. Aryati. Controllability and observability of non-autonomous Riesz-spectral systems. *Abstr. Appl. Anal.* **4210135** (2018) 1–10.

[10] S. Sutrima, C.R. Indrati and L. Aryati. Exact null controllability, stabilizability, and detectability of linear nonautonomous control systems: A quasisemigroup approach. *Abstr. Appl. Anal.* **3791609** (2018) 1–12.

[11] S. Sutrima, C.R. Indrati and L. Aryati. Contraction quasi semigroups and their applications in decomposing Hilbert spaces. *Azerb. J. Math.* **10** (2020) 57–74.

[12] S. Sutrima, M. Mardiyana and R. Setiyowati. Uniformly exponential dichotomy for strongly continuous quasi groups. *Oper. Matrices* **15** (1) (2021) 253–273.

[13] S. Sutrima, M. Mardiyana, Respatiwulan, W. Sulandari and M. Yunianto. Approximate controllability of nonautonomous mixed boundary control systems. *AIP Conference Proceedings* 2326(020037), 2021, 1–9.

[14] A. Gulisashvili. On the heat equation with a time-dependent singular potential. *J. Funct. Anal.* **194** (2002) 17–52.

[15] M. Maniruzzaman and R.D. Sisson, Jr. Heat transfer coefficients for quenching process simulation. *J. Phys. IV France* **120** (2004) 269–276.

[16] T.T.M. Onyango, D.B. Ingham, D. Lesnic and M. Slodicka. Determination of a time-dependent heat transfer coefficient from non-standard boundary measurements. *Math. Comput. Simul.* **79** (2009) 1577–1584.

[17] M. Slodicka, D. Lesnic and T.T.M. Onyango. Determination of a time-dependent heat transfer coefficient in a nonlinear inverse heat conduction problem. *Inverse Probl. Sci. Eng.* **18** (1) (2010) 65–81.

[18] A. Hazanee and D. Lesnic. Determination of a time-dependent heat source from nonlocal boundary conditions. *Eng. Anal. Bound. Elem.* **37** (2013) 936–956.

[19] A. Hazanee, M.I. Ismailovc, D. Lesnic and N.B. Kerimov. An inverse time-dependent source problem for the heat equation. *Appl. Numer. Math.* **69** (2013) 13–33.

[20] T. Wei, X.L. Li and Y.S. Li. An inverse time-dependent source problem for a time-fractional diffusion equation. *Inverse Probl.* **32** (2016) 1-24.

[21] A. Hazanee, D. Lesnic, M.I. Ismailovc and N.B. Kerimov. Inverse time-dependent source problems for the heat equation with nonlocal boundary conditions. *Appl. Math. Comput.* **346** (2019) 800–815.

[22] G.P. Menzala. Scattering properties of wave equations with time-dependent potentials. *Comp. & Maths. with Appls.* **12A** (4/5) (1986) 457–475.

[23] P.P. Stefanov. Inverse scattering problem for the wave equation with time-dependent potential. *J. Math. Anal. Appl.* **140** (1989) 351–362.

[24] J.A. Ferreira and G.P. Menzala. On scattering by time-dependent potentials: The 2-D Case. *J. Math. Anal. Appl.* **148** (1990) 131–146.

[25] G.F. Roach. Wave scattering by time-dependent potentials. *Chaos Solit. Fractals* **12** (2001) 2833–2847.

[26] A. Boumenir and V.K. Tuan. An inverse problem for the wave equation. *J. Inverse Ill-Posed Probl.* **19** (2011) 573–592.

[27] M. Vashisth. An inverse problem for the wave equation with source and receiver at distinct points. *J. Inverse Ill-Posed Probl.* **27** (6) (2019) 835–845.

[28] A. Friedman. *Partial Differential Equations of Parabolic Type*. Robert E. Krieger Publishing Company, Florida, 1983.

[29] A. Bensoussan, G. Da Prato, M.C. Delfour, and S.K. Mitter. *Representation and Control of Infinite Dimensional Systems*. Birkhaäuser, Boston, 2007.

[30] T. Cazenave, F. Dickstein and F.B. Weissler. Structural properties of the set of global solutions of the nonlinear heat equation. *GAKUTO Internat. Ser. Math. Sci. Appl.* **32** (2010) 13–23.

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Weighted Performance Measure and Generalized H_∞ Control Problem for Linear Descriptor Systems

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Abstract: In this paper, the generalized problem of H_∞ control with transients is investigated for linear descriptor systems using a weighted performance measure that describes a mixed attenuation level of exogenous and initial disturbances. Based on a generalization of the bounded real lemma, involving special matrix variables, new necessary and sufficient conditions for the existence of static and dynamic output-feedback controller are proposed to ensure the admissibility of a closed-loop system with prescribed estimate of the weighted performance measure. The corresponding synthesis techniques are reduced to solving the linear and quadratic matrix inequalities with rank constraints. A numerical example is included to demonstrate the applicability of the present approaches.

Keywords: *descriptor system; robust stability; admissible system; weighted performance measure; H_∞ control.*

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1 Introduction

Descriptor (differential algebraic) equations arise naturally in many significant applications, for example, in constrained mechanical systems, power generation, chemical processing, network fluid flow, vehicle dynamics, robotics etc. (see, e. g., [5,6,11]). Problems of sensitivity reduction and exogenous disturbance attenuation in descriptor control systems are very important and, at the same time, insufficiently studied for practical applications. These problems are solved by the H_2/H_∞ control design for state-space systems that provide internal stability and minimize the negative influence of exogenous

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disturbances on the dynamics of controlled objects (see, e.g., [4, 8, 12]). As a typical performance measure in H_∞ control design, one can use the H_∞-norm of the transfer function matrix corresponding to the maximum ratio of the L_2-norms for the regulated output and bounded disturbances of the system.

Recently, attention has been paid to the problem of H_∞ control with transients for state-space systems when the initial states are uncertain and might be non-zero. In this regard, more general performance measures characterizing the damping level of external and initial disturbances caused by the nonzero initial vector were used in [1, 13, 18] for standard state-space systems. Known methods of H_∞ control design are based on using the upper bounds for applicable performance measures established via linear matrix inequalities (LMI) and Riccati-type equations (Bounded Real Lemma type statements), see, e.g., [2, 8, 25]. Necessary and sufficient conditions for H_∞ control with transients for state-space systems were proposed in terms of algebraic and differential Riccati equations [13], and in terms of LMIs [1].

The Bounded Real Lemma and H_∞ control theory have been extended for a class of descriptor systems (e.g., [3, 7, 10, 14–16, 22, 24]). A state-feedback controller design approach based on LMIs was proposed in [7] for solving the H_∞ control with transients problem for descriptor systems. Many important control issues including the H_∞ optimization problem for descriptor systems can be formulated as dissipativity with general quadratic supply functions (e.g., [6, 15, 22]).

This paper is concerned with a non-standard H_∞ control problem for linear time-invariant descriptor systems. The purpose of this paper is to extend the results obtained in [1, 7, 20, 21] via using the weighted performance measure taking into account the influence evaluation of both exogenous and initial disturbances in control systems. The application of weight coefficients in the generalized performance measures enables one to establish priorities between the regulated output components and bounded disturbances. Compared with [1, 7], the control system and performance measure studied in this paper are more general. In contrast to [20, 21], we use a special parametrization of the desired solutions of linear and quadratic matrix inequalities, which simplifies the proposed controller synthesis procedure. Furthermore, in some cases, resulting conditions for the existence of the weighted state- and output-feedback H_∞ controller contain only LMIs, which can be solved by existing numerical tools.

This paper is organised as follows. Section 2 contains some basic definitions and lemmas for linear descriptor systems. In Section 3, new necessary and sufficient conditions are proposed for the existence of stabilizing static and dynamic output-feedback controllers solving the weighted H_∞ control problem for descriptor systems. These conditions guarantee a prescribed upper bound for the weighted performance measure of a closed-loop system and, in general, have the LMIs form with additional rank constraints, as well as the form of the generalized algebraic Riccati inequalities (GARIs). In Section 4, the effectiveness of the proposed methods is illustrated by means of a numerical example. After that, a conclusion is given in Section 5. Finally, the solvability criteria for some matrix inequalities are stated in Appendix. In particular, new necessary and sufficient conditions for the solvability of quadratic matrix inequalities arising in the proposed methods for the weighted H_∞ control are presented.

Notations: I_n is the identity $n\times n$ matrix; $0_{n\times m}$ is the $n\times m$ null matrix; $X = X^\top > 0$ (≥ 0) is a positive (nonnegative) definite symmetric matrix X; $\sigma(A)$ is the spectrum of A; $\operatorname{Ker} A$ is the kernel of A; A^{-1} (A^{+}) is the inverse (pseudo-inverse) of A; W_A is the right null matrix of $A \in \mathbb{R}^{m\times n}$, that is, $AW_A = 0$, $W_A \in \mathbb{R}^{n\times(n-r)}$, $\operatorname{rank} W_A = n - r$, where $r = \operatorname{rank} A < n$ ($W_A = 0$ if $r = n$); $\operatorname{Co}\{A_1, \dots, A_\nu\}$ is the convex polyhedron (polytope) with vertices $A_1, \dots, A_\nu$ in a matrix space; $\|x\|$ is the Euclidean norm of x; $\|w\|_P$ is the weighted L_2-norm of a vector function $w(t)$.

2 Basic Definitions and Lemmas

Consider the following continuous-time descriptor system:

$$E\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad x(0) = x_0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $w \in \mathbb{R}^s$ is the exogenous input (disturbances) and $z \in \mathbb{R}^k$ is the output, E, A, B, C and D are constant matrices with compatible dimensions and $\operatorname{rank} E = \rho \le n$.

Definition 2.1 A matrix pair (E, A) is said to be *admissible*, if it is *regular*, *impulse-free* and *stable*, i.e., $\det F(\lambda) \not\equiv 0$, $\deg F(\lambda) = \rho$ and $\sigma(F) \subset \{\lambda \in \mathbb{C} : \operatorname{Re} \lambda < 0\}$, respectively, where $F(\lambda) = A - \lambda E$ is a matrix pencil. Descriptor system (1) with the admissible pair (E, A) is *admissible*.

Lemma 2.1 (see [16]) *System* (1) *is admissible if and only if there exists a matrix* X *such that* $A^\top X + X^\top A < 0$ *and* $E^\top X = X^\top E \ge 0$.

A regular pair (E, A) can be transformed into the Weierstrass canonical form [9]

$$LER = \begin{bmatrix} I_r & 0 \\ 0 & N \end{bmatrix}, \quad LAR = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-r} \end{bmatrix},$$

where L and R are nonsingular matrices, $\sigma(F) = \sigma(A_1)$, $r \le \rho$ and N is a nilpotent matrix. A pair (E, A) is impulse-free if and only if [5]

$$\operatorname{rank} \begin{bmatrix} E & 0 \\ A & E \end{bmatrix} = n + \rho.$$

In this case, $N = 0$ and system (1) can be transformed into the following form:

$$\dot{x}_1 = A_1 x_1 + B_1 w, \quad x_2 = -B_2 w, \quad z = C_1 x_1 + D_1 w, \tag{2}$$

where

$$x_1 \in \mathbb{R}^r,\ x_2 \in \mathbb{R}^{n-r},\ x = R \begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\ LB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},\ CR = \begin{bmatrix} C_1, C_2 \end{bmatrix},\ D_1 = D - C_2 B_2.$$

Define the performance measure J for system (1) in the form

$$J = \sup_{(w,x_0)\in\mathcal{W}} \frac{\|z\|_Q}{\sqrt{\|w\|_P^2 + x_0^\top X_0 x_0}}, \quad \|z\|_Q^2 = \int_0^\infty z^\top Q z \, dt, \quad \|w\|_P^2 = \int_0^\infty w^\top P w \, dt, \tag{3}$$

where $\mathcal{W}$ is a set of pairs (w, x_0) such that $0 < \|w\|_P^2 + x_0^\top X_0 x_0 < \infty$ and system (1) has a solution, $P = P^\top > 0$, $Q = Q^\top > 0$ and $X_0 \ge 0$ are weight matrices. In the following, we consider $X_0 = E^\top H E$ with $H = H^\top > 0$.

The value J describes the weighted damping level of the external and initial disturbances in system (1). For example, if the weight matrices P and Q are diagonal, then their diagonal elements are the *priority coefficients* for the corresponding components of input w and output z in system (1) with respect to J. System (1) is *nonexpansive* if $J \le 1$. A pair (w, x_0) is the *worst* for system (1) with respect to J if in (3), a supremum is reached.

When $x_0 \in \operatorname{Ker} E$, we denote J as J_0. It is obvious that $J_0 \le J$. If $P = I_s$ and $Q = I_l$, then J_0 coincides with the H_∞-norm of the transfer matrix function $\mathcal{H}_1(\lambda) = C_1(\lambda I_n - A_1)^{-1}B_1 + D_1$ of the dynamical subsystem in (2) (see, e.g., [5,8]). In this case, we have a standard performance index J_0 used in the H_∞ control theory. Note that the performance measure (3) was introduced in [13] when $E = I_n$, $P = I_s$ and $Q = I_l$.

Lemma 2.2 (see [19]) *Given a scalar $\gamma > 0$, the descriptor system* (1) *is admissible and satisfies $J < \gamma$ if there exists a matrix X such that*

$$0 \le E^\top X = X^\top E \le \gamma^2 X_0, \ \ \operatorname{rank}(E^\top X - \gamma^2 X_0) = \rho, \tag{4}$$

$$\Psi(X) = \begin{bmatrix} A^\top X + X^\top A + C^\top QC & X^\top B + C^\top QD \\ B^\top X + D^\top QC & D^\top QD - \gamma^2 P \end{bmatrix} < 0. \tag{5}$$

The converse is true if

$$\operatorname{rank} \begin{bmatrix} E^\top & C^\top QD \end{bmatrix} = \rho. \tag{6}$$

Remark 2.1 Note that $E^\top X = X^\top E \ge 0$ if and only if the non-strict LMI

$$\begin{bmatrix} S_0 & S_0 - E^\top X \\ S_0 - X^\top E & 0 \end{bmatrix} \ge 0 \tag{7}$$

is feasible in the variables X and S_0, and moreover, $S_0 = E^\top X = X^\top E \ge 0$. It can be established that (5) is satisfied if and only if $D^\top QD < \gamma^2 P$ and $D_1^\top QD_1 < \gamma^2 P$, where $D_1 = D - CA^{-1}B$. Any matrix X satisfying (5) must be nonsingular.

Remark 2.2 If the LMIs (5) and (7) are feasible in the variables X and S_0, then system (1) is admissible with $J_0 < \gamma$. The converse is true under the additional condition (6). Moreover, if (5) and (7) hold, then system (1) with a structured uncertain input $w = \gamma^{-1}\Theta z$, where $\Theta^\top P \Theta \le Q$, is robust stable and $v(x) = x^\top S_0 x$ is a common Lyapunov function of the system (see [18]).

Note that the conditions of Lemma 2.2 can be used in calculating the values of J_0 and J as the solutions of optimization problems. In particular, we have

$$J = \inf \big\{ \gamma : \ \Psi(X) < 0, \ \ 0 \le E^\top X = X^\top E \le \gamma^2 X_0 \big\}.$$

Lemma 2.3 (see [20]) *Let system* (1) *be admissible and there exist matrices X and S_0 satisfying* (7) *and the Riccati-type equation*

$$A_1^\top X + X^\top A_1 + X^\top R_1 X + Q_1 = 0,$$

where $A_1 = A + BR^{-1}D^\top QC$, $R_1 = BR^{-1}B^\top$, $Q_1 = C^\top\big(Q + QDR^{-1}D^\top Q\big)C$, $R = \gamma^2 P - D^\top QD > 0$ and $\gamma = J$. Then the structured input vector

$$w = K_0 x, \quad K_0 = R^{-1}(B^\top X + D^\top QC),$$

and any initial vector $x_0 \in \operatorname{Ker}(S_0 - J^2 X_0)$ form the worst pair for system (1) *with respect to J.*

3 Main Results

Consider the following descriptor system with constant coefficient matrices:

$$\begin{aligned} E\dot{x} &= Ax + B_1 w + B_2 u, \quad x(0) = x_0, \\ z &= C_1 x + D_{11} w + D_{12} u, \\ y &= C_2 x + D_{21} w + D_{22} u, \end{aligned} \tag{8}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $w \in \mathbb{R}^s$ is the exogenous input, $z \in \mathbb{R}^k$ is the regulated output and $y \in \mathbb{R}^l$ is the measured output. The rank of the matrix E is $\rho \le n$. We are interested in static and dynamic control laws that guarantee a desirable estimate for performance measure (3) with respect to the regulated output z of a resulting closed-loop system. Controllers that minimize the value of J are called *J-optimal.* If $P = I_s$ and $Q = I_k$, then the J_0-optimal controller is *H_∞-optimal.*

In studying a class of systems (8), their properties such as C-, R- and I-controllability, as well as adjoint C-, R- and I-observability, are important (see, e.g., [5,6]). Known H_∞ control methods for such systems use the stabilizability and I-controllability properties of the triple (E, A, B_2). It means that there exists a matrix K for which the pair $(E, A + B_2K)$ is admissible. The criteria for I-controllability of the triple (E, A, B_2) and I-observability of the triple (E, A, C_2) are the corresponding rank conditions [5]

$$\operatorname{rank} \begin{bmatrix} E & 0 & 0 \\ A & E & B_2 \end{bmatrix} = n + \rho, \quad \operatorname{rank} \begin{bmatrix} E^\top & 0 & 0 \\ A^\top & E^\top & C_2^\top \end{bmatrix} = n + \rho.$$

3.1 Static output-feedback controller

When we apply the static output-feedback controller

$$u = Ky, \quad K \in \mathbb{R}^{m \times l}, \tag{9}$$

with the condition $\det(I_m - KD_{22}) \ne 0$, the closed-loop system is given by

$$E\dot{x} = A_* x + B_* w, \quad z = C_* x + D_* w, \quad x(0) = x_0, \tag{10}$$

where $A_* = A + B_2 K_* C_2$, $B_* = B_1 + B_2 K_* D_{21}$, $C_* = C_1 + D_{12} K_* C_2$, $D_* = D_{11} + D_{12} K_* D_{21}$, $K_* = (I_m - KD_{22})^{-1} K$. Let, for simplicity, $D_{22} = 0$, then $K_* = K$.

Applying Lemma 2.2 for system (10), we will use the special structure of a matrix X in (4), (5) and the skeletal decomposition $E = E_l E_r^\top$, where E_l and E_r are full column rank ρ matrices.

Lemma 3.1 *Given a scalar $\gamma > 0$ and matrices X and Y satisfying $XY = \gamma^2 I_n$, the following statements are equivalent:*

(i) *the conditions* (4) *of Lemma* 2.2 *hold;*

(ii) *there are matrices $S = S^\top$ and G such that*

$$X = SE + W_{E^\top} G, \quad 0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l; \tag{11}$$

(iii) *there are matrices $T = T^\top$ and F such that*

$$Y = TE^\top + W_E F, \quad E_r^\top T E_r > (E_l^\top H E_l)^{-1}. \tag{12}$$

Proof. (ii) $\Rightarrow$ (i) Considering (11), we have

$$E^\top X = E^\top SE \geq 0, \quad E^\top X - \gamma^2 X_0 = E_r\big(E_l^\top SE_l - \gamma^2 E_l^\top HE_l\big)E_r^\top.$$

Hence, (4) hold true.

(i) $\Rightarrow$ (ii) Let L and R be nonsingular matrices such that

$$E = L^{-1}\begin{bmatrix} I_\rho & 0 \\ 0 & 0 \end{bmatrix} R^{-1}, \quad E_l = L^{-1}\begin{bmatrix} I_\rho \\ 0 \end{bmatrix}, \quad E_r = R^{-1\top}\begin{bmatrix} I_\rho \\ 0 \end{bmatrix},$$

$$W_E = R\begin{bmatrix} 0 \\ I_{n-\rho} \end{bmatrix}, \quad W_{E^\top} = L^\top\begin{bmatrix} 0 \\ I_{n-\rho} \end{bmatrix}.$$

Then any matrix X in (4) can be expressed as

$$X = L^\top\begin{bmatrix} X_1 & 0 \\ X_2 & X_3 \end{bmatrix} R^{-1}, \quad 0 < X_1 = X_1^\top < \gamma^2 E_l^\top HE_l. \tag{13}$$

Assuming $S_1 = X_1$, $S_2 = X_2 - G_1$, $G_2 = X_3$, $S_3 = S_3^\top$, $G_1 \in \mathbb{R}^{(n-\rho)\times\rho}$ and

$$S = L^\top\begin{bmatrix} S_1 & S_2^\top \\ S_2 & S_3 \end{bmatrix} L, \quad G = \begin{bmatrix} G_1 & G_2 \end{bmatrix} R^{-1},$$

we get (11).

(iii) $\Rightarrow$ (i) Note that conditions (4) hold if and only if

$$0 \leq EY = Y^\top E^\top \leq Y^\top X_0 Y, \quad \operatorname{rank}\big(EY - Y^\top X_0 Y\big) = \rho, \tag{14}$$

where $Y = \gamma^2 X^{-1}$. Considering (12), we have

$$EY = ETE^\top \geq 0, \quad EY - Y^\top X_0 Y = E_l T_1\big(T_1^{-1} - E_l^\top HE_l\big)T_1 E_l^\top,$$

where $T_1 = E_r^\top TE_r$. Besides, $T_1^{-1} < E_l^\top HE_l$ if and only if $T_1 > (E_l^\top HE_l)^{-1}$. Hence, (14) and (4) hold true.

(i) $\Rightarrow$ (iii) Suppose that (4) hold true. Using (13), we have

$$Y = \gamma^2 X^{-1} = \gamma^2 R\begin{bmatrix} X_1^{-1} & 0 \\ -X_3^{-1}X_2X_1^{-1} & X_3^{-1} \end{bmatrix} L^{-1\top}.$$

Let $T_1 = \gamma^2 X_1^{-1}$, $T_2 = -F_1 - \gamma^2 X_3^{-1}X_2X_1^{-1}$, $F_2 = \gamma^2 X_3^{-1}$, $T_3 = T_3^\top$, $F_1 \in \mathbb{R}^{(n-\rho)\times\rho}$ and

$$T = R\begin{bmatrix} T_1 & T_2^\top \\ T_2 & T_3 \end{bmatrix} R^\top, \quad F = \begin{bmatrix} F_1 & F_2 \end{bmatrix} L^{-1\top}.$$

Since $T_1 = E_r^\top TE_r$, we obtain (12) using the equivalence of the matrix inequalities $X_1 < \gamma^2 E_l^\top HE_l$ and $\gamma^2 X_1^{-1} > (E_l^\top HE_l)^{-1}$. This completes the proof.

