Nonlinear Dynamics and Systems Theory, **22** (4) (2022) 414–423

New Design of Stability Study for Linear and Nonlinear Feedback Control of Chaotic Systems

W. Laouira * and N. Hamri

Laboratory of Mathematics and their interactions, Department of Science and Technology, University Center of Mila, Mila 43000, Algeria.

Received: May 11, 2022; Revised: September 4, 2022

Abstract: This paper presents the control of chaotic dynamical systems by designing linear and nonlinear feedback controllers, the stability of chaotic systems has been studied by three methods, the Lyapunov function, Routh-Hurwitz criteria and finally, a new method which is based on the Jacobian matrix conditions, we proved that we can find stability by the third method and not by the Lyapunov function and Routh-Hurwitz methods, we have also found a good interval or exact value for the parametric control which stabilises the chaotic system at its equilibruim point. Numerical simulations show the effectiveness or non-effectiveness of the results for the three different methods, we apply the feedback control to the Sprott J system,a novel chaotic system and the Genesio system.

Keywords: *Lyapunov function; Routh-Hurwitz theorem; Jacobian matrix conditions; feedback control; chaotic systems.*

Mathematics Subject Classification (2010): 93B52, 37N35, 93C10, 93D05, 93D15, 65P20, 65P40, 93D20, 93C95.

1 Introduction

The term "control of chaos" is used mostly to denote the area of studies lying at the interface between the control theory and the theory of dynamic systems studying the methods of control of deterministic systems with non-regular, chaotic behavior [16]. Several techniques have been devised for chaos control, but most are the developments of two basic approaches: the OGY (Ott, Grebogi and Yorke) method [17], and Pyragas continuous control [18]. Both methods require a previous determination of unstable periodic orbits of the chaotic system before the controlling algorithm can be designed. Different

* Corresponding author: mailto:w.laouira@centre-univ-mila.dz

control strategies for stabilizing chaos [11] have been proposed, namely, an adaptive control [10, 14], time delay control [3], and fuzzy control [7]. Generally speaking, there are two main approaches for controlling chaos: a feedback control [9, 12] and nonfeedback control. The feedback control approach offers many advantages such as robustness and computational complexity over the non-feedback control method.

We generally study stability for feedback control by two methods: the function of Lyapunov [1] and the criterion of Routh-Huritz, but we fail in the cases when we cannot assure the existence of stability for all the control laws. In this work, we show that we can use the third method which is based on the Jacobian matrix conditions, and we can also choose the function of feedback control.

2 Stability Condition

Suppose that B is an $n \times n$ matrix of real constants, its characteristic polynomial is

$$f(\lambda) = \lambda^n + a\lambda^{n-1} + b\lambda^{n-2} + c\lambda^{n-3} + \dots, \quad n = 1, 2, 3, 4.$$

The Routh-Hurwitz theorem [4–6] is as follows.

Theorem 2.1 *All the roots of the caracteristic polynomial have negative real parts precisely when the given conditions are satisfied:*
$\lambda^2 + a\lambda + b:\ a > 0,\ b > 0.$
$\lambda^3 + a\lambda^2 + b\lambda + c:\ a > 0,\ c > 0,\ ab - c > 0.$
$\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d:\ a > 0,\ ab - c > 0,\ (ab - c)c - a^2 d > 0,\ d > 0.$

Jacobian matrix conditions. We consider A is the Jacobian matrix at a fixed point [19],

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{1}$$

and $t = a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32}$, where

$$A_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix},\ A_{22} = \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix},\ A_{33} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}, \end{vmatrix}.$$

Theorem 2.2 *If $t \geq 0$, all the roots of the characteristic polynomial of A have negative real parts when the given conditions are satisfied:*
$\det(A) < 0$, $a_{ii} < 0$ *and* $A_{ii} > 0$, *for* $i = 1, 2, 3$.

