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Application of Accretive Operators Theory to Linear SIR Model

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Abstract: In this paper, we discuss the existence and uniqueness results for a linear SIR (Susceptible-Infected-Recovered) model on L^p-spaces, for $1 \leq p < +\infty$. This work represents two extensions of the basic static linear model presented in [4]. Our analysis is fundamentally based on the accretive operators theory.

Keywords: *SIR; epidemic models; accretive operators; existence result; mild solution.*

Mathematics Subject Classification (2010): 92D30, 47H06, 35F10.

1 Introduction

In epidemiology, mathematical models have become important tools in analyzing the spread and control of infectious diseases caused by bacteria, viruses and fungi through a direct transmission from individual-to-individual: through a sneeze, cough, skin-skin contact and exchange of body fluids. Some examples of the diseases are: Coronavirus desease (Covid-19), Acquired Immune Deficiency Syndrome (AIDS), Ebola, Dengue fever, etc. The first mathematician who proposed a mathematical model describing an infectious disease is Daniel Bernoulli. In 1760, he modelled the spread of smallpox [8]. In our case, we are interested in the SIR model which can model Coronavirus desease. This model was first used by Kermack and McKendrick in 1927, and has subsequently been applied to a variety of diseases [13]. They have considered a constant total population and assumed that the interaction between the groups was determined by the disease transmission and removal rates. They have classified the population into three groups: susceptible (S), infected (I) and recovered (R). There have been many variations such as

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classical epidemiological models. These models are based on the standard Susceptible-Infectious-Recovered (SIR) compartments segmented in the model. Susceptible is a group of people who are vulnerable to infection when contacting with infectious people, see [11] and the references therein.

The SIR model was discussed by many authors. Diekmann et al. [10] studied epidemic models with one strain. However, Ackleh and Allen [1] studied SIR-type models of disease with n strains and vertical transmission. In 2009, Hina Khan et al. [14] solved the SIR model by means of an analytic technique for nonlinear problems and the homotopy analysis method. After two years, Bain et al. studied the existence of at least two positive periodic solutions of the SIR model in [5]. They based on the continuation theorem of coincidence degree theory. Moreover, in 2016, I. Al-Darabsah and Y. Yuan proposed the mathematical model for the transmission by SIR for Ebola [2]. In the same year, I. Ameen and P. Novati studied the fractional SIR model with constant population [3], they obtained a numerical solution using discrete methods.

The aim of this paper is to study the problem (1) on L^p spaces, for $1 \leq p < \infty$. We note that our SIR model is linear because we have ignored the transmission of the epidemic disease from one person to another person. We note that this model was investigated theoretically in a number of papers. For example, in [16], the authors studied a stochastic epidemic-type model with enhanced connectivity, and they obtained an exact solution of the model. Our objective in this work is to discuss the existence and uniqueness result for the problem (1). In fact, although this model is standard, in our situation, we have encountered some difficulties lying in the fact that the problem is composed of three equations that are strongly coupled. To overcome these difficulties, we first rewrite our system as a Cauchy problem involving two matrix operators, and we show that the latter one has a unique solution using the accretive theory. We note that the solution of this system gives more information on the propagation of the epidemic. In general, it is difficult to compute the analytical solution of the problem. On the other hand, it is usually impossible to obtain the exact solution for the general case. Therefore, our approach guarantees the existence and uniqueness of the solution, we can approximate the solution using numerical methods.

The rest of this paper is organized as follows. In the next section, we present the mathematical formulation of the SIR model. In Section 3, we introduce the functional setting and gather some preliminary facts in connection with the problem. The existence and uniqueness for the problem (Theorem 4.1) is stated in Section 4 by the accretive theory.

2 Model Formulation

In this section, we give the mathematical formulation describing the mechanism of the SIR model. The following diagram represents the SIR model. In this diagram:

- b : Immigration rate of susceptible.
- c : Specific rate of contact with pathogen.
- β : Probability of infection when there is direct contact.
- μ : Probability of illness in case of infection.
- ξ : 1/shedding period.

