Nonlinear Dynamics and Systems Theory, **22** (4) (2022) 407–413

Chaos Synchronization between Fractional-Order Lesser Date Moth Chaotic System and Integer-Order Chaotic System via Active Control

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Received: January 26, 2022; Revised: September 19, 2022

Abstract: This paper investigates the phenomenon of chaos synchronization between the fractional-order lesser date moth and the integer-order chaotic systems. Based on the Lyapunov stability theory and numerical differentiation, an active control is obtained to achieve the synchronization between the fractional-order and the integer-order chaotic systems. Numerical examples are implemented to illustrate and validate the results.

Keywords: *chaos; synchronization; active control; fractional-order chaotic system; integer-order chaotic system.*

Mathematics Subject Classification (2010): 34H10, 37N35, 93C10, 93C15, 93C95.

1 Introduction

Chaos is a very interesting nonlinear phenomenon that has been intensively studied over the past two decades. The chaos theory is found to be useful in many areas such as data encryption [19], financial systems [17,18], biology [22] and biomedical engineering [2], etc. Fractional-order chaotic dynamical systems have begun to attract a lot of attention in recent years and can be seen as a generalization of chaotic dynamic integer-order systems. The synchronization between the fractional-order chaotic system and the integer-order chaotic system is thoroughly a new domain and it began to attract much attention in

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recent years [9, 20] because of its potential applications in secure communication and cryptography [11, 12]. Obviously, the synchronization between a fractional-order chaotic system and an integer-order chaotic system is more difficult than the synchronization between fractional-order chaotic systems or integer-order chaotic systems for the different order of their error dynamical system. The synchronization between a fractional-order system and an integer-order system was first studied by Zhou et al. [20] In the past twenty years, different synchronization types have been proposed, e.g., complete synchronization [24], lag synchronization [4], phase synchronization [10], project synchronization [21], generalized synchronization [6], etc. In this research work, we apply the active control theory to synchronize two chaotic systems when a fractional-order system is chosen as the drive system and an integer-order system serves as the response system, we demonstrate the technique capability by the synchronization between a fractional-order lesser date moth chaotic system and an integer-order chaotic system [15]. The paper is arranged in the following manner. In Section 2, we describe the problem formulation for the fractional-order and the integer-order chaotic systems. In Section 3, we discuss the synchronisation between a fractional-order lesser date moth chaotic system and an integer-order chaotic system by using the active control. Section 4 gives the brief conclusion.

2 Problem Formulation for Fractional-Order and Integer-Order Chaotic System

Consider the following fractional-order chaotic system as a drive (master) system:

$$D^\alpha x_1 = Ax_1 + g(x_1), \tag{1}$$

where $x_1 \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is the linear part, $g(x_1)$ is a continuous nonlinear function, and D^α is the Caputo fractional derivative. Also, the response system (slave) can be described as

$$\dot{x}_2 = Ax_2 + g(x_2) + u(t), \tag{2}$$

where $x_2 \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n\times n}$ is the linear part, and $g(x_2)$ is a continuous nonlinear function and $u(t) \in \mathbb{R}^n$ is the control.

Define the synchronous errors as $e = x_2 - x_1$. Our aim is to determine the controller $u(t) \in \mathbb{R}^n$ such that the drive system and response system are synchronized (i.e., $\lim_{t\to\infty} \|e(t)\| = 0$).

The synchronisation error system between the driving system (1) and the response system (2) can be expressed as

$$\dot{e} = \dot{x}_2 - \dot{x}_1,$$

where $\dot{x}_2$ is obtained from the response system (2), while no exact expressions of $\dot{x}_1$ can be obtained from the driving system (1). Therefore, the numerical differentiation method is used to obtain $\dot{x}_1$. According to the definition of derivative, the derivative is approximately expressed using the difference quotient as

$$g'(a) \approx \frac{g(a+h) - g(a)}{h}, \tag{3}$$

$$g'(a) \approx \frac{g(a) - g(a-h)}{h}, \tag{4}$$

where ($h > 0$) is a small increment. Formulae (3) and (4) are called the pre-difference formula and the post-difference formula, respectively. The post-difference formula is used in this paper.

3 Synchronisation of Fractional-Order Lesser Date Moth Chaotic System and Integer-Order Chaotic System by Active Control

In this section, to validate the active control method proposed in [5], we take the fractional-order lesser date moth chaotic system [15] as a drive system and the integer-order chaotic system as a response system.