3 Control of Sprott J System

Theorem 3.1 *The controlled Sprott J system [15] is*

$$\begin{cases} \dot{x} = 2z - u_1, \\ \dot{y} = -2y + z - u_2, \\ \dot{z} = -x + y + y^2 - u_3, \end{cases} \tag{2}$$

where $u_1 = kx, u_2 = 0, u_3 = y^2 + kz$ and k is the feedback coefficient, the system (2) will gradually converge to the equilibrium point $(0; 0; 0)$ when $k > 1,5$ for the Lyapunov method and when $k > 0,5$ for the Jacobian matrix conditions.

Proof. For non linear feedback system (2) consider a quadratic Lyapunov function as $v = \frac{1}{2}(x^2 + y^2 + z^2)$, then

$$\begin{aligned}\dot{v} &= -kx^2 - kz^2 - 2y^2 + xz + 2yz \\ &< -kx^2 - kz^2 - 2y^2 + \frac{1}{2}(y^2 + z^2) + y^2 + z^2 \\ &< (-k + \frac{1}{2})x^2 - y^2 + (-k + \frac{3}{2})z^2.\end{aligned}$$

So, if $k > 1,5$, we can obtain $\dot{v} < 0$.
For the Jacobian matrix conditions, the Jacobian matrix is as follows:

$$A = \begin{pmatrix} -k & 0 & 2 \\ 0 & -2 & 1 \\ -1 & 1 & -k \end{pmatrix} \Leftrightarrow \det(A) = -2k^2 + k - 4,$$

$$\begin{aligned}A_{11} &= \begin{vmatrix} -2 & 1 \\ 1 & -k \end{vmatrix} = 2k - 1, \\ A_{22} &= \begin{vmatrix} -k & 2 \\ -1 & -k \end{vmatrix} = k^2 + 2, \\ A_{33} &= \begin{vmatrix} -k & 0 \\ 0 & -2 \end{vmatrix} = 2k.\end{aligned}$$

According to the previous Theorem 2, we have $t = 0$, then $\det(A) < 0$, $a_{ii} < 0$ and $A_{ii} > 0$ for $i = 1, 2, 3$ if $k > 0.5$.

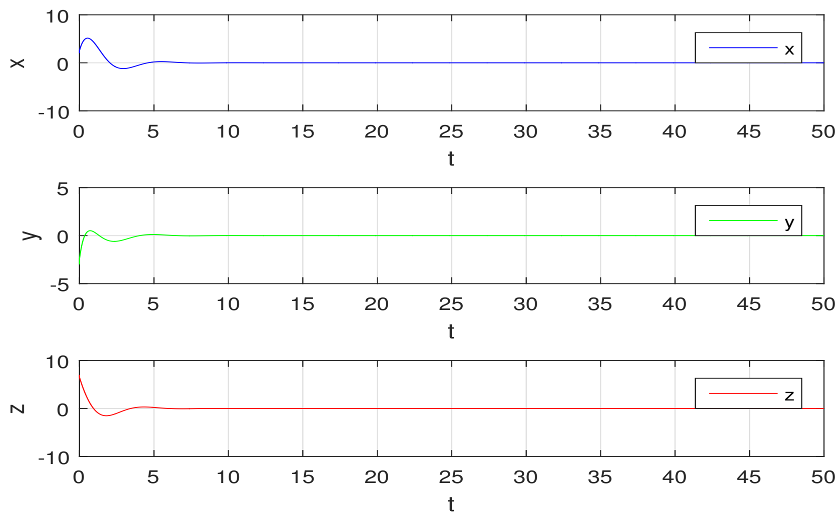

Figure 1: Control of the Sprott J system at the equilibrium point $(0; 0; 0)$ when $k = 0, 8$.

Remark 3.1 For the Routh-Hurwitz method we have not solutions for the same feedback control of the Sprott J system.

