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On the Dynamics of a Class of Planar Differential Systems

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Abstract: In this work, we discuss the existence of the first integral and no-existence of limit cycles for a class of Kolmogorov differential systems. As an application, we give an example to illustrate our results.

Keywords: *Kolmogorov systems; periodic orbits; limit cycle; integrability.*

Mathematics Subject Classification (2010): 37C07, 34C05, 34C07, 37K10.

1 Introduction

By definition, a two-dimensional real planar Kolmogorov system will be a differential system of the form

$$\begin{cases} \dot{x} = xf_1(x, y), \\ \dot{y} = yf_2(x, y), \end{cases} \tag{1}$$

where f_1, f_2 are real functions in the two variables x and y and the dot denotes derivative with respect to the time (t) variable. There are many natural phenomena which can be modelled by the Kolmogorov systems in mathematical ecology and population dynamics, see for example [5, 10].

Kolmogorov models are widely used in ecology to describe the interaction between two populations, and a limit cycle corresponds to an equilibrium state of the system. In the qualitative theory of dynamical systems, see [2, 4, 5, 11], one of the most important problems is the study of the limit cycles of planar dynamical systems (1). The definition of limit cycles appeared in the works of Poincaré [9], the statement of the 16-th Hilbert's problem, and the discovery by Liénard [8]. A limit cycle of a planar vector field given

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by (1) is an isolated periodic trajectory (isolated compact leaf of the corresponding foliation), in other words, a periodic trajectory of a vector field is a limit cycles, see for instance [1, 6, 7].

Let D be a non–empty open and dense subset of $\mathbb{R}^2$. We say that a non–locally constant C^1 function $\varphi : D \to \mathbb{R}$ is a first integral of the polynomial differential (1) in D if φ is constant on the trajectories of the polynomial differential system (1) contained in D, i.e., if

$$\frac{\partial\varphi(x,y)}{\partial x}f_1(x,y) + \frac{\partial\varphi(x,y)}{\partial y}f_2(x,y) = \frac{d\varphi(x,y)}{dt} \equiv 0, \text{ at the points of } D.$$

For a planar vector field, the existence of a first integral totally determines its phase portrait. The simplest planar vector fields having a first integral are the Hamiltonian ones. The integrable planar vector fields which are not Hamiltonian are, in general, very difficult to detect, see [3].

In this paper, we study the existence of first integrals and the non existence of limit cycle of Kolmogorov differential systems of the form

$$\begin{cases} \dot{x} = x\left(F(x,y)^p + k(x,y)\frac{\sum\limits_{i=1}^{m_1}\exp P_i(x,y)}{\sum\limits_{j=1}^{m_2}\exp Q_j(x,y)}\right), \\ \dot{y} = y\left(G(x,y)^p + k(x,y)\frac{\sum\limits_{i=1}^{m_1}\exp P_i(x,y)}{\sum\limits_{j=1}^{m_2}\exp Q_j(x,y)}\right), \end{cases} \tag{2}$$

where m_1, m_2 are positive integers and $p \in \mathbb{Q}^*$, $F(x,y)$, $G(x,y)$, $k(x,y)$, $P_i(x,y)$, $Q_j(x,y)$ are homogeneous polynomials of degree n, n, δ, m, m, respectively.

We define the trigonometric functions

$$\begin{aligned} f(\theta) &= \cos^2\theta F(\cos\theta,\sin\theta)^p + \sin^2\theta G(\cos\theta,\sin\theta)^p. \\ g(\theta) &= k(\cos\theta,\sin\theta)\left(\frac{\sum\limits_{i=1}^{m_1}\exp P_i(\cos\theta,\sin\theta)}{\sum\limits_{j=1}^{m_2}\exp Q_j(\cos\theta,\sin\theta)}\right). \\ h(\theta) &= \sin\theta\cos\theta\left(G(\cos\theta,\sin\theta)^p - F(\cos\theta,\sin\theta)^p\right). \end{aligned}$$

2 Main Result

Our main result on the integrability and the periodic orbits of the Kolmogorov system (2) is the following.

