Nonlinear Dynamics and Systems Theory, **22** (5) (2022) 473–488

Analysis of an Antiplane Thermo-Electro-Viscoelastic Contact Problem with Long-Term Memory

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Received: May 6, 2022; Revised: October 10, 2022

Abstract: We study a mathematical problem modeling the antiplane shear deformation of a cylinder in frictionless contact with a rigid foundation. The material is assumed to be thermo-electro-viscoelastic with long-term memory, the friction is modeled by Tresca's law and the foundation is assumed to be electrically conductive. We derive a variational formulation for the model which is in the form of a system involving the displacement field, the electric potential field and the temperature field. We prove the existence of a unique weak solution to the problem. The proof is based on the arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and a fixed point theorem.

Keywords: *weak solution; variational formulation; antiplane shear deformation; thermo-electroviscoelastic material; Tresca's friction law; fixed point; variational inequality.*

Mathematics Subject Classification (2010): 74M10, 49J40, 70K70, 70K75.

1 Introduction

Anti-plane shear deformation problems arise naturally from many real world applications such as rectilinear steady flow of simple fluids [6], interface stress effects of nanostructured materials [10], structures with cracks [16], layered/composite functioning materials [15], and phase transitions in solids [17]. Considerable attention has been paid to the modelling of such kind of problems, see for instance [8] and the references therein. In particular, the review paper [8] deals with modern developments for the antiplane shear model involving linear and nonlinear solid materials, various constitutive settings and applications. Antiplane frictional contact problems are used in geophysics in order to

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describe the pre-earthquake evolution of the regions of high tectonic activity, see for instance [5] and the references therein. The mathematical analysis of models for antiplane frictional contact problems can be found in [2, 8, 18].

Currently, there is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, i.e., materials characterized by the coupling of mechanical and electrical properties. This coupling, in a piezoelectric material, leads to the appearance of electric potential when mechanical stress is present, and conversely, mechanical stress is generated when electric potential is applied. The first effect is used in mechanical sensors, and the reverse effect is used in actuators, in engineering control equipment. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials and piezoelectric materials for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. General models for piezoelectric materials can be found in [3, 4, 12]. Static frictional contact problems for elastic and viscoelastic materials were studied in [11, 13, 14], under the assumption that the foundation is insulated. Contact problems with normal compliance for electro-viscoelastic materials were investigated in [9, 19]. There, variational formulations of the problems were considered and their unique solvability was proved. Antiplane problems for piezoelectric materials were considered in [18].

In paper [20], the authors have studied an antiplane contact problem for viscoelastic materials with long-term memory. This mechanical problem leads to an integro-differential variational inequality. In the present paper, we deal with an antiplane contact problem for a thermo-electro-viscoelastic cylinder, which leads to a new mathematical model, different from the one presented in [20]. The novelty of this paper consists in the fact that we model the friction by Tresca's law and the material's behavior by a thermo-viscoelastic constitutive law with long-term memory. We neglect the inertial term in the equation of motion to obtain a quasistatic approximation of the process.

Thermal effects in contact processes affect the composition and stiffness of the contacting surfaces, and cause thermal stresses in the contacting bodies. Moreover, the contacting surfaces exchange heat, and energy is lost to the surroundings. Our interest is to describe a simple physical process in which the frictional contact, viscosity and piezoelectric effects are involved, and to show that the resulting model leads to a well-posed mathematical problem. Taking into account the frictional contact between a viscous piezoelectric body and an electrically conductive foundation in the study of an antiplane problem leads to a new and interesting mathematical model which has the virtue of relative mathematical simplicity without loss of essential physical relevance.The main result we provide concerns the existence of a unique weak solution to the model. Its proof is carried out in several steps, and is based on the arguments of evolutionary variational inequalities and Banach's fixed-point theorem.

