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Functional Differential Inclusions with Unbounded Right-Hand Side in Banach Spaces

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Abstract: In this work, we provide a reduction method that solves functional differential inclusion in Banach spaces, that is, when the right-hand side contains a finite delay. We consider the case when the set-valued mapping takes nonempty closed nonconvex and unnecessarily bounded values, we use the notion of $\lambda - H$ Lipschitzness assumption instead of the standard Lipschitz condition, known as a truncation. An application to a dynamical system governed by a delayed perturbed sweeping process is given, such problems are well-posed for differential complementarity systems and vector hysteresis problems.

Keywords: *nonconvex differential inclusion; reduction; delay; unboundedness; λ-Hausdorff distance.*

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1 Introduction

Let τ, T be two non-negative real numbers, E be a separable Banach space equipped with the norm $\|\cdot\|$, $\mathcal{C}_0 := \mathcal{C}_E([-\tau, 0])$ (resp. $\mathcal{C}_T := \mathcal{C}_E([-\tau, T])$) be the Banach space of all continuous mappings from $[-\tau, 0]$ (resp. $[-\tau, T]$) to E equipped with the norm of uniform convergence. Let $\Pi : [0, T] \times \mathcal{C}_0 \rightrightarrows E$ be a set-valued mapping with nonempty closed values. In this work, we study the existence of solutions for the following differential inclusion with delay:

$$(DP) \quad \begin{cases} \dot{u}(t) \in \Pi(t, Z(t)u) & \text{a.e. } t \in [0, T], \\ u(t) \ = \ \psi(t), & t \in [-\tau, 0], \end{cases}$$

where $\psi \in \mathcal{C}_0$ and $Z(t) : \mathcal{C}_T \longrightarrow \mathcal{C}_0$ is defined by $(Z(t)u)(s) = u(t+s), \forall s \in [-\tau, 0]$. In [9], Fryszkowski proved an existence result for (DP) when Π is an integrably bounded

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and lower semicontinuous set-valued mapping with nonconvex values, the proof is based on the construction of a continuous selection for a class of nonconvex decomposable sets. Many other results have been obtained using a fixed point or discretization approach, see for instance [2, 4–6, 8] and the references therein. In [5], a discretization technique was initiated, it consists in subdividing the interval $[0,T]$ into a sequence of subintervals and reformulating the problem with delay to a sequence of problems without delay and then applying the results known in this case. Our goal in this work is to prove the existence of a global solution to (DP) for a general class of unbounded sets thanks to a recent result for the undelayed problem due to Tolstonogov [11]. We weaken the standard Lipschitz condition by a truncated one. Then, we use this result to present an application for functional differential inclusions governed by time and state dependent nonconvex sweeping process. This kind of problems corresponds to several important mechanical problems and nonsmooth dynamical systems. When external forces (perturbations) are applied to the system described by the sweeping process, the problem found many applications in resource allocation in economics, nonregular electrical circuits, crowd motion modeling and hysteresis. We propose here a new variant of the existence result which generalizes the previous results. The paper is organized as follows. In Section 2, we prepare some material which will be needed later in our proof. Section 3 is devoted to the main result. An application is given in Section 4 for a dynamical system governed by a sweeping process subject to external forces containing a finite delay.

2 Preliminaries

Throughout the paper, we will use the following notation and definitions. Let E be a separable Banach space, $\|\cdot\|$ be its norm and $\ominus$ be its zero element. We denote by $\mathcal{C}_E([0,T])$ the Banach space of all continuous mappings from $[0,T]$ to E, $L^1_E([0,T])$ is the Banach space of all measurable mappings from $[0,T]$ to E. Let $\mathcal{B}(\mathcal{C}_0)$ be the σ-algebra of Borel sets of $\mathcal{C}_0$ and $\mathcal{L}$ be the σ-algebra of Lebesgue measurable subsets of $[0,T]$, $d(u,A)$ means the usual distance from a point u to a set A, i.e., $d(u,A) := \inf_{v\in A}\|u-v\|,\ u \in E$. We denote by $\lambda\mathbb{B}$ the closed ball with radius λ in E centered at $\ominus$, and $\overline{\mathbb{B}}_{\mathcal{C}_0}$ is the closed unit ball of $\mathcal{C}_0$ with the center 0. A set-valued mapping $\Lambda : [0,T] \times E \rightharpoonup E$ is integrally bounded if there exists an integrable function $\xi : [0,T] \to \mathbb{R}^+$ such that

$$\|\Lambda(t,u)\| := sup\{\|v\|;\ v \in \Lambda(t,u)\} \leq \xi(t), t \in [0,T],\ u \in E.$$

A set-valued mapping with closed values is measurable whenever $\Lambda^{-1}(U) = \{t \in [0,T] : \Lambda(t) \cap U \neq \emptyset\}$ belongs to $\mathcal{L}$ for every closed set $U \subset E$.

Following [3], for any set $A \subset E$ and $\lambda > 0$, we put $A_\lambda = A \cap \lambda\overline{\mathbb{B}}$. For $A,\ B \subset E$, the excess, the Hausdorff distance and the λ-Hausdorff distance between A and B are defined, respectively, by $e(A,B) := \sup_{a\in A} d(a,B)$, $haus(A,B) = \max\{e(A,B),\ e(B,A)\}$ and

$$haus_\lambda(A,B) = \max\{e(A_\lambda,B),\ e(B_\lambda,A)\}.$$

If A is a nonempty closed subset of E, then $\delta^*(u,A) = \sup\limits_{v\in A}\langle u,v\rangle$ is the support function of A at $u \in E$, and $\overline{co}(A)$ stands for the closed convex hull of A, characterized by

$$\overline{co}(A) = \{u \in E : \forall v \in E, \langle v,u\rangle \leq \delta^*(v,A)\}.$$

The projection of u on A is the element of A denoted by $Proj_A(u)$ and satisfying $Proj_A(u) = \{v \in A\ :\ d(u,A) = \|u-v\|\}$.

