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Stabilization of Chaotic h-Difference Systems with Fractional Order

Hasna Yousfi [1], Ahlem Gasri [1] and Adel Ouannas [2*]

[1] *Laboratory of Mathematics, Informatics and Systems (LAMIS), Department of Mathematics and Computer Science, University of Larbi Tebessi, Tebessa 12002 Algeria.*
[2] *Department of Mathematics and Computer Science, University of Larbi Ben M'hidi, Oum El Bouaghi, Algeria.*

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Abstract: Based on the Lyapunov approach as well as on the properties of the Caputo h-difference operator, a one-dimensional linear control law is intoduced to stabilize the chaotic fractional discrete-time Ushio system. Numerical results are presented throughout the paper to illustrate the findings.

Keywords: *discrete fractional calculus; fractional discrete Ushio system; linear stabilization; Lyapunov approach.*

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1 Introduction

Fractional discrete calculus is a very interesting topic in mathematics with several potential applications in many fields [1]. Namely, since fractional discrete operators are non local, they are suitable for constructing models characterized by memory effect [2]. This is the reason why fractional-order difference systems, when describing engineering phenomena over large periods of time, perform better with respect to integer-order discrete-time systems [3]. Recently, attention has been focused on the presence of chaotic phenomena in fractional-order systems, described by difference equations [4, 5].

One of the important aspects in the study of chaotic systems is the development of control strategies to achieve stabilization. The aim of the stabilization of chaotic systems is to derive a one-dimensional control law such that both of the map trajectories are controlled to zero asymptotically. Recently, the topic of stabilization of fractional discrete chaotic systems started to attract increasing attention [6–10].

This study presents a novel contribution to the topic of stabilization of chaos in Caputo h-difference chaotic systems. After investigating the existence of chaotic behaviors in the fractional Ushio system, a linear scheme is introduced to control the fractional Ushio system.

* Corresponding author: mailto:adel.ouannas@yahoo.com

2 The Fractional Chaotic Ushio System

Referring to the discrete-time Ushio system, it was introduced in [11]. Herein, by exploiting the Caputo h-difference operator, the following fractional Ushio system is proposed:

$$\begin{cases} {}_{h}^{C}\Delta_{a}^{\nu}x\left(t\right)=\left(d-1\right)x\left(t+h\nu\right)-x^{3}\left(t+h\nu\right)+y\left(t+h\nu\right), \\ {}_{h}^{C}\Delta_{a}^{\nu}y\left(t\right)=0.5x\left(t+h\nu\right)-y\left(t+h\nu\right), \end{cases} \tag{1}$$

where ${}_{h}^{C}\Delta_{a}^{\nu}$ denotes the Caputo h-difference operator, $0\leq\nu\leq1$, $t\in\left(h\mathbb{N}\right)_{a+(1-\nu)h}$, $\left(h\mathbb{N}\right)_{a+(1-\nu)h}=\left\{a+\left(1-\nu\right)h,a+\left(2-\nu\right)h,...\right\}$, a is the starting point and d is a system parameter.

The Caputo h-difference operator ${}_{h}^{C}\Delta_{a}^{\nu}X\left(t\right)$ of a function $X\left(t\right)$ [12] is defined as

$${}_{h}^{C}\Delta_{a}^{\nu}X\left(t\right)=\Delta_{a}^{-(n-\nu)}\Delta^{n}X\left(t\right),\qquad t\in\left(h\mathbb{N}\right)_{a+(n-\nu)h}, \tag{2}$$

where $\Delta X\left(t\right)=\frac{X(t+h)-X(t)}{h}$, $n=\lceil\nu\rceil+1$, and the ν-th order h–sum [13] is given by

$${}_{h}\Delta_{a}^{-\nu}X\left(t\right)=\frac{h}{\Gamma\left(\nu\right)}\sum_{s=\frac{a}{h}}^{\frac{t}{h}-\nu}\left(t-\sigma\left(sh\right)\right)^{(\nu-1)}x\left(sh\right),\sigma\left(sh\right)=\left(s+1\right)h,a\in\mathbb{R},t\in\left(h\mathbb{N}\right)_{a+\nu h}, \tag{3}$$

where the h–falling factorial function is defined as

$$t_{h}^{(\nu)}=h^{\nu}\frac{\Gamma\left(\frac{t}{h}+1\right)}{\Gamma\left(\frac{t}{h}+1-\nu\right)},\quad t,\nu\in\mathbb{R}. \tag{4}$$

Now, according to [14], the equivalent implicit discrete formula can be written in the form

$$\begin{cases} x(n+1)=x(0)+\frac{h^{\nu}}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}((d-1)\,x\left(j+1\right)-x^{3}\left(j+1\right)+y\left(j+1\right)), \\ y(n+1)=y(0)+\frac{h^{\nu}}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(0.5x\left(j+1\right)-y\left(j+1\right)), \end{cases} \tag{5}$$

where $x(0)$, $y(0)$ are the initial states. Here, the implicit system given in (5) is employed to explore the chaotic behavior of the Ushio system in its fractional order. When $(x(0),y(0))=(0.1,-0.3))$ and $d=1.9$, then the fractional-order Ushio system will show chaotic behaviour. Figure 1, however, shows the chaotic attractor obtained by simulating the system (5) with the predictor-corrector method, along with the Largest Lyapunov Exponents (LLEs) and the bifurcation diagram that are obtained.