Thus, the drive and response systems are as follows:

$$\begin{cases} D^{\alpha}x_1 = x_1(1-x_1) - \frac{x_1y_1}{\beta+x_1}, \\ D^{\alpha}y_1 = -\delta y_1 + \frac{\gamma x_1 y_1}{\beta+x_1} - y_1z_1, \\ D^{\alpha}z_1 = -\eta z_1 + \sigma y_1 z_1, \end{cases} \tag{5}$$

and

$$\begin{cases} \dot{x}_2 = x_2(1-x_2) - \frac{x_2y_2}{\beta+x_2} + u_1(t), \\ \dot{y}_2 = -\delta y_2 + \frac{\gamma x_2 y_2}{\beta+x_2} - y_2z_2 + u_2(t), \\ \dot{z}_2 = -\eta z_2 + \sigma y_2 z_2 + u_3(t), \end{cases} \tag{6}$$

where $u_1(t)$, $u_2(t)$, $u_3(t)$ are the active controls.

It is reported that the fractional-order lesser date moth system (5) with the fractional order of $\alpha = 0.95$ can behave chaotically [15]. The three-dimensional (3D) phase portraits of the lesser date moth chaotic system with fractional order and integer order, respectively, are shown in Figure 1 and Figure 2 .

Subtracting (6) from (5) gives the error system as below:

$$\begin{cases} \dot{e}_1 = e_1 - x_1^2 + x_2^2 - (\frac{y_2}{\beta+x_2})e_1 + \frac{x_1y_2}{(\beta+x_1)(\beta+x_2)}e_1 - \frac{x_1}{\beta+x_1}e_2 + x_1 - x_1^2 \\ -\frac{x_1y_1}{\beta+x_1} - \dot{x}_1 + u_1(t), \\ \dot{e}_2 = -\delta e_2 + [\frac{\gamma y_2}{\beta+x_2} - \frac{\gamma x_1 y_2}{(\beta+x_1)(\beta+x_2)}]e_1 - (z_2 - \frac{x_1}{\beta+x_1})e_2 - y_1e_3 - \delta y_1 \\ +\frac{\gamma x_1 y_1}{\beta+x_1} - y_1z_1 - \dot{y}_1 + u_2(t), \\ \dot{e}_3 = -\eta e_3 + \sigma(y_1e_3 + z_2e_2) - \eta z_1 + \sigma y_1 z_1 - \dot{z}_1 + u_3(t), \end{cases} \tag{7}$$

where $e_1 = x_2 - x_1, \quad e_2 = y_2 - y_1, \quad e_3 = z_2 - z_1$.

We introduce a quadratic Lyapunov function

$$V(e) = \frac{1}{2}\sum_{i=1}^{3} e_i^2, \tag{8}$$

and calculate the derivative of $V(e)$ to obtain

$$\begin{aligned} \dot{V}(e) &= e_1[e_1 - (x_2+x_1)e_1 - (\frac{y_2}{\beta+x_2})e_1 + \frac{x_1y_2}{(\beta+x_2)(\beta+x_1)}e_1 - \frac{x_1}{\beta+x_1}e_2 && (9) \\ &+ x_1 - x_1^2 - \frac{x_1y_1}{\beta+x_1} - \dot{x}_1] + e_2[-\delta e_2 + (\frac{\gamma y_2}{\beta+x_2}e_1 - \frac{\gamma x_1 y_2}{(\beta+x_1)(\beta+x_2)})e_1 && (10) \\ &- (z_2e_2 + \frac{\gamma x_1}{\beta+x_1})e_2 - y_1e_3 - \delta y_1 + \frac{\gamma x_1 y_1}{\beta+x_1} - y_1z_1 - \dot{y}_1] \\ &+ e_3[-\eta e_3 + \sigma z_2 e_2 + \sigma y_1 e_3 - \eta z_1 + \sigma y_1 z_1 - \dot{z}_1] + \sum_{i=1}^{3} u_i(t)e_i(t). && (11) \end{aligned}$$

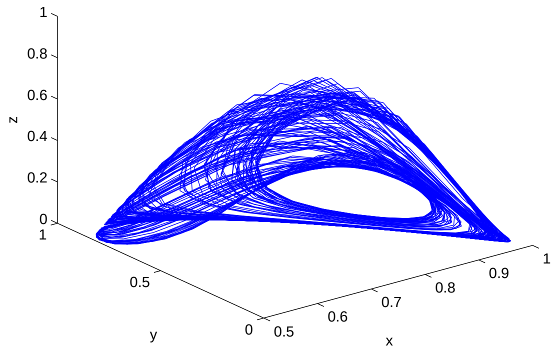

Figure 1: The 3D phase portrait of the fractional-order lesser date moth system.