Theorem 3.1 *Let there exist matrices X and Y such that* (11) *and the following conditions hold:*

$$W_R^\top\begin{bmatrix} A^\top X + X^\top A + C_1^\top QC_1 & X^\top B_1 + C_1^\top QD_{11} \\ B_1^\top X + D_{11}^\top QC_1 & D_{11}^T QD_{11} - \gamma^2 P \end{bmatrix} W_R < 0, \tag{15}$$

$$W_L^\top \begin{bmatrix} AY + Y^\top A^\top + B_1 P^{-1} B_1^\top & Y^\top C_1^\top + B_1 P^{-1} D_{11}^\top \\ C_1 Y + D_{11} P^{-1} B_1^\top & D_{11} P^{-1} D_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0, \tag{16}$$

$$\operatorname{rank} \begin{bmatrix} X & \gamma I_n \\ \gamma I_n & Y \end{bmatrix} = n, \tag{17}$$

where $R = \begin{bmatrix} C_2 & D_{21} \end{bmatrix}$, $L = \begin{bmatrix} B_2^\top & D_{12}^\top \end{bmatrix}$. *Then there exists a static output-feedback controller* (9) *such that closed-loop system* (10) *is admissible and its performance measure* $J < \gamma$*. Conversely, if system* (10) *is admissible with* $J < \gamma$ *and*

$$\operatorname{rank} \begin{bmatrix} E^\top & C_*^\top Q D_* \end{bmatrix} = \rho \tag{18}$$

for some controller (9)*, then conditions* (11) *and* (15) *–* (17) *are feasible in* X *and* Y*.*

Proof. Taking into account the Schur complement, we rewrite matrix inequality (5) in Lemma 2.2 for a closed-loop system (10) as the LMI with respect to K_*:

$$\begin{bmatrix} A_*^T X + X^\top A_* & X^\top B_* & C_*^\top \\ B_*^\top X & -\gamma^2 P & D_*^\top \\ C_* & D_* & -Q^{-1} \end{bmatrix} = \widehat{L}^\top K_* \widehat{R} + \widehat{R}^\top K_*^\top \widehat{L} + \Omega < 0, \tag{19}$$

where $\widehat{R} = \begin{bmatrix} R & 0_{l\times k} \end{bmatrix}$, $\widehat{L} = \begin{bmatrix} L & 0_{m\times s} \end{bmatrix} \widetilde{X}$, and

$$\widetilde{X} = \begin{bmatrix} X & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_s & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} A^\top X + X^\top A & X^\top B_1 & C_1^\top \\ B_1^\top X & -\gamma^2 P & D_{11}^\top \\ C_1 & D_{11} & -Q^{-1} \end{bmatrix}.$$

There exists K_* satisfying (19) if and only if (see the condition (d) in Lemma A.1)

$$W_{\widehat{R}}^\top \Omega W_{\widehat{R}} < 0, \quad W_{\widehat{L}}^\top \Omega W_{\widehat{L}} < 0. \tag{20}$$

Since

$$W_{\widehat{R}} = \begin{bmatrix} W_R & 0 \\ 0 & I_k \end{bmatrix}, \quad W_{\widehat{L}} = \widetilde{X}^{-1} \begin{bmatrix} W_L & 0 \\ 0 & I_s \end{bmatrix},$$

the conditions (20) are reduced to (15) and (16) with $Y = \gamma^2 X^{-1}$, respectively. The last equality is equivalent to the rank condition (17).

Thus, if (11) and (15) – (17) hold for some matrices $S = S^\top$, $G \in \mathbb{R}^{(n-\rho)\times n}$ and Y, then taking into account the equivalence of statements (i) and (ii) in Lemma 3.1, we can construct a controller (9) provided the admissibility of system (10) with $J < \gamma$. The gain matrix K of the controller can be defined as any solution $K = K_*$ of LMI (19).

Conversely, if system (10) is admissible with $J < \gamma$ and (18) holds for some controller (9), then (11) and (15) – (17) are feasible in X and Y (see Lemma 2.2).

Note that the rank constraint (18) does not depend on K if one of the following conditions is satisfied:

$$D_{11} = 0, \quad D_{21} = 0; \tag{21}$$

$$D_{12} = 0, \quad \operatorname{rank} \begin{bmatrix} E^\top & C_1^\top Q D_{11} \end{bmatrix} = \rho. \tag{22}$$

It can also be established that (18) follows from

$$\operatorname{rank} \begin{bmatrix} E^\top & C_1^\top Q D_{11} & C_1^\top Q D_{12} & C_2^\top \end{bmatrix} = \rho.$$

Corollary 3.1 *Assume that*

$$C_2 = I_n, \quad D_{11}^\top Q D_{11} < \gamma^2 P, \quad D_{21} = 0, \quad D_{22} = 0. \tag{23}$$

Then there is a static state-feedback controller $u = Kx$ *such that a closed-loop system* (10) *is admissible and its performance measure* $J < \gamma$ *if two LMIs* (12) *and* (16) *with nonsingular* Y *are feasible in the variables* $T = T^\top$ *and* F. *The converse statement is true if* (23) *and either* (21) *or* (22) *hold.*

Proof. Considering (23), we have $y \equiv x$, $W_R = \left[0_{s\times n}, I_s\right]^T$. In this case, matrix inequality (15) holds in Theorem 3.1 and does not depend on X. Taking into account the equivalence of the statements (i) and (iii) in Lemma 3.1, a sufficient condition for the existence of a static state-feedback controller in Theorem 3.1 is the solvability of (12) and (16) with respect to $T = T^\top$ and F. The gain matrix K of the controller can be defined as a solution $K = K_*$ of the LMI (19) with $X = \gamma^2 Y^{-1}$.

Given (23), we also have $C_* = C_1 + D_{12}K$ and $D_* = D_{11}$. Therefore, in the converse statement of Theorem 3.1, the rank condition (18) is true and does not depend on K if either (21) or (22) holds.

Remark 3.1 Note that Y in (12) is nonsingular if such is $FW_{E^\top}$. In particular, we can search for F in the form $F = \widetilde{F}E^\top + CW_{E^\top}^\top$, where $\widetilde{F}$ is a new required matrix and C is nonsingular. Then Y in Corollary 3.1 is nonsingular (see the proof of Lemma 3.1).

Theorem 3.2 *Assume that*

$$R_0 = D_{12}^\top Q D_{12} > 0, \quad R_1 = \gamma^2 P - D_{11}^\top Q_1 D_{11} > 0, \quad Q_1 = Q - QD_{12}R_0^{-1}D_{12}^\top Q \tag{24}$$

and there exist matrices $S = S^\top$ *and* G *such that* (11), (15) *and the GARI*

$$A_2^\top X + X^\top A_2 + X^\top R_2 X + Q_2 < 0 \tag{25}$$

hold with $A_2 = A - B_2R_0^{-1}D_{12}^\top QC_1 + B_{11}R_1^{-1}D_{11}^\top Q_1C_1$, $R_2 = B_{11}R_1^{-1}B_{11}^\top - B_2R_0^{-1}B_2^\top$, $B_{11} = B_1 - B_2R_0^{-1}D_{12}^\top QD_{11}$, $Q_2 = C_1^\top(Q_1 + Q_1D_{11}R_1^{-1}D_{11}^\top Q_1)C_1$. *Then there exists a static output-feedback controller* (9) *such that closed-loop system* (10) *is admissible and its performance measure* $J < \gamma$.

Proof. To apply Lemmas 2.2 and 3.1, we rewrite the expression $\Psi(X) < 0$ for system (10) in the form of a quadratic matrix inequality with respect to K_*:

$$A_0 + B_0^\top K_* C_0 + C_0^\top K_*^\top B_0 + C_0^\top K_*^\top R_0 K_* C_0 < 0, \tag{26}$$

where

$$A_0 = \left[\begin{array}{cc} A^\top X + X^\top A + C_1^\top QC_1 & X^\top B_1 + C_1^\top QD_{11} \\ B_1^\top X + D_{11}^\top QC_1 & D_{11}^\top QD_{11} - \gamma^2 P \end{array}\right],$$

$$B_0 = \left[\begin{array}{cc} B_2^\top X + D_{12}^\top QC_1 & D_{12}^\top QD_{11} \end{array}\right], \quad C_0 = \left[\begin{array}{cc} C_2 & D_{21} \end{array}\right].$$

Since $R_0 > 0$, the solvability conditions for (26) are of the form $W_{C_0}^\top A_0 W_{C_0} < 0$ and $A_0 < B_0^\top R_0^{-1} B_0$ (see the conditions (a) and (b) in Lemma A.2). The first inequality coincides with (15), and the second inequality takes the form (25) via the Schur complement.

Note that on the basis of Lemmas 2.2 and 3.1, as well as the generalized uncertainty lemma for inequality (26) (see [17]), we can construct an ellipsoidal set of gain matrices $\mathcal{K} = \left\{K : (K - K_*)^\top P_0 (K - K_*) \le Q_0\right\}$, where $P_0 = P_0^\top > 0$ and $Q_0 = Q_0^\top > 0$, for which closed-loop system (10) is admissible and its performance measure $J < \gamma$.

3.2 Dynamic output-feedback controller

Consider system (8) with the dynamic output-feedback controller

$$\dot{\xi} = Z\xi + Vy, \quad u = U\xi + Ky, \quad \xi(0) = 0, \tag{27}$$

where $\xi \in \mathbb{R}^p$, Z, V, U and K denote constant matrices with appropriate dimensions to be determined. The combined system in an extended state space $\mathbb{R}^{n+p}$ is represented by

$$\begin{aligned} \widehat{E}\dot{\widehat{x}} &= \widehat{A}\widehat{x} + \widehat{B}_1 w + \widehat{B}_2 \widehat{u}, \quad \widehat{x}(0) = \widehat{x}_0, \\ z &= \widehat{C}_1 \widehat{x} + D_{11} w + \widehat{D}_{12} \widehat{u}, \\ \widehat{y} &= \widehat{C}_2 \widehat{x} + \widehat{D}_{21} w, \end{aligned} \tag{28}$$

using the static output-feedback controller

$$\widehat{u} = \widehat{K}_* \widehat{y}, \quad \widehat{K}_* \in \mathbb{R}^{(m+p)\times(l+p)}, \tag{29}$$

where

$$\widehat{x} = \begin{bmatrix} x \\ \xi \end{bmatrix}, \ \widehat{x}_0 = \begin{bmatrix} x_0 \\ 0 \end{bmatrix}, \ \widehat{y} = \begin{bmatrix} y - D_{22}u \\ \xi \end{bmatrix}, \ \widehat{u} = \begin{bmatrix} u \\ \dot{\xi} \end{bmatrix},$$

$$\widehat{E} = \begin{bmatrix} E & 0 \\ 0 & I_p \end{bmatrix}, \quad \widehat{A} = \begin{bmatrix} A & 0_{n\times p} \\ 0_{p\times n} & 0_{p\times p} \end{bmatrix}, \quad \widehat{B}_1 = \begin{bmatrix} B_1 \\ 0_{p\times s} \end{bmatrix}, \quad \widehat{B}_2 = \begin{bmatrix} B_2 & 0_{n\times p} \\ 0_{p\times m} & I_p \end{bmatrix},$$

$$\widehat{C}_1 = \begin{bmatrix} C_1 & 0_{k\times p} \end{bmatrix}, \ \widehat{D}_{12} = \begin{bmatrix} D_{12} & 0_{k\times p} \end{bmatrix}, \ \widehat{C}_2 = \begin{bmatrix} C_2 & 0_{l\times p} \\ 0_{p\times n} & I_p \end{bmatrix}, \ \widehat{D}_{21} = \begin{bmatrix} D_{21} \\ 0_{p\times s} \end{bmatrix},$$

$$\widehat{K}_* = \begin{bmatrix} K_* & U_* \\ V_* & Z_* \end{bmatrix} = (I_{m+p} - \widehat{K}\widehat{D}_{22})^{-1}\widehat{K}, \quad \widehat{D}_{22} = \begin{bmatrix} D_{22} & 0_{l\times p} \\ 0_{p\times m} & 0_{p\times p} \end{bmatrix},$$

$$\widehat{K} = \begin{bmatrix} K & U \\ V & Z \end{bmatrix} = (I_{m+p} + \widehat{K}_*\widehat{D}_{22})^{-1}\widehat{K}_*. \tag{30}$$

Here $\det(I_m - KD_{22}) \neq 0$. Let, for simplicity, $D_{22} = 0$, then $\widehat{K}_* = \widehat{K}$.

The closed-loop system has the form

$$\widehat{E}\dot{\widehat{x}} = \widehat{A}_*\widehat{x} + \widehat{B}_* w, \quad z = \widehat{C}_*\widehat{x} + \widehat{D}_* w, \quad \widehat{x}(0) = \widehat{x}_0, \tag{31}$$

where $\widehat{A}_* = \widehat{A} + \widehat{B}_2\widehat{K}_*\widehat{C}_2$, $\widehat{B}_* = \widehat{B}_1 + \widehat{B}_2\widehat{K}_*\widehat{D}_{21}$, $\widehat{C}_* = \widehat{C}_1 + \widehat{D}_{12}\widehat{K}_*\widehat{C}_2$, $\widehat{D}_* = D_{11} + \widehat{D}_{12}\widehat{K}_*\widehat{D}_{21}$. Since $\xi_0 = 0$, the performance measure $\widehat{J}$ of the form (3) for system (31) with the weight matrices P, Q, and

$$\widehat{X}_0 = \widehat{E}^\top \widehat{H}\widehat{E}, \quad \widehat{H} = \begin{bmatrix} H & H_1^\top \\ H_1 & H_2 \end{bmatrix} > 0,$$

does not depend on H_1 and H_2, and its value coincides with J.

Lemma 3.2 *Given a scalar $\gamma > 0$ and matrices*

$$X = SE + W_{E^\top}G, \quad Y = TE^\top + W_E F, \tag{32}$$

where $S = S^\top$, $T = T^\top$ and $G, F \in \mathbb{R}^{(n-\rho)\times n}$, the following statements are equivalent:

(i) *X and Y are nonsingular, $\Delta = \gamma^2 I_n - XY \neq 0$ and*

$$0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l, \quad \Gamma = \begin{bmatrix} E_l^\top S E_l & \gamma I_\rho \\ \gamma I_\rho & E_r^\top T E_r \end{bmatrix} \geq 0, \tag{33}$$

$$\operatorname{rank} \Gamma = \rho + \delta, \quad \delta = \operatorname{rank} \Delta; \tag{34}$$

(ii) *there are matrices $S_1 \in \mathbb{R}^{\delta \times n}$, $S_2 = S_2^\top \in \mathbb{R}^{\delta \times \delta}$, $G_1 \in \mathbb{R}^{(n-\rho)\times\delta}$, $T_1 \in \mathbb{R}^{\delta\times n}$, $T_2 = T_2^\top \in \mathbb{R}^{\delta\times\delta}$, $F_1 \in \mathbb{R}^{(n-\rho)\times\delta}$, $H_1 \in \mathbb{R}^{\delta\times n}$ and $H_2 = H_2^\top \in \mathbb{R}^{\delta\times\delta}$ such that*

$$0 < \widehat{E}_l^\top \widehat{S} \widehat{E}_l < \gamma^2 \widehat{E}_l^\top \widehat{H} \widehat{E}_l, \quad \widehat{X}\widehat{Y} = \gamma^2 I_{n+\delta}, \tag{35}$$

where

$$\widehat{X} = \begin{bmatrix} X & X_3 \\ X_1 & X_2 \end{bmatrix} = \widehat{S}\widehat{E} + W_{\widehat{E}^\top}\widehat{G}, \quad \widehat{S} = \begin{bmatrix} S & S_1^\top \\ S_1 & S_2 \end{bmatrix}, \quad \widehat{G} = \begin{bmatrix} G & G_1 \end{bmatrix}, \tag{36}$$

$$\widehat{Y} = \begin{bmatrix} Y & Y_3 \\ Y_1 & Y_2 \end{bmatrix} = \widehat{T}\widehat{E}^\top + W_{\widehat{E}}\widehat{F}, \quad \widehat{T} = \begin{bmatrix} T & T_1^\top \\ T_1 & T_2 \end{bmatrix}, \quad \widehat{F} = \begin{bmatrix} F & F_1 \end{bmatrix}, \tag{37}$$

$$\widehat{E} = \begin{bmatrix} E & 0 \\ 0 & I_\delta \end{bmatrix} = \widehat{E}_l \widehat{E}_r^\top, \quad W_{\widehat{E}} = \begin{bmatrix} W_E \\ 0 \end{bmatrix}, \quad W_{\widehat{E}^\top} = \begin{bmatrix} W_{E^\top} \\ 0 \end{bmatrix},$$

$$\widehat{E}_l = \begin{bmatrix} E_l & 0 \\ 0 & I_\delta \end{bmatrix}, \quad \widehat{E}_r = \begin{bmatrix} E_r & 0 \\ 0 & I_\delta \end{bmatrix}, \quad \widehat{H} = \begin{bmatrix} H & H_1^\top \\ H_1 & H_2 \end{bmatrix} > 0.$$

Proof. (ii) $\Rightarrow$ (i) We rewrite (35) as

$$0 < \begin{bmatrix} E_l^\top S E_l & E_l^\top S_1^\top \\ S_1 E_l & S_2 \end{bmatrix} < \gamma^2 \begin{bmatrix} E_l^\top H E_l & E_l^\top H_1^\top \\ H_1 E_l & H_2 \end{bmatrix}, \tag{38}$$

$$XY + X_3 Y_1 = \gamma^2 I_n, \quad XY_3 + X_3 Y_2 = 0, \quad X_1 Y + X_2 Y_1 = 0, \quad X_1 Y_3 + X_2 Y_2 = \gamma^2 I_\delta. \tag{39}$$

Obviously, (38) implies $0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l$ and $X_2 = S_2 > 0$. In addition, $Y_2 = T_2 > 0$ because $\widehat{E}_r^\top \widehat{T} \widehat{E}_r > (\widehat{E}_l^\top \widehat{H} \widehat{E}_l)^{-1}$ (see Lemma 3.1).

From (39) it follows that $X(Y - Y_3 Y_2^{-1} Y_1) = (X - X_3 X_2^{-1} X_1)\, Y = \gamma^2 I_n$. Therefore, X and Y must be nonsingular. Besides, $E^\top X = X^\top E \geq 0$, $EY = Y^\top E^\top \geq 0$, $X_1 = X_3 E$ and $Y_1 = Y_3^\top E^\top$. Next, we use the following transformation of Γ:

$$\Phi^\top \Gamma \Phi = \begin{bmatrix} E^\top S E & 0 \\ 0 & \Xi \end{bmatrix} \geq 0, \quad \Phi = \begin{bmatrix} E_r^\top & -\gamma E_r^\top X^{-1} \\ 0 & E_l^\top \end{bmatrix}, \tag{40}$$

where $E^\top S E \geq 0$ and $\Xi = E(Y - \gamma^2 X^{-1}) = Y_1^\top Y_2^{-1} Y_1 \geq 0$. Since Φ is the full row rank matrix, it yields $\Gamma \geq 0$. Moreover, the rank conditions (34) hold because $\operatorname{rank}(E^\top S E) = \rho$, $\Xi = -EX^{-1}\Delta$, $\Delta = X_3 Y_1$, $\operatorname{rank}\Delta \leq \operatorname{rank} Y_1 = \operatorname{rank}\Xi \leq \operatorname{rank}\Delta$ and, hence, $\operatorname{rank}\Xi = \delta$ and $\operatorname{rank}\Gamma = \rho + \delta$.

(i) $\Rightarrow$ (ii) Assume that (33) and (34) hold with nonsingular X and Y. Given (33) and (40), we have the decomposition $\Xi = E(Y - \gamma^2 X^{-1}) = \Lambda^\top \Lambda \geq 0$, where $\Lambda \in \mathbb{R}^{\delta\times n}$ is a certain full row rank matrix. Then there exists $\Upsilon \in \mathbb{R}^{n\times\delta}$ such that $\Upsilon\Lambda = \Delta$. Indeed,

$$\begin{aligned} \operatorname{rank}\Lambda &\leq \operatorname{rank}\begin{bmatrix} \Lambda^\top & \Delta^\top \end{bmatrix} = \operatorname{rank}(\Lambda^\top\Lambda + \Delta^\top\Delta) = \operatorname{rank}\left[(\Delta^\top - EX^{-1})\Delta\right] \\ &\leq \operatorname{rank}\Delta = \operatorname{rank}\Xi = \operatorname{rank}\Lambda \end{aligned}$$

and hence rank $\begin{bmatrix} \Lambda^\top & \Delta^\top \end{bmatrix} = \operatorname{rank}\Lambda$. Moreover, $\Xi = -EX^{-1}\Upsilon\Lambda = \Lambda^\top\Lambda$ implies $\Lambda^\top = -EX^{-1}\Upsilon$.

Setting in (36) and (37) $S_1 = \Upsilon^\top - G_1^\top W_{E^\top}^\top$, $S_2 = \gamma^2 I_\delta - \Lambda\Upsilon$, $T_1 = -\Upsilon^\top X^{-1\top} - F_1^\top W_E^\top$ and $T_2 = I_\delta$, where $F_1 \in \mathbb{R}^{(n-\rho)\times\delta}$ and $G_1 \in \mathbb{R}^{(n-\rho)\times\delta}$ are arbitrary matrices, we have $X_1 = \Upsilon^\top E$, $X_2 = \gamma^2 I_\delta - \Lambda\Upsilon$, $X_3 = \Upsilon$, $Y_1 = \Lambda$, $Y_2 = I_\delta$ and $Y_3 = -X^{-1}\Upsilon$.

Considering $\Lambda = -\Upsilon^\top X^{-1\top} E^\top$ and $EX^{-1} = X^{-1\top}E^\top$, it is easy to verify (39). The first matrix inequality in (38) follows from the Schur complement. Indeed, $E_l^\top S E_l > 0$ and

$$\begin{aligned}
&S_2 - S_1 E_l (E_l^\top S E_l)^{-1} E_l^\top S_1^\top \\
&= \gamma^2 I_\delta - \Lambda\Upsilon - \Upsilon^\top E_l (E_l^\top S E_l)^{-1} E_l^\top \Upsilon \\
&= \gamma^2 I_\delta + \Upsilon^\top X^{-1\top}\big[E^\top - X^\top E_l S E_l (E_l^\top S E_l)^{-1} E_l^\top\big]\Upsilon \\
&= \gamma^2 I_\delta + \Upsilon^\top X^{-1\top} E_r \big[I_\rho - E_l^\top S E_l (E_l^\top S E_l)^{-1}\big] E_l^\top \Upsilon = \gamma^2 I_\delta > 0.
\end{aligned}$$

Here, it is also taken into account that $W_{E^\top} = W_{E_l^\top}$ and $X = SE + W_{E^\top}G$.