Theorem 2.1 *Consider a planar Kolmogorov system (2), then the following statements hold:*

(i) *If* $h(\theta) \neq 0$, $F(\cos\theta,\sin\theta)^p > 0$, $G(\cos\theta,\sin\theta)^p > 0$, *for* $\theta \in \left[0, \frac{\pi}{2}\right]$ *and* $\delta - np \neq 0$, *then system (2) has the first integral*

$$\begin{aligned} I(x,y) &= (x^2+y^2)^{\frac{np+\delta}{2}}\exp\left(-(np+\delta)\int_{\theta_*}^{\arctan\frac{y}{x}}\frac{f(s)}{h(s)}ds\right) - \\ &\quad (np+\delta)\int_{\theta_*}^{\arctan\frac{y}{x}}\exp\left(-(np+\delta)\int_{v_0}^{v}\frac{f(s)}{h(s)}ds\right)\frac{g(v)}{h(v)}dv, \end{aligned}$$

where $\theta_* \in \left[0, \frac{\pi}{2}\right]$. Additionally, the system (2) has no limit cycle at the interior of the first quadrant on the plane.

(ii) *If $h(\theta) \neq 0$, $F(\cos\theta, \sin\theta)^p > 0$, $G(\cos\theta, \sin\theta)^p > 0$, for $\theta \in \left[0, \frac{\pi}{2}\right]$ and $\delta - np = 0$, then system (2) has the first integral*

$$L(x, y) = (x^2 + y^2)^{\frac{1}{2}} \exp\left(-\int_{\theta*}^{\arctan\frac{y}{x}} \left(\frac{f(u)}{h(u)} + \frac{g(u)}{h(u)}\right) du\right),$$

where $\theta_* \in \left[0, \frac{\pi}{2}\right]$. Additionally, the system (2) has no limit cycle at the interior of the first quadrant on the plane.

(iii) *If $h(\theta) = 0$ for all $\theta \in [0, 2\pi]$, then system (2) has the first integral $T(x, y) = \frac{y}{x}$. Also , the system (2) has no limit cycle.*

Proof. In order to demonstrate our results, we write the polynomial differential system (2) in polar coordinates (r, θ), defined by $x = r\cos\theta$ and $y = r\sin\theta$, then system (2) becomes

$$\begin{cases} \dot{r} = f(\theta) r^{np+1} + g(\theta) r^{\delta+1}, \\ \dot{\theta} = h(\theta) r^{np}, \end{cases} \tag{3}$$

where $\dot{r} = \frac{dr}{dt}$, $\dot{\theta} = \frac{d\theta}{dt}$.

(i) If $h(\theta) \neq 0$, $F(\cos\theta, \sin\theta)^p > 0$, $G(\cos\theta, \sin\theta)^p > 0$, for $\theta \in \left[0, \frac{\pi}{2}\right]$ and $\delta - np \neq 0$.

Take as an independent variable the coordinate θ, then differential system (3) writes

$$\frac{dr}{d\theta} = \frac{f(\theta)}{h(\theta)} r + \frac{g(\theta)}{h(\theta)} r^{\delta - np + 1}, \tag{4}$$

which is a Bernoulli equation. We take a new variable $\rho = r^{np+\delta}$ and we obtain the linear equation

$$\frac{d\rho}{d\theta} = (np + \delta)\left(\frac{f(\theta)}{h(\theta)}\rho + \frac{g(\theta)}{h(\theta)}\right). \tag{5}$$