The rest of the paper is structured as follows. In Section 2, we describe the model of the frictional contact process between a thermo-electro-viscoelastic body and a conductive deformable foundation. In Section 3, we derive the variational formulation. It consists of a variational inequality for the displacement field coupled with a time-dependent variational equation for the electric potential and the heat equation for the temperature. We state our main result, the existence of a unique weak solution to the model, in Theorem 3.1. The proof of the theorem is provided in Section 4, where it is based on the arguments of evolutionary inequalities, an ordinary differential equation and a fixed-point theorem.

2 Mathematical Model

We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$, it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axis with a cross-section which is a regular region Ω in the x_1x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times\ (-\infty, +\infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density $\mathbf{q}_0$. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2, and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that the one-dimensional measures of Γ_1 and Γ_a, denoted meas Γ_1 and meas Γ_a, are positive. The cylinder is clamped on $\Gamma_1 \times\ (-\infty, +\infty)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times\ (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma a \times\ (-\infty, +\infty)$ and a surface electrical charge of density $\mathbf{q}_2$ is prescribed on $\Gamma_b \times\ (-\infty, +\infty)$. The cylinder is in contact over $\Gamma_3 \times\ (-\infty, +\infty)$ with a conductive obstacle, the so-called foundation. The contact is frictional and is modeled by Tresca's law. We are interested in the deformation of the cylinder on the time interval $[0, T]$. We assume that

$$\mathbf{f_0} = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t) : \Omega \times [0; T] \to \mathbb{R}, \tag{1}$$

$$\mathbf{f_2} = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times [0; T] \to \mathbb{R}, \tag{2}$$

$$q_0 = q_0\,(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{3}$$

$$q_2 = q_2\,(x_1, x_2, t) : \Gamma_b \times [0, T] \to \mathbb{R}. \tag{4}$$

The forces (1), (2) and the electric charges (3), (4) are expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and to an electric potential field φ which are independent of x_3 and have the form

$$\mathbf{u} = (0, 0, u) \quad \text{with} \quad u = u(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{5}$$

$$\varphi = \varphi(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}. \tag{6}$$

Such kind of deformation, associated to a displacement field of the form (5), is called an antiplane shear, see for instance [8] for details.

Below, the indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows the comma represents the partial derivative with respect to the corresponding spatial variable; also, the dot above represents the time derivative. We use S^3 for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and "$.$", $\|.\|$ will represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and S^3; we have

$$\begin{aligned} \mathbf{u}.\mathbf{v} &= u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v}.\mathbf{v})^{\frac{1}{2}} \quad \forall \mathbf{u} = (u_i)\,,\ \mathbf{v} = (v_i) \in \mathbb{R}^3, \\ \sigma.\tau &= \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma = (\sigma_{ij})\,,\ \ \tau = (\tau_{ij}) \in S^3. \end{aligned}$$

The infinitesimal strain tensor is denoted by $\varepsilon\,(\mathbf{u}) = (\varepsilon_{ij}\,(\mathbf{u}))$ and the stress field by $\sigma = (\sigma_{ij})$. We also denote by $\mathbf{E}\,(\varphi) = (E_i\,(\varphi))$ the electric field and by $\mathbf{D} = (D_i)$ the electric displacement field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad E_i\left(\varphi\right) = -\varphi_{,i}.$$

The material's behavior is modeled by the following thermal electro-viscoelastic constitutive law with long-term memory

$$\sigma = \lambda(tr\varepsilon(\mathbf{u}))\mathbf{I} + 2\mu\varepsilon(\mathbf{u}) + 2\int_0^t \mathcal{G}(t-s)\varepsilon(\mathbf{u}(s))ds - \mathcal{E}^*\mathbf{E}(\varphi) - M_c\theta, \tag{7}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \alpha\mathbf{E}(\varphi) - \mathcal{P}\theta, \tag{8}$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients, $tr(\varepsilon(\mathbf{u})) = \sum_{i=1}^{3}\varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, $\mathcal{G} : [0,T] \to \mathbb{R}$ is the relaxation function, $\mathcal{E}$ represents the third-order piezoelectric tensor, and $\mathcal{E}^*$ is its transpose, θ is the temperature field and $M_c := (m_{ij})$, $\mathcal{P}\,(p_i)$ are, respectively, the thermal expansion and the pyroelectric tensor which have the forms

$$M_c = \begin{pmatrix} 0 & 0 & \mathcal{M}_{c_1} \\ 0 & 0 & \mathcal{M}_{c_2} \\ \mathcal{M}_{c_1} & \mathcal{M}_{c_2} & 0 \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix}.$$