The second matrix inequality in (38) holds, if, for instance, $H_1 = \gamma^{-2} S_1$ and $H_2 > \gamma^{-2} S_2$. This completes the proof.

Theorem 3.3 *Let the LMIs* (15) *and* (16) *with* (32) *as well as* (33) *and the rank conditions* (34) *are feasible in the variables* $S = S^\top$, $T = T^\top$, G *and* F. *Then there exists a dynamic controller* (27) *of the order* $p = \delta$ *such that closed-loop system* (31) *is admissible and its performance measure* $J < \gamma$. *Conversely, if system* (31) *is admissible with* $J < \gamma$ *and satisfies* (18) *for some controller* (27)*, then* (15), (16) *and* (32)–(34) *are feasible.*

Proof. According to Theorem 3.1, we can find a static controller (29) for extending system (28) such that closed-loop system (31) is admissible and its performance measure $J < \gamma$ if there exist matrices $\widehat{S} = \widehat{S}^\top$, $\widehat{T} = \widehat{T}^\top$, $\widehat{G}$ and $\widehat{F}$ satisfying (35) – (37) and

$$W_{\widehat{R}}^\top \begin{bmatrix} \widehat{A}^\top \widehat{X} + \widehat{X}^\top \widehat{A} + \widehat{C}_1^\top Q \widehat{C}_1 & \widehat{X}^\top \widehat{B}_1 + \widehat{C}_1^\top Q D_{11} \\ \widehat{B}_1^\top \widehat{X} + D_{11}^\top Q \widehat{C}_1 & D_{11}^\top Q D_{11} - \gamma^2 P \end{bmatrix} W_{\widehat{R}} < 0, \tag{41}$$

$$W_{\widehat{L}}^\top \begin{bmatrix} \widehat{A}\widehat{Y} + \widehat{Y}^\top \widehat{A}^\top + \widehat{B}_1 P^{-1} \widehat{B}_1^\top & \widehat{Y}^\top \widehat{C}_1^\top + \widehat{B}_1 P^{-1} D_{11}^\top \\ \widehat{C}_1 \widehat{Y} + D_{11} P^{-1} \widehat{B}_1^\top & D_{11} P^{-1} D_{11}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_{\widehat{L}} < 0, \tag{42}$$

where $\widehat{R} = \begin{bmatrix} \widehat{C}_2 & \widehat{D}_{21} \end{bmatrix}$, $\widehat{L} = \begin{bmatrix} \widehat{B}_2^\top & \widehat{D}_{12}^\top \end{bmatrix}$. Moreover, all diagonal blocks of the matrices $\widehat{X}$ and $\widehat{Y}$ are nonsingular. Using the block structure of coefficient matrices of the system and the following matrix representations:

$$W_{\widehat{R}} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_s \end{bmatrix} W_R, \quad W_{\widehat{L}} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \\ 0 & I_k \end{bmatrix} W_L,$$

one can establish the equivalence of matrix inequalities (15) and (41), as well as (16) and (42). Considering the equivalence of statements (i) and (ii) in Lemma 3.2, there exists a dynamic controller (27) such that closed-loop system (31) is admissible and its performance measure $J < \gamma$ if (15), (16) and (32) – (34) hold for some $S = S^\top$, $T = T^\top$, G and F. An additional constraint in the converse statement of Lemma 2.2 for system (31) has the form rank $\begin{bmatrix} \widehat{E}^\top & \widehat{C}_*^\top Q \widehat{D}_* \end{bmatrix} = \rho + p$. This equality is equivalent to (18).

Remark 3.2 Note that the required matrices of dynamic controller (27) in Theorem 3.3 can be determined according to (30), where $\widehat{K}_*$ is the gain matrix of a static controller (29) found for extending system (28) via the LMI technique (see the proof of Theorem 3.1). In the case $\delta = 0$, Theorem 3.3 yields sufficient and necessary conditions for the existence of a static controller in Theorem 3.1.

Corollary 3.2 *Let the LMIs* (15) *and* (16) *with*

$$X = \big(S + W_{E^\top}\widetilde{G}\big)E + \gamma W_{E^\top} C W_E^+, \quad Y = \big(T + W_E\widetilde{F}\big)E^\top + \gamma W_E C^{-1} W_{E^\top}^+, \tag{43}$$

where C denotes any nonsingular matrix, and

$$0 < E_l^\top S E_l < \gamma^2 E_l^\top H E_l, \quad \Gamma = \begin{bmatrix} E_l^\top S E_l & \gamma I_\rho \\ \gamma I_\rho & E_r^\top T E_r \end{bmatrix} > 0 \tag{44}$$

hold for some $S = S^\top$, $T = T^\top$, $\widetilde{G}$ and $\widetilde{F}$. Then there exists a dynamic controller (27) *of the order $p = \rho$ such that closed-loop system* (31) *is admissible with $J < \gamma$.*

Proof. Denote $\mathcal{L} = \begin{bmatrix} E & W_{E^\top} \end{bmatrix}$, $\mathcal{R} = \begin{bmatrix} E^\top & W_E \end{bmatrix}$. Since $\mathcal{L}$ and $\mathcal{R}$ are full row rank matrices, $\operatorname{rank}(\mathcal{L}^\top X \mathcal{R}) = n$ and $\operatorname{rank}(\mathcal{R}^\top Y \mathcal{L}) = n$, where

$$\mathcal{L}^\top X \mathcal{R} = \begin{bmatrix} E^\top S E E^\top & 0 \\ W_{E^\top}^\top X E^\top & \gamma W_{E^\top}^\top W_{E^\top} C \end{bmatrix}, \quad \mathcal{R}^\top Y \mathcal{L} = \begin{bmatrix} E T E^\top E & 0 \\ W_E^\top Y E & \gamma W_E^\top W_E C^{-1} \end{bmatrix},$$

$$E^\top S E E^\top = E_r E_l^\top S E_l E_r^\top E_r E_l^\top, \quad E T E^\top E = E_l E_r^\top T E_r E_l^\top E_l E_r^\top,$$

$$E_l^\top S E_l > 0,\ E_r^\top T E_r > 0,\ W_{E^\top}^\top W_{E^\top} > 0,\ W_E^\top W_E > 0,$$

X and Y in (43) are nonsingular.

Next, we use the following transformation of matrix $\Delta = \gamma^2 I_n - XY$:

$$\mathcal{L}^\top \Delta \mathcal{L} = \begin{bmatrix} E_r & 0 \\ 0 & I_{n-\rho} \end{bmatrix} \begin{bmatrix} D E_l^\top E_l & 0 \\ -W_{E^\top}^\top X Y E_l & 0 \end{bmatrix} \begin{bmatrix} E_r^\top & 0 \\ 0 & I_{n-\rho} \end{bmatrix},$$

where $D = \gamma^2 I_\rho - E_l^\top S E_l E_r^\top T E_r$. Then, due to (44), $\det D \neq 0$ and $\operatorname{rank}\Delta = \rho$. Hence, the rank conditions in (34) hold with $\operatorname{rank}\Gamma = 2\rho$ and $\delta = \rho$. The statement of Corollary 3.2 follows from Theorem 3.3.

Note that Theorems 3.1 and 3.3 can be extended to a class of descriptor systems (8) with the following polyhedral uncertainties:

$$A \in \operatorname{Co}\big\{A_1, \ldots, A_\alpha\big\}, \quad B_1 \in \operatorname{Co}\big\{B_{11}, \ldots, B_{1\beta}\big\},$$

$$C_1 \in \operatorname{Co}\big\{C_{11}, \ldots, C_{1\mu}\big\}, \quad D_{11} \in \operatorname{Co}\big\{D_{111}, \ldots, D_{11\nu}\big\},$$

where $\operatorname{Co}\{A_1, \ldots, A_\alpha\} = \Big\{ \sum\limits_{i=1}^{\alpha} a_i A_i :\ a_i \geq 0,\ i = \overline{1,\alpha},\ \sum\limits_{i=1}^{\alpha} a_i = 1 \Big\}$. For this, instead of (15) and (16), we can use the corresponding LMIs systems

$$W_R^\top \begin{bmatrix} A_i^\top X + X^\top A_i + C_{1p}^\top Q C_{1p} & X^\top B_{1j} + C_{1p}^\top Q D_{11q} \\ B_{1j}^\top X + D_{11q}^\top Q C_{1p} & D_{11q}^\top Q D_{11q} - \gamma^2 P \end{bmatrix} W_R < 0,$$

$$W_L^\top \begin{bmatrix} A_i Y + Y^\top A_i^\top + B_{1j} P^{-1} B_{1j}^\top & Y^\top C_{1p}^\top + B_{1j} P^{-1} D_{11q}^\top \\ C_{1p} Y + D_{11q} P^{-1} B_{1j}^\top & D_{11q} P^{-1} D_{11q}^\top - \gamma^2 Q^{-1} \end{bmatrix} W_L < 0$$

for $i = \overline{1, \alpha}$, $j = \overline{1, \beta}$, $p = \overline{1, \mu}$, $q = \overline{1, \nu}$.

Considering formulas (28) and (29) with constraints (24) and the conditions (a) and (d) in Lemma A.2, we can formulate an analog of Theorem 3.2 that yields conditions for the existence of a dynamic controller (27) in terms of LMIs and GARIs such that closed-loop system (31) is admissible and its performance measure $J < \gamma$.

4 Numerical Example: Controlled Electrical Circuit

Consider the electrical circuit given in Fig. 1. The dynamics of this system is described

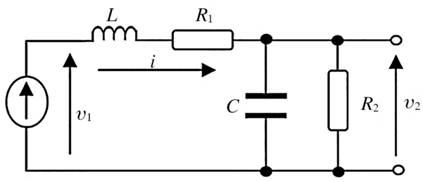

Figure 1: The electrical circuit.

by descriptor form (8) with the following data [23]:

$$E = \begin{bmatrix} L & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -R_1 & -1 & 1 \\ 0 & -1/R_2 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad B_1 = B_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha \end{bmatrix}, \; C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \; D_{11} = D_{21} = D_{22} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

where $x = \begin{bmatrix} i & v_2 & v_1 \end{bmatrix}^\top$, $i(t)$ denotes the current flow, v_1 and v_2 are the voltages, $i(t) = u(t) + w(t)$, $u(t)$ is the control input, $w(t)$ is the bounded disturbance, L and C are the inductance and the capacitance, respectively, R_1 and R_2 are the resistances, α is a constant parameter. In this example, $n = 3$, $m = s = 1$ and $k = l = 2$. Regulated and measured outputs of the system have the form $z = \begin{bmatrix} v_2 & \alpha v_1 + u \end{bmatrix}^\top$ and $y = \begin{bmatrix} v_2 & v_1 \end{bmatrix}^\top$, respectively.

Let $L = 3$, $C = 2$, $R_1 = 2$, $R_2 = 1$ and $\alpha = 1$. Then the pair (E, A) is impulsive, the triple (E, A, B_2) is I-controllable and the triple (E, A, C_2) is I-observable.

Assume that the standard performance index J_0 and performance measure J of the form (3) are defined by the weight matrices $P = 1$, $Q = I_2$ and $X_0 = E^\top E$. Let $\gamma = 0.5$, then conditions (24) of Theorem 3.2 are satisfied. Using the Mathcad Prime 6.0 system, we found the matrices

$$S = \begin{bmatrix} 0.06402 & 0.03218 & -0.02942 \\ 0.03218 & 0.23409 & 0.00678 \\ -0.02942 & 0.00678 & 0.16182 \end{bmatrix}, \quad X = \begin{bmatrix} 0.19206 & 0.06436 & 0 \\ 0.09654 & 0.46818 & 0 \\ 0.07355 & 0.46119 & -0.33001 \end{bmatrix},$$

$$G = \begin{bmatrix} 0.16181 & 0.44763 & -0.33001 \end{bmatrix},$$

satisfying (11), (15) and (25). Further, we determine the gain matrix $K = \begin{bmatrix} 0.01512 & -1.81255 \end{bmatrix}$ of controller (9) for which $J_0 = 0.44823$, $J = 0.48232 < \gamma$ and a closed-loop system (10) with the finite spectrum $\{-0.67528 \pm 0.36681i\}$ is admissible. Also, by using Lemma 2.3, the structured worst disturbance

$$w = K_0 x, \quad K_0 = \left[\begin{array}{ccc} -0.72013 & 0.28805 & 1.14308 \end{array}\right], \tag{45}$$

and the worst initial vector $x_0 = \left[\begin{array}{ccc} 0.08668 & 0.88792 & 0.17938 \end{array}\right]^\top$ are determined.

The finite spectrum of system (10) with the worst pair (w, x_0)

$$E\dot{x} = (A_* + B_* K_0)x, \quad x(0) = x_0, \tag{46}$$

is computed as $\{-0.59864, -1.42448\}$. Figs. 2 and 3 show the behavior of system (46) and function $w(t)$ in (45), respectively.

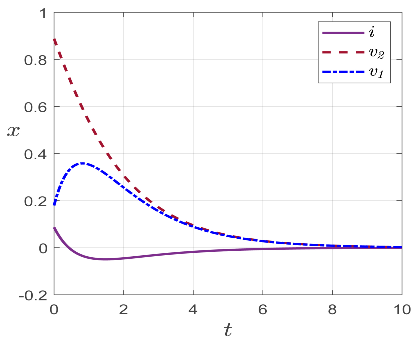

Figure 2: Closed-loop system behavior with the worst pair (w, x_0).

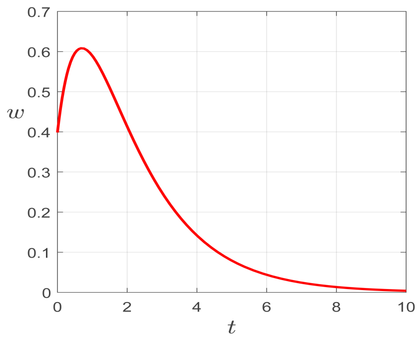

Figure 3: The worst disturbance.

Computational experiments have shown that the decrease of parameter α in the interval $[0, 1]$ leads to the increase of the minimum possible characteristics J_0 and J for a closed-loop system using static controllers of the form (9).

Also, on the basis of Theorem 3.3 (see Remark 3.2), a dynamic controller (27) with

$$\left[\begin{array}{cc} K & U \\ V & Z \end{array}\right] = \left[\begin{array}{cc|cc} -0.08957 & -0.96093 & 0.57234 & 2.61512 \\ \hline 0.03600 & -0.07555 & -0.38338 & -0.00721 \\ -0.00232 & -0.00068 & 0.08507 & -0.41471 \end{array}\right]$$

is determined, for which system (31) is admissible with $J_0 = 0.25070$ and $J = 0.47138$.

5 Conclusion

This paper presents new approaches to the generalized problem of H_∞ control with transients for continuous-time descriptor systems. The weighted performance measure used takes into account the influence of both exogenous disturbances and initial states. Necessary and sufficient conditions for the solvability of this problem via static and dynamic controllers have been proposed in terms of LMIs and GARIs with special matrix variables.

New auxiliary Lemmas 3.1 and 3.2 are obtained here, and used in the synthesis of static and dynamic controllers, respectively. These lemmas make it possible to search for solutions of the arising linear and quadratic matrix inequalities in parametric form (11) and (12) with considering the skeletal decomposition of system matrix E. This makes the main results (Theorems 3.1, 3.2 and 3.3) more constructive in comparison with [20, 21]. Moreover, with this approach, it is possible to formulate necessary and sufficient conditions for the existence of generalized H_∞ controllers exclusively in terms of LMIs (see Corollaries 3.1 and 3.2). Projection Lemma (Lemma A.1) and its new generalization (Lemma A.2) gives criteria for the solvability of linear and quadratic matrix inequalities, respectively. The presented synthesis approaches have been illustrated by a numerical example of the controlled electrical circuit.

A Solvability of Some Matrix Inequalities

Lemma A.1 (see [8]) *Given matrices* $A = A^\top \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{p\times n}$ *and* $C \in \mathbb{R}^{q\times n}$, *the LMI*

$$A + B^\top XC + C^\top X^\top B < 0 \tag{47}$$

is solvable for $X \in \mathbb{R}^{p\times q}$ *if and only if one of the following conditions holds:*
(a) $\operatorname{rank} B = n$, $\operatorname{rank} C = n$; (b) $\operatorname{rank} B < n$, $\operatorname{rank} C = n$, $W_B^\top A W_B < 0$;
(c) $\operatorname{rank} B = n$, $\operatorname{rank} C < n$, $W_C^\top A W_C < 0$;
(d) $\operatorname{rank} B < n$, $\operatorname{rank} C < n$, $W_B^\top A W_B < 0$, $W_C^\top A W_C < 0$.

Consider the following quadratic matrix inequality:

$$A + B^\top XC + C^\top X^\top B + C^\top X^\top RXC < 0, \tag{48}$$

where $A = A^\top \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{p\times n}$, $C \in \mathbb{R}^{q\times n}$ and $R \in \mathbb{R}^{p\times p}$. Suppose that matrices C and R are nonzero and $R = R^\top \geq 0$.

Lemma A.2 (see [21]) *There exist a matrix* $X \in \mathbb{R}^{p\times q}$ *satisfying* (48) *if and only if*
(a) *either* $\operatorname{rank} C = n$ *or* $\operatorname{rank} C < n$ *and* $W_C^\top A W_C < 0$;
and one of the following conditions hold:
(b) $R > 0$, $A < B^\top R^{-1} B$; (c) $\operatorname{rank} R < p$, $\operatorname{rank} B_0 = n$;
(d) $\operatorname{rank} R < p$, $\operatorname{rank} B_0 < n$, $W_{B_0}^\top \big(A - B^\top R^+ B\big) W_{B_0} < 0$;
where $B_0 = W_R^\top B$ *and* R^+ *is a pseudo-inverse of* R.

References

[1] D. V. Balandin and M. M. Kogan. LMI-based H_∞-optimal control with transients. *Internat. J. Control* **83** (2010) 1664–1673.

[2] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishman. *Linear Matrix Inequalities in System and Control Theory.* SIAM Studies in Applied Mathematics, Vol. 15, SIAM, Philadelphia, USA, 1994.

[3] M. Chadli, P. Shi, Z. Feng, and J. Lam. New bounded real lemma formulation and H_∞ control for continuous-time descriptor systems. *Asian J. Control* **20** (2018) 1–7.

[4] J. C. Doyle, K. Glover, P. Khargonekar, and B. Francis. State-space solutions to standard H_2 and H_∞ control problems. *IEEE Trans. Automat. Control* **34** (8) (1989) 831–847.

[5] G.-R. Duan. *Analysis and Design of Descriptor Linear Systems.* New York, Springer, 2010.

[6] Y. Feng and M. Yagoubi. *Robust Control of Linear Descriptor Systems.* Singapore: Springer Nature Singapore Pte Ltd, 2017.

[7] Z. Feng, J. Lam, S. Xu, and S. Zhou. H_∞ control with transients for singular systems. *Asian J. Control* **18** (2016) 817–827.

[8] P. Gahinet and P. Apkarian. A linear matrix inequality approach to H_∞ control. *Internat. J. Robust Nonlinear Control* **4** (4) (1994) 421–448.

[9] F. R. Gantmakher. *Theory of Matrices.* Moscow: Nauka, 1988. [Russion]

[10] F. Gao, W. Q. Liu, V. Sreeram, and K. L. Teo. Bounded real lemma for descriptor systems and its application. In: *IFAC 14th Triennial World Congress*, Beijing, P. R., China, 1999, 1631–1636.

[11] A. Ilchmann and T. Reis, eds. *Surveys in Differential-Algebraic Equations IV / Differential-Algebraic Equations Forum.* Cham, Switzerland: Springer, 2017.

[12] T. Iwasaki and R. E. Skelton. All controllers for the general H_∞ control problem: LMI existence conditions and state space formulas. *Automatica* **30** (8) (1994) 1307–1317.

[13] P. Khargonekar, K. Nagpal and K. Poolla. H_∞ control with transients. *SIAM J. Control Optim.* **29** (1991) 1373–1393.

[14] P. Losse, V. Mehrmann, L. Poppe and T. Reis. The modified optimal H_∞ control problem for descriptor systems. *SIAM J. Control Optim.* **47** (2008) 2795–2811.

[15] I. Masubushi. Output feedback controller synthesis for descriptor systems satisfying closed-loop dissipativity. *Automatica* **43** (2007) 339–345.

[16] I. Masubushi, Y. Kamitane, A. Ohara and N. Suda. H_∞ control for descriptor systems: A matrix inequalities approach. *Automatica* **33** (1997) 669–673.

[17] A. G. Mazko. Robust stability and evaluation of the quality functional for nonlinear control systems. *Autom. Remote Control* **76** (2) (2015) 251–263.

[18] A. G. Mazko. *Robust Stability and Stabilization of Dynamical Systems. Methods of Matrix and Cone Inequalities.* Kyiv: Institute of Mathematics of NAS of Ukraine, 2016. [Russion]

[19] A. G. Mazko. Evaluation of the weighted level of damping of bounded disturbances in descriptor systems. *Ukrainian Math. J.* **70** (11) (2019) 1777–1790.

[20] A. G. Mazko and T. O. Kotov. Robust stabilization and weighted damping of bounded disturbances in descriptor control systems. *Ukrainian Math. J.* **71** (10) (2020) 1572–1589.

[21] A. G. Mazko. Weighted estimation and reduction of the influence of bounded perturbations in descriptor control systems. *Ukrainian Math. J.* **72** (11) (2021) 1742–1757.

[22] A. Rehm and F. Allgöwer. General quadratic performance analysis and synthesis of differential algebraic equation (DAE) systems. *J. Process Control* **12** (2002) 467–474.

[23] K. Takaba. Robust H^2 control of descriptor system with time-varying uncertainty. *Internat. J. Robust Nonlinear Control* **71** (1998) 559–579.

[24] J.-L. Wu and C.-F. Yung. A new generalized bounded real lemma for continuous-time descriptor systems. *Asian J. Control* (2021) 1–9. https://doi.org/10.1002/asjc.2667.

[25] S. Xu, J. Lam and Y. Zou. New versions of bounded real lemmas for continuous and discrete uncertain systems. *Circuits, Syst. Signal Process.* **26** (2007) 829–838.

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PI-Fuzzy Control Applied to the Hybrid PV / Wind Pumping System with Energy Storage

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Abstract: The aim of this paper is to develop an autonomous hybrid energy management algorithm method with storage (lead-acid type battery bank), applied to the hybrid pumping system capable of ensuring the level regulation of water in the tank. The designed system is composed of a photovoltaic generator, connected to a booster chopper, a DC bus and a wind power system driving a permanent magnet synchronous generator. Two control strategies were implemented. The first, based on P and O, ensures the operation of the GPV with its maximum power via the control of a booster chopper. The second was applied to the turbine to extract the maximum power from the wind (MPPT). Two PI and fuzzy controllers to drive the operation of the asynchronous motor driving the centrifugal pump controlled the mechanical speed and the magnetic flux. In addition, we have established an explicit relation making it possible to adopt the set point rotation speed of the asynchronous machine and consequently, the pump output according to various climatic conditions. The results obtained show the validity, efficiency and robustness of the various techniques developed.