The general solution of linear equation (5) is

$$\begin{aligned} \rho(\theta) &= \left(k + (np+\delta)\int_{\theta_*}^{\theta} \exp\left(-(np+\delta)\int_{v_0}^{v} \frac{f(s)}{h(s)} ds\right) \frac{g(v)}{h(v)} dv\right) \times \\ &\quad \exp\left((np+\delta)\int_{\theta_*}^{\theta} \frac{f(s)}{h(s)} ds\right), \end{aligned}$$

where $k \in \mathbb{R}$, which has the first integral

$$\begin{aligned} I(x, y) &= (x^2 + y^2)^{\frac{np+\delta}{2}} \exp\left(-(np+\delta)\int_{\theta_*}^{\arctan\frac{y}{x}} \frac{f(s)}{h(s)} ds\right) - \\ &\quad (np+\delta)\int_{\theta_*}^{\arctan\frac{y}{x}} \exp\left(-(np+\delta)\int_{v_0}^{v} \frac{f(s)}{h(s)} ds\right) \frac{g(v)}{h(v)} dv. \end{aligned}$$

The curves $I = l$ with $l \in \mathbb{R}$, are created by the trajectories of the differential system (2). These trajectories equations can be written in Cartesian coordinates as follows:

$$x^2+y^2=\left(\begin{array}{c}\left(l+(np+\delta)\int_{\theta_*}^{\arctan\frac{y}{x}}\exp\left(-(np+\delta)\int_{v_0}^{v}\frac{f(s)}{h(s)}ds\right)\frac{g(v)}{h(v)}dv\right)\times\\ \exp\left((np+\delta)\int_{\theta*}^{\arctan\frac{y}{x}}\frac{f(s)}{h(s)}ds\right)\end{array}\right)^{\frac{2}{np+\delta}}.$$

So, the periodic orbit $\digamma$ is contained in the curve equation

$$(\Lambda):x^2+y^2=\left(\begin{array}{c}\left(l_\digamma+(np+\delta)\int_{\theta_*}^{\arctan\frac{y}{x}}\exp\left(-(np+\delta)\int_{v_0}^{v}\frac{f(s)}{h(s)}ds\right)\frac{g(v)}{h(v)}dv\right)\times\\ \exp\left((np+\delta)\int_{\theta_*}^{\arctan\frac{y}{x}}\frac{f(s)}{h(s)}ds\right)\end{array}\right)^{\frac{2}{(np+\delta)}}.$$

But the curve (Λ) cannot contain the periodic orbit $\digamma$ and hence no limit cycle is contained in the first quadrant on the plane because the curve (Λ) in the realistic quadrant contains only a single point or no point on every straight line $(\Delta_\lambda): y=\lambda x$ for all $\lambda>0$.

To be persuaded by this verity, let (x_0,y_0) be a point of intersection of this curve with the straight line $(\Delta_\lambda): y=\lambda x$ for all $\lambda>0$, then x_0 and y_0 must satisfy

$$\begin{cases} x_0^2+y_0^2=\left(\begin{array}{c}\left(l_\digamma+(np+\delta)\int_{\theta_*}^{\arctan\frac{y_0}{x_0}}\exp\left(-(np+\delta)\int_{v_0}^{v}\frac{f(s)}{h(s)}ds\right)\frac{g(v)}{h(v)}dv\right)\\ \times\exp\left((np+\delta)\int_{\theta_*}^{\arctan\frac{y}{x}}\frac{f(s)}{h(s)}ds\right)\end{array}\right)^{\frac{2}{(np+\delta)}}, \\ y_0=\lambda x_0, \end{cases}$$

hence

$$\begin{cases} x_0=\left(1+\lambda^2\right)^{\frac{-1}{2}}\left(\begin{array}{c}\left(l_\digamma+(np+\delta)\int_{\theta_*}^{\arctan\lambda}\exp\left(-(np+\delta)\int_{v_0}^{v}\frac{f(s)}{h(s)}ds\right)\frac{g(v)}{h(v)}dv\right)\\ \times\exp\left((np+\delta)\int_{\theta_*}^{\arctan\alpha}\frac{f(s)}{h(s)}ds\right)\end{array}\right)^{\frac{1}{(np+\delta)}}, \\ y_0=\lambda x_0. \end{cases}$$

There is at most a unique value of x_0 on every half straight OX^+. Consequently, there is at most a unique point in the first quadrant on the plane. So this curve cannot contain the periodic orbit and hence there is no limit cycle.