4 Control of Novel Chaotic System

Theorem 4.1 *The controlled novel chaotic system [13] is*

$$\begin{cases} \dot{x} = 0.2x - yz - u_1, \\ \dot{y} = -0.1y + xz - u_2, \\ \dot{z} = -z + xy - u_3, \end{cases} \tag{3}$$

where $u_1 = k(x-x^*), u_2 = x(z-z^*)+k(y-y^*), u_3 = k(z-z^*)$ *and* k *is the feedback coefficient, the system (3) will gradually converge to the equilibrium point* $E_2(0.31; 0.44; 0.14)$ *when* $k > 0.2$ *for the Jacobian matrix conditions.*

Proof. For non linear feedback system (3) consider a quadratic Lyapunov function as $v = \frac{1}{2}[(x-x^*)^2 + (y-y^*)^2 + (z-z^*)^2)$, then

$$\begin{aligned} &\dot{v} = 0.2x^2 - 0.2xx^* + yzx^* - k(x-x^*)^2 - 0.1y(y-y^*) + xz^*(y-y^*) - k(y-y^*)^2 \\ &-z(z-z^*) - (z-z^*)^2 \\ &< (0.7-k)x^2 + (0.4 - k + \tfrac{x^*}{2})y^2 + (-0.5 - k + \tfrac{x^*}{2})z^2 - kx^* - ky^* - kz^* \\ &+\tfrac{1}{2}(-0.2x^* + 2kx^* - z^*y^*)^2 + \tfrac{1}{2}(-0.1y^* + 2ky^*)^2 + \tfrac{1}{2}(z^* + 2kz^*)^2. \end{aligned}$$

So, if $E_2(0.31; 0.44; 0.14)$, we can obtain $\dot{v} < 0$ if

$$\begin{cases} 0.7 - k < 0 \\ 0,555 - k < 0 \\ -0,345 - k < 0 \\ 1.237\,2k^2 - 0.616\,02k + 3.681\,3 \times 10^{-2} < 0 \end{cases} \Leftrightarrow \begin{cases} k > 0.7 \\ k > 0,555 \\ k > -0,345 \\ k \in \left[6.944\,5 \times 10^{-2}, 0.428\,47\right]. \end{cases}$$

So, we have no solution.

For the Jacobian matrix conditions, the Jacobian matrix is as follows:

$$A = \begin{pmatrix} 0.2-k & -0.14 & 0 \\ 0.14 & -0.1-k & 0 \\ 0.44 & 0.31 & -k \end{pmatrix} \Leftrightarrow \det(A) = K((0.2-k)(0.1-k) - 0,0434),$$

$$\begin{aligned} A_{11} &= \begin{vmatrix} -0.1-k & 0 \\ 0.31 & -k \end{vmatrix} = k(0.1+k), \\ A_{22} &= \begin{vmatrix} 0.2-k & 0 \\ 0.44 & -k \end{vmatrix} = k(k-0.2), \\ A_{33} &= \begin{vmatrix} 0.2-k & -0.14 \\ 0.14 & -0.1-k \end{vmatrix} = k^2 - 0.1k - 0.000\,4. \end{aligned}$$

According to Theorem 2, we have $t = 0$, then $\det(A) < 0$, $a_{ii} < 0$ and $A_{ii} > 0$ for $i = 1, 2, 3$ if $k > 0.2$.

For the Routh-Hurwitz theorem, the characteristic polynomial is

$$p(\lambda) = \lambda^3 + (3k - 0.1)\lambda^2 + \left(3k^2 - 0.2k - 0.000\,4\right)\lambda + k^3 - 0.1k^2 - 0.000\,4k,$$

then

$$\begin{cases} a = 3k - 0.1, \\ b = 3k^2 - 0.2k - 0.000\,4, \\ c = k^3 - 0.1k^2 - 0.000\,4k, \\ ab - c = 8.0k^3 - 0.8k^2 + 0.019\,2k + 0.000\,04, \end{cases}$$

then $a > 0, c > 0$ and $ab - c > 0$ if $k \in]0.103\,85, +\infty[$.