Keywords: *centrifugal pump; hybrid energy; fuzzy logic control; PMSG.*

Mathematics Subject Classification (2010): 03B52, 93C42, 94D05.

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1 Introduction

The most common renewable energy sources for agricultural applications are solar energy and wind energy; photovoltaic pumping usually consists of solar panels, a control unit and a pump set. Depending on the sizing of the system, it is sometimes necessary to use storage batteries and a charge regulator (due to the intermittent nature of solar radiation). This makes it possible to avoid problems of adaptation between the photovoltaic generator and the pump set. However, it is better to store water in a tank rather than storing energy in batteries. The use of wind turbines to pump water is not new. Wind power is one of the most promising sources for water pumping applications. Recent and relevant research on wind pumping applications shows that:

- Wind power based water pumping systems are best suited for irrigation applications [1];
- It is an economically viable alternative technology for irrigation systems [2];
- The wind power system can be used successfully for groundwater pumping in remote areas, where wind resources are available [3].

Moreover, a comparative study between photovoltaic and wind power systems for water pumping systems in the Sahara regions was carried out in Algeria [4]. It emerges from this study that the cost per cubic meter of water produced by the wind pumping system is less expensive than that produced by the photovoltaic system. Over the past decade, very few studies have been done on hybrid renewable energy systems for pumping water, unlike systems on their own [4]. Hybrid systems designed to work in agricultural applications are more flexible and reliable, it is also concluded that hybrid pumping systems are suitable for small water pumping systems and are of considerable interest for their flexibility . The work presented consists of the modeling, the regulation using PI and fuzzy regulators and the optimization of a hybrid wind-photovoltaic energy system not connected to the electricity grid. The work aims mainly to bring to the energy management of hybrid autonomous systems with renewable energies with storage for agricultural applications in tropical environments, then we will present the results obtained comparatively by the fuzzy regulator and PI.

2 Modeling of the Hybrid Pumping System

Figure 1 shows the complete diagram of hybrid power generation. The system consists of a wind power system assisted by PMSG, a PV system assisted by MPPT technique, a water pump assisted by ASM and an energy management system.

2.1 Wind turbine model

The power of the wind is defined by [5]

$$P = 0.5\,\rho\,A\,V^3, \tag{1}$$

where ρ is the density of the area, A is the circular area (m^2), V is the wind speed in (m/s). The aerodynamic power produced by the turbine can be expressed by

$$\begin{aligned} P_m &= 0.5\,C_p\,(\lambda,\beta)\;\rho\,A\,V^3, \\ \lambda &= \frac{R.\,\Omega_{tur}}{V}, \end{aligned} \tag{2}$$

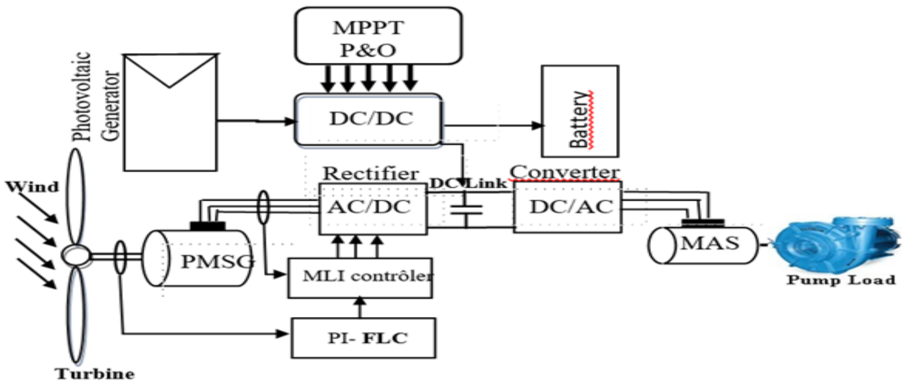


Figure 1: The block diagram of the hybrid water pumping system.

where C_p is the power coefficient, λ is the speed ratio, β is the blade pitch angle, Ω_{tur} is the turbine rotation speed (rd/s). The power coefficient can be expressed by

$$\begin{aligned} C_p(\lambda,\beta) \; &= \; 0.5176\left(\frac{116}{\lambda_i}-0.4\beta-5\right)e^{\frac{1}{\lambda_i}}+0.0068\,\lambda, \\ \frac{1}{\lambda_i} &= \frac{1}{\lambda+0.08\beta}-\frac{0.035}{\beta^3+1}. \end{aligned} \tag{3}$$

The aerodynamic torque is defined by

$$T_{aero}=\frac{P_{aero}}{\Omega_{tur}}=\frac{0.5\,C_p(\lambda,\beta)\;\rho\,A\,V^3}{\Omega_{tur}}. \tag{4}$$

The fundamental dynamic equation of the tree is described by the following equation:

$$\begin{aligned} J\frac{d\Omega}{dt} &= T_{aero}-T_{em}-f\Omega, \\ J &= \frac{J_{tur}}{G^2}+J_g, \end{aligned} \tag{5}$$

where T_{em} is the electromagnetic torque produced by the generator, T_{aero} is the turbine torque, f is the friction of the turbine rotor. J is the total inertia.

2.2 Model of the permanent magnet synchronous machine PMSG

The stator voltage equations dq of this generator are given by the following expressions [6]:

$$\begin{aligned} L_d\frac{di_d}{dt} &= V_d-R_s\,i_d+L_q\,\omega\,i_q, \\ L_q\frac{di_q}{dt} &= V_q-R_s\,i_d-L_d\,\omega\,i_d+\varphi_f, \\ \varphi_d &= L_d\,i_d+\varphi_f \qquad \varphi_q=L_q\,i_q. \end{aligned} \tag{6}$$

The electromagnetic torque is represented by [7, 8]

$$T_{em}=\frac{3}{2}\,p\,\left[(L_d-L_q)\;i_d i_q+i_q\,\varphi_f\right], \tag{7}$$

where p is the number of pole pairs. The PMSG is assumed to be a wound rotor, and the expression of the electromagnetic torque in the rotor can be described as follows:

$$T_{em} = \frac{3}{2}\, p\, \varphi_f. \tag{8}$$

2.3 Asynchronous machine model

The ASM modeling is represented by the following equations [9, 10]:

$$\begin{aligned} V_{ds} &= R_s\, i_{ds} + L_s\, \frac{di_{ds}}{dt} + M\, \frac{di_{dr}}{dt} - L_s\, \omega_s\, i_{qs} - M\, \omega_s\, i_{qr}, \\ V_{qs} &= R_s\, i_{qs} + L_s\, \frac{di_{qs}}{dt} + M\, \frac{di_{qr}}{dt} - L_s\, \omega_s\, i_{ds} - M\, \omega_s\, i_{dr}, \\ V_{dr} = 0 &= R_r\, i_{dr} + L_r\, \frac{di_{dr}}{dt} + M\, \frac{di_{ds}}{dt} - L_r\, \omega_r\, i_{qr} - M\, \omega_r\, i_{qs}, \\ V_{qr} = 0 &= R_r\, i_{qr} + L_r\, \frac{di_{qr}}{dt} + M\, \frac{di_{qs}}{dt} - L_r\, \omega_r\, i_{ds} - M\, \omega_r\, i_{dr}. \end{aligned} \tag{9}$$

Using the orientation of the rotor flux towards the d-axis equation 6 and substituting for the magnetic flux equations and the equations from 9, the differential equations of ASM can be obtained as follows:

$$\varphi_{dr} = \varphi_r, \quad \varphi_{qr} = 0, \quad \sigma = 1 - \left(\frac{M^2}{L_s L_r}\right) \quad T_r = \frac{L_r}{R_r}, \tag{10}$$

$$V_{ds} = (R_s + L_s P)\; i_{ds} - \left(\omega_s . L_s . \,\sigma\, .i_{qs} - \frac{M}{L_r}\frac{d\varphi_r}{dt}\right) = Vd^* - e_q, \tag{11}$$

$$V_{qs} = (R_s + L_s P)\; i_{qs} + \left(\omega_s . L_s . \,\sigma\, .i_{ds} + \frac{M.\omega_s}{L_r}\varphi_r\right) = Vq^* + e_d, \tag{12}$$

$$i_{ds}^* = (1 + T_r P)\, \frac{1}{M}\left(\varphi_{dr} - \varphi_{dr}^*\right), \quad i_{qs}^* = \frac{L_r . C_{em}^*}{p.M.\varphi_{dr}^*}. \tag{13}$$

The resistant torque is defined as follows:

$$Cr \;=\; k.\, \Omega. \tag{14}$$

2.4 Current model

The output of the speed regulator makes it possible to generate the reference current which is compared to the value of the current from the measurement of the real currents in Figure 2 and their error applied to the input of the current regulator. In parallel with this loop, there is a current regulation loop id, which is kept constant corresponding to the output of the flow regulator [12]. The outputs of the current regulators are applied to a decoupling block which generates reference voltages, and by the inverse Park transformation, we obtain the voltages which are the voltages of the inverter control with PWM control, i_{ds}, i_{qs}, V_{ds} and V_{qs}.

2.5 Photovoltaic generator model

To size the capacity of the system and predict the power of the PV generator, there are several energy models that take into consideration E_t (W/m^2), the solar radiation and

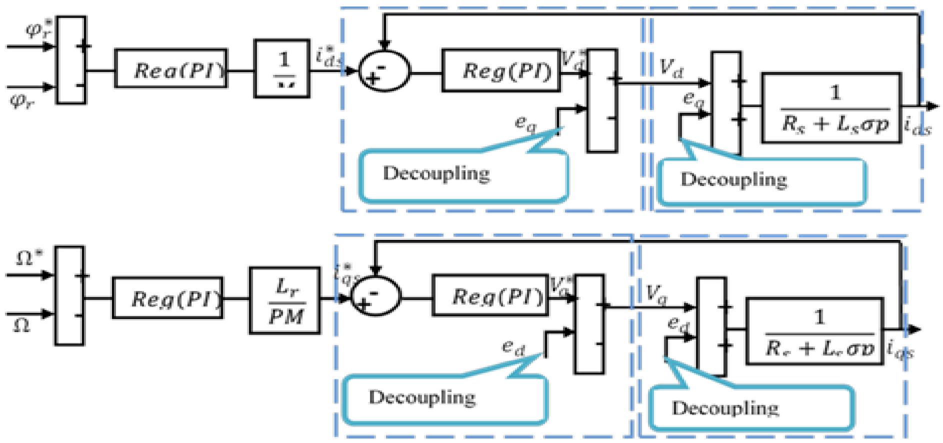


Figure 2: Block diagram of the currents including the MAS compensation terms.

the $A(m^2)$surface of the panel which is given as follows [13]:

$$P_{PV} = N_{PV}.A.E_t, \tag{15}$$

$$N_{PV} = N_r.N_{PC} + [1 - \beta (T_C - T_{C-ref})], \tag{16}$$

$$T_C = T_a \left[\frac{NOCT - 40}{1000}\right].E_t. \tag{17}$$

N_{PV} represents the efficiency of the PV generator, N_r is the efficiency of the reference module, NOCT is the nominal operating temperature of the cell, N_{PC} is the efficiency of the power conditioning, T_{C-ref} is the reference cell temperature ($degC$) and T_C is the cell temperature that can be calculated as in [14].

To achieve the desired pump power, the PV system is made up of $N_s = 12$ panels in series and $N_p = 2$ panels in parallel.

2.6 Centrifugal pump model

The sizing of a pumping system is according to field conditions, but the choice of the pump (centrifugal pump) is according to the real characteristics of the installation in which it is to be installed [15]. The following data will be required to size the pump: the water flow $Q(m^3/h)$ and the total anemometry head HMT. HMT is given as follows:

$$HTM = H_g + \Delta H = (H_{ga} + H_{gr})(1 + PC), \tag{18}$$

where ΔH are the head losses in the installation, H_{ga}, H_{gr} are the height of respiration and discharge, PC is the pipe loss = 10% for $H_{ga} = H_{gr} = 7m$, then $HMT = 15.4m$. For 3 hectares of agricultural land and 56 people, the water needs of the installation considered are estimated at $200m^3/day$, The pump operates 3 hours/day, we will have $Q = 0.185m^3/s$. The hydraulic power is defined by

$$P_H = \rho\, g\, H\, Q = 2.8\, kW. \tag{19}$$

The efficiency of the pump type $NM4125/250$ CALPEDA is 78%, the number of revolutions $n = 1450tr/min$.

$$\eta = \frac{P_H}{P_{mec}} \Longrightarrow P_{mec} = \frac{P_H}{\eta} = 3.6\,kW. \tag{20}$$

The performances (flow Q, height H and power P) are given by

$$\frac{Q}{Q_{ref}} = \frac{\Omega}{\Omega_{ref}}, \tag{21}$$

$$\frac{H}{H_{ref}} = \left(\frac{\Omega}{\Omega_{ref}}\right)^2, \tag{22}$$

$$\frac{P}{P_{ref}} = \left(\frac{\Omega}{\Omega_{ref}}\right)^3. \tag{23}$$

3 Hybrid Pumping System Control

The purpose of using an MPPT controller in such a system is to maximize the water flow. The water flow is related to the speed of the asynchronous motor by equation (20) so that maximizing the rotational speed of the motor pump, the group will maximize the water flow, it is doubled by maximizing the power absorbed by the motor which drives the centrifugal pump. The fuzzy logic controller (FLC) is another method used to extract

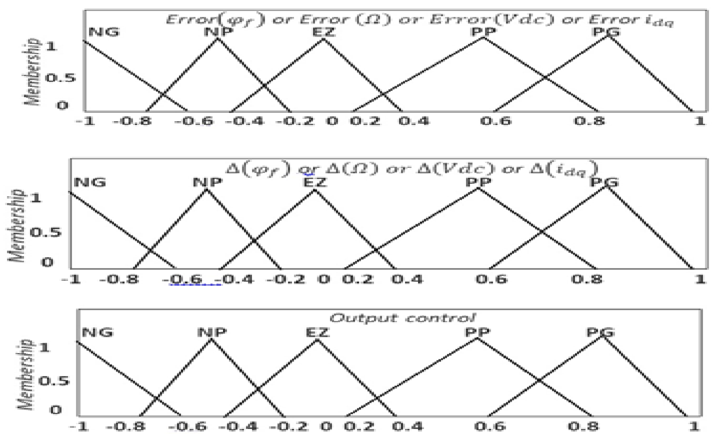


Figure 3: Membership functions for FLC input and output variables.

the maximum power point for either photovoltaic or wind power conversion systems [16, 17]. The FLC has many advantages such as the simplicity of designing it without any knowledge of the characteristics of the system, its high performance with lower oscillation around the point of maximum power, which means increasing the stability of the system, and also having a quick response. The FLC controller applied to the system inder study consists of two input variables [18]: the first input variable is the mechanical speed error of MAS or the magnetic flux error as shown in the figure, the second is the variation of the error, the inputs of the FLC can be calculated as follows:

$$E_\Omega = \Omega_{ref} - \Omega, \quad E_\varphi = \varphi_{f-ref} - \varphi_f, \quad E_{V_{dc}} = V_{dc-ref} - V_{dc}, \quad \Delta E = \frac{dE}{dt}. \tag{24}$$

Table 1: Inference matrix.

output control / Error for $\varphi_f, V_{dc}, \Omega$ and i_{dq}	NG	NP	EZ	PP	PG
NG	PG	PG	NG	NP	NG
NP	PG	PP	NP	NP	NG
EZ	NP	NP	EZ	PP	PP
PP	NG	NP	EZ	PP	PG
PG	NG	NG	NP	PG	PG

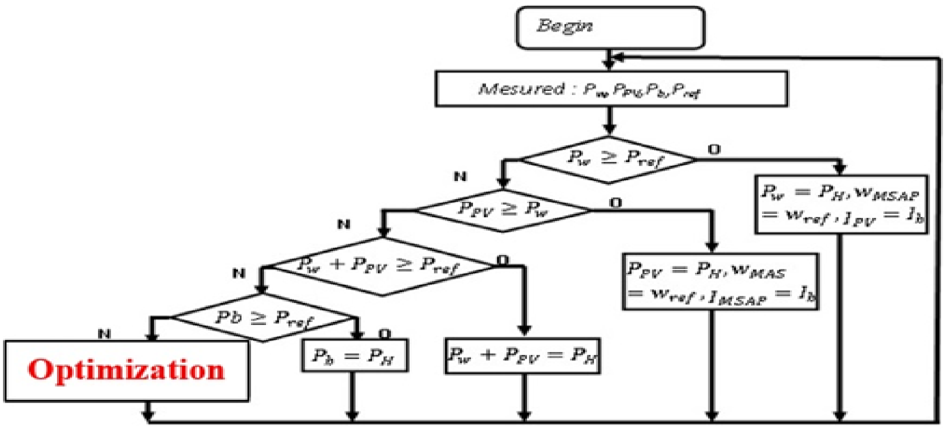
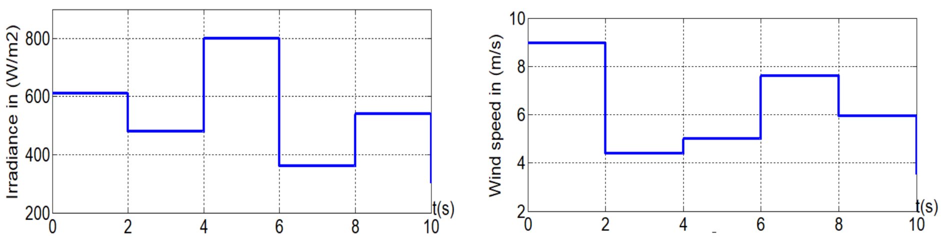


Figure 4: Hybrid management organization chart.

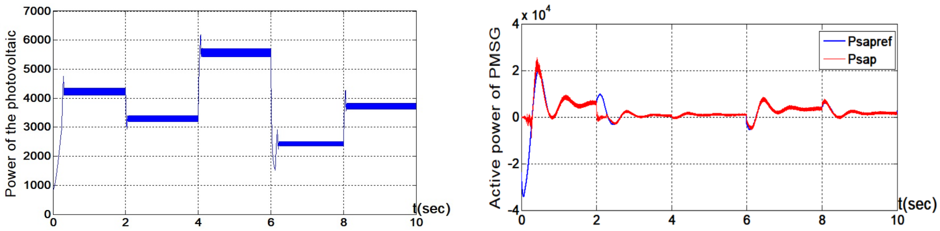
4 Simulation Results and Discussion



(a) (b)

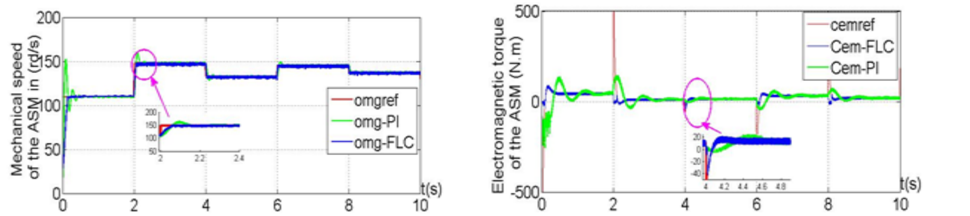
Figure 5: (a): Solar radiance, (b): Wind profile.

The evolution of the weather conditions influences the characteristic of the asynchronous machine and, consequently, the performance of the pump, the results obtained show the performance of the PI-Flow controllers and of the vector control applied to the



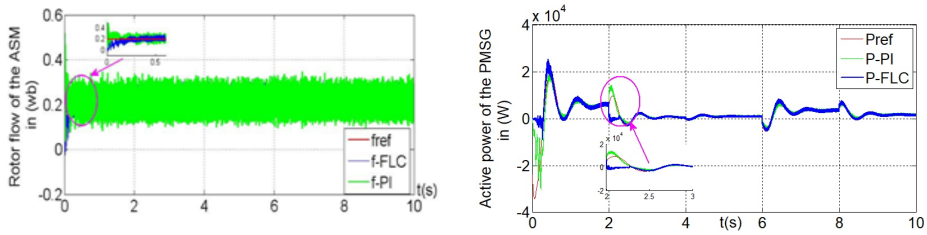
(a) (b)

Figure 6: (a): Power of the photovoltaic panel, (b): Active power of the PMSG.



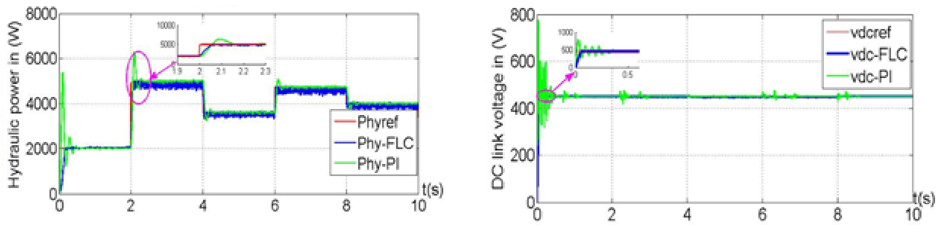
(a) (b)

Figure 7: (a): Mechanical speed of the ASM, (b): Electromagnetic torque of the ASM.



(a) (b)

Figure 8: (a): Rotor flow of the ASM, (b): Active power of the PMSG.

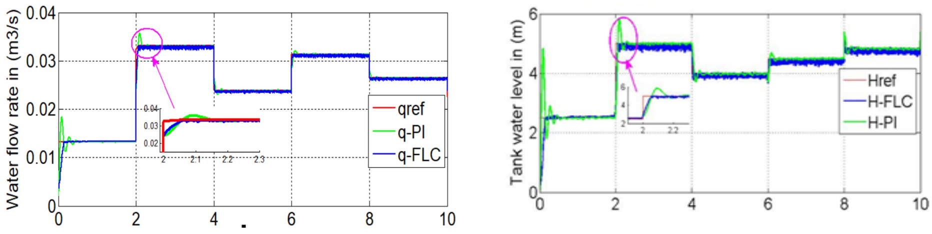


(a) (b)

Figure 9: (a): The Hydraulic power of the pump, (b): DC link voltage.