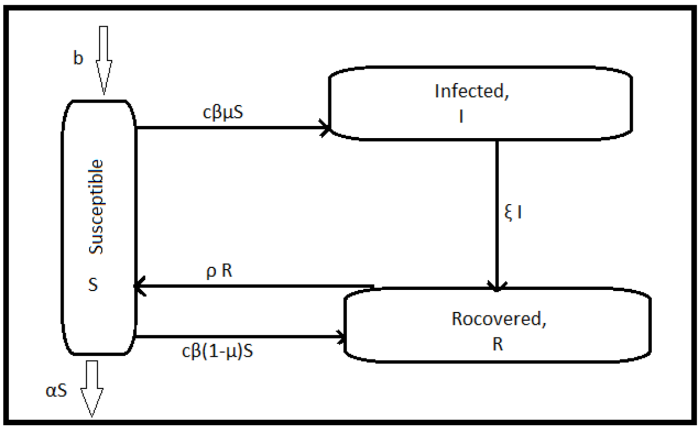

Figure 1: The mechanism of SIR model.

- α : Specific death rate in population.
- ρ : Specific immunity loss rate.

We denote the total population size by N, i.e., $N(t, a) = I(t, a) + S(t, a) + R(t, a)$. Now, in order to formulate the dynamics of the above diagram mathematically, the following assumptions have been adopted:

1. There is a constant number of the host populations entering into the system with the immigration rate $b > 0$.
2. Person-to-person transmission can be ignored.
3. α is the same for all $S - I - R$ classes.
4. The parameters ξ and ρ are constants.
5. Individuals can become infected and ill and then recover to become immune, or, on exposure, they may pass directly into the immune class.

Remark 2.1 a) Assumption (3) dictates a linear system, whereas much of the SIR model literature is concerned with nonlinear models, including an SI interaction term [15].

b) Assumption (4) is permissible because many zoonotic pathogens, and campylobacter in particular, cause much more mild illness rather than death.

c) Under these assumptions, our system represents an extension of the basic linear model presented in [4].

According to these assumptions, the SIR model can be represented mathematically by the following coupled system of partial differential equations:

$$\begin{cases} \dfrac{\partial S}{\partial t}(t,a) = t.\dfrac{\partial S}{\partial a}(t,a) + b(t,a) - (\alpha + c\beta)S(t,a) + \rho R(t,a), \\ \dfrac{\partial I}{\partial t}(t,a) = c\beta\mu S(t,a) - (\alpha+\xi)I(t,a), \\ \dfrac{\partial R}{\partial t}(t,a) = \xi I(t,a) + c\beta(1-\mu)S(t,a) - (\alpha+\rho)R(t,a), \\ S(0,a) = S_0(a), \quad I(0,a) = I_0(a) \ \text{ and } \ R(0,a) = R_0(a), \end{cases} \tag{1}$$

where $t \in [0,T]$, $a \in [0,\, L]$, $L > 0$ and $b(t,a) = \alpha N(t,a) = \alpha\,(S(t,a) + I(t,a) + R(t,a))$. The functions S, I and R are dependent on time t and age "a", and all others parameters are independent of time and age.

3 Notations and Preliminaries

In this section, we shall fix on the notations and introduce the functional framework, which will be used throughout this paper. Let X be a real Banach space with norm $\|\cdot\|$ and dual X^*.

We are going to introduce now the class of operators for which we could obtain existence and uniqueness results for solutions. Accretive operators were introduced by Browder [9] and Kato [12] independently.

Definition 3.1

- An operator $A : D(A) \subset X \longrightarrow 2^X$ is said to be accretive if the inequality $\|u - v + \lambda(\widehat{u} - \widehat{v})\| \geq \|u - v\|$ holds for all $\lambda \geq 0$, $u\,, v \in D(A)$ and $\widehat{u} \in Au$, $\widehat{v} \in Av$. If, in addition, $R(I + \lambda A)$ (i.e., the range of the operator $I + \lambda A$), is for some, hence for all, $\lambda > 0$, precisely X, then A is called m-accretive.
- An operator A is said to be quasi-accretive (quasi-m-accretive) if there exists $\omega \in \mathbb{R}$ such that $A + \omega I$ is accretive (respectively, m-accretive), in this case, we say also that A is ω-accretive (ω-m-accretive, respectively).