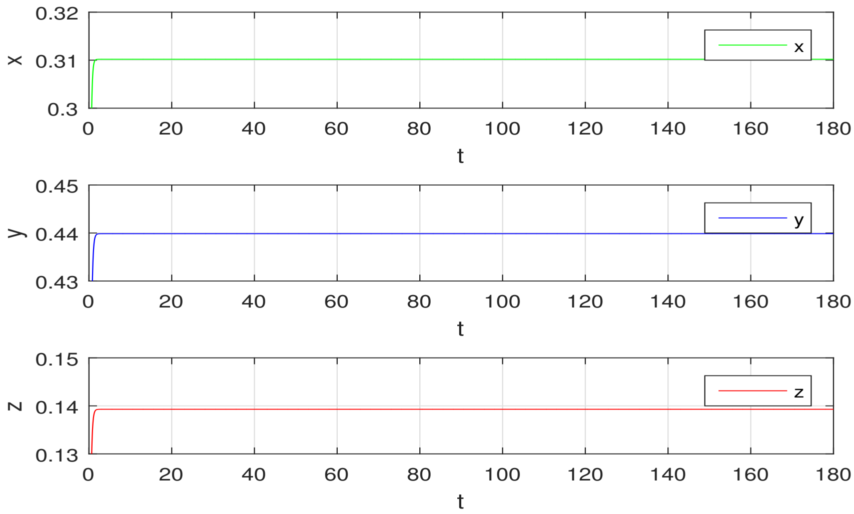

Figure 2: Control of a novel chaotic system at the equilibrium point $E_2(0.31; 0.44; 0.14)$ and $k = 4$.

5 Control of Modified Genesio System

Theorem 5.1 *The controlled modified Genesio system [8] is*

$$\begin{cases} \dot{x} = y - u_1, \\ \dot{y} = -0.5y + z - u_2, \\ \dot{z} = 3x^2 - 6x - 2.85y - 0.5z - u_3, \end{cases} \tag{4}$$

where $u_1 = k(x - x^*), u_2 = ky + z, u_3 = kz + 3(x - x^*)$ *and* k *is the feedback coefficient, the system (2) will gradually converge to the equilibrium point* $(x^*; 0; 0)$*when* $k \in]-0.25, 0[\cup]0.5, +\infty[$ *for the Routh-Hurwitz method and when* $k \in]0.5, +\infty[$ *for the Jacobian matrix conditions.*

Proof. For non linear feedback system (4) consider a quadratic Lyapunov function as $v = \frac{1}{2}[(x - x^*)^2 + y^2 + z^2)$, then

$$\dot{v} = -kx^2 + (0.5-k)y^2 + (-0.5-k)z^2 + xy + 6(x^*-1)xz - 2.85yz - yx^* - 3x^*z + 2kxx^* - kx^*$$

for $x^* = 2$,

$$\dot{v} < (k + 3.5)x^2 + (\frac{2.85}{2} - k)y^2 + (\frac{1.85}{2} - k)z^2 - 2k - 4.$$

So, we can obtain $\dot{v} < 0$, if

$$\begin{cases} k + 3.5 < 0 \\ \frac{2.85}{2} - k < 0 \\ \frac{1.85}{2} - k < 0 \\ -2k - 4 < 0 \end{cases} \Leftrightarrow \begin{cases} k < -3.5 \\ k > \frac{2.85}{2} \\ k > \frac{1.85}{2}, \\ k > -2 \end{cases}$$

so we have no solution.

For $x^* = 0$,

$$\dot{v} < (k+3)x^2 + (\frac{4.85}{2} - k)y^2 + (\frac{8.85}{2} - k)z^2,$$

then $\dot{v} < 0$

$$\begin{cases} k+3 \leq 0 \\ \frac{4.85}{2} - k < 0 \\ \frac{8.85}{2} - k < 0 \end{cases} \Leftrightarrow \begin{cases} k < -3 \\ k < \frac{4.85}{2} \\ k > \frac{8.85}{2} \end{cases},$$

so we have no solution.

For the Routh-Hurwitz method, the Jacobian matrix is as follows:

$$J_{(0;0;0)} = \begin{pmatrix} -K & 1 & 0 \\ 0 & -0.5-K & 0 \\ -6 & -2.85 & 0.5-K \end{pmatrix}, \text{ so } \det J_{(0;0;0)} = 0.25K - K^3,$$

$$\begin{aligned} A_{11} &= \begin{vmatrix} -0.5-K & 0 \\ -2.85 & 0.5-K \end{vmatrix} = K^2 - 0.25, \\ A_{22} &= \begin{vmatrix} -k & 0 \\ -6 & 0.5-k \end{vmatrix} = k^2 - 0.5k, \\ A_{33} &= \begin{vmatrix} -K & 1 \\ 0 & -0.5-K \end{vmatrix} = K^2 + 0.5K, \end{aligned}$$

we have $t = 0$, then $\det(j) < 0$, $a_{ii} < 0$ and $A_{ii} > 0$for $i = 1, 2, 3$ if $k \geq 0.5$.