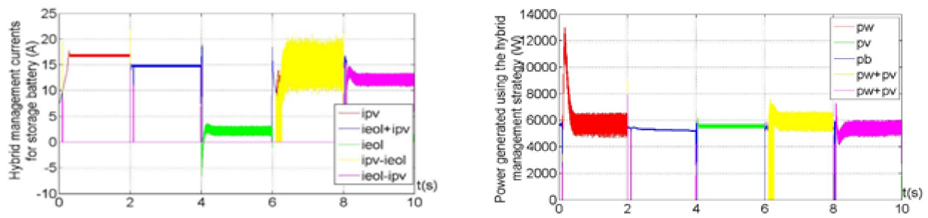
ASM, we consider, in this simulation, the evolution of the hydraulic power requested, see Figure 9(a), according to the above results, we notice that at an instant less than 2s, the wind speed is important, while the solar irradiation is low as shown in Figures 5(a) and

5(b), then the power is generated by the wind generator, see Figure 6(b), depending on the speed of the wind, the management system makes the power system to operate the pump as shown in Figure 11(b), and takes advantage of the energy supplied by the PV generator to recharge the battery, see Figure 11(a). The increase in the state of charge of the battery, see Figure 12, and at the instant between 2s and 4s, the available power (PV and wind turbine) is insufficient to satisfy the need for water, and at this moment, the pump operates from the battery, see Figure 11(b), this results in the reduction of the state of charge of the battery, see Figure 12, for the moment between 4s and 6s, at the strong solar irradiation but the weak wind, in this case, the energy of the PV generator is injected into the motor pump, see Figure 11(b), and exploiting the current produced by the wind turbine to recharge the battery as shown in Figure 11(a), subsequently, the PV and wind power is less than the hydraulic power, but the sum is greater.



(a) (b)

Figure 10: (a): Water flow rate, (b): Tank water level.



(a) (b)

Figure 11: (a): Hybrid management currents for charging and discharging the storage battery, (b): Power generated using the hybrid management strategy.

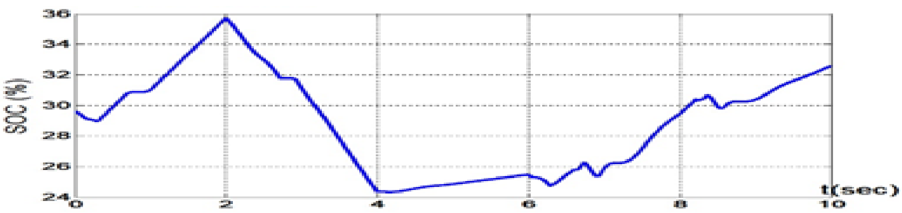


Figure 12: The state of charge and discharge of the battery.

In this case, the photovoltaic energy, as well as that produced by the wind turbine, is used to drive the motor pump, and the excess energy is sent to the battery, which leads to an increase in the state of charge of the battery, see Figure 11(b) and Figure 12, therefore, the electrical and mechanical reference quantities have been well followed by the pump set, clearly showing the fuzzy regulator performance, PI and vector control by the orientation of the rotor flux applied to the machine, thus the efficiency of the MPPT adapter. The response time obtained by the FLC controller is considerably reduced, the peak overrun values are limited as compared to the PI controller, the tank water level, see Figure 10(b), the water flow, see Figure 10(a) as well as the pump power, see Figure 9(a) follow their reference values successfully thanks to the hybridization of photovoltaic-wind energy, hence the satisfaction of daily water needs.

5 Conclusion

The main objective of this paper was to develop a complete model of the photovoltaic-wind hybrid system with storage applied to the pumping system, from this model; an energy management strategy was developed and analyzed. We then chose the strategy that allows maintaining both the state of charge of the batteries and the water level of the tank. To achieve our objective, we presented the modeling of the various components of the system, a good dimensioning of the storage allowed us to ensure the energy needs requested by the pump. Subsequently, we were able to optimize the system under study to guarantee the daily water requirements requested. In order to exploit the photovoltaic generators and the wind power to the maximum, the simulation results show the performance of fuzzy regulator and the vector control applied to the asynchronous machine having for principle, the decoupling between the torque and the flux. These tests in weather conditions show the efficiency of hybridization in a hybrid photovoltaic-wind pumping system to meet daily water needs.

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References

[1] A. Adallah and Z. Abdulqader. Performance analysis of a hybrid wind/photovoltaic power generation system for water pumping. *Springer, International Journal of Environmental Science and Technology* **16** (2019) 5295–5304.

[2] G. Misrak, M. Marta and A. Abebayehu. Feasibility study of a wind powered water pumping system for rural Ethiopia. *AIMS Energy* **3** (4) (2015) 851–868.

[3] Y. Chedni, Dj. Boudana, A. Moualdia, L. Nezli and P. Wira. Sensorless Two Series Connected Quasi Six-Phase IM Based Direct Torque Control for Torque Ripples Minimization. *Nonlinear Dynamics and Systems Theory* **20** (2) (2020) 153–167.

[4] A. Moualdia, M. O. Mahmoudi and L. Nezli. Direct Torque Control of the DFIG and Direct Power Control for Grid Side Converter in wind power generation system. *The Mediterranean Journal of Measurement and Control, MEDJMC* **9** (3) (2013) 101–108.

[5] S. Boulkhrachef, A. Moualdia, Dj. Boudana and P. Wira. Higher-order sliding mode control of a wind energy conversion system. *Nonlinear Dynamics and Systems Theory* **19** (4) (2019) 486–496.

[6] A. G. Mazko. Robust output feedback stabilization and optimization of discrete-time control systems. *Nonlinear Dynamics and Systems Theory* **18** (1) (2018) 92–106.

[7] T. Djellouli, S. Moulahoum, A. Moualdia, M. S. Bouchrit and P. Wira. Speed Sensorless Direct Torque Control Strategy of a Doubly Fed Induction Motor Using an ANN and an EKF. *Nonlinear Dynamics and Systems Theory* **20** (4) (2020) 374–387.

[8] A. Ansari and D. M. Deshpande. Mathematical Model of Asynchronous Machine in MATLAB Simulink. *International Journal of Engineering Science and Technology* **2** (5) (2010) 1260–1267.

[9] E. Benyoussef, A. Meroufel and S. Barkat. Neural network and fuzzy logic direct torque control of sensorless double star synchronous machine. *Rev. Roum. Sci. Techn. Ectrotechn. and Nerg.* **61** (3) (2016) 239–243.

[10] S. Bentouati, A. Tlemcani, N. Henini and H. Nouri. Adaptive Sliding Mode Control Based on Fuzzy Systems Applied to the Permanent Magnet Synchronous Machine. *Nonlinear Dynamics and Systems Theory* **21** (5) (2021) 457–470.

[11] A. Chalanga, S. Kamal and B. Moreno. *I*mplementation of super-twisting control: super-twisting and higher order sliding-mode observer-based approaches. *IEEE Trans. on Industrial Electronics* **63** (6) (2016) 3677–3685 .

[12] A. E. Leon and J. A. Solsona. Sub-Synchronous Interaction Damping Control for DFIG Wind Turbines. *IEEE Transactions on Power Systems* **30** (1) (2015) 419–428.

[13] L. Wang, M. Zhao, X. Wu, X. Gong and L. Yang. Fully integrated high-efficiency high step-down ratio DC/DC buck converter with predictive over-current protection scheme. *IET Power Electronics* **10** (14) (2017) 1959–1965.

[14] A. Ajami, H. Ardi and A. Farakhor. Design, analysis and Implementation of a Buck-Boost DC/DC Converter. *IET Power Electronics* **7** (12) (2014) 2902–2913.

[15] T. Ramji and N. Ramesh Babu. Fuzzy logic based MPPT for permanent magnet synchronous generator in wind energy conversion system. *IFAC-Papers* **49** (1) (2016) 462–467.

[16] N. Khattab, M. Badr, E. El Shenawy, H. Sharawy and M. Shalaby. Feasibility of Hybrid Renewable Energy Water Pumping System for a Small Farm in Egypt. *International Journal of Applied Engineering Research* **11** (11) (2016) 7406–7414.

[17] A. Gholamshahi and R. Fadaeinedjad. A New Control Strategy for Hybrid Water Pumping Systems Used by Utilities in Developing Countries *IEEE* (2019) 2406–2413.

[18] A. Nekkache, B. Bouzidi and S. Ait-Cheikh. Optimal Sizing of Hybrid PV/Wind Based Water Pumping System Considering Reliability and Economic Aspects. In: *International Conference on Wind Energy and Applications in Algeria (ICWEAA)*, 2018.

[19] L. Stoyanov, I. Bachev and Z. Zarkov. Multivariate Analysis of a Wind/PV-Based Water Pumping Hybrid System for Irrigation Purposes *Energie* **14** (2021) 3231–3246.

[20] J. C. Rosa, S. Uranio and N. Diana. Design and Performance of Photovoltaic Water Pumping Systems: Comprehensive Review towards a Renewable Strategy for Mozambique. *Journal of Power and Energy Engineering* **6** (2018) 32–63.

[21] T. Aized, M. Syed and K. Sajid. Design and analysis of wind pump for wind conditions in Pakistan. *Advances in Mechanical Engineering* **9** (11) (2019) 1–18.

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Some Results on Controllability for a Class of Non-Integer Order Differential Equations with Impulses

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Abstract: In this paper, we considered a class of impulsive fractional differential equations of order $1 < \alpha \leq 2$, in a Banach space. An associated integral equation is obtained by using the fractional integral and the cosine or sine family of linear operators. By using the measure of non-compactness and Mönch's condition, we prove that the problem under consideration is controllable. Abstract results are illustrated by an example in the last section.

Keywords: *controllability; non-integer order differential equation; impulsive condition; measure of non-compactness; Mönch's condition.*

Mathematics Subject Classification (2010): 93B05, 34A08, 35R12, 47H08.

1 Introduction

Controllability is a fundamental concept in the theory of control dynamic systems, which plays an important role in the investigations and design of various kinds of control dynamic processes in finite and infinite dimensional spaces. An extensive study on controllability of various types of differential equations in abstract spaces has been done by many authors [2, 3, 5–8, 11–13]. In papers [2, 6, 9], the authors proved the results on controllability for second order control systems. Controllability of damped second order integrodifferential systems with impulses has been studied by Arthi and Balachandran [6].

The present work has been motivated by the work of Ravichandran and Baleanu [3], in which a control problem involving non-integer order (Caputo) derivatives is studied by using the measure of non-compactness and Mönch's condition. There are only few papers dealing with the study of controllability for a dynamic system with impulses. Impulse conditions describe the dynamics of a process in which discontinuous jumps occur. Such

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processes are generally seen in biology, physics and engineering. For earlier works on impulsive differential equations, we refer the readers to [10, 13, 14] and references cited in these papers.

In this manuscript, we are concerned with the controllability of the following fractional impulsive differential equation in a Banach space $(Y, \|\cdot\|)$:

$$\begin{cases} {}_C D^{\alpha} y(t) = Ay(t) + Ew(t) + g(t, y(t), y(\nu(t))), \quad t \in [0, b_0],\ t \neq t_i, \\ \Delta y(t_i) = \hat{I}_i(y(t_i)), \quad \Delta y'(t_i) = \hat{J}_i(y(t_i)), \quad i \in \mathbb{N},\ 1 \leq i \leq q, \\ y(t) = h(t), \quad t \in [-\tau, 0], \end{cases} \tag{1}$$

where $1 < \alpha \leq 2$, ${}_C D^{\alpha}$, denotes the (Caputo) fractional derivative, A is a densely defined closed linear operator, which generates a strongly continuous cosine family in Y, E denotes the bounded linear operators defined on W, $w \in L^2([0, b_0], W)$ denotes the control function, which takes the values in a Banach space W. The maps $g : [0, b_0] \times Y^2 \to Y$ and the maps $\hat{I}_i$, $\hat{J}_i$ defined on Y satisfy some suitable conditions, and the function $\nu : [0, b_0] \to [0, b_0]$ is continuous such that $0 \leq \nu(t) \leq t$, $t_i \in [0, b_0]$ for all $i \in \mathbb{N},\ 1 \leq i \leq q$ such that $t_1 < t_2 < \cdots < t_q$, and $q \in \mathbb{N},\ b_0 > 0$. $h \in C^2([-\tau, 0], Y)$, i.e., h is twice continuously differentiable on $[-\tau, 0]$. Let $I_0 = [0, b_0]$.

The main aim of this paper is to prove the controllability of the problem (1) by using the measure of non-compactness and Mönch's condition.

2 Preliminaries and Assumptions

It is well known that if A generates a strongly continuous cosine family, then A also generates an analytic semigroup. The fractional power A^{β} of A from $D(A^{\beta}) \subset Y$ into Y is well defined for all $0 \leq \beta \leq 1$ (cf., A. Pazy [1], pp. 69-75). The space $Y_{\beta} = \big(D(A^{\beta}), \|.\|_{\beta}\big)$ is a Banach space, where

$$\|\psi\|_{\beta} = \|A^{\beta}\psi\|, \quad \psi \in D(A^{\beta}).$$

Let $PC([0, b_0], Y_{\beta})$ denote the set of all piecewise continuous functions on $[0, b_0]$, and $\Omega_{\beta}^{b_0} = \big\{y \mid y, y' \in PC([0, b_0], Y_{\beta})$ such that $y(t), y'(t)$ are left continuous at $t = t_i$ and the right-hand limit of $y(t), y'(t)$ exists at $t = t_i,\ i \in \mathbb{N},\ 1 \leq i \leq q\big\}$. Eventually, $\big(\Omega_{\beta}^{b_0}, \|.\|_{\beta, b_0}\big)$ is a Banach space, where

$$\|\psi\|_{\beta, b_0} = \sup_{s \in [0, b_0]} \|\psi(s)\|_{\beta}, \quad \psi \in \Omega_{\beta}^{b_0}.$$

For $R_0 > 0$, let

$$B_{R_0}(\Omega_{\beta}^{b_0}, \tilde{h}) = \{y \in \Omega_{\beta}^{b_0} \ :\ \|y - \tilde{h}\|_{\beta, b_0} \leq R_0\},$$

where

$$\tilde{h}(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0), & t \in [0, b_0]. \end{cases}$$

Let $\{C_{\alpha}(t) : t \geq 0\}$ denote the cosine family generated by A. For $t \geq 0$, we define

$$S_{\alpha}(t) = \int_0^t C_{\alpha}(s)ds, \quad \text{and} \quad P_{\alpha}(t) = \frac{1}{\Gamma(\alpha - 2)} \int_0^t (t - s)^{\alpha - 2} C_{\alpha}(s)ds.$$

Definition 2.1 [10] A measure of non-compactness defined on a Banach space Y is a function defined from Y to a positive cone of an ordered Banach space $(F, \leq)$ such that $\phi(\bar{ch}(B)) = \phi(B)$ for all bounded subset B of Y, where $\bar{ch}(B)$ denotes the closure of convex hull of B.

Lemma 2.1 *[10] Let Ω_0 be a closed convex subset of a Banach space Y and f_0 be a continuous map defined on Ω_0. If f_0 satisfies the (Mönch's) condition: $C_0 \subseteq \Omega_0$ is countable, $C_0 \subseteq \bar{ch}(\{0\} \cup f_0(C_0)) \Rightarrow \bar{C}_0$ is compact, then f_0 has a fixed point in Ω_0.*

Lemma 2.2 *[4] There are constants $T_C > 0, T_S > 0$ and $T_P > 0$ such that*

$$\|C_\alpha(s'') - C_\alpha(s')\| \leq T_C|s'' - s'|,$$

$$\|S_\alpha(s'') - S_\alpha(s')\| \leq T_S|s'' - s'|,$$

$$\|P_\alpha(s'') - P_\alpha(s')\| \leq T_P|s'' - s'|,$$

for $s', s'' \in I_0$.

Assumption 2.1 *Consider the following assumptions:*

(H1) There exists an increasing function $L_g : R^+ \to R^+$ such that

$$\|g(t, \hat{\phi}_1, \hat{\psi}_1) - g(s, \hat{\phi}_2, \hat{\psi}_2)\| \leq L_g(r)\Big[|t - s| + \|\hat{\phi}_1 - \hat{\phi}_2\|_\beta + \|\hat{\psi}_1 - \hat{\psi}_2\|_\beta\Big]$$

for all $\hat{\phi}_1, \hat{\phi}_2, \hat{\psi}_1, \hat{\psi}_2 \in B_{R_0}(\Omega_\beta^{b_0}, \tilde{h})$, and $t, s \in [0, b_0]$.

(H2) There are positive constants C_i, D_i and L_i, N_i, ($i \in \mathbb{N}$, $1 \leq i \leq q$) such that

(i) $\|\hat{I}_i(\hat{z})\|_\beta \leq C_i, \quad \|\hat{I}_i(\hat{z}_1) - \hat{I}_i(\hat{z}_2)\|_\beta \leq L_i\|\hat{z}_1 - \hat{z}_2\|_\beta,$

(ii) $\|\hat{J}_i(\hat{z})\|_\beta \leq D_i, \quad \|\hat{J}_i(\hat{z}_1) - \hat{J}_i(\hat{z}_2)\|_\beta \leq N_i\|\hat{z}_1 - \hat{z}_2\|_\beta$

for all $\hat{z}, \hat{z}_1, \hat{z}_2 \in B_{R_0}(\Omega_\beta^{b_0}, \tilde{h})$.

(H3) The linear operator $E : L^2(I_0, W) \to: L^1(I_0, W)$ is bounded. Also, the operator $Q : L^2(I_0, W) \to Y$ defined by

$$Qw = \int_0^{b_0} P_\alpha(b_0 - s)Ew(s)ds$$

has bounded inverse, i.e., $\|E\| \leq M_2$ and $\|Q^{-1}\| \leq M_3$, for some $M_2, M_3 > 0$.

3 Main Results

We assume that the families $\{C_\alpha(t)\}$, $\{S_\alpha(t)\}$, $\{P_\alpha(t)\}$ and $\{AP_\alpha(t)\}$ are unifromly bounded, i.e., there are constants r_1, r_2, r_3, r_4 such that

$$\|C_\alpha(t)\| \leq r_1, \quad \|S_\alpha(t)\| \leq r_2, \quad \|P_\alpha(t)\| \leq r_3 \quad \|AP_\alpha(t)\| \leq r_4, \quad t \in [0, b_0].$$

Lemma 3.1 *If $y(t)$ satisfies the control system (1), then $y(t)$ also satisfies the integral equation*

$$y(t) = \begin{cases} \tilde{h}(t), \quad -\tau \leq t \leq 0, \\ C_\alpha(t)h(0) + S_\alpha(t)h'(0) + \int_0^t P_\alpha(t-s)\big(g(s,y(s),y(\nu(s))) + Ew(s)\big)ds \\ \quad + \sum\limits_{1\leq i\leq q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum\limits_{1\leq i\leq q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \leq b_0. \end{cases}$$

Proof. If $t \in [-\tau, 0]$, then $u(t) = h(t) = \tilde{h}(t)$. If $t \in [0, t_1)$, then

$$\begin{aligned} {}_CD_t^\alpha y(t) &= Ay(t) + Ew(t) + g(t,y(t),y(\nu(t))), \\ y(0) &= h(0),\ y'(0) = h'(0). \end{aligned}$$

Integrating, we get

$$y(t) + c_1 + c_2 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[Ax(s) + Ew(s) + g(s,y(s),y(\nu(s)))\Big]ds.$$

Using $y(0) = h(0)$, $y'(0) = h'(0)$, we get $c_1 = -h(0)$, $\quad c_2 = -h'(0)$. Thus

$$y(t) = h(0) + h'(0)t + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[Ay(s) + Ew(s) + g(s,y(s),y(\nu(s)))\Big]ds.$$

If $t \in (t_1, t_2]$, then

$$\begin{aligned} {}_CD_t^\alpha y(t) &= Ay(t) + Ew(t) + g(t,y(t),y(\nu(t))) \\ y(t_1^+) &= y(t_1^-) + \hat{I}_1(y(t_1)) \\ y'(t_1^+) &= y'(t_1^-) + \hat{J}_1(y(t_1)). \end{aligned}$$

Again, integrating, we get

$$y(t) + c_3 + c_4 t = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[Ay(s) + Ew(s) + g(s,y(s),y(\nu(s)))\Big]ds.$$

Using $y(t_1^+) = y((t_1^-) + \hat{I}_1(y(t_1)))$ and $y'(t_1^+) = y'((t_1^-) + \hat{J}_1(y(t_1)))$, we get $c_3 = -h(0) - \hat{I}_1(y(t_1)) + t_1\hat{J}_1(y(t_1))$, $c_4 = -h'(0) - \hat{J}_1(y(t_1))$. Thus,

$$\begin{aligned} y(t) &= h(0) + h'(0)t + \hat{I}_1(y(t_1)) + (t-t_1)\hat{J}_1(y(t_1)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[Ax(s) + Ew(s) + g(s,y(s),y(\nu(s)))\Big]ds. \end{aligned}$$

Similarly, if $t \in (t_i, t_{i+1}]$, we have

$$\begin{aligned} x(t) &= h(0) + h'(0)t + \sum_{1\leq i\leq q}\hat{I}_i(y(t_i)) + \sum_{1\leq i\leq q}(t-t_i)\hat{J}_i(y(t_i)) \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\Big[Ay(s) + Ew(s) + g(s,y(s),y(\nu(s)))\Big]ds. \end{aligned}$$

Taking the Laplace transform, we get

$$\begin{aligned} \tilde{y}(\lambda) &= \frac{h(0)}{\lambda} + \frac{h'(0)}{\lambda^2} + \sum_{1\leq i\leq q}\frac{e^{-t_i\lambda}}{\lambda}\hat{I}_i(y(t_i)) + \sum_{1\leq i\leq q}\frac{e^{-t_i\lambda}}{\lambda^2}\hat{J}_i(y(t_i)) \\ &\quad - \frac{1}{\lambda^\alpha}A\tilde{y}(\lambda) + \frac{1}{\lambda^\alpha}\tilde{g}(\lambda) + \frac{1}{\lambda^\alpha}E\tilde{w}(\lambda), \end{aligned}$$

where $\tilde{y}(\lambda) = L[y(t)]$, $\tilde{g}(\lambda) = L[g(t, y(t), y(\nu(t)))]$, and $\tilde{w}(\lambda) = L[w(t)]$.

$$\begin{aligned}\Rightarrow \tilde{y}(\lambda) &= \lambda^{\alpha-1}(\lambda^\alpha I - A)^{-1}h(0) + \lambda^{\alpha-2}(\lambda^\alpha I - A)^{-1}h'(0)\\ &+\lambda^{\alpha-1}\sum_{1\le i\le q} e^{-t_i\lambda}(\lambda^\alpha I - A)^{-1}\hat{I}_i(y(t_i))\\ &+\lambda^{\alpha-2}\sum_{1\le i\le q} e^{-t_i\lambda}(\lambda^\alpha I - A)^{-1}\hat{J}_i(y(t_i)).\end{aligned}$$