3 Chaos Stabilization Scheme

This section intends to prove a novel result established for stabilizing the dynamics of the fractional Ushiou system at zero through establishing a linear control law. When we refer to stablization, what we are talking about is adding a new time varying parameter $\mathbf{C}\left(t\right)$ to one of the system's states and finding a closed form adaptive formula for these parameters to force the system states to zero in sufficient time. Before stating the proposed control law and establishing its stability, it is important to state the following theorem, which is essential for our proof. Interested readers are referred to [15] for the proof of this result.

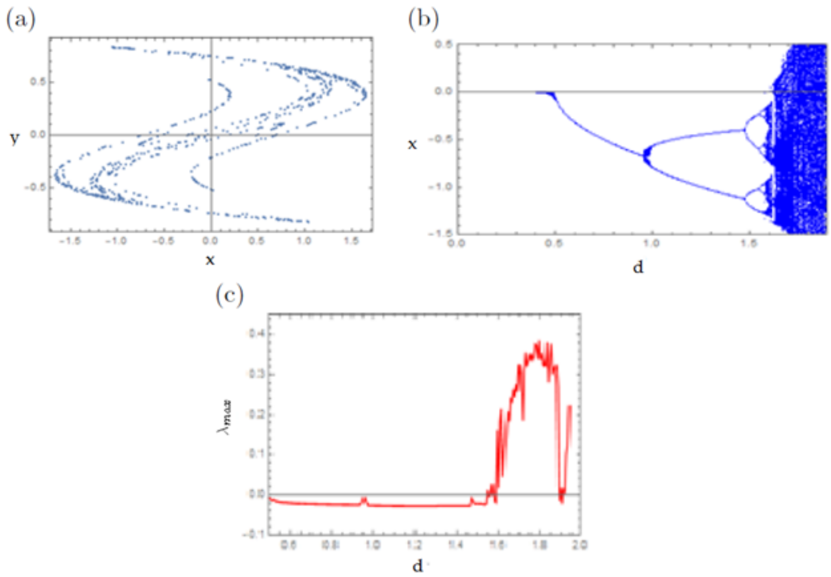

Figure 1: (a) Phase portrait of the Ushio system for $\upsilon = 0.95$ and $d = 19$, b) The bifurcation diagram versus d, (c) the largest Lyapunov exponents corresponding to (b).

Theorem 3.1 *Let $x = 0$ be an equilibrium point of the nonlinear discrete fractional system*

$$ {}_{h}^{C}\Delta_{a}^{\nu} X(t) = f(t + \nu h, X(t + \nu h)), \quad t\,(h\mathbb{N})_{a+(1-\nu)h}. \tag{6}$$

If there exists a positive definite and decrescent scalar function $V(t, X(t))$ such that ${}_{h}^{C}\Delta_{a}^{\nu} V(t, X(t)) \leq 0$, then the equilibrium point is asymptotically stable.

In the following, a useful inequality for Lyapunov functions is introduced.

Lemma 3.1 *[15] For any discrete time $t \in (h\mathbb{N})_{a+(1-\nu)h}$, the following inequality holds:*

$$ {}_{h}^{C}\Delta_{a}^{\nu} X^{2}(t) \leq 2X(t + \nu h){}_{h}^{C}\Delta_{a}^{\nu} X(t), \quad 0 < \nu \leq 1. \tag{7}$$

Theorem 3.2 *The two-dimesional fractional Ushio system can be stabilized under the one–dimensional control law*

$$ \mathbf{C}(t) = -dx(t) - 1.5y(t), \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{8}$$