For the second equilibrium point,

$$J_{(2;0;0)} = \begin{pmatrix} -K & 1 & 0 \\ 0 & -0.5-K & 0 \\ 6 & -2.85 & 0.5-K, \end{pmatrix},$$

so, $\det J_{(2;0;0)}$, A_{11}, A_{22} and A_{33} have the same value of the first point, then, if $k > 0.5$, the system (2) will gradually converge to the equilibrium point $(2; 0; 0)$. For the Routh-Hurwitz method, the characteristic polynomial for $E_1(0; 0; 0)$ and $E_2(2; 0; 0)$ equilibrium points is

$p(\lambda) = \lambda^3 + 3K\lambda^2 + ((K-0.5)(2K+0.5) + K(K+0.5))\lambda + K(K-0.5)(K+0.5)$,

$$\begin{aligned} a &= 3K, \\ b &= 3.0K^2 - 0.25, \\ ab - c &= K\left(8.0K^2 - 0.5\right), \end{aligned}$$

then, if $k \in]-0.25, 0[\cup]0.5, +\infty[$, we have $a > 0, b > 0$ and $ab - c > 0$.

Remark 5.1 For k with a negative value of the Routh-Hurwitz method, we have not a good result for the same feedback control of the Genesio system as we see in the figures.

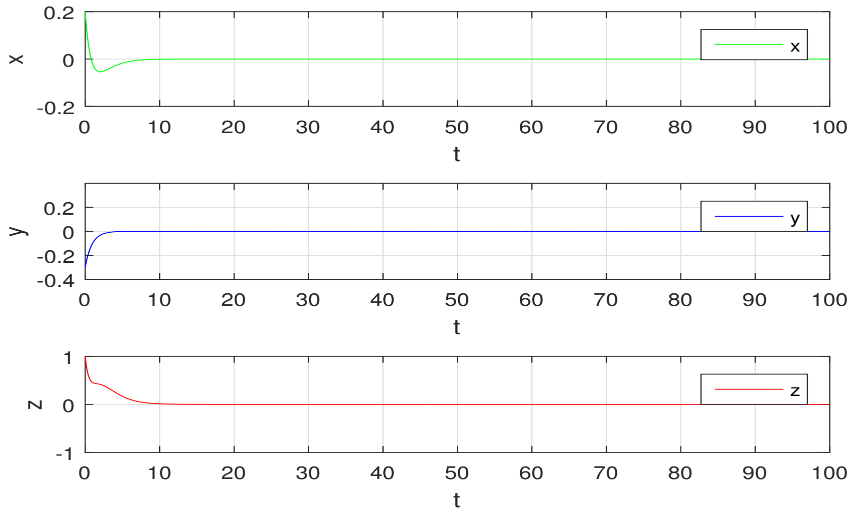

Figure 3: Control of the Genesio system at the equilibrium point $E_1(0; 0; 0)$.