Using the properties of resolvent operator [4], we get

$$\begin{aligned}\tilde{y}(\lambda) &= \left\{\int_0^\infty e^{-\lambda t}C_\alpha(t)dt\right\}h(0) + \left\{\int_0^\infty e^{-\lambda t}S_\alpha(t)dt\right\}h'(0)\\ &+\sum_{1\le i\le q} e^{-t_i\lambda}\int_0^\infty e^{-\lambda t}C_\alpha(t)\hat{I}_i(y(t_i))dt\\ &+\sum_{1\le i\le q} e^{-t_i\lambda}\int_0^\infty e^{-\lambda t}S_\alpha(t)\hat{J}_i(y(t_i))dt\\ &+\int_0^\infty e^{-\lambda t}P_\alpha(t)\tilde{g}(\lambda)dt + \int_0^\infty e^{-\lambda t}P_\alpha(t)E\tilde{w}(\lambda)dt. \end{aligned}\tag{2}$$

Consider

$$\begin{aligned}&\sum_{1\le i\le q} e^{-\lambda t_i}\int_0^\infty e^{-\lambda t}C_\alpha(t)\hat{I}_i(y(t_i))dt\\ &\quad= \int_0^\infty e^{-\lambda t}\left\{\sum_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i))\right\}dt.\end{aligned}\tag{3}$$

Similarly,

$$\begin{aligned}&\sum_{1\le i\le q} e^{-\lambda t_i}\int_0^\infty e^{-\lambda t}S_\alpha(t)\hat{J}_i(y(t_i))dt\\ &\quad= \int_0^\infty e^{-\lambda t}\left\{\sum_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i))\right\}dt,\end{aligned}\tag{4}$$

$$\begin{aligned}&\int_0^\infty e^{-\lambda t}P_\alpha(t)\tilde{g}(\lambda)dt\\ &\quad= \int_0^\infty e^{-\lambda t}P_\alpha(t)\int_0^\infty e^{-\lambda s}g(s, y(s), y(\nu(s)))dsdt\\ &\quad= \int_0^\infty e^{-\lambda t}\left\{\int_0^t P_\alpha(t-s)g(s, y(s), y(\nu(s)))ds\right\}dt,\end{aligned}\tag{5}$$

and

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} P_\alpha(t) E\tilde{w}(\lambda) dt \\
&= \int_0^\infty \int_0^\infty e^{-\lambda(t+s)} P_\alpha(t) E w(s) ds dt \\
&= \int_0^\infty e^{-\lambda t} \left\{ \int_0^t P_\alpha(t-s) E w(s) ds \right\} dt. \qquad (6)
\end{aligned}$$

Putting the values of (3), (4), (5) and (6) in (2), and by taking the inverse Laplace transform, we get

$$\begin{aligned}
y(t) &= C_\alpha(t)h(0) + S_\alpha(t)h'(0) + \int_0^t P_\alpha(t-s)(g(s, y(s), y(\nu(s))) + Ew(s))ds \\
&\quad + \sum_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \le b_0.
\end{aligned}$$

Definition 3.1 A mild solution of the problem (1) is a fucntion $y \in \Omega_\beta^{b_0}$ satisfying the integral equation

$$y(t) = \begin{cases} \tilde{h}(t), \quad -\tau \le t \le 0, \\ C_\alpha(t)h(0) + S_\alpha(t)h'(0) + \int_0^t P_\alpha(t-s)(g(s, y(s), y(\nu(s))) + Ew(s))ds \\ \quad + \sum\limits_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum\limits_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \le b_0. \end{cases} \qquad (7)$$

Definition 3.2 The system (1) is said to be controllable on the interval I_0 if for every $h(t) \in C^2([-\tau, 0], Y)$, $y_1 \in Y$, there is a control function $w \in L^2(I_0, W)$ such that the mild solution $y(t)$ of (1) satisfies $y(b_0) = y_1$.

For any $y \in \Omega_\beta^{b_0}$, we define the control function

$$\begin{aligned}
w_y(t) = Q^{-1}\Big\{ &y_1 - C_\alpha(b_0)h(0) - S_\alpha(b_0)h'(0) - \int_0^{b_0} P_\alpha(b_0 - t) g(t, y(t), y(\nu(t)))dt \\
&- \sum_{1\le i\le q} C_\alpha(b_0 - t_i)\hat{I}_i(y(t_i)) - \sum_{1\le i\le q} S_\alpha(b_0 - t_i)\hat{J}_i(y(t_i)) \Big\}.
\end{aligned}$$

Using (H1)-(H3), we can find a constant $K_w > 0$ s.t. $\|w_y(t)\| \le K_w$.

Theorem 3.1 *If (H1)-(H3) hold, then control system (1) is controllable.*

Proof. Using the control w_y, we show that the operator $F : B_{R_0}(\Omega_\beta^{b_0}, \tilde{h}) \to B_{R_0}(\Omega_\beta^{b_0}, \tilde{h})$, defined by

$$Fy(t) = \begin{cases} \tilde{h}(t), \quad -\tau \le t \le 0, \\ C_\alpha(t)h(0) + S_\alpha(t)h'(0) + \int_0^t P_\alpha(t-s)(g(s, y(s), y(\nu(s))) + Ew_y(s))ds \\ \quad + \sum\limits_{1\le i\le q} C_\alpha(t-t_i)\hat{I}_i(y(t_i)) + \sum\limits_{1\le i\le q} S_\alpha(t-t_i)\hat{J}_i(y(t_i)), \quad 0 < t \le b_0, \end{cases}$$

has a fixed point. This fixed point is then a solution of the given system. Clearly, $Fy(b_0) = y_1$, which shows that the given system is controllable on I_0.

We define

$$\hat{h}(t)=\begin{cases}\tilde{h}(t), & -\tau\leq t\leq 0,\\ C_\alpha(t)h(0)+S_\alpha(t)h'(0), & 0<t\leq b_0,\end{cases}$$

and

$$z(t)=\begin{cases}0, & -\tau\leq t\leq 0,\\ \int_0^t P_\alpha(t-s)(g(s,y(s),y(\nu(s)))+Ew(s))ds\\ \quad+\sum\limits_{1\leq i\leq q}C_\alpha(t-t_i)\hat{I}_i(y(t_i))+\sum\limits_{1\leq i\leq q}S_\alpha(t-t_i)\hat{J}_i(y(t_i)), & 0<t\leq b_0.\end{cases}$$

Let $y(t)=\hat{h}(t)+z(t)$, then $y(t)$ satisfies (7). Define

$$\Omega^{b_0}_{\beta,0}=\left\{y\in\Omega^{b_0}_\beta \mid y(t)=0,\quad -\tau\leq t\leq 0\right\},$$

and the operator $\tilde{F}:\Omega^{b_0}_{\beta,0}\to\Omega^{b_0}_{\beta,0}$ is

$$\tilde{F}z(t)=\begin{cases}0, & -\tau\leq t\leq 0,\\ \int_0^t P_\alpha(t-s)\left(g\big(s,\hat{h}(s)+z(s),\hat{h}(\nu(s))+z(\nu(s))\big)+Ew_{\hat{h}+z}(s)\right)ds\\ \quad+\sum\limits_{1\leq i\leq q}C_\alpha(t-t_i)\hat{I}_i\big(\hat{h}(t_i)+z(t_i)\big)\\ \quad+\sum\limits_{1\leq i\leq q}S_\alpha(t-t_i)\hat{J}_i\big(\hat{h}(t_i)+z(t_i)\big), & 0<t\leq b_0.\end{cases}$$

Obviously, to show that F has a fixed point, it is sufficient to show that $\tilde{F}$ has a fixed point. For this, we use Lemma 2.1. Let

$$B_R=\left\{x\in\Omega^{b_0}_{\beta,0}\mid \|x\|_{\beta,b_0}\leq R\right\}.$$

We prove this result in four steps.

Step 1: There is a number $R>0$ such that

$$\tilde{F}(B_R)\subseteq B_R.$$

Let $z\in B_R$, $t\in(0,b_0]$, we have

$$\begin{aligned}\|(\tilde{F}z)(t)\|_\beta &\leq r_4\|A^{\beta-1}\|\int_0^t\Big[\|g\big(s,\hat{h}(s)+z(s),\hat{h}(\nu(s))+z(\nu(s))\big)\|+\|Ew_{\hat{h}+z}(s)\|\Big]ds\\ &\quad+r_1\sum_{1\leq i\leq q}\|\hat{I}_i\big(\hat{h}(t_i)+z(t_i)\big)\|_\beta+r_2\sum_{1\leq i\leq q}\|\hat{J}_i\big(\hat{h}(t_i)+z(t_i)\big)\|_\beta.\end{aligned}$$

Using (H1)-(H3), and the inequality $\|w_y(t)\|\leq K_w$, and then taking R sufficiently large, we have

$$\|(\tilde{F}z)\|_{\beta,b_0}\leq R.$$

Thus there is a $R>0$ such that

$$\tilde{F}(B_R)\subseteq B_R.$$

Step 2: $\tilde{F}$ is continuous on B_R. We consider a sequence $\{z_n\}$ in B_R such that $z_n \to z \in B_R$. Then we have

$$\begin{aligned}
\|(\tilde{F}z_n)(t) - (\tilde{F}z)(t)\|_\beta \quad \leq \quad & r_4\|A^{\beta-1}\| \int_0^t \|g\big(s, \hat{h}(s) + z_n(s), \hat{h}(\nu(s)) + z_n(\nu(s))\big) \\
& \qquad - g\big(s, \hat{h}(s) + z(s), \hat{h}(\nu(s)) + z(\nu(s))\big)\|ds \\
& + r_4\|A^{\beta-1}\| \int_0^t \|E\|\|w_{\hat{h}+z_n}(s) - w_{\hat{h}+z}(s)\|ds \\
& + r_1 \sum_{1\leq i\leq q} \|\hat{I}_i\big(\hat{h}(t_i) + z_n(t_i)\big) - \hat{I}_i\big(\hat{h}(t_i) + z(t_i)\big)\|_\beta \\
& + r_2 \sum_{1\leq i\leq q} \|\hat{J}_i\big(\hat{h}(t_i) + z_n(t_i)\big) - \hat{J}_i\big(\hat{h}(t_i) + z(t_i)\big)\|_\beta.
\end{aligned} \tag{8}$$

Using (H1)-(H3), and taking supremum over $[0, b_0]$, we have

$$\|\tilde{F}z_n - \tilde{F}z\|_{\beta,b_0} \to 0 \quad \text{as} \quad n \to \infty,$$

which implies that $\tilde{F}$ is continuous on B_R.

Step 3: $\tilde{F}(B_R)$ is equicontinuous on I_0. For this, we assume $z \in \tilde{F}(B_R)$ and $0 \leq s' < s'' \leq b_0$. Then there is a $y \in B_R$ such that

$$\begin{aligned}
\|z(s'') - z(s')\| \quad \leq \quad & \int_0^{s'} \|P_\alpha(s'' - s) - P_\alpha(s' - s)\|_\beta \Big[\|g\big(s, \hat{h}(s) + y(s), \\
& \qquad \hat{h}(\nu(s)) + y(\nu(s))\big)\| + \|E\|\|w_{\hat{h}+y}(s)\|ds\Big] \\
& + \int_{s'}^{s''} \|P_\alpha(s'' - s)\|_\beta \Big[\|g\big(s, \hat{h}(s) + y(s), \\
& \qquad \hat{h}(\nu(s)) + y(\nu(s))\big)\| + \|E\|\|w_{\hat{h}+y}(s)\|\Big]ds \\
& + \sum_{1\leq i\leq q} \|C_\alpha(s'' - t_i) - C_\alpha(s' - t_i)\|\|\hat{I}_i\big(\hat{h}(t_i) + y(t_i)\big)\|_\beta \\
& + \sum_{1\leq i\leq q} \|S_\alpha(s'' - t_i) - S_\alpha(s' - t_i)\|\|\hat{J}_i\big(\hat{h}(t_i) + y(t_i)\big)\|_\beta.
\end{aligned}$$

Using (H1), (H2), (H3), Lemma 2.2 and $\|w_y(t)\| \leq K_w$, we can find a constant $G_1 > 0$ such that

$$\|z(s'') - z(s')\| \quad \leq \quad G_1|s'' - s'|.$$

From the above inequality, it is clear that $\|z(t'') - z(t')\| \to 0$ as $t'' \to t'$. Therefore $\tilde{F}(B_R)$ is equicontinuous on I_0.

Step 4: Next, we show that Mönch's condition is satisfied, i.e., if $V \subseteq B_R$ is countable and $V \subseteq \bar{ch}\Big(\{0\} \cup \tilde{F}(V)\Big)$, then $\bar{V}$ is compact. According to the idea used in [13], we can show that $\tilde{F}(V)$ is relatively compact, i.e., if ϕ is a monotone, nonsingular measure of non-compactness, then $\phi\big(\tilde{F}(V)\big) = 0$.

Since $V \subseteq \bar{ch}(\{0\} \cup \tilde{F}(V))$, by using the definition of ϕ, we have

$$\phi(V) \quad \leq \quad \phi\Big(\bar{ch}(\{0\} \cup \tilde{F}(V))\Big) = \phi\big(\tilde{F}(V)\big) = 0.$$

This implies that V is relatively compact, i.e., $\bar{V}$ is compact. Thus Mönch's condition is satisfied. Therefore, by Lemma 2.1, $\tilde{F}$ has a fixed point. This completes the proof.

4 Application

Consider the problem

$$\begin{cases} {}_C D^{\frac{4}{3}} w(y,s) = \dfrac{\partial^2 w}{\partial x^2} + \mu_0(y,s) + \mathcal{H}(y,s,w(y,s),w(y,s-\tau)), \\ \qquad\qquad\qquad\qquad y \in (0,\pi), \quad s \in [0,b_0],\ s \neq s_i, \\ w(0,s) = w(\pi,s) = 0, \quad s \in (0,b_0], \\ \Delta w(y,s_i) = \dfrac{3w(y,s_i)}{4 + w(y,s_i)}, \quad y \in (0,\pi), \\ \Delta w'(y,s_i) = \dfrac{5u_1(x,s_i)}{6 + w(y,s_i)}, \quad y \in (0,\pi), \\ w(y,s) = \chi(y,s), \quad y \in [0,\pi], \quad s \in [-\tau,0], \\ \qquad\qquad i \in \mathbb{N},\ 1 \leq i \leq q, \end{cases} \tag{9}$$

where $\mathcal{H}, \chi$ are sufficiently smooth real-valued functions. $\Delta w(y,s_i) = w(y,s_i^+) - w(y,s_i^-)$, $\Delta w'(y,s_i) = w'(y,s_i^+) - w'(y,s_i^-)$, where $w(y,s_i^-)(w(y,s_i^+))$ are the left-(right-) hand limits of w and w' at $(y,s) = (y,s_i)$, respectively. Let the control function $\mu_0 : [0,b_0] \times (0,\pi) :\to R$ be continuous on $[0,b_0]$.

Function $\mathcal{H}$, satisfies the condition

$$|\mathcal{H}(y,s_1,\phi_1,\psi_1) - \mathcal{H}(y,s_2,\phi_2,\psi_2)| \leq L[|s_1 - s_2| + |\phi_1 - \phi_2| + |\psi_1 - \psi_2|],$$

where $L > 0$ is a constant

System (9) is a generalization of the wave equation with impulsive conditions. This system represents the acoustic wave propagation through human tissues, sediments, rock layers etc.

To write the problem (9) in abstract form, we define an operator $\mathcal{A}$ by

$$\mathcal{A}w = w''.$$

The domain of $\mathcal{A}$, $\mathcal{D}(\mathcal{A})$ is given as follows. If $w \in \mathcal{D}(\mathcal{A})$, then $w \in L^2(0,\pi), w'' \in L^2(0,\pi)$, and $w(0) = w(\pi) = 0$. $\mathcal{A}$ generates a strongly continuous cosine family on $L^2(0,\pi)$ (see [4]). Therefore, $\mathcal{A}$ also generates an analytic semigroup (see [1]). If we take $\beta = \frac{1}{3}$, then the fractional power $\mathcal{A}^{\frac{1}{3}}$ is well defined (see [1]). $\Big(\mathcal{D}(\mathcal{A}^{\frac{1}{3}}), \|.\|_{\frac{1}{3}}\Big)$ is a Banach space, where for $w \in \mathcal{D}(\mathcal{A})$,

$$\|w\|_{\frac{1}{3}} = \|\mathcal{A}^{\frac{1}{3}} w\|.$$

We denote this Banach space by $Y_{\frac{1}{3}}$.

Let $PC\Big([0,b_0], Y_{\frac{1}{3}}\Big)$ denote the set of all piecewise continuous functions on $[0,b_0]$, and $\Omega_{\frac{1}{3}}^{b_0} = \Big\{y \mid y, y' \in PC\Big([0,b_0], Y_{\frac{1}{3}}\Big)$ such that $y(s), y'(s)$ are left-continuous at $s = s_i$

and the right-hand limit of $y(s), y'(s)$ exists at $s = s_i,\ i \in \mathbb{N},\ 1 \leq i \leq q\Big\}$. Eventually, $\big(\Omega^{b_0}_{\frac{1}{3}}, \|.\|_{\beta,b_0}\big)$ is a Banach space, where

$$\|\psi\|_{\frac{1}{3},b_0} = \sup_{s\in[0,b_0]} \|\psi(s)\|_{\frac{1}{3}}, \quad \psi \in \Omega^{b_0}_{\frac{1}{3}}.$$

If we also define $w(s)(y) = w(y,s),\ \chi(s)(y) = \chi(y,s),\ \nu(t) = t-\tau,\ h(s, w(s), w(s-\tau))(t) = \mathcal{H}(y, s, w(y,s), w(y, s-\tau))$, and $Ev : [0, b_0] \to L^2(0,\pi)$, by $(Ev)(t)(y) = \mu_0(s,y)$, then the abstract formulation of the problem (9) is

$$\begin{cases} {}_CD^{\frac{4}{3}}w(s) = Aw(s) + Ev(s) + h(s, w(s), w(\nu(s))), \quad t \in [0, b_0],\ s \neq s_i, \\ \Delta w(s_i) = \hat{I}_i(w(s_i)), \quad \Delta w'(s_i) = \hat{J}_i(w(s_i)), \quad i \in N,\ 1 \leq i \leq q, \\ u(s) = \chi(s), \quad s \in [-\tau, 0]. \end{cases}$$

It can be easily shown that all the assumptions of Theorem 3.1 are satisfied. Therefore, we conclude that the control system (9) is controllable.

5 Conclusion

In this paper, we proved the controllability for a class of fractional impulsive differential equations in a Banach space X. An associated integral equation is obtained by using the fractional integral and the family of cosines of linear operators, and then by using the measure of non-compactness and Mönch's fixed point theorem, we proved the existence of mild solution and controllability of the problem. In the last section, we presented an example to illustrate the abstract results.

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References

[1] A. Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations*. Springer-Verlag, New York, 1983.

[2] A. Shukla, N. Sukavanam, D. N. Pandey and U. Arora. Approximate controllability of second order semilinear control system. *International Journal of Circuits, Systems and Signal Processing* **35** (2016) 3339–3354.

[3] C. Ravichandran and D. Baleanu. On the controllability of fractional functional integro-differential systems with an infinite delay in Banach spaces. *Advances in Differential Equations* **291** (2013) 1–13.

[4] E. Bazhlekova. *Fractional Evolution Equations in Bannch Spaces*. University Press Facilities, Eindhoven University of Technology, 2001.

[5] E. H. Lakhel. Controllability of neutral functional differential equations driven by fractional brownian motion with infinite delay. *Nonlinear Dynamics and Systems Theory* **17** (3) (2017) 291–302.

[6] G. Arthi and K. Balachandran. Controllability of impulsive second-order nonlinear system with nonlocal conditions in Banach spaces. *Journal of Control and Decision* **2** (3) (2015) 203–218.

[7] J. R. Wang, Z. Fan and Y. Zhou. Nonlocal controllability of semilinear dynamic system with fractional derivative in Banach spaces. *Journal of Optimization Theory and Applications* **154** (2012) 292–302.

[8] L. Kexue, P. Jigen and G. Jinghuai. Controllability of nonlocal fractional differential systems of order $\alpha \in (1,2]$ in Banach spaces. *Reports on Mathematical Physics* **71** (1) (2013) 33–43.

[9] L. Muglia and P. Pietramala. Second order impulsive differential equations with functional initial condition on unbounded intervals. *Journal of Function Spaces and Applications* (2013), Article ID 479049, 1–14.

[10] S. Ji, F. Yang and M. Wang. Controllability of impulsive differential system with nonlocal conditions. *Applied Mathematics and Computation* **217** (2011) 6981–6989.

[11] S. Kumar. Approximate controllability of nonlocal impulsive fractional order semilinear time varying delay systems. *Nonlinear Dynamics and Systems Theory* **16** (4) (2016) 420–430.

[12] U. Arora and N. Sukavanam. Approximate controllability of non-densely defined semilinear control system with nonlocal conditions. *Nonlinear Dynamics and Systems Theory* **17** (1) (2017) 5–18.

[13] Z. Tai and X. Wang. Controllability of fractional-order impulsive neutral functional infinite delay integrodifferential systems in Banach spaces. *Applied Mathematics Letters* **22** (2009) 1760–1765.

[14] Z. Tai. Controllability of fractional impulsive neutral integrodifferential system with a nonlocal Cauchy condition in Banach spaces. *Applied Mathematics Letters* **24** (2011) 2158–2161.

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Equivalent Conditions and Persistence for Uniformly Exponential Dichotomy

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Abstract: The purpose of this paper is to provide equivalence conditions of existing conditions for the uniformly exponential dichotomy of strongly continuous quasi groups (C_0-quasi groups) on Banach spaces. There are four equivalent conditions for the existence of uniformly exponential dichotomy in the used classes of continuous and integrable function spaces over $\mathbb{R}$. Each condition emphasizes the existence and uniqueness of mild solutions of the corresponding inhomogeneous equation on the corresponding space in the C_0-quasi group term. The results are parallel with the dichotomy for the evolution family. Moreover, a small time-dependent perturbation of the infinitesimal generator of the C_0-quasi groups persists the uniformly exponential dichotomy. The results are also motivated by illustrative examples.

Keywords: *strongly continuous quasi semigroup; uniformly exponential dichotomy; mild solution; time-dependent perturbation; persistence.*

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1 Introduction

As a generalization of exponential stability and dichotomy for the evolution family [1, 2], the Dichotomy Theorem of C_0-quasi groups on Banach spaces has just been developed in [3], see Theorem 4. The theorem implies that a uniformly exponential dichotomy of the C_0-quasi groups on Banach spaces X is equivalent to the spectral property of the corresponding evolution semigroup on $L_p(\mathbb{R}, X)$. Besides, the uniformly exponential dichotomy is also equivalent to the existence and uniqueness of Green's function for the quasi group, Theorem 9 of [3]. The uniformly exponential stability in this paper refers to the term in [4–6].