Remark 3.1 An operator A is accretive if and only if A is quasi-accretive with $\omega = 0$.

In order to verify accretivity of a given operator, it is useful to take into account alternative characterizations of this property. To do that, we need to introduce the bracket and the duality map.

Let $u \in X$. We denote by $[v, u]_s$ the function defined from $X \times X$ into $\mathbb{R}$ by

$$[v,u]_s = \sup\{u^*(v) \,:\, u^* \in \Gamma_1(u)\},$$

where $\Gamma_1(\cdot)$ denotes the duality map from X into 2^{X^*} given by

$$\Gamma_1(u) = \{u^* \in X^* \,:\, \langle u^*\,, u\rangle = \|u\| \text{ and } |u^*\| = 1\}.$$

We also define the duality map Γ from X into 2^{X^*} by

$$\Gamma(u) = \{u^* \in X^* \,:\, \langle u^*\,, u\rangle = \|u\|^2 \text{ and } |u^*\| = \|u\|\}.$$

We recall that the function $sgn_0(\cdot)$ is defined by

$$sgn_0(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0. \end{cases}$$

Now, we recall some important facts regarding accretive operators which will be used in our paper, we have the following proposition [6].

Proposition 3.1 *Let $A : D(A) \subset X \longrightarrow 2^X$ be an operator on X. The following conditions are equivalent:*

1. *A is an ω-accretive operator.*
2. *the inequality $[\widehat{u} - \widehat{v},\, u - v]_s \geq -\omega \|u - v\|$ holds for every u, $v \in D(A)$ and $\widehat{u} \in Au$, $\widehat{v} \in Av$.*
3. *for each $\lambda > 0$ with $\lambda\omega < 1$, the resolvent $(I + \lambda A)^{-1} : R(I + \lambda A) \longrightarrow D(A)$ is a single-valued $\frac{1}{1-\lambda\omega}$-Lipschitzian mapping.*

The quasi-m-accretive operators play an important role in the study of the Cauchy problem.

Consider the following Cauchy problem:

$$\begin{cases} u'(t) + A(u(t)) \ni f(t), \;\; t \in (0, T), \\ u(0) = u_0 \in \overline{D(A)}, \end{cases} \tag{2}$$

where A is quasi-m-accretive on X and $f \in L^1(0, T, X)$.

Let $\epsilon > 0$. An ϵ-discretization on $[0, T]$ of the equation $u'(t)+A(u(t)) \ni f(t)$ consists of a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_N$ of the interval $[0, t_N]$ and a finite sequence$(f)_{i=1}^N \subseteq X$ such that

$$\begin{cases} t_i - t_{i-1} < \epsilon \quad \text{for } i = 1, \cdots, N, \quad T - \epsilon < t_N \leq T, \\ \displaystyle\sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \|f(s) - f_i\| ds < \epsilon. \end{cases}$$

A $D_{A;\epsilon} = (t_0 \leq t_1 \leq \cdots \leq t_N\, ;\, f_1, \cdots, f_N)$ solution of (2) is a piecewise constant function $x : [0,\, t_N] \longrightarrow X$ whose values x_i on $(t_{i-1},\, t_i]$ satisfy the finite difference equation

$$\frac{x_i - x_{i-1}}{t_i - t_{i-1}} + A(x_i) \ni f_i, \;\; i = 1, \cdots, N.$$

Such a function $x = (x)_{i=1}^N$ is called an ϵ-approximate solution to the Cauchy problem (2) if it further satisfies

$$\|x(0) - u_0\| \leq \epsilon.$$

The following theorem is known (see [4, Theorem 4.5] or [7, p.108]) and deals with the existence of strong solutions.

