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# A New Fractional-Order Three-Dimensional Chaotic Flows with Identical Eigenvalues

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**Abstract:** This paper deals with certain new fractional-order three-dimensional chaotic systems. These autonomous systems are the fractional version of dynamical systems introduced recently by Faghani et al. [6]. The feature property of these systems is the presence of fractional order derivatives as well as equality of their eigenvalues. Numerical investigations on the dynamics of these systems have been carried out using a systematic computer search. Some simple fractional chaotic systems with identical eigenvalues were obtained, and their dynamical properties have been analyzed by means of the Lyapunov exponents.

**Keywords:** *fractional order derivative; chaotic system; Lyapunov exponents.*

**Mathematics Subject Classification (2010):** 34C28, 37D45, 37M22, 70K42, 93D05.

## 1 Introduction

Chaos systems have been receiving much attention from scientific community in the study of dynamical systems due to their applications in ecology, engineering and secure communications [3, 20]. Since the publication of Lorenz's seminal paper in 1963, there is no theory that allows us to predict chaotic solutions.The relationship between chaotic systems and their strange attractors is still unknown. Thanks to numerical simulations, we have been studying chaos, it was the essential tool by which many works have been done to study chaos in dynamical systems. Chaotic systems can be categorized as systems with self-excited attractors and systems with hidden attractors. The basin of attraction for the chaotic system with self-excited attractor intersects with an unstable equilibrium, while the chaotic system with hidden attractors has a basin of attraction which does not

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intersect with the neighborhoods of the equilibrium. It is well known that if real parts of all eigenvalues at the equilibrium point are negative, then there exist stable manifolds in a small neighborhood of an equilibrium point, whereas the existence of a positive real part in at least one eigenvalue of them shows the unstable manifolds.

Recently, fractional calculus, which is a mathematical topic whose history goes back more than 300 years, has received a considerable attention. It has been found that many systems can be described by fractional differential equations. For instance, fractional derivatives have been widely used in viscoelasticity, anomalous diffusion phenomena, electromagnetism, digital cryptography and many other phenomena [13, 15]. Some fractional-order dynamical systems have been investigated since the seminal paper of Grigorenko and Grigorenko [7], which demonstrated the existence of chaotic solutions in the fractional-order Lorenz dynamical system, see [2, 4, 5, 8, 9, 11, 18, 19].

In 2011, Sprott presented criteria for proposing new systems with strange attractors. To date, many new chaotic systems which satisfy Sprott's criteria are proposed, among which we cite chaotic systems without any equilibria, with a line, curve, and surface equilibria [1, 10, 12, 14].

Recently, Faghani et al. [6] defined a new category of chaotic systems with identical eigenvalues, proposed three general structures with special conditions and described their chaotic attractors.

In this paper, we propose the fractional version of systems studied by Faghani et al. [6]. These systems have the features of the presence of fractional derivatives and the equality of eigenvalues. The paper is organized as follows. In Section 2, the fractional systems are defined with their conditions. From defined systems, 14 simple chaotic flows are proposed according to the initial conditions, parameters, and fractional orders. The paper is concluded in Section 3.

## 2 Proposed Fractional Systems

First of all, we define the Caputo fractional derivative. The reader can refer to [13], for more details.

**Definition 2.1** The $\alpha$th-order Caputo fractional derivative of function $f(t)$ with respect to $t$ and the terminal 0 is given by

$$ {}_0D_t^{\alpha} f = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, $$

where $m$ is an integer such that $m-1 \leq \alpha < m$, and $\Gamma$ is the well-known Gamma function.

We consider now the fractional version of systems proposed in [6] with the Caputo fractional derivatives as follows:

$$ \begin{aligned} D^{\alpha_1} x &= y, \\ D^{\alpha_2} y &= z, \\ D^{\alpha_3} z &= a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 xz + a_9 yz + a_{10}, \end{aligned} \tag{1} $$

$$ \begin{aligned} D^{\alpha_1} x &= -z, \\ D^{\alpha_2} y &= b_1 x + b_2 z, \\ D^{\alpha_3} z &= a_1 x + a_2 y + a_3 z + a_4 x^2 + a_5 y^2 + a_6 z^2 + a_7 xy + a_8 xz + a_9 yz + a_{10}, \end{aligned} \tag{2} $$

$$\begin{aligned} D^{\alpha_1}x &= z, \\ D^{\alpha_2}y &= z-y \\ D^{\alpha_3}z &= a_1x+a_2y+a_3z+a_4x^2+a_5y^2+a_6z^2+a_7xy+a_8xz+a_9yz+a_{10}, \end{aligned} \tag{3}$$

where $D^{\alpha_i}$ denotes the derivatives of order $\alpha_i$ $(0<\alpha_i<1,\ i=\overline{1,3}\ )$ in the sense of Caputo, $a_i$, $i=\overline{1,10}$, are the real parameters of the systems, and $(b_1,b_2)=(-1,1)$ or $(b_1,b_2)=(1,-1)$.