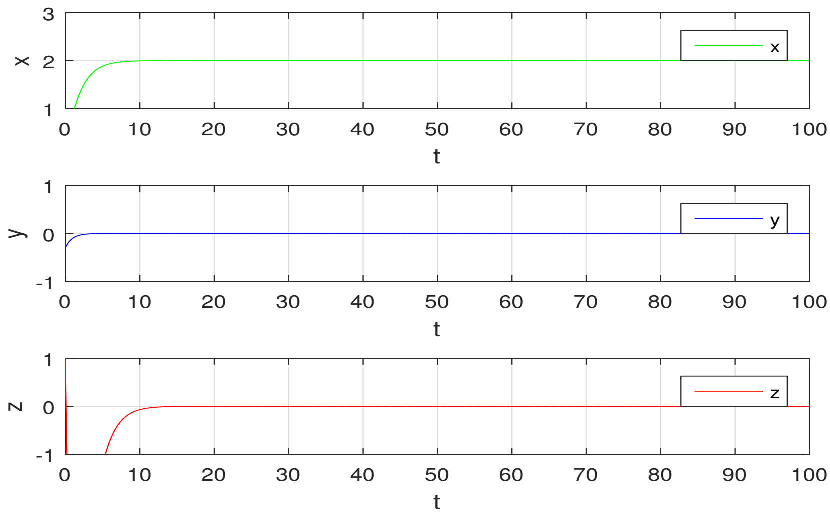

Figure 4: Control of the Genesio system at the equilibrium point $E_2(2; 0; 0)$.

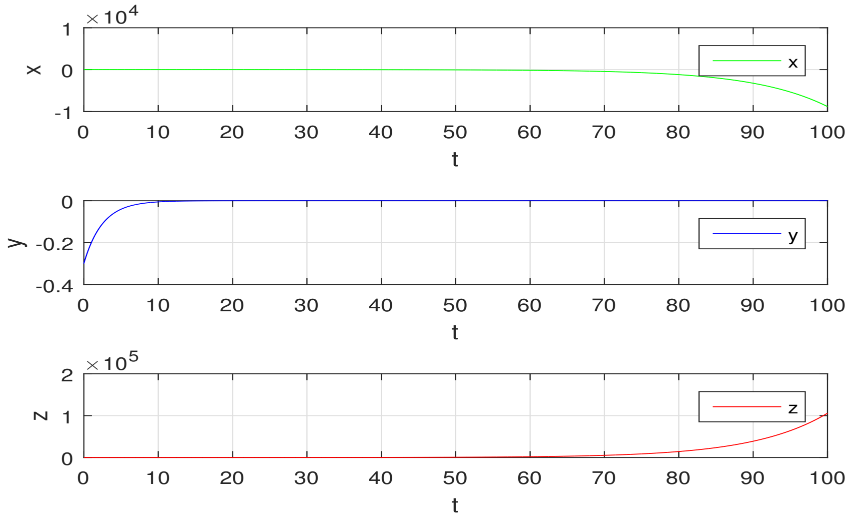

Figure 5: Control of the Genesio system at the equilibrium point $E_1(0; 0; 0)$ when $k = -0.1$.

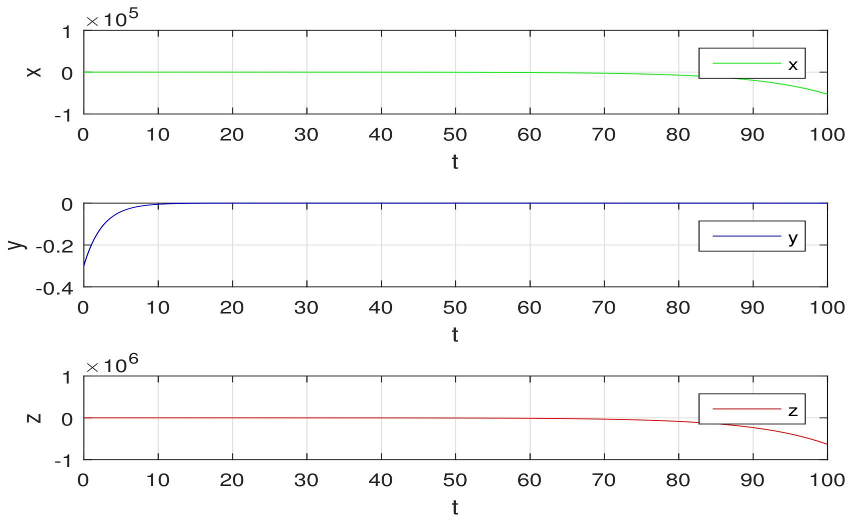

Figure 6: Control of the Genesio system at the equilibrium point $E_2(2; 0; 0)$ when $k = -0.1$.