(ii) If $h(\theta)\neq 0$, $F(\cos\theta,\sin\theta)^p>0$, $G(\cos\theta,\sin\theta)^p>0$, for $\theta\in\left[0,\frac{\pi}{2}\right]$ and $\delta-np=0$.

Take as an independent variable the coordinate θ, then the differential system (3) becomes

$$\frac{dr}{d\theta}=\left(\frac{f(\theta)}{h(\theta)}+\frac{g(\theta)}{h(\theta)}\right)r. \tag{6}$$

The general solution of equation (6) is

$$r(\theta)=k\exp\left(\int_{\theta_*}^{\theta}\left(\frac{f(u)}{h(u)}+\frac{g(u)}{h(u)}\right)du\right),$$

where $k\in\mathbb{R}$, which has the first integral

$$L(x,y)=(x^2+y^2)^{\frac{1}{2}}\exp\left(-\int_{\theta*}^{\arctan\frac{y}{x}}\left(\frac{f(u)}{h(u)}+\frac{g(u)}{h(u)}\right)du\right).$$

The curves $L=l$ with $l\in\mathbb{R}$ are created by the trajectories of the differential system (2). These trajectories can be written in Cartesian coordinates as follows:

$$(x^2+y^2)^{\frac{1}{2}}=k\exp\left(\int_{\theta_*}^{\arctan\frac{y}{x}}\left(\frac{f(u)}{h(u)}+\frac{g(u)}{h(u)}\right)du\right).$$

Therefore the periodic orbit (Σ) is contained in the curve

$$(C): x^2+y^2 = k_\Sigma^2 \exp\left(\int_{\theta_*}^{\arctan\frac{y}{x}}\left(\frac{f(u)}{h(u)}+\frac{g(u)}{h(u)}\right)du\right)^2.$$

But the curve (C) cannot contain the periodic orbit (Σ) and hence no limit cycle contained in the first quadrant on the plane, because the curve (C) in the realistic quadrant has at most a unique point on every straight line $y = \lambda x$ for all $\lambda > 0$.

To be persuaded by this verity, let (x_0, y_0) be a point of intersection of this curve with the straight line $(\Delta_\lambda): y = \lambda x$ for all $\lambda > 0$, then x_0 and y_0 must satisfy

$$\begin{cases} (x_0^2+y_0^2)^{\frac{1}{2}} = k_\Sigma \exp\left(\int_{\theta_*}^{\arctan\frac{y_0}{x_0}}\left(\frac{f(u)}{h(u)}+\frac{g(u)}{h(u)}\right)du\right), \\ y_0 = \lambda x_0, \end{cases}$$

hence

$$\begin{cases} x_0 = k_\Sigma(1+\lambda^2)^{\frac{-1}{2}} \exp\left(\int_{\theta_*}^{\arctan\lambda}\left(\frac{f(u)}{h(u)}+\frac{g(u)}{h(u)}\right)du\right). \\ y_0 = \lambda x_0, \end{cases}$$

There is at most a unique value of x_0 on every half straight OX^+. Consequently, there is at most a unique point in the first quadrant on the plane. So this curve cannot contain the periodic orbit and consequently, there is no limit cycle.

(iii) If $h(\theta) = 0$ for all $\theta \in [0, 2\pi]$, then from (3), it follows that $\dot{\theta} = 0$. So the straight lines through the origin of coordinates of the differential system (2) are invariant by the flow of this system. Hence, $T(x, y) = \frac{y}{x}$ is a first integral of the system. Then all straight lines through the origin are created by the trajectories, which can be written in Cartesian coordinates as $y = \gamma x$, where $\gamma \in \mathbb{R}$. Hence, there is no limit cycle. This completes the proof of the theorem.