Our next step is to make the three eigenvalues equal, we do this by putting some suitable conditions on systems parameters. The equilibrium points of the above systems are calculated as follows:

$D^{\alpha_1}x=0$,

$D^{\alpha_2}y=0$,

$D^{\alpha_2}z=0$.

For the system (1) and system (3), the equilibrium point is

$$(x^*,y^*,z^*)=(m,0,0)\,,m=\frac{-a_1\pm\sqrt{a_1^2-4a_4\,a_{10}}}{2a_4}$$

if $a_4\neq 0$ and $a_1^2-4a_4\,a_{10}\geq 0$.

For the system (2), the equilibrium point is

$$(x^*,y^*,z^*)=\left(0,\frac{-a_2\pm\sqrt{a_2^2-4a_5a_{10}}}{2a_5},0\right)$$

for $a_5\neq 0$ and $a_2^2-4a_5a_{10}\geq 0$.

The eigenvalues of the equilibrium points for the systems are determined by setting the determinant of the matrix $\lambda I-J$ to zero, where $J$ is the Jacobien matrix defined as

$$J=\begin{pmatrix} \delta_x f_1(q) & \delta_y f_1(q) & \delta_z f_1(q) \\ \delta_x f_2(q) & \delta_y f_2(q) & \delta_z f_2(q) \\ \delta_x f_3(q) & \delta_y f_3(q) & \delta_z f_3(q) \end{pmatrix},$$

where $D^{\alpha_i}x_i=f_i(x,y,z)\,,\ 1\leq i\leq 3$, $(x_1,x_2,x_3)=(x,y,z)$ and $q=(x^*,y^*,z^*)$ is the equilibrium point.

We obtain the characteristic equation for each equilibrium point. For example, the characteristic equation for the equilibrium point $\left(\dfrac{-a_1+\sqrt{a_1^2-4a_4\,a_{10}}}{2a_4},0,0\right)$ is

$$\lambda^3-(a_3+a_8m)\,\lambda^2-(a_2+a_7m)\,\lambda-(a_1+2a_4m)=0,$$

where

$$m=\frac{-a_1+\sqrt{a_1^2-4a_4\,a_{10}}}{2a_4}.$$

Eigenvalues are solutions of the characteristic equation, if they are equal, we have identical eigenvalues. Under the following conditions, the eigenvalues are equal:

$$a_2=\frac{-1}{3}\left(a_3+a_8\frac{-a_1+\sqrt{a_1^2-4a_4\,a_{10}}}{2a_4}\right)^2-a_7\frac{-a_1+\sqrt{a_1^2-4a_4\,a_{10}}}{2a_4}, \tag{4}$$

and

$$a_3 = 3\left(\sqrt{a_1^2 - 4a_4a_{10}}\right)^{\frac{1}{3}} - a_8\frac{-a_1 + \sqrt{a_1^2 - 4a_4a_{10}}}{2a_4}. \tag{5}$$

In a similar manner, we found conditions for the second equilibrium point of Eq. (1) and for the equilibrium points in the other structures in Eqs. (2) and (3).

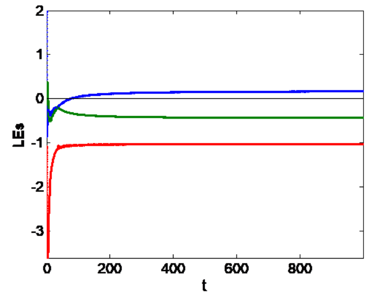
Under these conditions, we search for systems according to the fractional-order, parameters, and initial conditions which show chaotic dynamics.