6 Conclusion

This work presents a linear and nonlinear feedback control for the Sprott J system, a novel chaotic system and the Genesio system, stabilizing the systems at equilibrium points we use three different methods: the Lyapunov function, the Routh-Hurwitz criterion and a new method based on the Jacobian matrix, which is a modification of Routh-Hurwitz conditions. We proved that the stability by the new method is satisfied while we do not have it for the others, and we can get a good interval or an exact value for the gain matrix where the stability is satisfied.

References

[1] A. Ayadi, K. Dchich, B. Sfaihi and M. Benrejeb. Unified Continuous and Discrete Lur'e Systems Stability Analysis Based on Augmented Model Description. *Nonlinear Dynamics and Systems Theory* **20** (3) (2020) 242—252.

[2] B. R. Andrievskii and A. L. Fradkov. Control of Chaos: Methods and Applications. I. Methods 1. *Automation and Remote Control* **64** (5) (2003) 673–713.

[3] M. E. Brandt, H. T. Shih and G. Chen. Linear Time-delay Feedback Control of a Pathological Rhythm in a Cardiac Conduction Model. *Phys. Rev. E* **56** (1997) 1334–1337.

[4] S. Barnett and R. G. Cameron. *Introduction to Mathematical Control Theory.* Clarendon Press. OXFORD, Second Edition (1985).

[5] R. L. Borrelli and C. S. Coleman. *Differential Equations* (fourth ed.). John Wiley and Sons, Inc. New York, (1998).

[6] W. E. Boyce and R. C. DiPrima. *Elementary Differential Equations and Boundary Value Problems* (eighth ed.). John Wiley and Sons, Inc. United States of America, (2010).

[7] W. Chang, J. B. Park, Y. H. Joo and G. R. Chen. Outout feedback control of Chen chaotic attractor using fuzzy logic. In: *Proceedings of IECON 2000*, Japan, 2159–2164.

[8] K. Deng, J. Li and S. Yu. Dynamics analysis and synchronization of a new chaotic attractor. *Opt Int J. Light Electron Opt.* **125** (13) (2014) 3071–3075.

[9] W. Q. Feng Gao, V. Liu and K. L. Teo. Sreeram. Nonlinear Feedback Control for the Lorenz System. *Dynamics and Control* **11** (2001) 57–69.

[10] W. He and J. Cao. Adaptive synchronization of a class of chaotic neural networks with known or unknown parameters. **Phys. Lett. A 372** (2008) 408–416.

[11] H. Norouzi and D. Younesian. Chaos control for the plates subjected to subsonic flow. *Regular and Chaotic Dynamics* **21** (2016) 437–454.

[12] J. Zhang and W. Tang. Control and synchronization for a class of new chaotic systems via linear feedback. *Nonlinear Dyn.* **58** (2009) 675.

[13] K. Rajagopal, A. Akgul, I. M. Moroz, Z. Wei, S. Jafari and I. Hussain. A Simple Chaotic System With Topologically Different Attractors. IEEE Access 10.1109/ACCESS.2019.2922164.

[14] J. A. Lu, X. P. Han, Y. T. Li and M. H. Yu. Adaptive coupled synchronization among multi-Lorenz systems family. *Chaos Solitons Fractals* **31** (2007) 866–878.

[15] M. Islam, N. Islam and S. Nikolov. Adaptive Control and Synchronization of Sprott J System With Estimation Of Fully Unknown Parameters. *Journal of Theoretical and Applied Mechanics* **45** (2) (2015) 45–58.

[16] L. Meddour and K. Belakroum. On the Equivalence of Lorenz System and Li System. *Nonlinear Dynamics and Systems Theory* **22** (1) (2022) 58—65.

[17] E. Ott, C. Grebogi and J. A. Yorke. Controlling chaos. *Phys. Rev. Lett.* **64** (11) (1990) 1196–1199.

[18] K. Pyragas. Continuous control of chaos by self-controlling feedback. *Phys. Lett. A* **170** (6) (1992) 421–428.

[19] W. Laouira, and N. Hamri. Feedback Control of Chaotic Systems by Using Jacobian Matrix Conditions. *Nonlinear Dynamics and Systems Theory* **18** (3) (2018) 285–296.