We assume that $\mathcal{M}_{c_i}(x_1,x_2) : \Omega \to \mathbb{R}$, and $\quad p_i : \Omega \to \mathbb{R}$.

In the antiplane context (5), (6), when using the constitutive equations (7), (8), it follows that the stress field and the electric displacement field are given by

$$\sigma = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}, \tag{9}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \alpha\varphi_{,1} - p_1\theta \\ eu_{,2} - \alpha\varphi_{,2} - p_2\theta \\ 0 \end{pmatrix}, \tag{10}$$

where α is the electric permittivity constant, e is a piezoelectric coefficient,

$$\sigma_{13} = \sigma_{31} = \mu u_{,1} + \int_0^t \mathcal{G}(t-s)u_{,1}(s)ds + e\varphi_{,1} - \mathcal{M}_{c_1}.\theta,$$

$$\sigma_{23} = \sigma_{32} = \mu u_{,2} + \int_0^t \mathcal{G}(t-s)u_{,2}(s)ds - e\varphi_{,2} - \mathcal{M}_{c_2}.\theta.$$

We assume that

$$\mathcal{E}\varepsilon = \begin{pmatrix} e\left(\varepsilon_{13} + \varepsilon_{31}\right) \\ e\left(\varepsilon_{23} + \varepsilon_{32}\right) \\ e\varepsilon_{33} \end{pmatrix} \qquad \forall\varepsilon = (\varepsilon_{ij}) \in S^3. \tag{11}$$

We also assume that the coefficients $\mathcal{G}$, μ, α, and e depend on the spatial variables x_1, x_2, but are independent of the spatial variable x_3. Since $\mathcal{E}\varepsilon.\mathbf{v} = \varepsilon.\mathcal{E}^*\mathbf{v}$ for all $\varepsilon \in S^3, \mathbf{v}$

$\in \mathbb{R}^3$, it follows from (11) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall \mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{12}$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\operatorname{Div} \sigma + \mathbf{f}_0 = 0, \quad \operatorname{div} D - q_0 = 0 \ \text{ in } \mathcal{B} \times (0, T)\,,$$

where $\operatorname{Div} \sigma = (\sigma_{ij,j})$ represents the divergence of the tensor field σ. When taking into account (1), (3), (5), (6), (9), and (10), the equilibrium equations above reduce to the following scalar equations:

$$\operatorname{div}(\mu \nabla u) + \int_0^t \mathcal{G}(t-s) \ \operatorname{div}(\nabla u(s))\, ds + \ \operatorname{div}(e \nabla \varphi) - \ \operatorname{div}(\theta\, \mathcal{M}_c) + f_0 = 0 \quad \text{in } \Omega \times (0, T), \tag{13}$$

$$\operatorname{div}(e \nabla u - \alpha \nabla \varphi) - \ \operatorname{div}(\theta \mathcal{P}) = q_0 \text{ in } \Omega \times (0, T) \tag{14}$$

with

$$\mathcal{M}_c = \begin{pmatrix} \mathcal{M}_{c_1} \\ \mathcal{M}_{c_2} \\ 0 \end{pmatrix}.$$

Here and below we use the notation

$$\operatorname{div} \tau = \tau_{1,1} + \tau_{1,2} \qquad \text{for} \quad \tau = (\tau_1(x_1, x_2, t), \tau_2(x_1, x_2, t))\,,$$
$$\nabla v = (v_{,1}, v_{,2})\,, \qquad \partial_\nu v = v_{,1}\nu_1 + v_{,2}\nu_2 \qquad \text{for} \qquad v = v(x_1, x_2, t)\,.$$