So, using a systematic computer search, fourteen systems with chaotic dynamics were found by combining the parameters $a_1$ through $a_{10}$, which satisfy the constraints in Eq. (4)-(5) (and similar constraints for systems (2) and (3)) on the fractional orders and initial conditions. The found simple chaotic systems are listed in Table 1 as $FE_1 - FE_{14}$. The equilibriums of all these systems are at the origin. The systems $FE_1 - FE_9$ have three zero eigenvalues, then the stability of the equilibrium point is not determined, while the systems $FE_{10} - FE_{14}$ have positive identical eigenvalues, thus the equilibrium point is unstable. The Lyapunov exponents of the systems are calculated by Wolf's method [17]. The chaotic solutions are determined by the positivity of at least one Lyapunov exponent, which is the case in all systems $FE_1 - FE_{14}$. Also, the Lyapunov exponents with respect to time of some proposed systems are presented in Fig.1. Attractors projected onto the $xy-$plane for all proposed systems are shown in Fig. 2.

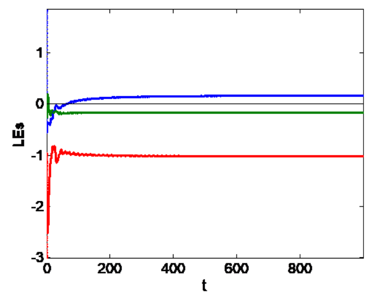
| Case | Equations | F.O ($\alpha$) | Parameters | Equilibrium | Eigenvalues | LEs | $(x_0, y_0, z_0)$ |
|---|---|---|---|---|---|---|---|
| $FE_1$ | $D^{\alpha_1}x = y$ | 0.99 | $a = 0.78$ | 0 | 0 | 0.0103 | $-48.73$ |
| | $D^{\alpha_1}y = z$ | | | 0 | 0 | 0.0007 | $-30.86$ |
| | $D^{\alpha_1}z = x^2 - y^2 + axz$ | | | 0 | 0 | $-16.6512$ | 63.52 |
| $FE_2$ | $D^{\alpha_1}x = y$ | 0.98 | $a = 0.78$ | 0 | 0 | 0.0085 | $-48.73$ |
| | $D^{\alpha_1}y = x - z$ | | | 0 | 0 | $-0.0011$ | $-30.86$ |
| | $D^{\alpha_1}z = -x^2 + ay^2 + byz$ | | | 0 | 0 | $-16.9313$ | 63.52 |
| $FE_3$ | $D^{\alpha_1}x = -z$ | 0.95 | $a = 6$ | 0 | 0 | 0.0207 | $-2.64$ |
| | $D^{\alpha_1}y = x - z$ | | $b = 9$ | 0 | 0 | $-0.0006$ | 0.91 |
| | $D^{\alpha_1}z = x^2 - y^2 + axz$ | | | 0 | 0 | $-0.8690$ | $-4.14$ |
| $FE_4$ | $D^{\alpha_1}x = -z$ | 0.9 | $a = 6$ | 0 | 0 | 0.0212 | $-2.64$ |
| | $D^{\alpha_1}y = x - z$ | | $b = 9$ | 0 | 0 | $-0.0003$ | 0.91 |
| | $D^{\alpha_1}z = x^2 - y^2 + axz$ | | | 0 | 0 | $-1.0683$ | $-4.14$ |
| $FE_5$ | $D^{\alpha_1}x = z$ | 0.9 | $a = 0.3$ | 0 | 0 | 0.1450 | $-39.56$ |
| | $D^{\alpha_1}y = z - y$ | | $b = -1.5$ | 0 | 0 | 0.0002 | $-2.85$ |
| | $D^{\alpha_1}z = -y + z + ax^2$ | | $c = 0.6$ | 0 | 0 | $-37.5602$ | $-41.22$ |
| | $+bxy + cxz$ | | | | | | |
| $FE_6$ | $D^{\alpha_1}x = z$ | 0.8 | $a = 0.3$ | 0 | 0 | 0.1914 | $-39.56$ |
| | $D^{\alpha_1}y = z - y$ | | $b = -1.5$ | 0 | 0 | $-0.0008$ | $-2.85$ |
| | $D^{\alpha_1}z = -y + z + ax^2$ | | $c = 0.6$ | 0 | 0 | $-44.2027$ | $-41.22$ |
| | $+bxy + cxz$ | | | | | | |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| $FE_7$ | $D^{\alpha_1}x = z$ | 0.9 | $a = -0.4364$ | 0 | 0 | 0.1812 | $-3.87$ |