Let $g : E \to \mathbb{R} \cup \{+\infty\}$ be a proper convex continuous function on E and $u \in E$ with $g(u) < +\infty$, the subdifferential of g is the set defined by

$$\partial g(u) = \{z \in E \ : \ < z, v - u > \leq g(v) - g(u), \ \forall v \in E\},$$

if $g(u)$ is not finite, we set $\partial g(u) = \emptyset$, $\partial g(u)$ is a closed convex set if g is convex.
Let $A \subset E$ and $u \in A$, the normal cone to A at u is defined by

$$N_A(u) = \{v \in E \ : \ < v, c - u > \leq 0, \text{ for all } c \in A\}.$$

A vector $\omega \in E$ is said to be in the Fréchet subdifferential of g at u, denoted by $\partial^F g(u)$, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $u' \in B(u, \delta)$, we have

$$\langle \omega, u' - u \rangle \leq g(u') - g(u) + \varepsilon \|u' - u\|.$$

The Fréchet normal cone $N_A^F(u)$ of A at $u \in A$ is given by $N_A^F(u) = \partial^F \chi_A(u)$, where χ_A is the indicator function of A, so we have the inclusion $N_A^F(u) \subset N_A(u)$ for all $u \in A$.
On the other hand, the Fréchet normal cone is also related to the Fréchet subdifferential of the distance function since for all $u \in A$,

$$N_A^F(u) = \mathbb{R}_+ \partial^F d(u, A); \quad \text{and} \quad \partial^F d_A(u) = N^F(A; u) \cap \mathbb{B}.$$

We now recall the definition of subsmooth sets.

Definition 2.1 Let A be a closed subset of E, we say that A is subsmooth at $u_0 \in A$ if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\langle \zeta_2 - \zeta_1; u_2 - u_1 \rangle \geq -\epsilon \|u_2 - u_1\| \tag{1}$$

whenever $u_1, u_2 \in B(u_0; \delta) \cap A$ and $\zeta_i \in N_A(u_i) \cap \mathbb{B}$. The set A is subsmooth if it is subsmooth at each point of A.

Let A be a closed subset in E and $u_0 \in A$. Then, if A is subsmooth at u_0, then it is normally Fréchet regular at u_0, that is, $N_A^F(u_0) = N_A(u_0)$ and $\partial d(u_0, A) = \partial^F d(u_0, A)$.

Definition 2.2 A family $(S(q))_{q \in Q}$ of closed sets in E with parameter $q \in Q$, is called equi-uniformly subsmooth if for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $q \in Q$, the inequality (1) holds for all $u_1, u_2 \in S(q)$ satisfying $\|u_1 - u_2\| < \delta$ and all $\xi_i \in N(S(q); u_i) \cap \mathbb{B}$.

Proposition 2.1 *[10] Let $\{C(t, u) : (t, u) \in [0; T] \times E\}$ be a family of nonempty closed sets of E which is equi-uniformly subsmooth and let $\nu \geq 0$, assume that there exist positive real constants L_1, L_2 such that for any $t, s \in [0, T]$ and $u, v, z \in E$,*

$$|d(z, C(t, u)) - d(z, C(s, v))| \leq L_1 |t - s| + L_2 \|u - v\|.$$

Then the following assertions hold:

(a) For all $(t, u, v) \in gph(C)$, we have $\nu \partial d(v, C(t, u)) \subset \nu \mathbb{B}$;

(b) For any sequences $(t_n)_n$ in $[0, T]$ converging to t, $(u_n)_n$ converging to u, $(v_n)_n$ converging to $v \in C(t, u)$ with $v_n \in C(t_n, u_n)$, and any $\zeta \in H$, we have

$$\limsup_{n \to +\infty} \sigma\big(\zeta, \nu \partial d(v_n, C(t_n, u_n))\big) \quad \leq \quad \sigma\big(\zeta, \nu \partial d(v, C(t, u))\big) \ .$$

Lemma 2.1 *[1] Let $m > 0$, (ω_i) and (υ_i) be nonnegative sequences satisfying $\omega_i \leq m + \sum_{j=0}^{i-1} \upsilon_j \omega_j$ for any $i \in \mathbb{N}$, then $\omega_i \leq m \exp\left(\sum_{j=0}^{i-1} \upsilon_j \right)$, $\forall i \in \mathbb{N}$.*

3 Main Result

We begin this section by listing the hypotheses used throughout the paper.

Hypotheses $\mathcal{H}_{(\mathbf{\Pi})}$: For every $\beta > 0$ and $\mathcal{C}_0^\beta = \mathcal{C}_0 \cap \beta\overline{\mathbb{B}}_{\mathcal{C}_0}$, let $\Pi : [0,T] \times {\mathcal{C}_0}^\beta \rightharpoonup E$ be a set-valued mapping with nonempty closed values satisfying:

(i) for every $\psi \in \mathcal{C}_0^\beta$, $\Pi(\cdot, \psi)$ is measurable;

(ii) for some functions $\eta(\cdot),\ \xi(\cdot) \in L^1_{\mathbb{R}^+}([0,T])$ such that for all $t \in [0,T]$ and for all $\psi \in \mathcal{C}_0^\beta$,
$$d(\ominus, \Pi(t,\psi)) < \xi(t) + \eta(t)\|\psi\|_{\mathcal{C}_0},$$
and $d(\ominus, \Pi(t,\ominus)) = 0$ for $\xi(t) = 0$;

(iii) $\forall\ \psi, \phi \in \mathcal{C}_0^\beta$, *with* $\phi \neq \psi$, we have
$$haus_\lambda\big(\Pi(t,\phi), \Pi(t,\psi)\big) \le \eta(t)\|\phi - \psi\|_{\mathcal{C}_0}.$$

For the proof of our theorem we need the following result for the undelayed problem.