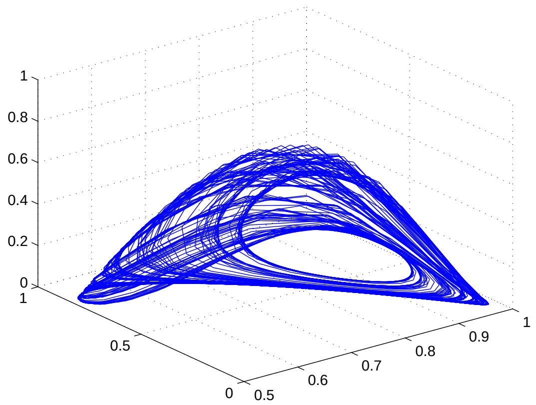

Figure 2: The 3D phase portrait of the integer-order lesser date moth system.

From the above equation, we deduce that if the active control functions $u_i(t)$ are chosen such that

$$
\begin{aligned}
u_1(t) &= -[2e_1 - (x_1 + x_2)e_1 - (\frac{y_2}{\beta + x_2})e_1 + \frac{x_1 y_2}{(\beta + x_1)(\beta + x_2)}e_1 - \frac{x_1}{\beta + x_1}e_2 \\
&+ x_1 - x_1^2 - \frac{x_1 y_1}{\beta + x_1} - \dot{x}_1] \\
u_2(t) &= -[(\frac{\gamma y_2}{\beta + x_2} - \frac{\gamma x_1 y_2}{(\beta + x_1)(\beta + x_2)})e_1 - (z_2 - \frac{\gamma x_1}{\beta + x_1})e_2 - y_1 e_3 - \delta y_1 \\
&+ \frac{\gamma x_1 y_1}{\beta + x_1} - y_1 z_1 - \dot{y}_1], \\
u_3(t) &= -[\sigma z_2 e_2 + \sigma y_1 e_3 - \eta z_1 + \sigma y_1 z_1 - \dot{z}_1],
\end{aligned}
$$

equation (11) becomes

$$\dot{V}(e) = -(e_1^2 + \delta e_2^2 + \eta e_3^2) < 0. \tag{12}$$

According to the inequality (8), the system (7) is asymptotically stable.

For the numerical simulations, we use some documented data for some parameters such as $\gamma = 3$, $\delta = \eta = 1$, $\sigma = 3$, $\beta = 1.15$, $h = 0.85$, $\alpha = 0.95$, then we have $(x_1, y_1, z_1) = (0.7, 0.3, 0.8)$ and $(x_2, y_2, z_2) = (1.2, 0.12, 2.0)$. The simulation results are illustrated in Figure 3.

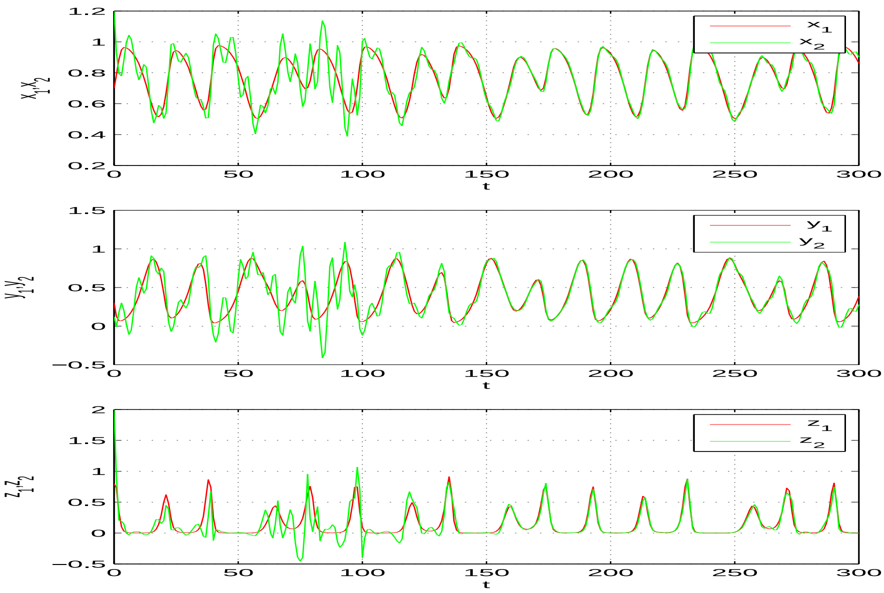

Figure 3: Synchronization between response system (6) and drive system (5)

4 Conclusion

In this paper, we have studied the phenomenon of chaos synchronization between a fractional-order lesser date moth chaotic system and an integer-order chaotic system. Our results demonstrate that if one uses the technique of active control, chaos synchronization can be achieved between a fractional-order chaotic system and an integer-order chaotic system. The numerical results are in good accordance with the theoretical analyses.

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