Theorem 3.1 *If X is a Banach space with the Radon-Nikodym property, $A : D(A) \subseteq X \longrightarrow 2^X$ is a quasi-m-accretive operator, and $f \in BV(0, T; X)$, i.e., f is a function of bounded variation on $[0, T]$, then problem (2) has a unique strong solution whenever $u_0 \in D(A)$.*

Let $f,\, g \in L^1(0,T;X)$ and A be a ω-accretive operator; if u and v are integral solutions of $u'(t)+Au(t) \ni f(t)$ and $u'(t)+Au(t) \ni g(t)$, respectively, then

$$\|u(t)-v(t)\| \le e^{\omega t}\|u(0)-v(0)\| + \int_0^t e^{\omega(t-s)}\|f(s)-g(s)\|ds. \tag{3}$$

The following theorem plays an important role in our results.

Theorem 3.2 *Let X be a reflexive Banach space and let A be a quasi-m-accretive operator in X. Let $F : X \longrightarrow X$ be locally Lipschitz. Then, for each $u_0 \in D(A)$, there is a local strong solution to the problem*

$$\begin{cases} u'(t) + A(u(t)) \ni F(u(t)), \\ u(0) = u_0 \in \overline{D(A)}. \end{cases}$$

Assume further that

$$\langle -Fu,\, w\rangle \ge -k_1\|u\|^2 + k_2, \quad (u,\, w) \in \Gamma,$$

then the solution is global.

We have the following definition.

Definition 3.2 We say that $u \in C(0,T;X)$ is a weak solution of problem (2) if there are sequences $(u_n) \subseteq W^{1,\infty}(0,T;X)$ and $(f_n) \subseteq L^1(0,T;X)$ satisfying the following conditions:

1. $u_n'(t) + A(u_n(t)) \ni f_n(t)$ for almost all $t \in [0,\, T]$, $n = 1, 2, \cdots$;
2. $\lim\limits_{n\to\infty} \|u_n - u\|_\infty = 0$;
3. $u(0) = u_0$;
4. $\lim\limits_{n\to\infty} \|f_n - f\|_1 = 0$.

The following result, which is an easy consequence of Theorem 3.1, is important.

Theorem 3.3 *Let X be a Banach space with the Radon-Nikodym property. Then problem* (2) *admits a unique weak solution which is the unique integral solution of this problem.*

Remark 3.2 The results stated above for quasi-m-accretive operators with $\omega \neq 0$ are also valid for m-accretive operators.

Let $p \in [1\,, +\infty)$, we denote by X_p the following space:

$$X_p := L^p([0,T] \times [0,L]\,, dtda).$$

We also consider the following product space:

$$\mathcal{H}_p := X_p \times X_p \times X_p$$

equipped with the norm

$$\|v\|_{\mathcal{H}_p} = \|(v_0, v_1, v_2)^T\|_{\mathcal{H}_p} = \|v_0\|_{X_p} + \|v_1\|_{X_p} + \|v_2\|_{X_p}.$$

4 Existence Result

In this section, we are concerned with the existence and uniqueness result for problem (1). For our subsequent analysis, we need the following hypothesis:

$$\mathcal{A} \;:\; \text{The parameters } c,\ \alpha,\ \beta,\ \mu,\ \xi \text{ and } \rho \text{ are positive.}$$

For $i = 1, 2, \cdots, 5$, let F_i denote the bounded multiplication operators from X_p into itself. We define the matrix operator

$$F = \begin{pmatrix} 0 & F_1 & F_2 \\ F_3 & 0 & 0 \\ F_4 & F_5 & 0 \end{pmatrix},$$

where

$$F_1(u_1) = \alpha u_1, \quad F_2(u_2) = (\alpha + \rho)u_2, \quad F_3(u_3) = c\beta\mu_3,$$
$$F_4(u_4) = c\beta(1-\mu)u_4 \ \text{ and } \ F_5(u_5) = \xi u_5.$$

The operators F_i, $i = 1, \cdots, 5$, are bounded on the space X_p, therefore F is also bounded on the product space $\mathcal{H}_p$.

Remark 4.1 The boundedness of the operator F implies that it is a Lipschitz operator with a Lipschitz constant $\|F\|_{\mathcal{L}(\mathcal{H}_p)}$.