| | $D^{\alpha_1}y = z - y$ | | $b = 2$ | 0 | 0 | 0.0010 | $-0.7$ |
| | $D^{\alpha_1}z = -y + z + ax^2$ | | $c = -0.7229$ | 0 | 0 | $-36.7658$ | 2.31 |
| | $+bxy + cxz$ | | | | | | |
| $FE_8$ | $D^{\alpha_1}x = z$ | 0.85 | $a = -0.4364$ | 0 | 0 | 0.1743 | $-3.87$ |
| | $D^{\alpha_1}y = z - y$ | | $b = 2$ | 0 | 0 | 0.0009 | $-0.7$ |
| | $D^{\alpha_1}z = -y + z + ax^2$ | | $c = -0.7229$ | 0 | 0 | $-41.3095$ | 2.31 |
| | $+bxy + cxz$ | | | | | | |
| $FE_9$ | $D^{\alpha_1}x = -z$ | 0.99 | $a = 0.128$ | 0 | 0.2 | 0.0323 | $-21.36$ |
| | $D^{\alpha_1}y = -x + z$ | | $b = 0.008$ | 0 | 0.2 | 0.0001 | $-18.43$ |
| | $D^{\alpha_1}z = a\,x + b\,y + cz$ | | $c = 0.6$ | 0 | 0.2 | $-1.7123$ | $-11.03$ |
| | $+d\,x^2 + e\,y^2 + fzy$ | | $d = -0.16$ | | | | |
| | | | $e = 0.01$ | | | | |
| | | | $f = 0.1$ | | | | |
| $FE_{10}$ | $D^{\alpha_1}x = -z$ | 0.88 | $a = 0.128$ | 0 | 0.2 | 0.0538 | $-21.36$ |
| | $D^{\alpha_1}y = -x + z$ | | $b = 0.008$ | 0 | 0.2 | 0.0002 | $-18.43$ |
| | $D^{\alpha_1}z = a\,x + b\,y + cz$ | | $c = 0.6$ | 0 | 0.2 | $-2.5653$ | $-11.03$ |
| | $+d\,x^2 + e\,y^2 + fzy$ | | $d = -0.16$ | | | | |
| | | | $e = 0.01$ | | | | |
| | | | $f = 0.1$ | | | | |
| $FE_{11}$ | $D^{\alpha_1}x = -z$ | 0.99 | $a = 0.544$ | 0 | 0.4 | 0.1754 | $-21.36$ |
| | $D^{\alpha_1}y = -x + z$ | | $b = 0.64$ | 0 | 0.4 | $-0.1676$ | $-18.43$ |
| | $D^{\alpha_1}z = a\,x + b\,y + cz$ | | $c = 1.2$ | 0 | 0.4 | $-1.0270$ | $-11.03$ |
| | $+d\,x^2 + e\,y^2 + f\,zy$ | | $d = -0.16$ | | | | |
| | | | $e = 0.01$ | | | | |
| | | | $f = 0.1$ | | | | |
| $FE_{12}$ | $D^{\alpha_1}x = -z$ | 0.89 | $a = 0.544$ | 0 | 0.4 | 0.2714 | $-21.36$ |
| | $D^{\alpha_1}y = -x + z$ | | $b = 0.64$ | 0 | 0.4 | $-0.2575$ | $-18.43$ |
| | $D^{\alpha_1}z = a\,x + b\,y + cz$ | | $c = 1.2$ | 0 | 0.4 | $-1.5700$ | $-11.03$ |
| | $+d\,x^2 + e\,y^2 + f\,zy$ | | $d = -0.16$ | | | | |
| | | | $e = 0.01$ | | | | |
| | | | $f = 0.1$ | | | | |
| $FE_{13}$ | $D^{\alpha_1}x = -z$ | 0.97 | $a = 0.875$ | 0 | 0.5 | 0.17.86 | $-21.36$ |
| | $D^{\alpha_1}y = -x + z$ | | $b = 0.125$ | 0 | 0.5 | $-0.4383$ | $-18.43$ |
| | $D^{\alpha_1}z = a\,x + b\,y + cz$ | | $c = 1.5$ | 0 | 0.5 | $-1.0196$ | $-11.03$ |
| | $+d\,x^2 + e\,y^2 + f\,zy$ | | $d = -0.16$ | | | | |
| | | | $e = 0.01$ | | | | |
| | | | $f = 0.1$ | | | | |
| $FE_{14}$ | $D^{\alpha_1}x = -z$ | 0.99 | $a = 0.875$ | 0 | 0.5 | 0.1636 | $-21.36$ |
| | $D^{\alpha_1}y = -x + z$ | | $b = 0.125$ | 0 | 0.5 | $-0.4013$ | $-18.43$ |
| | $D^{\alpha_1}z = a\,x + b\,y + cz$ | | $c = 1.5$ | 0 | 0.5 | $-0.9368$ | $-11.03$ |
| | $+d\,x^2 + e\,y^2 + f\,zy$ | | $d = -0.16$ | | | | |
| | | | $e = 0.01$ | | | | |
| | | | $f = 0.1$ | | | | |