Theorem 3.1 *[11] For every $\beta > 0$, let $\Lambda : [0,T] \times \beta\overline{\mathbb{B}} \rightharpoonup E$ be a set-valued mapping with nonempty closed values satisfying:*

(1) for every $u \in \mathcal{C}_E([0,T])$ and $t \in [0,T]$, the mapping $t \longrightarrow \Lambda(t, u(t))$ is measurable;

(2) for some functions $\eta(\cdot),\ \xi(\cdot) \in L^1_{\mathbb{R}^+_*}([0,T])$,
$$d(\ominus, \Lambda(t, u(t))) < \xi(t) + \eta(t)\|u(t)\| \ a.e.,\ \|u(t)\| \le \beta,$$
$d(\ominus, \Lambda(t,\ominus)) = 0$ for $\xi(t) = 0$;

(3) for $\|u(t)\| \le \beta$, $\|v(t)\| \le \beta$, $u(t) \neq v(t)$, we have
$$haus_\lambda(\Lambda(t,u(t)), \Lambda(t,v(t))) \le \eta(t)\|u(t) - v(t)\| \ a.e.$$
with $0 \le \lambda \le \dot{m}(t)$ for $t \in [0,T]$, where $m(\cdot) : [0,T] \longrightarrow \mathbb{R}$ is the absolutely continuous solution to the differential equation
$$\begin{cases} \dot{m}(t) &= \eta(t)\, m(t) + \xi(t) \ a.e.\ in\ [0,T], \\ m(0) &= m_0 \ge 0. \end{cases}$$

Then, $\forall\ u_0 \in \mathcal{C}_E([0,T])$ with $\|u_0\| < \beta$, the problem
$$\begin{cases} \dot{u}(t) \in \Lambda(t, u(t)), & a.e.\ on\ [0,T], \\ u(0) = u_0 \end{cases} \tag{2}$$
admits a solution u such that $\|u(t)\| \le m(t)$, $\|\dot{u}(t)\| \le \dot{m}(t)$ a.e. for $t \in [0,T]$ with $m(t) \le \beta$.

Now, we are able to give the existence result for the delayed problem.

Theorem 3.2 *Let $\Pi : [0,T] \times \mathcal{C}_0^\beta \rightharpoonup E$ be a set-valued mapping satisfying $\mathcal{H}_{(\mathbf{\Pi})}$. Then, $\forall\ \psi \in \mathcal{C}_0$, the problem (DP) admits at least one continuous solution $u : [-\tau, T] \to E$, absolutely continuous on $[0,T]$. Furthermore, $\|\dot{u}(t)\| \le \dot{m}(t)$ a.e. $t \in [0,T]$.*

Proof. We will reduce our problem to a problem without delay and apply Theorem 3.1. For every $n \in \mathbb{N}$, consider a partition of $[0,T]$ defined by $t_i^n = i\varpi_n$, $\varpi_n = Tn^{-1}$, $i = 0,1,...,n$.

Step 1 *(Construction of approximate solutions)*: for every $(t,u) \in [-\tau, t_1^n] \times E$, we define $p_0^n : [-\tau, t_1^n] \times E \longrightarrow E$ by

$$p_0^n(t,u) = \begin{cases} \psi(t) & \text{if } t \in [-\tau, 0]; \\ \psi(0) + \frac{t}{\varpi_n}(u - \psi(0)) & \text{if } t \in \,]0, t_1^n]; \end{cases}$$

clearly, $p_0^n(t_1^n, u) = u, \;\; \forall u \in E$.

We define the set-valued mapping Λ_0^n on $[0,t_1^n] \times E$ with closed values in E by

$$\Lambda_0^n(t,u) := \Pi(t, \mathcal{Z}(t_1^n)p_0^n(\cdot,u)), \;\; \forall (t,u) \in [0,t_1^n] \times E.$$

Let us show that Λ_0^n satisfies the conditions of Theorem 3.1. Note first that the function $u \longmapsto \mathcal{Z}(t_1^n)p_0^n(\cdot,u)$ is Lipschitz. Indeed, for every $u,v \in E$, we have

$$\begin{aligned} \|Z(t_1^n)p_0^n(\cdot,u) - Z(t_1^n)p_0^n(\cdot,v)\|_{\mathcal{C}_0} &= \sup_{s\in[-\tau,0]} \|p_0^n(t_1^n+s,u) - p_0^n(t_1^n+s,v)\| \\ &= \sup_{s\in[-\varpi_n,0]} \|p_0^n(t_1^n+s,u) - p_0^n(t_1^n+s,v)\| \\ &= \sup_{s\in[-\varpi_n,0]} \|\tfrac{t_1^n+s}{\varpi_n}(u-v)\| \\ &= \|u-v\|. \end{aligned}$$

So the mapping $t \longrightarrow \Lambda_0^n(t,u)$ is measurable. On the other hand,

$$\begin{aligned} \|Z(t_1^n)p_0^n(\cdot,u)\|_{\mathcal{C}_0} &= \sup_{s\in[-\tau+t_1^n,t_1^n]} \|p_0^n(s,u)\| \\ &\leq \max\{\|\psi\|_{\mathcal{C}_0}, \sup_{s\in[0,t_1^n]} \|\psi(0) + \tfrac{s}{\varpi_n}(u-\psi(0))\|\} \\ &\leq \max\{\|\psi\|_{\mathcal{C}_0}, \sup_{s\in[0,t_1^n]} \big((1-\tfrac{s}{\varpi_n})\|\psi(0)\| + \tfrac{s}{\varpi_n}\|u\|\big)\} \\ &\leq \max\{\|\psi\|_{\mathcal{C}_0}, \|\psi(0)\| + \|u\|\}. \end{aligned}$$