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Consider a non-autonomous abstract Cauchy problem

$$\dot{u}(t) = A(t)u(t), \quad u(r) = u_r, \quad u_r \in \mathcal{D}, \quad t \geq r, \quad t, r \in \mathbb{R}, \tag{1}$$

where $A(t)$ is a linear closed operator in X with the domain $\mathcal{D}(A(t)) = \mathcal{D}$ being independent of t and dense in X. Assume that (1) is well-posed in the sense that there exists a quasi group $\{R(t,s\}_{t,s\in\mathbb{R}}$ which gives a differential function u [7,8]. In fact, if $u_r \in \mathcal{D}$, then $u(t) = R(r, t-r)u_r$, $t \geq r$, is a solution of (1) and $u(t) \in \mathcal{D}$. This confirms that the uniformly exponential dichotomy is a fundamental asymptotic property of the solutions of (1). The important examples of the finite cases of (1) are given in [9,10].

Next, consider the inhomogeneously non-autonomous abstract Cauchy problem

$$\dot{u}(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R}, \tag{2}$$

where f is a locally integrable X-valued function on $\mathbb{R}$. It can be verified that the function u, which satisfies the integral equation

$$u(t) = R(r, t-r)u(r) + \int_r^t R(s, t-s)f(s)ds, \quad t \geq r, \tag{3}$$

is a solution (mild solution) of (2). In particular, this confirms that the uniformly exponential dichotomy of solutions of the non-autonomous abstract Cauchy problem (1) is equivalent to the existence and uniqueness of the mild solution of the inhomogeneous equation (2) for some integrable functions f. In other words, it allows to characterize the uniformly exponential dichotomy for the quasi groups in terms of "Perron-type" theorems of classes of integrable function spaces over $\mathbb{R}$. These questions are the counterpart of the classical theorems of the Perron type for the evolutionary families with varying $A(t)$ and classes of f's are discussed in [11–17].

In [7, 18], a bounded time-dependent perturbation under certain conditions of an infinitesimal generator of C_0-quasi semigroups produces a perturbed C_0-quasi semigroup on the same space. The classical and mild solutions of the new non-autonomous abstract Cauchy problem induced by the perturbed infinitesimal generator retain dependence on the similar solutions of the old problem. A question arises whether this situation applies to the quasi groups. Further, if the old quasi groups have a uniformly exponential dichotomy, whether the perturbed quasi group also persists the uniformly exponential dichotomy. As a comparison, under time-dependent Miyadera-type perturbations, the evolution family persists the uniformly exponential dichotomy [1,2].

This paper focuses on characterizations of the equivalent conditions for the uniformly exponential dichotomy of the C_0-quasi group using classes of integrable function spaces and investigates the persistence of the uniformly exponential dichotomy due to the time-dependent perturbations. The organization of this paper is as follows. In Section 2, re-exposure of the existing results for the uniformly exponential dichotomy of the C_0-quasi groups on a Banach space is considered. Characterizations for the uniformly exponential dichotomy using four spaces of $C_b(\mathbb{R}, X)$, $C_0(\mathbb{R}, X)$, $L_p(\mathbb{R}, X)$, and a scale space of continuous functions $\mathcal{F}_\alpha$ are considered in Section 3. Section 4 investigates the persistence for the uniformly exponential dichotomy under a bounded time-dependent perturbation of the infinitesimal generator.

2 Preliminaries

In this section, we recall the results about the sufficient and necessary conditions for the uniformly exponential dichotomy of the strongly continuous quasi groups on Banach spaces [3]. The quasi group itself is a generalization of the strongly continuous quasi semigroup [19].

Definition 2.1 (Definition 1 [3]) *Let $\mathcal{L}(X)$ be the set of all bounded linear operators on a Banach space X. A two-parameter commutative family $\{R(t,s)\}_{s,t\in\mathbb{R}}$ in $\mathcal{L}(X)$ is called a strongly continuous quasi group (C_0-quasi group) on X if for each $r,s,t \in \mathbb{R}$ and $x \in X$:*

(a) $R(t,0) = I$, *the identity operator on* X,

(b) $R(t,s+r) = R(t+r,s)R(t,r)$,

(c) $\lim_{s\to 0} \|R(t,s)x - x\| = 0$,

(d) there is a continuous increasing function $M : \mathbb{R} \to [1,\infty)$ *such that*

$$\|R(t,s)\| \le M(t+s).$$

Let $\mathcal{D}$ be the set of all $x \in X$ such that the following limits exist:

$$\lim_{s\to 0} \frac{R(t,s)x - x}{s}, \quad s,t \in \mathbb{R}.$$

For $t \in \mathbb{R}$, we define an operator $A(t)$ on $\mathcal{D}$ as

$$A(t)x = \lim_{s\to 0} \frac{R(t,s)x - x}{s}.$$

The family of operators $\{A(t)\}_{t\in\mathbb{R}}$ is called an infinitesimal generator of the C_0-quasi group $\{R(t,s)\}_{s,t\in\mathbb{R}}$. In what follows, for simplicity, we denote the quasi group $\{R(t,s)\}_{s,t\in\mathbb{R}}$ and the family $\{A(t)\}_{t\in\mathbb{R}}$ by $R(t,s)$ and $A(t)$, respectively.

We have identified the dichotomy for the C_0-quasi groups using uniformly exponential stability, an extension of the similar term for C_0-quasi semigroups [18].

Definition 2.2 (Definition 2 [3]) *A C_0-quasi group $R(t,s)$ is said to be uniformly exponentially stable on a Banach space X if there exist constants $\gamma > 0$ and $N \ge 1$ such that*

$$\|R(t,s)x\| \le Ne^{-\gamma|s|}\|x\|, \quad t,s \in \mathbb{R}, \quad x \in X. \tag{4}$$

Definition 2.3 *The C_0-quasi group $R(t,s)$ is said to be exponentially bounded on a Banach space X if there exist a constant $\omega \in \mathbb{R}$ and a function $N_\omega : \mathbb{R}^+ \to [1,\infty)$ such that*

$$\|R(t,s)x\| \le N_\omega(t)e^{\omega|s|}\|x\|, \quad t,s \in \mathbb{R}, \quad x \in X.$$

Sometimes, we have to convert a quasi-group to be an evolution semigroup. For example, the uniformly exponential stability for a quasi-group is more easily identified by the spectrum of the infinitesimal generator of the corresponding evolution semigroup. For a Banach space X, $L_p(\mathbb{R},X)$, $1 \le p < \infty$, denotes the space of all functions $f : \mathbb{R} \to X$ with the norm $\|f\|_{L_p(\mathbb{R},X)} = \left(\int_{-\infty}^{\infty} \|f(t)\|_X^p dt\right)^{\frac{1}{p}}$. Henceforth, in this paper we always assume that $L_p(\mathbb{R},X)$ with $1 \le p < \infty$.

Definition 2.4 (Definition 3 [3]) *Let $R(t,s)$ be a C_0-quasi group on a Banach space X. The evolution semigroup associated with $R(t,s)$ on $L_p(\mathbb{R},X)$ is a family of operators $\{E^s\}_{s\geq 0}$ given by*

$$(E^s f)(t) = R(t-s,s)f(t-s), \quad s \geq 0, \quad t \in \mathbb{R}, \quad f \in L_p(\mathbb{R},X). \tag{5}$$

For simplicity, the evolution semigroup $\{E^s\}_{s\geq 0}$ is simply written as E^s. We see that E^s is strongly continuous on $L_p(\mathbb{R},X)$. Moreover, if $A(t)$ is the infinitesimal generator of the C_0-quasi group $R(t,s)$ with domain $\mathcal{D}$, then an operator Γ defined by

$$(\Gamma f)(t) = -\frac{df}{dt} + A(t)f(t), \quad t \in \mathbb{R}, \tag{6}$$

is the infinitesimal generator of E^s with the domain

$$\mathcal{D}(\Gamma) = \{f \in L_p(\mathbb{R},X) : f \text{ is absolutely continuous}, \ f(t) \in \mathcal{D}\}.$$

The uniformly exponential dichotomy for the C_0-quasi groups is an extension of the similar term for the C_0-quasi semigroups introduced by Cuc [4]. Let $P : \mathbb{R} \to \mathcal{L}(X)$ be a projection-valued function, the complementary projection is given by $Q(t) = I - P(t)$ for all $t \in \mathbb{R}$. If $P(t+s)R(t,s) = R(t,s)P(t)$, then

$$R_P(t,s) := P(t+s)R(t,s)P(t) \quad \text{and} \quad R_Q(t,s) := Q(t+s)R(t,s)Q(t)$$

are the restrictions of $R(t,s)$ on $\operatorname{ran} P(t)$ and $\operatorname{ran} Q(t)$, respectively. The $R_P(t,s)$ is the operator from $\operatorname{ran} P(t)$ to $\operatorname{ran} P(t+s)$, while $R_Q(t,s)$ maps $\operatorname{ran} Q(t)$ to $\operatorname{ran} Q(t+s)$.

Definition 2.5 (Definition 4 [3]) *The C_0-quasi group $R(t,s)$ is said to have a uniformly exponential dichotomy on X if there exist constants $N \geq 1$, $\gamma > 0$ and a projection-valued function $P : \mathbb{R} \to \mathcal{L}(X)$ such that for each $x \in X$, the function $x \mapsto P(t)x$ is continuous and bounded, and, for all $t, s \in \mathbb{R}$, the following conditions hold:*

(a) $P(t+s)R(t,s) = R(t,s)P(t)$,

(b) $R_Q(t,s)$ *is invertible as an operator from* $\operatorname{ran} Q(t)$ *to* $\operatorname{ran} Q(t+s)$,

(c) $\|R_P(t,s)\| \leq Ne^{-\gamma|s|}$,

(d) $\|[R_Q(t,s)]^{-1}\| \leq Ne^{-\gamma|s|}$.

The pair of γ and N in Definition 2.5 is called the dichotomy constants of $R(t,s)$. Definition 2.5 states that if the quasi group $R(t,s)$ has a uniformly exponential dichotomy on X, then $R(t,s)$ and $R^{-1}(t,s)$ are uniformly exponentially stable on $\operatorname{ran} P(t)$ and on $\operatorname{ran} Q(t)$, respectively. The dichotomy bound of $R(t,s)$ is defined as

$$\begin{aligned}\gamma(R) := \sup\{\gamma > 0 : R(t,s) &\text{ has exponential dichotomy} \\ &\text{with constants } \gamma \text{ and } N = N(\gamma)\}.\end{aligned} \tag{7}$$

The sufficient and necessary conditions for the uniformly exponential dichotomy of the C_0-quasi groups are given by the following theorems.

Theorem 2.1 (Dichotomy Theorem, Theorem 4 [3]) *Assume that $R(t,s)$ is a C_0-quasi group on a Banach space X. Let E^s be the corresponding evolution semigroup given by* (5) *on $L_p(\mathbb{R},X)$ and let Γ denote its infinitesimal generator given by* (6). *The following statements are equivalent:*

(a) The quasi group $R(t,s)$ has a uniformly exponential dichotomy on X.

(b) For each $s > 0$, $\sigma(E^s) \cap \mathbb{T} = \emptyset$, where $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

(c) $0 \in \rho(\Gamma)$.

Let $C_b(\mathbb{R}, X)$ be the space of all bounded continuous functions $f : \mathbb{R} \to X$ with the supremum norm. Let $P(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$ be the projection that satisfies (a) and (b) of Definition 2.5. Green's function for $R(t,s)$ is a map $G_P : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathcal{L}_s(X)$ defined by

$$G_P(t,s) = R_P(t,s)P(t), \quad t > s,$$
$$G_P(t,s) = -[R_Q(t,s)]^{-1}Q(t), \quad t < s.$$

Green's operator $\mathbb{G}$ associated with G_P on $L_p(\mathbb{R}, X)$ is defined by

$$(\mathbb{G}f)(t) = \int_{-\infty}^{\infty} G_P(s, t-s)f(s)ds, \quad f \in L_p(\mathbb{R}, X). \tag{8}$$

Theorem 2.2 (Theorem 9 [3]) *Let Γ be the infinitesimal generator of the evolution semigroup E^s corresponding to a C_0-quasi group $R(t,s)$ defined by (5) on $L_p(\mathbb{R}, X)$. The quasi group $R(t,s)$ has a uniformly exponential dichotomy on X if and only if there exists a unique Green's function G_P for $R(t,s)$. Moreover, if the associated Green's operator is given by (8), then $\mathbb{G} = -\Gamma^{-1}$ on $L_p(\mathbb{R}, X)$.*

We summarize that the sufficient and necessary conditions for a C_0-quasi group to have a uniformly exponential dichotomy are that the corresponding evolution semigroup is hyperbolic. Moreover, the dichotomy is equivalent to the uniqueness of Green's function for the C_0-quasi group.

3 Equivalent Conditions for Uniformly Exponential Dichotomy

In the section, we shall characterize the others equivalent conditions for the uniformly exponential dichotomy of the C_0-quasi groups. The characterizations refer to the method used in [1, 13] for the family of the evolution operators.

We start with defining Green's operator $\mathbb{G}$ for the C_0-quasi group $R(t,s)$ as in (8) on $C_b(\mathbb{R}, X)$ by

$$(\mathbb{G}f)(t) = \int_{-\infty}^{\infty} G_P(s, t-s)f(s)ds, \quad f \in C_b(\mathbb{R}, X). \tag{9}$$

We see that $\mathbb{G}$ is a bounded operator on $C_b(\mathbb{R}, X)$.

Condition (M). For each $g \in C_b(\mathbb{R}, X)$, there exists a unique function $u \in C_b(\mathbb{R}, X)$ such that

$$u(t) = R(r, t-r)u(r) + \int_r^t R(s, t-s)g(s)ds, \quad t \geq r. \tag{10}$$

Remark 3.1 *Condition (M) states that for each $g \in C_b(\mathbb{R}, X)$, there exists a unique mild solution $u \in C_b(\mathbb{R}, X)$ of the integral equation (10). Thus, if we define an operator*

$Gg = u$ on $C_b(\mathbb{R}, X)$, then G is closed. In fact, if $g_n \to g$ and $u_n := Gg_n \to u$ in $C_b(\mathbb{R}, X)$, then for each $t \in \mathbb{R}$,

$$u(t) = \lim_{n\to\infty} u_n(t) = \lim_{n\to\infty}\left(R(r, t-r)u_n(r) + \int_r^t R(s, t-s)g_n(s)ds\right)$$
$$= R(r, t-r)u(r) + \int_r^t R(s, t-s)g(s)ds.$$

This gives $u = Gg$.

In particular, if $R(t, s)$ is uniformly exponentially dichotomic, then G is equal to Green's operator $\mathbb{G}$ in (9).

Lemma 3.1 *If Green's operator $\mathbb{G}$ defined in* (9) *is bounded on $C_b(\mathbb{R}, X)$, then for each $g \in C_b(\mathbb{R}, X)$, there exists a solution $u \in C_b(\mathbb{R}, X)$ of* (10).

Proof. For $g \in C_b(\mathbb{R}, X)$, we set $u := \mathbb{G}g$. For $t \geq r$, we show that u satisfies (10). In this proof, we use the fact that $R^{-1}(k, l-k) = R(l, k-l)$. For $t \geq r$,

$$\begin{aligned} u(t) - R(r, t-r)u(r) &= (\mathbb{G}g)(t) - R(r, t-r)(\mathbb{G}g)(r) \\ &= \int_r^t P(t)R(s, t-s)P(s)g(s)ds - \int_t^\infty R_Q^{-1}(s, t-s)Q(s)g(s)ds \\ &\quad + \int_r^t R(s, t-s)R(r, s-r)R_Q^{-1}(s, r-s)Q(s)g(s)ds \\ &\quad + \int_t^\infty R(r, t-r)[R_Q(t, r-t)R_Q(s, t-s)]^{-1}Q(s)g(s)ds \\ &= \int_r^t P(t)R(s, t-s)P(s)g(s)ds + \int_r^t R(s, t-s)Q(s)g(s)ds \\ &= \int_r^t R(s, t-s)g(s)ds. \end{aligned}$$

As a generalization of Theorem 10 from [18], we have the following lemma which implies that the infinitesimal generator Γ is invertible on $L_p(\mathbb{R}, X)$.

Lemma 3.2 *Let E^s be the evolution semigroup defined in* (5) *on $L_p(\mathbb{R}, X)$ with its infinitesimal generator Γ in* (6)*. If $u, g \in L_p(\mathbb{R}, X)$, then the following statements are equivalent.*

(a) $u \in \mathcal{D}(\Gamma)$ dan $\Gamma u = -g$.

(b) u is a solution of the integral equation (10) *that corresponds to g.*

Proof. (a) $\Rightarrow$ (b). Assume that (a) holds. By an elementary property of C_0-semigroup, we have

$$E^s u - u = \int_0^s E^r \Gamma u dr = -\int_0^s E^r g\, dr, \quad s \geq 0. \tag{11}$$

Substituting $(E^s u)(t) = R(t-s, s)u(t-s)$ (definition of E^s) into (11) gives

$$R(t-s, s)u(t-s) - u(t) = -\int_0^s R(t-v, v)g(t-v)\, dv.$$

The transformation of variable $r = t - s$ gives statement (b).

(b)$\Rightarrow$ (a). Assume that (b) holds. If $s \geq 0$, $t - s \geq r$, and u is a solution of (10), then

$$\begin{aligned}(E^s u)(t) &= R(t-s,s)\left[R(r,t-s-r)u(r) + \int_r^{t-s} R(v,t-s-v)g(v)\,dv\right]\\ &= R(r,t-r)u(r) + \int_r^{t-s} R(v,t-v)g(v)\,dv.\end{aligned}$$

Consequently, for $s > 0$, we obtain

$$\begin{aligned}s^{-1}\left[(E^s u)(t) - u(t)\right] &= s^{-1}\Big[R(r,t-r)u(r) + \int_r^{t-s} R(v,t-v)g(v)\,dv\\ &\qquad - \left(R(r,t-r)u(r) + \int_r^{t} R(v,t-v)g(v)\,dv\right)\Big]\\ &= -s^{-1}\int_{t-s}^{t} R(v,t-v)g(v)\,dv = -s^{-1}\int_0^s R(t-v,v)g(t-v)\,dv.\end{aligned}$$

Therefore,

$$s^{-1}(E^s u - u) = -s^{-1}\int_0^s E^v g\,dv.$$

Passing to the limit as $s \to 0^+$ proves that $u \in \mathcal{D}(\Gamma)$ and $\Gamma u = -g$.

Remark 3.2 *Lemma 3.2 remains valid if $L_p(\mathbb{R}, X)$ is replaced by $C_0(\mathbb{R}, X)$, the space of all continuous functions $f : \mathbb{R} \to X$ such that $\lim_{t\to\pm\infty} f(t) = 0$ with the supremum norm. Moreover, Condition (M) holds for some $g, u \in L_p(\mathbb{R}, X)$.*

Theorem 3.1 *An exponentially bounded C_0-quasi group $R(t, s)$ on a Banach space X has a uniformly exponential dichotomy if and only if Condition (M) is satisfied.*

Proof. ($\Rightarrow$). Let $R(t, s)$ be uniformly exponentially dichotomic. By Theorem 9 of [3], there exists Green's operator $\mathbb{G}$ as defined in (9) corresponding to Green's function G_P and dichotomy projection P. Lemma 3.1 guarantees the existence of a solution $u \in C_b(\mathbb{R}, X)$ of (10) for each $g \in C_b(\mathbb{R}, X)$.

To prove the uniqueness of the solution of (10), let $g = 0$ and suppose there exists $u \in C_b(\mathbb{R}, X)$ such that $u(t) = R(r, t - r)u(r), t \geq r$. It suffices to prove that $u = 0$. The uniformly exponential dichotomy of $R(t, s)$ implies

$$P(t)u(t) = R_P(r,t-r)P(r)u(r) \quad \text{and} \quad Q(t)u(t) = R_Q(r,t-r)Q(r)u(r), \quad t \geq r.$$

The boundedness of $\|u(\cdot)\|$ and condition (c) of Definition 2.5 give

$$\|P(t)u(t)\| \leq Ne^{-\gamma(t-r)}\|u(r)\|.$$

Passing to the limit as $r \to -\infty$ provides that $P(t)u(t) = 0$ for all $t \in \mathbb{R}$. On the other hand, condition (d) of Definition 2.5 forces

$$\|Q(r)u(r)\| = \|[R_Q(r,t-r)]^{-1}Q(t)u(t)\| \leq Ne^{-\gamma(t-r)}\|u(t)\|.$$

Passing to the limit as $t \to \infty$ implies that $Q(r)u(r) = 0$ for all $r \in \mathbb{R}$. Therefore, $u = 0$.

($\Leftarrow$). Let Condition (M) be satisfied. We define an operator G on $C_b(\mathbb{R}, X)$ by $Gg = u$. By Theorem 2.1, it suffices to show that Γ is invertible on $C_b(\mathbb{R}, X)$. Since $u = Gg$ and $g \in L_p(\mathbb{R}, X)$, Lemma 3.2 implies that $u \in \mathcal{D}(\Gamma)$ and $\Gamma(-G)g = \Gamma(-u) = g$. Thus, Γ is right invertible. On the other hand, the linearity of G implies that $(-G)\Gamma u = (-G)(-g) = u$. This proves the left invertibility of Γ. Thus, Γ is invertible with $\Gamma^{-1} = -G$.

We shall characterize the other conditions for the uniformly exponential dichotomy of the quasi groups. We start with defining the scale of function space $\mathcal{F}_\alpha$, $\alpha > 0$, by

$$\mathcal{F}_\alpha := \{f \in C(\mathbb{R}, X) : e^{-\alpha|\cdot|} f(\cdot) \in C_b(\mathbb{R}, X)\}.$$

Thus, $\mathcal{F}_\alpha$ is the space of continuous, exponentially bounded functions with exponent α. These spaces provide three conditions formulated as follows.

Condition ($\mathbf{M}_{C_0}$). For each $g \in C_0(\mathbb{R}, X)$, the integral equation (10) has a unique solution $u \in C_0(\mathbb{R}, X)$.

Condition ($\mathbf{M}_{L_p}$). For each $g \in L_p(\mathbb{R}, X)$, $1 \le p \le \infty$, the integral equation (10) has a unique solution $u \in L_p(\mathbb{R}, X)$.

Condition ($\mathbf{M}_{\mathcal{F}_\alpha}$). For each $g \in \mathcal{F}_\alpha$, the integral equation (10) has a unique solution $u \in \mathcal{F}_\alpha$.

Theorem 3.2 *Let $R(t, s)$ be an exponentially bounded C_0-quasi group on X.*

(a) The following statements are equivalent:

(i) $R(t, s)$ has uniformly exponential dichotomy.

(ii) Condition (M) holds.

(iii) Condition (M_{C_0}) holds.