Define the following linear operator T by

$$T : D(T) \subseteq L^p(D) \longrightarrow L^p(D)$$
$$\psi \longrightarrow T\psi(t, a) = -t\frac{\partial \psi}{\partial a}(t, a).$$

Remark 4.2 The operator T is usually called the free streaming operator. It is a closed densely defined linear operator. Its resolvent set $\rho(T)$ contains the half plane

$$\{\lambda \in \mathbb{C} \,:\, \mathbf{Re}\lambda > 0\}.$$

We also define the matrix operator

$$A = \begin{pmatrix} T + c\beta\mu & 0 & 0 \\ 0 & \alpha + \xi & 0 \\ 0 & 0 & \alpha + \rho \end{pmatrix}$$

with the domain $D(A)$ given by $D(A) = D(T) \times X_p \times X_p$.

Now, we establish some auxiliary results required in the proof of our existence and uniqueness result. In the following lemma, we prove that A is an m-accretive operator.

Lemma 4.1 *If the hypothesis $\mathcal{A}$ is true, then the operator A is m-accretive on $\mathcal{H}_p$.*

Proof. In the first step, we prove that A is accretive on $\mathcal{H}_p$. Indeed, let g_1, $g_2 \in D(A)$ and let $u = (u_0, u_1, u_2) \in \Gamma_1(g_1 - g_2)$. If we note $g_1 - g_2 = (g_1^0 - g_2^0, g_1^1 - g_2^1, g_1^2 - g_2^2)$, then, for $i = 0, 1, 2$, we have

$$u_i = \|g_1^i - g_2^i\|^{1-p}|g_1^i - g_2^i|sgn_0(g_1^i - g_2^i).$$

So, we have

$$\begin{aligned}
&[A(g_1)-A(g_2),u]_s\\
\geq\ & \|g_1^0-g_2^0\|^{1-p}\int_0^T\int_0^L |g_1^0-g_2^0|^{p-1}t.\frac{\partial}{\partial a}(g_1^0-g_2^0)(t,a))sgn_0(g_1^0-g_2^0)dadt\\
&+c\beta\mu\|g_1^0-g_2^0\|^{1-p}\int_0^T\int_0^L |g_1^0-g_2^0|^{p-1}((g_1^0-g_2^0)(t,a))sgn_0(g_1^0-g_2^0)dadt\\
&+(\alpha+\xi)\|g_1^1-g_2^1\|^{1-p}\int_0^T\int_0^L |g_1^1-g_2^1|^{p-1}((g_1^1-g_2^1)(t,a))sgn_0(g_1^1-g_2^1)dadt\\
&+(\alpha+\rho)\|g_1^2-g_2^2\|^{1-p}\int_0^T\int_0^L |g_1^2-g_2^2|^{p-1}((g_1^2-g_2^2)(t,a))sgn_0(g_1^2-g_2^2)dadt\\
=\ & \|g_1^0-g_2^0\|^{1-p}\frac{1}{p}\int_0^T\int_0^L t.\frac{\partial}{\partial a}(|(g_1^0-g_2^0)(t,a)|^p)dadt\\
&+c\beta\mu\int_0^l |(g_1^0-g_2^0)(t,a)|^p dadt+(\alpha+\xi)\int_0^T\int_0^L |(g_1^1-g_2^2)(t,a)|^p dadt\\
&+(\alpha+\rho)\int_0^T\int_0^L |(g_1^2-g_2^2)(t,a)|^p dadt\\
=\ & c\beta\mu\|g_1^0-g_2^0\|_{X_p}+(\alpha+\xi)\|g_1^1-g_2^1\|_{X_p}+(\alpha+\rho)\|g_1^2-g_2^3\|_{X_p}\geq 0.
\end{aligned}$$

This proves that the operateur A is accretive on $\mathcal{H}_p$.