Furthermore, by the condition (ii) of $\mathcal{H}_{(\mathbf{\Pi})}$, we have, for every $t \in [0,t_1^n]$ and $u \in E$ such that $\|u\| \leq \beta$,

$$\begin{aligned} d(\ominus, \Lambda_0^n(t,u)) = d(\ominus, \Pi(t, Z(t_1^n)p_0^n(\cdot,u))) &\leq \xi(t) + \eta(t)\,\|Z(t_1^n)p_0^n(\cdot,u)\| \\ &\leq \xi(t) + \eta(t)(\|\psi\|_{\mathcal{C}_0} + \|u\|), \\ &\leq \zeta(t)(1 + \|\psi\|_{\mathcal{C}_0}) + \eta(t)\|u\|, \end{aligned}$$

where $\zeta(t) := \max\{\xi(t), \eta(t)\}$.

For $\zeta(t) = 0$, we have $d(\ominus, \Lambda_0^n(t,\ominus)) = d(\ominus, \Pi(t, Z(t_1^n)p_0^n(\cdot,\ominus))) = 0$. Finally, according to (iii), one obtains

$$\begin{aligned} haus_\lambda(\Lambda_0^n(t,u), \Lambda_0^n(t,v)) &= haus_\lambda\big(\Pi(t, Z(t_1^n)p_0^n(\cdot,u)), \Pi(t, Z(t_1^n)p_0^n(\cdot,v))\big) \\ &\leq \eta(t)\,\|Z(t_1^n)p_0^n(\cdot,u) - Z(t_1^n)p_0^n(\cdot,v)\| \\ &= \eta(t)\,\|u-v\|, \end{aligned}$$

$\|u\| \leq \beta$ and $\|v\| \leq \beta$, $u \neq v$. Hence Λ_0^n verifies the conditions of Theorem 3.1, this provides an absolutely continuous solution $\vartheta_0^n : [0,t_1^n] \longrightarrow E$ to the problem

$$\begin{cases} \dot{\vartheta}_0^n(t) & \in \Lambda_0^n(t, \vartheta_0^n(t)) & \text{a.e. on } [0,t_1^n]; \\ \vartheta_0^n(t) & = \psi(0) + \int_0^t \dot{\vartheta}_0^n(s)ds & \forall\, t \in \,]0,t_1^n]; \\ \vartheta_0^n(0) & = \psi(0) \end{cases}$$

with $\|\vartheta_0^n(t)\| \leq m(t)$ and $\|\dot{\vartheta}_0^n(t)\| \leq \dot{m}(t)$. That is, ϑ_0^n is a solution to

$$\begin{cases} \dot{\vartheta}_0^n(t) & \in \Pi(t, Z(t_1^n)p_0^n(\cdot, \vartheta_0^n)) \qquad \text{a.e. on } [0, t_1^n]; \\ \vartheta_0^n(0) & = \psi(0). \end{cases}$$

Put

$$u_n(t) = \begin{cases} \psi(t) & if\ t \in [-\tau, 0]; \\ \vartheta_0^n(t) & if\ t \in\ ]0, t_1^n]. \end{cases}$$

As before, for every $(t, u) \in [-\tau, t_1^n] \times E$, we define $p_1^n : [-\tau, t_2^n] \times E \longrightarrow E$ by

$$p_1^n(t, u) = \begin{cases} u_n(t) & \text{if } t \in [-\tau, t_1^n]; \\ u_n(t_2^n) + \frac{t - t_2^n}{\varpi_n}(u - u_n(t_2^n)) & \text{if } t \in\ ]t_1^n, t_2^n] \end{cases}$$

with $p_1^n(t_2^n, u) = u, \quad \forall u \in E$. Hence, we can define similarly the set-valued mapping Λ_1^n on $[t_1^n, t_2^n] \times E$ with closed values of E by

$$\Lambda_1^n(t, u) := \Pi(t, Z(t_2^n)p_1^n(\cdot, u)), \ \ \forall (t, u) \in [t_1^n, t_2^n] \times E.$$

The function $u \longmapsto Z(t_2^n)p_1^n(\cdot, u)$ is Lipschitz since for all $u, v \in E$, we have

$$\begin{aligned} \|Z(t_2^n)p_1^n(\cdot, u) - Z(t_2^n)p_1^n(\cdot, v)\| &= \sup_{s \in [-\tau, 0]} \|p_1^n(t_2^n + s, u) - p_1^n(t_2^n + s, v)\| \\ &= \sup_{s \in [-\varpi_n, 0]} \|p_1^n(t_2^n + s, u) - p_1^n(t_2^n + s, v)\| \\ &= \sup_{s \in [-\varpi_n, 0]} \|u_n(t_1^n) + \tfrac{t_2^n + s - t_1^n}{\varpi_n}(u - u_n(t_1^n)) \\ &\quad -(u_n(t_1^n) + \tfrac{t_2^n + s - t_1^n}{\varpi_n}(v - u_n(t_1^n)))\| \\ &= \sup_{s \in [-\varpi_n, 0]} \|\tfrac{t_2^n + s - t_1^n}{2^{-n}}(u - v)\| \\ &= \|\tfrac{t_2^n - t_1^n}{\varpi_n}(u - v)\| \\ &= \|u - v\| \end{aligned}$$