Example 2.1 If we take $F(x,y) = \frac{1}{9}x^2y^2\left(x^2+y^2\right)$, $G(x,y) = y^4 + x^2y^2$, $\sum_{i=1}^{2} \exp P_i(x,y) = e^x - e^{-x}$, $\sum_{j=1}^{2} \exp Q_j(x,y) = e^x + e^{-x}$, $k(x,y) = x^3+xy^2+x^2y+y^3$, and $p = -\frac{1}{2}$, then system (2) becomes

$$\begin{cases} \dot{x} = x\left(\left(\frac{1}{9}x^2y^2\left(x^2+y^2\right)\right)^{\frac{-1}{2}} + (x^3+xy^2+x^2y+y^3)\tanh(x)\right), \\ \dot{y} = y\left(\left(y^4+x^2y^2\right)^{\frac{-1}{2}} + (x^3+xy^2+x^2y+y^3)\tanh(x)\right), \end{cases} \tag{7}$$

where $x(t)$ and $y(t)$ represent the population density of two species at time t, and

$$\begin{aligned} f_1(x,y) &= \left(\frac{1}{9}x^2y^2\left(x^2+y^2\right)\right)^{\frac{-1}{2}} + (x^3+xy^2+x^2y+y^3)\tanh(x), \\ f_2(x,y) &= \left(y^4+x^2y^2\right)^{\frac{-1}{2}} + (x^3+xy^2+x^2y+y^3)\tanh(x), \end{aligned}$$

are the capita growth rate of each species.

The Kolmogorov system (7) in polar coordinates (r, θ) is written as

$$\begin{cases} \dot{r} = (\cos\theta(\frac{1}{9}\sin^2\theta)^{\frac{-1}{2}} + \sin\theta)r^{-1} + ((\cos\theta+\sin\theta)\tanh(\theta))r^4, \\ \dot{\theta} = (\cos\theta - 3)r^{-2}, \end{cases}$$

accordingly, $f(\theta)=\cos\theta(\frac{1}{9}\sin^2\theta)^{\frac{-1}{2}}+\sin\theta$, $g(\theta)=(\cos\theta+\sin\theta)\tanh(\theta)$, $h(\theta)=\cos\theta-3$. This corresponds to the case (i) of Theorem 2.1. Then the system (7) has the first integral

$$\begin{aligned} I(x,y) &= (x^2+y^2)^{\frac{1}{2}}\exp\left(\int_{\theta_*}^{\arctan\frac{y}{x}}\frac{\cos s(\frac{1}{9}\sin^2 s)^{\frac{-1}{2}}+\sin s}{3-\cos s}ds\right)- \\ & \int_{\theta_*}^{\arctan\frac{y}{x}}\exp\left(\int_{v_0}^{v}\frac{\cos s(\frac{1}{9}\sin^2 s)^{\frac{-1}{2}}+\sin s}{3-\cos s}ds\right)\frac{(\cos v+\sin v)\tanh(v)}{\cos v-3}dv. \end{aligned}$$

The curves $I=l$ with $l\in\mathbb{R}$, which are created by the trajectories of the differential system (7), in Cartesian coordinates are written as

$$x^2+y^2=\left(\begin{array}{c}\left(l+\int_{\theta_*}^{\arctan\frac{y}{x}}\exp\left(-\int_{v_0}^{v}\frac{\cos s(\frac{1}{9}\sin^2 s)^{\frac{-1}{2}}+\sin s}{\cos s-3}ds\right)\frac{(\cos v+\sin v)\tanh(v)}{\cos v-3}dv\right)\times \\ \exp\left(\int_{\theta*}^{\arctan\frac{y}{x}}\frac{\cos s(\frac{1}{9}\sin^2 s)^{\frac{-1}{2}}+\sin s}{\cos s-3}ds\right)\end{array}\right)^2,$$

where $l\in\mathbb{R}$. Then the system (7) has no periodic orbits, and consequently, no limit cycle.

3 Conclusion

In this paper, we proposed a special form of Kolmogorov differential system, where just selecting the parameters satisfying the conditions of Theorem 2.1, we obtain explicit expression for a first integral and characterize its trajectories, this is one of the classical tools in the classification of all trajectories of dynamical systems.

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