Proof. The controlled fractional Ushio system involves the time–varying control parameter $\mathbf{C}(t)$ and is given by

$$ \begin{cases} {}_{h}^{C}\Delta_{a}^{\nu} x(t) = (d-1)\,x(t + h\nu) - x^{3}(t + h\nu) + y(t + h\nu) + \mathbf{C}(t + h\nu), \\ {}_{h}^{C}\Delta_{a}^{\nu} y(t) = 0.5x(t + h\nu) - y(t + h\nu), \end{cases} \tag{9}$$

where $t \in (h\mathbb{N})_{a+(1-\nu)h}$. Substituting the proposed control law (8) into (9) yields the simplified dynamics

$$ \begin{cases} {}_{h}^{C}\Delta_{a}^{\nu} x(t) = -x(t + h\nu) - x^{3}(t + h\nu) - 0.5y(t + h\nu), \\ {}_{h}^{C}\Delta_{a}^{\nu} y(t) = 0.5x(t + h\nu) - y(t + h\nu). \end{cases} \tag{10}$$

Now, we should prove that the trivial solution of (10) is globally asymptotically stable. If so, we will deduce immediately that all the states of the controlled system given in (9) will definitely converge towards zero. Actually, this task can be performed using the Lyapunov method that was summarized earlier by Theorem 3.1. To see this, the following Lyapunov function has to be considered:

$$V\left(x\left(t\right),y\left(t\right)\right)=\frac{1}{2}\left(x^{2}\left(t\right)+y^{2}\left(t\right)\right),\quad t\in\left(h\mathbb{N}\right)_{a+(1-\nu)h}. \tag{11}$$

Consequently, applying the fractional Caputo h-difference operator to (11) leads us to the following assertion:

$$ {}_{h}^{C}\Delta_{a}^{\nu}V\left(x\left(t\right),y\left(t\right)\right)=\frac{1}{2}\left({}_{h}^{C}\Delta_{a}^{\nu}x^{2}\left(t\right)+{}_{h}^{C}\Delta_{a}^{\nu}y^{2}\left(t\right)\right). \tag{12}$$

Using Lemma 1 yields

$$\begin{aligned}
{}_{h}^{C}\Delta_{a}^{\nu}V\left(x\left(t\right),y\left(t\right)\right) &\leq x(t+\nu h){}_{h}^{C}\Delta_{a}^{\nu}x(t)+y(t+\nu h){}_{h}^{C}\Delta_{a}^{\nu}y(t)\\
&= -x^{2}\left(t+h\nu\right)-x^{4}\left(t+h\nu\right)-0.5x(t+\nu h)y\left(t+h\nu\right)\\
&\quad +0.5y\left(t+h\nu\right)x\left(t+h\nu\right)-y^{2}\left(t+h\nu\right)\\
&= -\left(x^{2}\left(t+h\nu\right)+x^{4}\left(t+h\nu\right)+y^{2}\left(t+h\nu\right)\right)\\
&< 0.
\end{aligned}$$

This means that an efficient stabilization for all states of system (1) occurs at the origin when using the linear control law (8).

For the purpose of confirming the validity of the established controller, the phase-space and the evolution of all states of the controlled system (9) are plotted as shown in Figure 2. Such plots clearly show a stabilization at zero occurs for all chaotic dynamics of the fractional Ushio system given in (1) when using the linear control law given in (8).

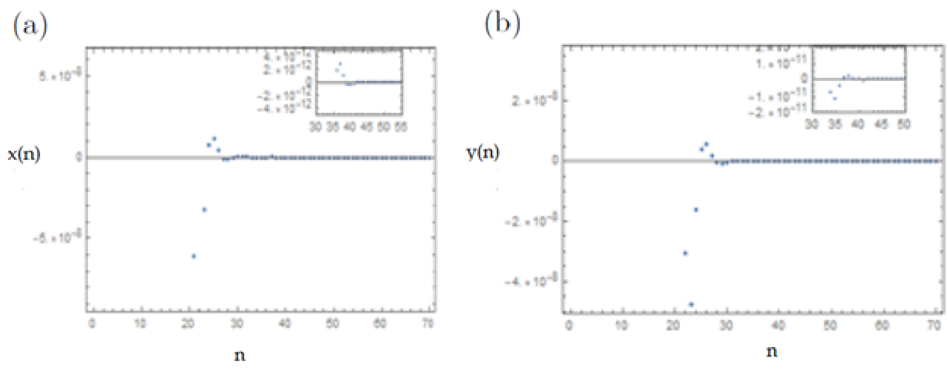

Figure 2: Stabilization of all states of the fractional chaotic Ushio system (1) by using the control law (8) with $\nu = 0.95$ and $d = 1.9$.

4 Conclusion

This work has made a contribution to this research field by proposing simple linear control laws for stabilizing the dynamics of some types of those fractional maps which have been established in view of the Caputo h-difference operator. The objective has been achieved by proving a new theorem based on assuming suitable Lyapunov functions. Since the designed control law is one-dimensional and linear, it is inexpensive and easy to implement. Finally, simulation findings have been implemented with the aim of highlighting the validity of all proposed control schemes.

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