and

$\|Z(t_2^n)p_1^n(\cdot, u)\|_{\mathcal{C}_0}$

$$\begin{aligned} &= \sup_{s \in [-\tau + t_2^n, t_2^n]} \|p_1^n(s, u)\| \\ &\leq \max\{\|\psi\|_{\mathcal{C}_0}, \sup_{s \in [0, t_1^n]} \|v_0^n(s)\|\} + \sup_{s \in [t_1^n, t_2^n]} \big((1 - \tfrac{t-s}{\varpi_n})\|u_n(t_2^n)\| + \tfrac{t-s}{\varpi_n}\|u\|\big) \\ &\leq \max\{\|\psi\|_{\mathcal{C}_0}, \sup_{s \in [0, t_1^n]} \|v_0^n(s)\|\} + \|u\|. \end{aligned}$$

For every $t \in [t_1^n, t_2^n]$ and $u \in E$, with $\|u\| \leq \beta$

$$d(\ominus, \Lambda_1^n(t, u)) = d(\ominus, \Pi(t, Z(t_2^n)p_1^n(\cdot, u))) \leq \xi(t) + \eta(t)\, \|Z(t_2^n)p_1^n(\cdot, u)\|$$

$$\leq \zeta(t)(1 + \max\{\|\psi\|_{\mathcal{C}_0}, \sup_{s \in [0, t_1^n]} \|v_0^n(s)\|\}) + \eta(t)\, \|u\|,$$

for $\zeta(t) = 0$

$$d(\ominus, \Lambda_1^n(t, \ominus)) = d(\ominus, \Pi(t, Z(t_2^n)p_1^n(\cdot, \ominus))) = 0.$$

Furthermore, by condition (iii) of $\mathcal{H}_{(\mathbf{\Pi})}$, we have for every $t \in [0, t_1^n]$ and $u, v \in E$ such that $\|u\| \leq \beta$, and $\|v\| \leq \beta$, $u \neq v$,

$$\begin{array}{rcl} haus_\lambda(\Lambda_1^n(t,u), \Lambda_1^n(t,v)) & = & haus_\lambda\big(\Pi(t, Z(t_2^n)p_1^n(\cdot,u)), \Pi(t, Z(t_2^n)p_1^n(\cdot,v))\big) \\ & \leq & \eta(t)\,\|Z(t_2^n)p_1^n(0,u) - Z(t_2^n)p_1^n(0,v)\| \\ & = & \xi(t) + \eta(t)\,\|p_1^n(t_2^n,u) - p_1^n(t_2^n,v)\| \\ & = & \xi(t) + \eta(t)\,\|u-v\|. \end{array}$$

Hence Λ_1^n verifies the conditions of Theorem 3.1, this provides an absolutely continuous solution $\vartheta_1^n : [t_1^n, t_2^n] \longrightarrow E$ to the problem

$$\begin{cases} \dot{\vartheta}_1^n(t) & \in \Lambda_1^n(t, \vartheta_1^n(t)) \qquad\qquad a.\ e.\ on\ \ [t_1^n, t_2^n]; \\ \vartheta_1^n(t) & = u_n(t_2^n) + \int_{t_1^n}^t \dot{\vartheta}_1^n(s)ds \qquad \forall\ t \in\,]t_1^n, t_2^n]; \\ \vartheta_1^n(t_2^n) & = u_n(t_2^n), \end{cases}$$

$\|\vartheta_1^n(t)\| \leq m(t)$ and $\|\dot{\vartheta}_1^n(t)\| \leq \dot{m}(t)$. So v_1^n is a solution of

$$\begin{cases} \dot{\vartheta}_1^n(t) & \in \Pi(t, Z(t_2^n)p_1^n(\cdot, \vartheta_1^n)) \qquad \text{a.e. on } \ [t_1^n, t_2^n]; \\ \vartheta_1^n(t) & = u_n(t_1^n) + \int_{t_1^n}^t \dot{\vartheta}_1^n(s)ds \qquad \forall\ t \in\,]t_1^n, t_2^n]; \\ \vartheta_1^n(0) & = \psi(0). \end{cases}$$

By induction, suppose that u_n is defined on $[-\tau, t_k^n]$, absolutely continuous on $[0, t_k^n]$, and satisfies

$$\begin{cases} \dot{u}_n(t) & \in \Pi(t, Z(t_{k-1}^n)p_{k-1}^n(\cdot,u)) \qquad \text{a.e. on } \ [t_{k-1}^n, t_k^n]; \\ u_n(t) & = u_n(t_{k-1}^n) + \int_{t_{k-1}^n}^t \dot{u}_n(s)ds \qquad \forall\ t \in\,]t_{k-1}^n, t_k^n]; \end{cases}$$

and build a solution on $[t_k^n, t_{k+1}^n]$. For every $(t,u) \in [-\tau, t_1^n] \times E$, we defined $p_k^n : [-\tau, t_{k+1}^n] \times E \longrightarrow E$ by

$$p_k^n(t,u) = \begin{cases} u_n(t) & \text{if } \ t \in [-\tau, t_k^n]; \\ u_n(t_k^n) + \frac{t - t_k^n}{\varpi_n}(u - u_n(t_k^n)) & \text{if } \ t \in\,]t_k^n, t_{k+1}^n]; \end{cases}$$

with $p_k^n(t_{k+1}^n, u) = u$ and $p_k^n \in \mathcal{C}_E([-\tau, t_{k+1}^n])$. The function $u \longmapsto Z(t_{k+1}^n)p_k^n(\cdot,u)$ is Lipschitz. Indeed, for all $u, v \in E$, we have

$$\begin{array}{c} \|Z(t_{k+1}^n)p_k^n(\cdot,u) - Z(t_{k+1}^n)p_k^n(\cdot,v)\| = \\ \sup\limits_{s\in[-\tau,0]} \|p_k^n(t_{k+1}^n + s, u) - p_k^n(t_{k+1}^n + s, v)\| \\ = \sup\limits_{t \in [-\tau + t_{k+1}^n, t_{k+1}^n]} \|p_k^n(t,u) - p_k^n(t,v)\|. \end{array}$$