To complete the proof, it suffices to establish that $R(I+A)=\mathcal{H}_p$, where $R(I+A)$ denotes the range of the operator $I+A$. Indeed, let (v_0,v_1,v_2) be an element of $\mathcal{H}_p$, we seek for an element $(u_0,u_1,u_3)\in D(A)$ such that

$$\begin{pmatrix} T+c\beta\mu & 0 & 0\\ 0 & 1+(\alpha+\xi) & 0\\ 0 & 0 & 1+(\alpha+\rho)\end{pmatrix}\begin{pmatrix}u_0\\u_1\\u_2\end{pmatrix}=\begin{pmatrix}v_0\\v_1\\v_2\end{pmatrix}$$

or equivalently, we look for a solution of the following system:

$$\begin{cases} Tu_0+c\beta\mu u_0=v_0,\\ u_1+(\alpha+\xi)u_1=v_1,\\ u_2+(\alpha+\rho)u_2=v_2.\end{cases}$$

It is clear that

$$\begin{cases} u_1=\dfrac{v_1}{1+\alpha+\xi},\\[2ex] u_2=\dfrac{v_2}{1+\alpha+\rho}.\end{cases}$$

Hence, it remains to solve the equation

$$Tu_0+c\beta\mu u_0=v_0. \tag{4}$$

According to Remark 4.2, equation (4) has a unique solution because $1\in\rho(T)$. This yields that $R(I+A)=\mathcal{H}_p$ and completes the proof.

We introduce the following lemma which shows that the operator F is Lipschitzian.

Lemma 4.2 *If F maps $\mathcal{H}_p$ into itself, then there exists a constant $\lambda > 0$ such that, for all u, $v \in \mathcal{H}_p$, we have*

$$\|F(u) - F(v)\|_{\mathcal{H}_p} \leq \lambda \|u - v\|_{\mathcal{H}_p}.$$

Proof. Let u, $v \in \mathcal{H}_p$, we have

$$\begin{array}{rl} & \|F(u) - F(v)\|_{\mathcal{H}_p} \\ = & \|(F_1u_1 - F_1v_1 + F_2u_2 - F_2v_2, F_3u_0 - F_3v_0, F_4u_0 - F_4v_0 + F_5u_1 - F_5v_1\|_{\mathcal{H}_p} \\ = & \|F_1u_1 - F_1v_1 + F_2u_2 - F_2v_2\|_{X_p} + \|F_3u_0 - F_3v_0\|_{X_p} \\ & +\|F_4u_0 - F_4v_0 + F_5u_1 - F_5v_1\|_{X_p} \\ \leq & \alpha\|u_1 - v_1\|_{X_p} + (\alpha+\rho)\|u_2 - v_2\|_{X_p} + c\beta\mu\|u_0 - v_0\|_{X_p} \\ & +c\beta(1-\mu)\|u_0 - v_0\|_{X_p} + \xi\|u_1 - v_1\|_{X_p} \\ = & c\beta\|u_0 - v_0\|_{X_p} + (\alpha+\xi)\|u_1 - v_1\|_{X_p} + (\alpha+\rho)\|u_2 - v_2\|_{X_p} \\ \lambda \leq & \|u - v\|_{\mathcal{H}_p}, \end{array}$$

where $\lambda = \max(c\beta, \alpha + \xi, \alpha + \rho)$. This completes the proof.

Now, using the operators A and F, problem (1) may be written in the form

$$\begin{cases} U'(t) + AU(t) = FU(t), \ \ t \in [0,\, T], \\ U(0) = U_0, \end{cases} \tag{5}$$

where

$$U(t) = \begin{pmatrix} S(t) \\ I(t) \\ R(t) \end{pmatrix} \quad \text{and } U_0 = \begin{pmatrix} S_0 \\ I_0 \\ R_0 \end{pmatrix}.$$

In the following result, we try to show that if assumption $\mathcal{A}$ holds, then equation (5) has a unique solution. Hence the main result of this section reads as follows.

Theorem 4.1 *Let $1 \leq p < +\infty$. We assume that the condition $\mathcal{A}$ holds true and F maps $\mathcal{H}_p$ into itself, then the problem (5) has a unique mild solution for all initial data $(S_0,\, I_0,\, R_0)$ belonging to $\mathcal{H}_p$.*

If $1 < p < +\infty$, it is a weak solution. Moreover, if $(S_0,\, I_0,\, R_0) \in \mathcal{H}_p$, then it is a strong solution.