We now describe the boundary condition. During the process, the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and the electric potential vanishes on $\Gamma_1 \times (-\infty, +\infty)$. Thus, (5) and (6) imply that

$$u = 0 \qquad \text{on } \Gamma_1 \times (0, T), \tag{15}$$

$$\varphi = 0 \qquad \text{on } \Gamma_a \times (0, T). \tag{16}$$

Let ν denote the unit normal on $\Gamma \times (-\infty, +\infty)$. We have

$$\nu = (\nu_1, \nu_2, 0) \quad \text{with} \quad \nu_i = \nu_i(x_1, x_2) : \Gamma \to \mathbb{R},\ i = 1, 2. \tag{17}$$

For a vector $\mathbf{v}$, we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary, given by

$$v_\nu = \mathbf{v}.\nu \ , \ \mathbf{v}_\tau = \ \mathbf{v} - v_\nu \nu. \tag{18}$$

For a given stress field σ, we denote by σ_ν and σ_τ the normal and the tangential components on the boundary, that is,

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu. \tag{19}$$

From (9), (10), and (17), we deduce that the Cauchy stress vector and the normal component of the electric diplacement field are given by

$$\begin{gathered} \sigma\nu = (0, 0, \mu \partial_\nu u + \int_0^t \mathcal{G}(t-s) \partial_\nu u(s) ds + e \partial_\nu \varphi - \theta \mathcal{M}_c.\nu), \\ \mathbf{D}.\nu = e \partial_\nu u - \alpha \partial_\nu \varphi - \theta \mathcal{P}.\nu. \end{gathered} \tag{20}$$

Taking into account (2), (4), and (20), the traction condition on $\Gamma_2 \times (-\infty, +\infty)$ and the electric conditions on $\Gamma_b \times (-\infty, +\infty)$ are given by

$$\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu = f_2 \text{ on } \Gamma_2 \times (-\infty, +\infty), \quad (21)$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = q_2 \text{ on } \Gamma_b \times (-\infty, +\infty). \quad (22)$$

Now, we describe the frictional contact condition and electric conditions on $\Gamma_3 \times (-\infty, +\infty)$. First, from (5) and (17), we infer that the normal displacement vanishes, $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during the whole process. Using now (5) and (17)-(19), we conclude that

$$\mathbf{u}_\tau = (0, 0, u), \ \sigma_\tau = (0, 0, \sigma_\tau), \quad (23)$$

where

$$\sigma_\tau = (0, 0, \mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu).$$

We assume that the friction is invariant with respect to the x_3-axis and is modeled by Tresca's friction law, that is,

$$\begin{cases} |\sigma_\tau| \leq g, \\ |\sigma_\tau| < g \Rightarrow \dot{\mathbf{u}}_\tau = 0, \\ |\sigma_\tau| = g \Rightarrow \exists\beta \geq 0, \text{ such that } \sigma_\tau = -\beta\dot{\mathbf{u}}_\tau, \end{cases} \quad \text{on } \Gamma_3 \times (0, T). \quad (24)$$

Here $g : \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{\mathbf{u}}_\tau$ represents the tangential velocity on the contact boundary. Using now (23), it is straightforward to see that the conditions (24) imply

$$\begin{cases} |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| \leq g, \\ |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| < g \Rightarrow \dot{u}(t) = 0, \\ |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| = g \Rightarrow \exists\beta \geq 0, \\ \text{such that } \mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu = -\beta\dot{u}. \end{cases} \quad \text{on } \Gamma_3 \times (0, T). \quad (25)$$

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D}.\nu = k(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0, T),$$

where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient. We use (20) and the previous equality to obtain

$$e\partial_\nu u - \alpha\partial_\nu\varphi - \theta\mathcal{P}.\nu = k(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0, T). \quad (26)$$

Finally, we prescribe the initial displacement

$$u(0) = u_0 \qquad \text{in } \Omega, \tag{27}$$

where u_0 is the given function on Ω.