We distinguish two cases:

(1) if $-\tau + t_{k+1}^n \leq t_k^n$, we have

$$\begin{array}{rcl} \sup\limits_{t \in [-\tau + t_{k+1}^n, t_{k+1}^n]} \|p_k^n(t,u) - p_k^n(t,v)\| & = & \sup\limits_{t \in [t_k^n, t_{k+1}^n]} \|p_k^n(t,u) - p_k^n(t,v)\| \\ & = & \sup\limits_{t \in [t_k^n, t_{k+1}^n]} \|\frac{t - t_k^n}{\varpi_n}(u-v)\| \\ & = & \|u - v\|. \end{array}$$

(2) if $t_k^n \leq -\tau + t_{k+1}^n \leq t_{k+1}^n$, we have

$$\begin{aligned}\sup_{t\in[-\tau+t_{k+1}^n, t_{k+1}^n]} \|p_k^n(t,u) - p_k^n(t,v)\| &\leq \sup_{t\in[t_k^n, t_{k+1}^n]} \|p_k^n(t,u) - p_k^n(t,v)\| \\ &= \sup_{t\in[t_k^n, t_{k+1}^n]} \|\tfrac{t-t_k^n}{\varpi_n}(u-v)\| \\ &= \|u-v\|.\end{aligned}$$

$$\begin{aligned}\|Z(t_{k+1}^n)p_k^n(\cdot,u)\|_{\mathcal{C}_0} &= \sup_{s\in[-\tau+t_{k+1}^n, t_{k+1}^n]} \|p_k^n(s,u)\| \\ &\leq \max\Big\{\|\psi\|_{\mathcal{C}_0}, \max_{0\leq k\leq i-1} \sup_{s\in[t_k^n, t_{k+1}^n]} \|v_k^n(s)\|\Big\} + \|u\|.\end{aligned}$$

Similarly, we can define Λ_k^n on $[t_k^n, t_{k+1}^n] \times E$ with closed values of E by

$$\Lambda_k^n(t,u) := \Pi(t, Z(t_{k+1}^n)p_k^n(\cdot,u)), \ \ \forall (t,u) \in [t_k^n, t_{k+1}^n] \times E,$$

satisfying conditions of Theorem 3.1. Hence, there exists an absolutely continuous solution $\vartheta_k^n : [t_k, t_{k+1}] \longrightarrow E$ to

$$\begin{cases} \dot{\vartheta}_k^n(t) & \in \Lambda_k^n(t, \vartheta_k^n(t)) & \text{a.e. on } [t_k^n, t_{k+1}^n]; \\ \vartheta_k^n(t) & = u_n(t_k^n) + \int_{t_k^n}^t \dot{\vartheta}_k^n(s)ds & \forall\, t \in]t_k^n, t_{k+1}^n]; \\ \vartheta_k^n(t_k^n) & = u_n(t_k^n), \end{cases}$$

$\|\vartheta_k^n(t)\| \leq m(t)$ and $\|\dot{\vartheta}_k^n(t)\| \leq \dot{m}(t)$. So ϑ_k^n is a solution of

$$\begin{cases} \dot{\vartheta}_k^n(t) & \in \Pi(t, Z(t_{k+1}^n)p_k^n(\cdot, \vartheta_k^n)) & \text{a.e. on } [t_k^n, t_{k+1}^n]; \\ \vartheta_k^n(t) & = u_n(t_k^n) + \int_{t_k^n}^t \dot{\vartheta}_k^n(s)ds & \forall\, t \in]t_k^n, t_{k+1}^n]; \\ \vartheta_k^n(t_k^n) & = u_n(t_k^n). \end{cases}$$

Putting $u_n(t) = \vartheta_k^n(t)$ on $[t_k^n, t_{k+1}^n]$, we obtain

$$u_n(t) = \begin{cases} \vartheta_0^n(t) = \psi(0) + \int_0^t \dot{u}_n(s)ds & if \ \ t \in [0, t_1^n]; \\ \vartheta_1^n(t) = u_n(t_1^n) + \int_{t_1^n}^t \dot{u}_n(s)ds & if \ \ t \in]t_1^n, t_2^n]; \\ \quad \cdot \ \cdot \ \cdot & \\ \vartheta_k^n(t) = u_n(t_k^n) + \int_{t_k^n}^t \dot{u}_n(s)ds & if \ \ t \in]t_k^n, t_{k+1}^n]; \end{cases}$$

and $\|u_n(t)\| \leq m(t)$. For every $t \in [0,T]$, we set $\theta_n(t) = t_i^n$, $\delta_n(t) = t_{i+1}^n$, $\forall\, t \in]t_i^n, t_{i+1}^n]$ and $\theta_n(0) = 0$ and define $p_{\varpi_n\theta_n(t)}^n \in \mathcal{C}_E([-\tau, \delta_n(t)])$ by

$$p_{\varpi_n\theta_n(t)}^n(t,x) = \begin{cases} u_n(t) & \text{if } \ t \in [-\tau, \theta_n(t)]; \\ u_n(\theta_n(t)) + \frac{t-\theta_n(t)}{\varpi_n}(u - u_n(\theta_n(t))) & \text{if } \ t \in]\theta_n(t), \delta_n(t)]. \end{cases}$$