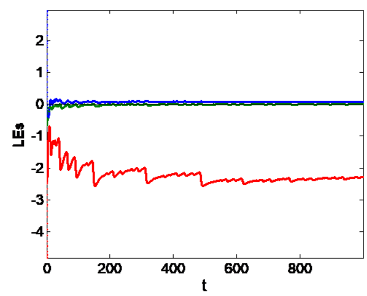
**Table 1**: Fourteen fractional-order three-dimensional chaotic systems with identical eigenvalues.



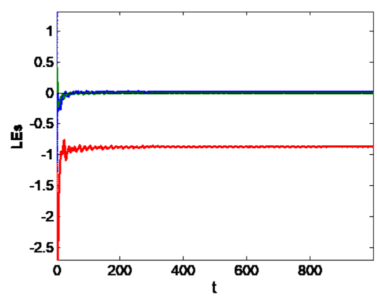

FE13




FE11




FE10




FE3

**Figure 1**: Lyapunov Exponents of some systems in Table 1 with respect to time.

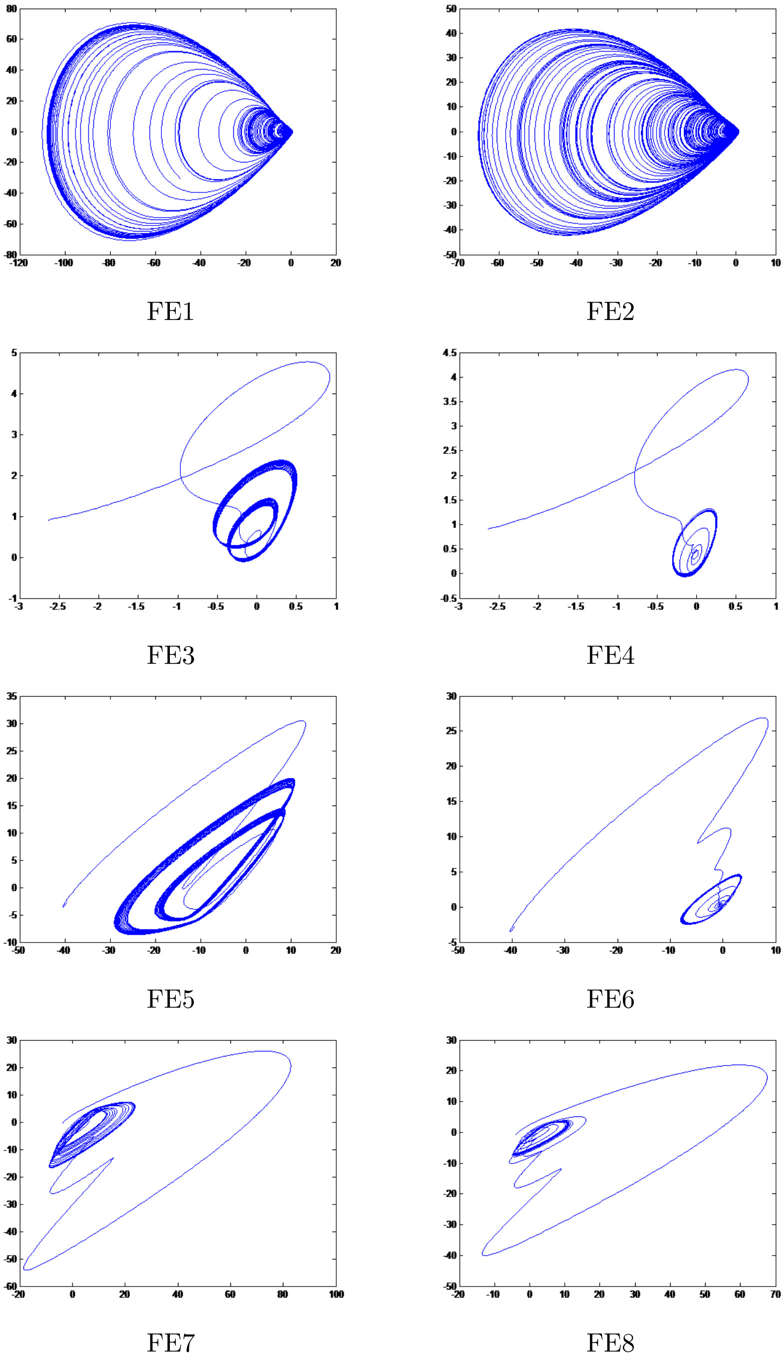



**Figure 2**: Attractors for 14 fractional-order systems in the xy-plane with initial conditions given in Table 1.

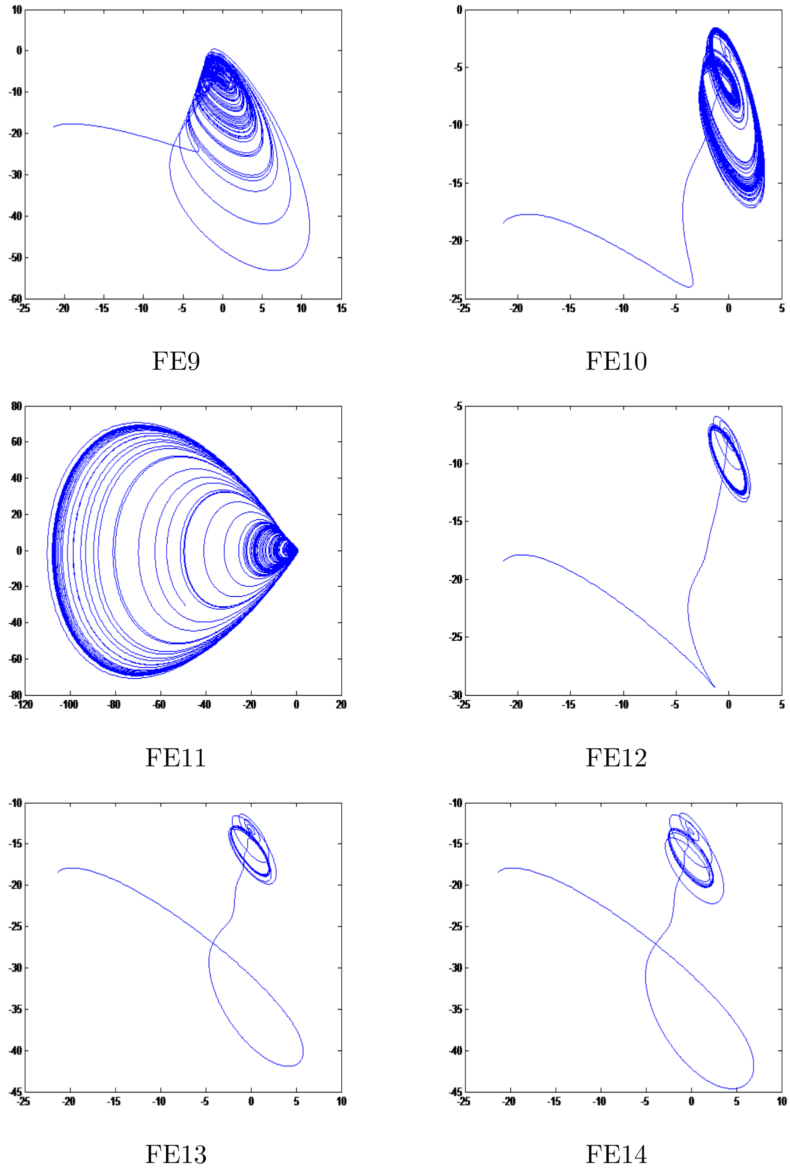



**Figure 3**: Attractors for 14 fractional-order systems in the xy-plane with initial conditions given in Table 1 (continued).

## 3 Conclusion

This paper introduces new fractional-order three-dimensional chaotic systems which have identical eigenvalues as a particular property. Using an exhaustive computer search, we proposed 14 fractional-order systems which show chaotic dynamics, where the origin was the equilibrium of these systems. Eight of the proposed systems have zero identical eigenvalues, while six of the other systems have three positive and equal eigenvalues.

For all fractional-order chaotic systems proposed, the attractors were projected onto the xy-plane, and the Lyapunov exponents were calculated.

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