Proof. It follows from Lemma 4.1 that the operator A is m-accretive on $\mathcal{H}_p$. Further, Remark 4.1 together with Lemma 4.2 show that F is λ-Lipschitz on $\mathcal{H}_p$ and therefore the operator $A-F$ is λ-m-accretive on $\mathcal{H}_p$. Applying Corollary 4.1 from [6], we conclude that problem (5) has a unique mild solution. Moreover, since the spaces X_p, for $1 < p < +\infty$, are Banach spaces with the Radon-Nikodym property, applying Theorem 3.3, we infer that it is a weak solution on $\mathcal{H}_p$. Next, if $U_0 \in \mathcal{H}_p$, then applying Theorem 3.1, we infer that this solution is a strong solution.

The next result shows that the solution depends continuously on the initial data. To this end, let us introduce the Banach space $\mathcal{C}_p := C([0,T]^3;\, \mathcal{H}_p)$ endowed with the norm

$$\|u\|_\infty := \{\max \|u_i\|_{X_p} \,:\, i = 0,\, 1,\, 2\}.$$

Proposition 4.1 *Let* $1 \leq p < \infty$ *and* U_1, $U_2 \in \mathcal{C}_p$ *be two mild solutions of problem* (5). *Given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that if* $|U_1(0)-U_2(0)| \leq \delta$, *then* $\|U_1-U_2\|_\infty \leq \epsilon$.

Proof. Since A is an m-accretive operator on $\mathcal{H}_p$ (see Lemma 4.1) and $F : \mathcal{H}_p \longrightarrow \mathcal{H}_p$ is λ-Lipschitzian, where $\lambda = \max(c\beta, \alpha+\xi, \alpha+\rho)$ (see Lemma 4.2), we have $A - F$ is a λ-m-accretive operator on $\mathcal{H}_p$. So, for $i \in 1, 2$, U_i is the unique solution of the problem

$$\begin{cases} U'(t) + AU(t) - FU(t) = 0, \\ U(0) = U_i(0) \in \mathcal{H}_p. \end{cases} \tag{6}$$

Hence, using (3), we have

$$|U_1(t) - U_2(t)| \leq e^{\lambda t}|U_1(0) - U_2(0)|.$$

The above inequality implies that, for every $t \in [0, T]$,

$$|U_1(t) - U_2(t)| \leq e^{\lambda t}|U_1(0) - U_2(0)|,$$

therefore,

$$\|U_1 - U_2\|_\infty \leq e^{\lambda T}|U_1(0) - U_2(0)|.$$

It suffices to take $\delta = \frac{\epsilon}{e^{\lambda T}}$, this completes the proof.

Remark 4.3 We note that we can extend the result obtained above to prove the existence and uniqueness of the solution of the SEIR (Susceptible, Exposed, Infectious and Recovered) model presented in [17].

5 Conclusion

In the present work, we have considered a linear SIR model, describing the propagation of an epidemic in given population. The existence and uniqueness results for this problem were obtained in L^p spaces, for $1 \leq p < \infty$, by using the accretive theory. The solution of this model is important because biologists could use it to observe the spread of infectious diseases by introducing natural initial conditions. Therefore they can learn the ways of how to control the propagation of epidemics. In the future works, we will consider the nonlinear SIR model to explain how epidemic diseases can be eradicated by vaccination. Our approach may be extended to the following model:

$$\begin{cases} \dfrac{\partial S}{\partial t}(t,a) = t.\dfrac{\partial S}{\partial a}(t,a) + (\alpha - \sigma)N(t,a) - \beta F(I,S), \\ \dfrac{\partial I}{\partial t}(t,a) = \beta f(I,S) - (\alpha + \xi)I(t,a), \\ \dfrac{\partial R}{\partial t}(t,a) = \xi G(I,R) + c\beta(1-\mu)S(t,a), \end{cases}$$

where F and G are nolinear operators. A new parameter σ is introduced in the model and represents the specific vaccination rate of the new infected.

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