Clearly, u_n is continuous on $[-\tau, T]$, absolutely continuous on $[0,T]$ and satisfies

$$\begin{cases} \dot{u}_n(t) & \in \Pi(t, Z(\delta_n(t))p_{\varpi_n\theta_n(t)}^n(\cdot, u_n(t))) & a.\ e.\ on \ \ [0,T]; \\ u_n(t) & = \psi(0) + \int_0^t \dot{u}_n(s)ds & \forall\, t \in [0,T]; \\ u_n(t) & = \psi(t) & \forall\, t \in [-\tau, 0]. \end{cases} \tag{3}$$

Step 2 *(Uniform convergence):* by condition (2) of Theorem 3.1, for almost every $t \in [0, T]$, one has

$$\dot{u}_n(t) \in \Pi(t, Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(\cdot, u_n(t))) \tag{4}$$

with $Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(0, u_n(t)) = u_n(t)$ and

$$d\big(\ominus, \Pi(t, Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(\cdot, u_n(t)))\big) \leq \xi(t) + \eta(t)\|u_n(t)\|.$$

Further, since $\|u_n(t)\| \leq m(t)$, we have

$$d\big(\ominus, \Pi(t, Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(\cdot, u_n(t)))\big) \leq \xi(t) + \eta(t)m(t).$$

Hence for almost every $t \in [0, T]$,

$$\|\dot{u}_n(t)\| \leq \dot{m}(t). \tag{5}$$

This shows that $\dot{u}_n(\cdot)$ is uniformly bounded by $\dot{m}(\cdot)$, By extracting a subsequence, we may assume that $(\dot{u}_n)_n$ converges $\sigma(L^1, L^\infty)$ to some $v \in L^1_E([0, T])$. So $(u_n(\cdot))$ is a bounded sequence of $\mathcal{C}_E([0, 1])$ since for every $t \in [0, T]$,

$$\|u_n(t)\| = \|\psi(0)\| + \int_0^t \|\dot{u}_n(s)\|ds \leq \|\psi(0)\| + \int_0^t \dot{m}(s)ds = \gamma(t)$$

and it is clear that $(u_n(\cdot))$ is equicontinuous. By Ascoli's theorem, we get that $(u_n)_n$ is relatively compact. By extracting a subsequence (that we do not relabel), we conclude that $(u_n)_n$ converges uniformly to some mapping u and

$$u(t) = \psi(0) + \int_0^t v(s)ds, \ \forall t \in [0, T],$$

hence $\dot{u}(t) = v(t)$ almost everywhere.

Now, let us show that

$$\|Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(\cdot, u_n(t)) - Z(t)u\| \longrightarrow 0, \text{ when } n \longrightarrow \infty.$$

$$\sup_{s\in[-\tau,0]} \|Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(s, u_n(t)) - Z(t)u(s)\|_{\mathcal{C}_0} =$$

$$\sup_{s\in[-\tau,0]} \|p^n_{\varpi_n\theta_n(t)}(\delta_n(t) + s, u_n(t)) - u(s + t)\|$$

$$= \sup_{s\in[-\tau,0]} \|p^n_{\varpi_n\theta_n(t)}(\delta_n(t) + s, u_n(t)) - u(\delta_n(t) + s) + u(\delta_n(t) + s) - u(s + t)\|$$

$$\leq \sup_{s\in[-\tau,0]} \|p^n_{\varpi_n\theta_n(t)}(\delta_n(t) + s, u_n(t)) - x(\delta_n(t) + s)\| +$$

$$\sup_{s\in[-\tau,0]} \|u(\delta_n(t) + s) - u(s + t)\|.$$

First,

$$\sup_{s\in[-\tau,0]} \|p^n_{\varpi_n\theta_n(t)}(\delta_n(t) + s, u_n(t)) - x(\delta_n(t) + s)\|$$

$$\leq \sup_{s\in[-\tau,-\varpi_n]} \|p^n_{\varpi_n\theta_n(t)}(\delta_n(t) + s, u_n(t)) - u(\delta_n(t) + s)\|$$

$$+\sup_{s\in[-\varpi_n,0]}\|p^n_{\varpi_n\theta_n(t)}(\delta_n(t)+s,u_n(t))-u(\delta_n(t)+s)\|$$
$$=\sup_{s\in[-\tau,-\varpi_n]}\|u_n(\delta_n(t)+s)-u(\delta_n(t)+s)\|+$$
$$\sup_{s\in[-\varpi_n,0]}\|u_n(\theta_n(t))+\frac{\delta_n(t)+s-\theta_n(t)}{\mu_n}(u_n(t)-u_n(\theta_n(t))-u(\delta_n(t)+s))\|$$
$$=\sup_{s\in[-\tau,-\varpi_n]}\|u_n(\delta_n(t)+s)-u(\delta_n(t)+s)\|$$
$$+\sup_{s\in[-\varpi_n,0]}\|\frac{s}{\varpi_n}(u_n(t)-u_n(\theta_n(t)))+u_n(t)-u(\delta_n(t)+s)\|$$
$$=\|u_n(\theta_n(t))-u(\theta_n(t))\|+\|u_n(t)-u_n(\delta_n(t))\|.$$

On the other hand,

$$\begin{aligned}\sup_{s\in[-\tau,0]}\|u(\delta_n(t)+s)-u(s+t)\| &\leq \sup_{s\in[-\tau,-\varpi_n]}\|u(\delta_n(t)+s)-u(s+t)\|\\ &\quad+\sup_{s\in[-\varpi_n,0]}\|u(\delta_n(t)+s)-u(s+t)\|\\ &= \sup_{s\in[-\tau,-\varpi_n]}\|u(\delta_n(t)+s)-u(s+t)\|\\ &\quad+\|u(\delta_n(t))-u(t)\|.\end{aligned}$$