(iv) Condition (M_{L_p}) holds.

(b) The operator G defined by Conditions (M), (M_{C_0}), or (M_{L_p}) as in Remark 3.1, is equal to Green's operator $\mathbb{G}$ as in (9)*. Further, if E^s is the evolution semigroup on the space $C_0(\mathbb{R}, X)$ or $L_p(\mathbb{R}, X)$ with the infinitesimal generator Γ, then $G = -\Gamma^{-1}$.*

Proof. Theorem 3.1 guarantees that Condition (M) is equivalent to (i).

Let G be an operator defined using Condition (M_{C_0}) (resp. (M_{L_p})) as in Remark 3.1. Lemma 3.2 together with Dichotomy Theorem 2.1 implies the uniformly exponential dichotomy for $R(t, s)$. These show that (iii)(resp. (iv)) is equivalent to (i).

If $R(t, s)$ has a uniformly exponential dichotomy, then by Theorem 2.2, Green's operator $\mathbb{G}$ is defined on $L_p(\mathbb{R}, X)$ or $C_0(\mathbb{R}, X)$ satisfies $\mathbb{G} = -\Gamma^{-1}$. Moreover, using the same argument as in the proof of the necessity of Theorem 3.1, we conclude that (M_{C_0}) and (M_{L_p}) hold, and $G = \mathbb{G}$.

Lemma 3.3 *Condition ($M_{\mathcal{F}_\alpha}$) holds for $R(t, s)$ if and only if Condition (M) holds for $R_\alpha(t, s)$, where $R_\alpha(t, s) = e^{-\alpha(|t+s|-|t|)} R(t, s)$ and $\alpha \in [0, \beta)$ for some $\beta > 0$.*

Proof. If Condition (M) holds for $R_\alpha(t, s)$, there exists a bounded operator G_α on $C_b(\mathbb{R}, X)$ defined by $G_\alpha g = u$. We define an operator $J_\alpha : \mathcal{F}_\alpha \to C_b(\mathbb{R}, X)$ by $(J_\alpha f)(t) = e^{-\alpha|t|} f(t), t \in \mathbb{R}$. Similarly, if Condition ($\mathrm{M}_{\mathcal{F}_\alpha}$) holds for $R(t, s)$, then there exists a bounded operator $G \in \mathcal{L}(\mathcal{F}_\alpha)$ defined by $Gg = u$. We see that $G_\alpha = J_\alpha G J_\alpha^{-1}$. Thus, Condition (M) holds for $R_\alpha(t,s)$ if and only if $G_\alpha \in \mathcal{L}(\mathbb{R}, X)$. However, $G \in \mathcal{L}(\mathcal{F}_\alpha)$ if and only if Condition ($\mathrm{M}_{\mathcal{F}_\alpha}$) holds for $R(t,s)$.

Theorem 3.3 *Let $R(t,s)$ be an exponentially bounded C_0-quasi group on X. The quasi group $R(t,s)$ has a uniformly exponential dichotomy if and only if there exists $\beta > 0$ such that if $\alpha \in [0, \beta)$, then Condition ($M_{\mathcal{F}_\alpha}$) holds for $R(t,s)$. Moreover, for each $\alpha > 0$ and $g \in \mathcal{F}_\alpha$, the solution of the integral equation* (10) *is given by $u = Gg$, where $G \in \mathcal{L}(\mathcal{F}_\alpha)$ is equal to Green's operator $\mathbb{G}$ on $\mathcal{F}_\alpha$ as defined in* (9).

Proof. ($\Leftarrow$). If $\alpha = 0$, then Condition ($\mathrm{M}_{\mathcal{F}_\alpha}$) and Condition (M) are identical.

($\Rightarrow$). Assume that $R(t,s)$ has a uniformly exponential dichotomy with the dichotomy bound $\gamma > 0$. If $\beta \in (0, \gamma)$, then $R(t,s)$ has a uniformly exponential dichotomy with constants β and $N = N(\beta)$, see (7). Consequently, if $\alpha \in [0, \beta)$, then the quasi group $R_\alpha(t,s)$ defined in Lemma 3.3 has a uniformly exponential dichotomy with constants $N(\beta)$ and $\beta - \alpha$. Theorem 3.1 provides that Condition (M) holds for $R_\alpha(t,s)$. Let $G \in \mathcal{L}(\mathcal{F}_\alpha)$ be the operator defined by $Gg = u$. Since $G_\alpha = \mathbb{G}_\alpha$, where $\mathbb{G}_\alpha$ is Green's operator for the dichotomic quasi group $R_\alpha(t,s)$ and G_α is as in the proof of Lemma 3.3, the assertions follow.

Remark 3.3 *We note that conditions (M), (M_{C_0}), (M_{L_p}), and ($M_{\mathcal{F}_\alpha}$) for the uniformly exponential dichotomy of the C_0-quasi groups are parallel with the similar conditions for exponential dichotomy of the evolution family, see [1, 13].*

Example 3.1 Let $X = \mathbb{R}^2$ and $\varphi : \mathbb{R} \to \mathbb{R}$ be a continuous increasing function such that $\lim_{t\to\pm\infty} \varphi(t) < \infty$. Define a C_0-quasi group on X by

$$R(t,s)x = \left(e^{-(v(t+s)-v(t))}x_1, e^{-s\varphi(0)+v(t+s)-v(t)}x_2\right), \quad t, s \in \mathbb{R},$$

where $v(t) = \int_0^t \varphi(s)ds$ and $x = (x_1, x_2)$. The quasi group $R(t,s)$ has a uniformly exponential dichotomy on X.

Similar to Example 3 of [3], we have the evolution semigroup E^s in (5) on the space $L_p(\mathbb{R}, X)$ given by

$$(E^s f)(t) = \left(e^{-(v(t)-v(t-s))}f_1(t-s), e^{-s\varphi(0)+v(t)-v(t-s)}f_2(t-s)\right),$$

where $f(t) = (f_1(t), f_2(t))$, $s \geq 0$, and $t \in \mathbb{R}$ with the infinitesimal generator

$$(\Gamma f)(t) = (-f_1'(t) - \varphi(t)f_1(t), -f_2'(t) + [-\varphi(0) + \varphi(t)]f_2(t)).$$

Moreover,

$$(\Gamma^{-1} f)(t) = -(h_1(t), h_2(t)),$$

where

$$h_1(t) = e^{-\phi(t)} \int f_1(t)e^{\phi(t)}dt, \qquad h_2(t) = e^{-\varphi(0)t+\phi(t)} \int f_2(t)e^{\varphi(0)t-\phi(t)}dt,$$

$$\phi(t) = \int \varphi(t)\, dt.$$

By Condition (M), for each $g \in C_b(\mathbb{R}, X)$, there exists a unique solution $u \in C_b(\mathbb{R}, X)$ satisfying the integral equation (10). In fact, we have $u = -\Gamma^{-1}g$. Therefore, $R(t,s)$ has a uniformly exponential dichotomy on X.

Remark 3.4 *We can easily verify that Example 3.1 fulfills Conditions (M_{C_0}), (M_{L_p}) and ($M_{\mathcal{F}_\alpha}$). It is possible that Condition ($M_{\mathcal{F}_\alpha}$) holds for some $\alpha \in (0,\gamma)$, but the quasi group $R(t,s)$ has no uniformly exponential dichotomy, as shown by the following example.*

Example 3.2 Let X be a Banach space of $\mathbb{R}^2$ with the norm $\|x\| = |x_1| + |x_2|$, where $x = (x_1, x_2)$. The quasi group $R(t,s)$ defined on X by

$$R(t,s)x = \left(e^{(t+s)\cos(t+s)-t\cos t-s}x_1, e^s x_2\right), \qquad t,s \in \mathbb{R},$$

has no uniformly exponential dichotomy, but it satisfies Condition ($\mathrm{M}_{\mathcal{F}_\alpha}$) for all $g \in M_{\mathcal{F}_\alpha}$ and $0 < \alpha < 2$.

From Lemma 3.3, it suffices to show that $R_\alpha(t,s)$ satisfies Condition (M) for all $g \in C_b(\mathbb{R}, X)$. In fact, for $g = (g_1, g_2) \in C_b(\mathbb{R}, X)$ and $P(t)x = (x_1, 0)$, we can set $u = \mathbb{G}g$, where $\mathbb{G}$ is Green's operator defined in (9) with respect to $R_\alpha(t,s)$. For $0 < \alpha < 2$, we verify that

$$u(t) = (\mathbb{G}g)(t) = (u_1(t), u_2(t)) \in C_b(\mathbb{R}, X),$$

where

$$u_1(t) = e^{-\alpha|t|-t+t\cos t}\int_{-\infty}^{t} e^{\alpha|s|+s-s\cos s}g_1(s)ds,$$
$$u_2(t) = -e^{-\alpha|t|}\int_{t}^{\infty} e^{-\alpha|s|-s}g_2(s)ds.$$

Suppose that $R(t,s)$ has uniformly exponential dichotomy with respect to the family of projections $P(t)$ above. If N, $\gamma > 0$ are the constants satisfying Definition 2.5, i.e., $\|R_P(t,s)\| \le Ne^{-\gamma|s|}$, then

$$e^{(t+s)\cos(t+s)-t\cos t-s} \le Ne^{-\gamma|s|}$$

for all $t, s \in \mathbb{R}$. But for $t = (2n-1)\pi$ and $s = \pi$, we have $e^{2(2n-1)\pi} \le Ne^{-\gamma\pi}$, which is absurd for large enough n.

4 Persistence under Perturbation

Theorem 2.1 implies that the existence of a dichotomy for a strongly continuous quasi group $R(t,s)$ is a spectral property. It persists under small perturbations. We shall first consider the bounded perturbation.

Theorem 4.1 *Let $R(t,s)$ and $R_1(t,s)$ be the C_0-quasi groups on a Banach space X. If $R(t,s)$ has a uniformly exponential dichotomy on X, then for each $r > 0$, there exists an $\epsilon > 0$ such that $R_1(t,s)$ has a uniformly exponential dichotomy and*

$$\sup_{t\in\mathbb{R}} \|R_1(t,r) - R(t,r)\|_{\mathcal{L}(X)} \le \epsilon.$$

Proof. From (5), for $f \in L_p(\mathbb{R}, X)$, we have

$$(E^r f)(t) = R(t-r,r)f(t-r) \quad \text{and} \quad (E_1^r f)(t) = R_1(t-r,r)f(t-r).$$

We obtain the estimate

$$\begin{aligned}\|E_1^r f - E^r f\|_{L_p}^p &= \int_{\mathbb{R}} \|R_1(t-r,r)f(t-r) - R_1(t-r,r)f(t-r)\|^p dt \\ &= \int_{\mathbb{R}} \|[R_1(t,r) - R(t,r)]f(t)\|^p dt \le \epsilon^p \|f\|_{L_p}^p.\end{aligned}$$

This implies that $\|E_1^r - E^r\|_{\mathcal{L}(L_p(\mathbb{R},X))} \le \epsilon$.

The equivalence of (a) and (b) in the Dichotomy Theorem 2.1 gives $\sigma(E^r) \cap \mathbb{T} = \emptyset$. The semicontinuity of the spectrum implies that $\sigma(E_1^r) \cap \mathbb{T} = \emptyset$ for a sufficiently small ϵ. Therefore, $R_1(t,s)$ has a uniformly exponential dichotomy.

Theorem 4.1 describes that a dichotomy persists under small perturbation of the C_0-quasi groups. The similar result of the additive perturbation is given by the following theorem. The theorem refers to the perturbed generator of the C_0-quasi groups given below.

Theorem 4.2 *Let $A(t)$ be the infinitesimal generator of a C_0-quasi group $R(t,s)$ on a Banach space X. If $B \in C_b(\mathbb{R}, \mathcal{L}(X))$, then there exists a unique C_0-quasi group $R_B(t,s)$ with the infinitesimal generator $A(t) + B(t)$ such that*

$$R_B(t,r)x = R(t,r)x + \int_0^r R(t+s, r-s)B(t+s)R_B(t,s)x ds \tag{12}$$

for all $t \in \mathbb{R}$, $r > 0$, and $x \in X$. Moreover, if $\|R(t,r)\| \le M(r)$, then

$$\|R_B(t,r)\| \le M(r)e^{\|B\|M(r)r}.$$

Proof. The proof is similar to the proof of Theorem 3 of [18].

Theorem 4.3 *Let $R(t,s)$ be the C_0-quasi group with the infinitesimal generator $A(t)$ which has a uniformly exponential dichotomy on a Banach space X. Then, there exists $\epsilon > 0$ such that for each $B \in C_b(\mathbb{R}, \mathcal{L}(X))$ with $\|B\|_\infty := \sup_{t\in\mathbb{R}} \|B(t)\|_{\mathcal{L}(X)} \le \epsilon$, there exists a C_0-quasi group $R_B(t,s)$ with the infinitesimal generator $A(t) + B(t)$ which has a uniformly exponential dichotomy on X.*

Proof. From Theorem 4.2, there exists a C_0-quasi group $R_B(t,s)$ with the infinitesimal generator $A(t) + B(t)$. Further, by (12), for $t > r$ and $x \in X$, we have

$$R_B(r, t-r)x = R(r, t-r)x + \int_0^{t-r} R(r+s, t-r-s)B(r+s)R_B(r,s)x ds. \tag{13}$$

Let Γ and Γ_B be the infinitesimal generators of the evolution semigroups corresponding to the C_0-quasi groups $R(t,s)$ and $R_B(t,s)$, respectively.

We consider the operator $\Gamma + \mathcal{B}$, where $(\mathcal{B}f)(t) = B(t)f(t)$, $t \in \mathbb{R}$. Since $\mathcal{B}$ is a bounded operator, the operator $\Gamma + \mathcal{B}$ generates a unique C_0-semigroup $T(s)$ satisfying the equation

$$T(s)f = E^s f + \int_0^s E^{s-w}\mathcal{B}T(w)f\, dw, \quad E^s = e^{s\Gamma}, \quad s \ge 0. \tag{14}$$

The implication $(a) \Rightarrow (c)$ of Theorem 2.1 gives $0 \in \rho(\Gamma)$. Consequently, if $\|\mathcal{B}\| = \|B\|_\infty \le \epsilon$, then $0 \in \rho(\Gamma + \mathcal{B}) = \rho(\Gamma_B)$. The implication $(c) \Rightarrow (a)$ of Theorem 2.1 concludes that $R_B(t,s)$ has an exponential dichotomy.

From (13), with $s = t - r$ and $x = f(t-r)$, we have

$$(e^{s\Gamma_B} f)(t) = (E^s f)(t) + \int_0^s (E^{s-w} B e^{w\Gamma_B} f)(t)\, dw, \quad t \in \mathbb{R}.$$

In this case, we have proved that $e^{s\Gamma_B} = T(s)$ satisfies (14) and $\Gamma_B = \Gamma + \mathcal{B}$.

Next, we shall prove the persistence of a uniformly exponential dichotomy for a C_0-quasi group $R(t,s)$ with the infinitesimal generator $A(t)$ relative to the class of perturbations that satisfy the Miyadera condition. Theorem 2.1 implies that if Γ is the infinitesimal generator of the evolution semigroup E^s associated with a uniformly exponentially dichotomic C_0-quasi group $R(t,s)$, then Γ is invertible on $L_p(\mathbb{R}, X)$. Dichotomy Theorem 2.1 implies the following result.

Theorem 4.4 *Let $R(t,s)$ be a uniformly exponentially dichotomic C_0-quasi group with the infinitesimal generator $A(t)$ and $R_1(t,s)$ be a C_0-quasi group with the infinitesimal generator $A(t) + B(t)$. Assume that $\mathcal{B}$ is an operator on the domain $\mathcal{D}(\Gamma) \cap \mathcal{D}(\mathcal{B})$, which has an extension $\hat{\mathcal{B}}$ on $\mathcal{D}(\Gamma)$ such that the operator $\Gamma_1 := \Gamma + \mathcal{B}$ on $\mathcal{D}(\Gamma_1) = \mathcal{D}(\Gamma)$ generates the evolution semigroup associated with $R_1(t,s)$. If there exist constants a and b such that*

$$\|\hat{\mathcal{B}} f\| \le a\|f\| + b\|\Gamma f\| \quad \textit{for} \quad f \in \mathcal{D}(\Gamma) \quad \textit{and} \quad a\|\Gamma^{-1}\| + b < 1,$$

then the perturbed quasi group $R_1(t,s)$ has a uniformly exponential dichotomy.

Proof. Theorem IV.1.16 [20] implies that Γ_1 is invertible on $L_p(\mathbb{R}, X)$. Since Γ_1 is the infinitesimal generator of the evolution semigroup associated with $R_1(t,s)$, the assertion follows from the implication (c) $\Rightarrow$ (a) of Dichotomy Theorem 2.1.

Example 4.1 Consider the quasi group $R(t,s)$ in Example 3.1, which has a uniformly exponential dichotomy on $X = \mathbb{R}^2$ with the norm $\|x\| = |x_1| + |x_2|$ and $\varphi(0) < -1$. Under a perturbation

$$B(t) = \begin{cases} 0, & t < 0, \\ -t, & 0 \le t \le 1, \\ -1, & t > 1, \end{cases}$$

$R(t,s)$ persists the uniformly exponential dichotomy on X.

We notice that $R(t,s)$ has the infinitesimal generator

$$A(t)x = \left(-\varphi(t)x_1, [-\varphi(0) + \varphi(t)]x_2\right), \quad x \in X.$$

Given $\epsilon = 1$, we verify that $B \in C_b(\mathbb{R}, \mathcal{L}(X))$ with $\|B\|_\infty = \epsilon$. By Theorem 4.3, there exists a uniformly exponentially dichotomic quasi group $R_B(t,s)$ on X generated by $A(t) + B(t)$. Indeed, we have $R_B(t,s) = \mathcal{B}(t,s)R(t,s)$, where

$$\mathcal{B}(t,s) = \begin{cases} 1, & t,s < 0, \\ e^{-\frac{1}{2}(s^2+2st)}, & 0 \le t,s \le 1, \\ e^{-s}, & t,s > 1. \end{cases}$$

Moreover, by the mean value theorem for the integral with respect to φ, we obtain the dichotomy constants $N = \max\left\{1, e^{\frac{3}{2}+\varphi(0)}\right\}$ and $\gamma = \inf_{t\in\mathbb{R}} \varphi(t)$ in Definition 2.5 for $R_B(t,s)$, where $\beta = \sup_{t\in\mathbb{R}} \varphi(t)$.

5 Conclusions

In this paper, we provide four equivalent conditions for uniformly exponential dichotomy of C_0-quasi groups on Banach spaces. They base on the existence and uniqueness of mild solutions of the inhomogeneous equations on $C_b(\mathbb{R}, X)$, $C_0(\mathbb{R}, X)$, $L_p(\mathbb{R}, X)$, $1 \leq p < \infty$, and $\mathcal{F}_\alpha$, respectively. The equivalent conditions are parallel with the exponential dichotomy for the evolution family. A small time-dependent perturbation of the infinitesimal generator of the C_0-quasi groups persists the uniformly exponential dichotomy.

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References

[1] C. Chicone and Y. Latushkin. *Evolution Semigroups in Dyamical Systems and Differential Equations.* American Mathematical Society, Rhode Island, 1999.

[2] R. Schnaubelt. Sufficient conditions for exponential stability and dichotomy of evolution equations. *Forum Math.* **11** (1999) 543–566.

[3] S. Sutrima, M. Mardiyana and R. Setiyowati. Uniformly exponential dichotomy for strongly continuous quasi groups. *Oper. Matrices* **15** (1) (2021) 253–273.

[4] V. Cuc. Generalization of a theorem of Datko and Pazy for C_0-quasi semigroups. *Demonstr. Math.* **34** (3) (2001) 577–586.

[5] M. Megan and V. Cuc. On exponential stability of C_0-quasi semigroups in Banach spaces. *Le Matematiche* **Liv(1999)** (1999) 229–241.

[6] S. Sutrima, C.R. Indrati and L. Aryati. Stability of C_0-quasi semigroups in Banach spaces. *J. Phys. Conf.* 943(012021), 2017, 1–14.

[7] S. Sutrima, M. Mardiyana, Respatiwulan, W. Sulandari and M. Yunianto. Approximate controllability of nonautonomous mixed boundary control systems. *AIP Conference Proceedings* 2326(020037), 2021, 1–9.

[8] S. Sutrima, M. Mardiyana, R. Setiyowati and R. Respatiwulan. A new approach on well-posedness of nonautonomous Cauchy problems: An application in population growth. *J. Nonlinear Funct. Anal.* Article ID 29 (2021) 1–15.

[9] E. Zerrik, A.A. Aadi and R. Larhrissi. On the stabilization of infinite dimensional bilinear systems with unbounded control operator. *Nonlinear Dyn. Syst.*, **18**(4) (2018) 418–425.

[10] A. Rodriguez and J. Collado. Periodic solutions in non-homogeneous Hill equation. *Nonlinear Dyn. Syst.* **20** (1) (2020) 78–91.

[11] A. Ben-Artzi and I. Gohberg. Dichotomy of systems and invertibility of linear ordinary differential operators. *Oper. Theory Adv. Appl.* **56** (1992) 90–119.

[12] G. Dore. Lp regularity for abstract differential equations. In: *Functional Analysis and Related Topics, Lecture Notes Math.*, No. 1540. Springer-Verlag, Berlin, 1993.

[13] Y. Latushkin, T. Randolph and R. Schnaubelt. Exponential dichotomy and mild solutions of nonautonomous equations in Banach spaces. *J. Dyn. Differ. Equ.* **10** (3) (1998) 489–510.

[14] R. Rau. Hyperbolic evolutionary semigroups on vector-valued function spaces. *Semigroup Forum* **48** (1994) 107–118.

[15] M.M. Rodrigues and J.G.Ruas-Filho. Evolution equations: Dichotomies and the Fredholm alternative for bounded solutions. *J. Diff. Eq.* **119** (1995) 263-–283.

[16] A.L. Sasu and B. Sasu. Integral equations, dichotomy of evolution families on the half-Line and applications. *Integr. Eqns. Oper. Th.* **66** (2010) 113-–140.

[17] W. Zhang. The Fredholm alternative and exponential dichotomies for parabolic equations. *J. Math. Anal. Appl.* **191** (1995) 180–201.

[18] S. Sutrima, C.R. Indrati and L. Aryati. Exact null controllability, stabilizability, and detectability of linear non-autonomous control systems: A quasi semigroup approach. *Abstr. Appl. Anal.* **3791609** (2018) 1–12.

[19] S. Sutrima, C.R. Indrati, L. Aryati and M. Mardiyana. The fundamental properties of quasi semigroups. *J. Phys. Conf.*, 855(012052), (2017), 1–9.

[20] T. Kato. *Perturbation theory for linear operators*. Springer-Verlag, New York, 1966.