Then

$$\sup_{s\in[-\tau,0]}\|Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(s,u_n(t))-Z(t)u(s)\|_{\mathcal{C}_0}\leq$$
$$\|u_n(\theta_n(t))-u(\theta_n(t))\|+\|u_n(t)-u_n(\delta_n(t))\|+$$
$$\sup_{s\in[-\tau,-\varpi_n]}\|u(\delta_n(t)+s)-u(s+t)\|+\|u(\delta_n(t))-u(t)\|.$$

As $|\theta_n(t)-t|\leq\varpi_n$ and $|\delta_n(t)-t|\leq\varpi_n,\ \forall t\in[0,T]$, then $\theta_n(t)\longrightarrow t$ and $\delta_n(t)\longrightarrow t$ for n large enough. Furthermore, $(u_n)_n$ converges uniformly to u, $\|u(\delta_n(t))-u(t)\|\longrightarrow 0$, $\|u_n(\delta_n(t))-u_n(t)\|\longrightarrow 0$ and $\|u_n(\theta_n(t))-u(\theta_n(t))\|\longrightarrow 0$. As u is uniformly continuous, there is $\lambda>0$ such that $|s-t|\leq\lambda$ implies $\|u(s)-u(t)\|\leq\epsilon$. But we have $|\delta_n(t)+s-(s+t)|\leq\varpi_n$ for all $s\in[-\tau,\varpi_n]$. Hence

$$\sup_{s\in[-\tau,-\varpi_n]}\|u(\delta_n(t)+s)-u(s+t)\|\leq\epsilon \text{ for } \lambda\leq\varpi_n.$$

We can conclude that

$$Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(\cdot,u_n(t))\longrightarrow Z(t)u \text{ in } \mathcal{C}_0. \tag{6}$$

Finally, $\dot{u}(t)\in\Pi(t,Z(t)u)$. Indeed, by (4), (6) and condition (iii), we infer that

$$d\big(\dot{u}_n(t),\Pi(t,Z(t)u)\big)\leq\eta(t)\|Z(\delta_n(t))p^n_{\varpi_n\theta_n(t)}(\cdot,u_n(t))-Z(t)u\|\ a.e.$$

Passing to the limit in this inequality as $n\longrightarrow\infty$, we have

$$d\big(\dot{u}(t),\Pi(t,Z(t)u)\big)=0\ \ a.e.$$

So, u satisfies

$$\begin{cases}\dot{u}(t) &\in\Pi(t,Z(t)u) & \text{a.e. on } [0,T];\\ u(t) &=\psi(0)+\int_0^t\dot{u}(s)ds & \forall\ t\in[0,T];\\ u(t) &=\psi(t) & \forall\ t\in[-\tau,0].\end{cases}$$

The proof is then complete.

4 Application: a Delay Perturbed Sweeping Process

In this section, we present an application for functional differential inclusions governed by time and state-dependent nonconvex sweeping process. The sweeping process is a constrained differential inclusion involving normal cones, which appears naturally in several applications such as elastoplasticity, electrical circuits, hysteresis, crowd motion, etc. This kind of problems corresponds to several important mechanical problems, planning procedures in mathematical economy and nonsmooth dynamical systems. We propose here a new variant of the existence result which generalizes the previous results.

Theorem 4.1 *Let H be a separable Hilbert space and let $C : [0,T] \times H \rightrightarrows H$ be a set-valued mapping with nonempty closed values satisfying the following assumptions:*

($\mathcal{H}_1^C$) *for all $(t,u) \in [0,T] \times H, C(t,u)$ is uniformly subsmooth;*

($\mathcal{H}_2^C$) *there are real constants $L_1 > 0$ and $0 < L_2 < 1$ such that for all $t, s \in [0,T]$, and $u, v, z \in H$,*

$$|d(z, C(t,u)) - d(z, C(s,v))| \leq L_1|t-s| + L_2\|u-v\|;$$

($\mathcal{H}_3^C$) *for any bounded subset $A \in H$, the set $C(t,A)$ is ball-compact.*

Assume that ($\mathcal{H}_1^C$), ($\mathcal{H}_2^C$), ($\mathcal{H}_3^C$) and hypotheses $\mathcal{H}_{(\mathbf{\Pi})}$ are satisfied. Then, for any $\psi \in \mathcal{C}_0$ with $\psi(0) = u_0 \in C(0,u_0)$, there exists a continuous solution $u : [-\tau, T] \to H$, Lipschitz on $[0,T]$ to the problem

$$(R) \quad \begin{cases} \dot{u}(t) \in -N_{C(t,u(t))}(u(t)) + \Pi(t, Z(t)u), \quad a.e. \ in \ [0,T]; \\ u(t) \in C(t,u(t)), \ \forall \ t \in [0,T]; \\ \psi(s) = Z(0)u(s), \ \forall \ s \in [-\tau, 0]. \end{cases}$$

Proof. By using the discretization approach based on Moreau's catching-up algorithm, the proof is a careful adaptation of Theorem 3.5 in [7].

5 Conclusion

In this paper, we established an existence result to first order functional differential inclusions for a general class of unbounded nonconvex sets. The approach used is an adaptation of a reduction method which consists of replacing the problem with delay with a problem without delay and applying the known results in this case. As an application, we stated also a new version of the existence result for a first order perturbed nonconvex sweeping process that finds several applications in nonsmooth dynamical systems such as differential complementarity systems and vector hysteresis problems. This will be the subject of forthcoming works.

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