We collect the above equations and conditions to obtain the classical formulation of the antiplane problem for thermo-electro-viscoelastic materials with long-term memory, in frictional contact with a foundation.

Problem $\mathcal{P}$: Find the displacement field $u : \Omega \times (0,T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}_+$ and the electric potential $\varphi : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\text{div}\,(\mu\nabla u) + \int_0^t \mathcal{G}(t-s)\ \text{div}\,(\nabla u(s))\,ds + \ \text{div}\,(e\nabla\varphi) - \text{div}\,(\theta\,\mathcal{M}_c) + f_0 = 0 \text{ in } \Omega \times (0,T), \tag{28}$$

$$\text{div}\,(e\nabla u - \alpha\nabla\varphi) - \ \text{div}\,(\theta\mathcal{P}) = q_0 \text{ in } \Omega \times (0,T), \tag{29}$$

$$\dot{\theta} - \ \text{div}(K\nabla\theta) = -\mathcal{M}_c\nabla\dot{u} + h(t) \text{ in } \Omega \times (0,T), \tag{30}$$

$$u = 0 \qquad \text{on } \Gamma_1 \times (0,T), \tag{31}$$

$$\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu = f_2 \ \text{ on } \Gamma_2 \times (0,T), \tag{32}$$

$$\begin{cases} |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| \le g, \\ |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| < g \Rightarrow \dot{u} = 0, \\ |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| = g \Rightarrow \exists\beta \ge 0, \\ \text{such that } \ \mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu = -\beta\dot{u}, \end{cases} \quad on\ \Gamma_3 \times (0,T), \tag{33}$$

$$\theta = 0 \qquad \text{on } \Gamma_1 \cup \Gamma_2 \times (0,T), \tag{34}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = q_2 \ \text{ on } \Gamma_b \times (0,T), \tag{35}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi - \theta\mathcal{P}.\nu{=}k\,(\varphi - \varphi_F) \ \text{ on } \Gamma_3 \times (0,T), \tag{36}$$

$$-\,k_{ij}\frac{\partial\theta}{\partial x_j}n_i = k_e(\theta - \theta_R) \qquad \text{on } \Gamma_3 \times (0,T), \tag{37}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{38}$$

The differential equation (30) describes the evolution of the temperature field, where $K := (k_{ij})$ represents the thermal conductivity tensor, $h(t)$ is the density of volume heat sources. The associated temperature boundary condition is given by (37), where θ_R is the temperature of the foundation, and k is the heat exchange coefficient between the body and the obstacle. Finally, u_0, θ_0 represent the initial displacement and temperature, respectively.

3 Variational Formulation and Main Result

We derive now the variational formulation of Problem $\mathcal{P}$. To this end we introduce the function spaces

$$V = \left\{ v \in H^1(\Omega) \mid v = 0 \text{ on } \Gamma_1 \right\}, \qquad W = \left\{ \psi \in H^1(\Omega) \mid \psi = 0 \text{ on } \Gamma_a \right\},$$

and we assume that

$$E = \left\{ \eta \in H^1(\Omega) \mid \eta = 0 \text{ on } \Gamma_1 \cup \Gamma_2 \right\}.$$

Similarly, we write ζ for the trace $\gamma\zeta$ of the function $\zeta \in H^1(\Omega)$ on Γ. Since $meas\,\Gamma_1 > 0$ and $meas\,\Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u, v)_V = \int_\Omega \nabla u.\nabla v \; dx \quad \forall\; u, v \;\in V, \quad (\varphi, \psi)_W = \int_\Omega \nabla\varphi.\nabla\psi \; dx \quad \forall\; \varphi, \psi \;\in W.$$

Moreover, the associated norms

$$\| v \|_V = \| \nabla v \|_{L^2(\Omega)^2} \quad \forall\, v \in V, \quad \| \psi \|_W = \| \nabla\psi \|_{L^2(\Omega)^2} \quad \forall\, \psi \in W, \tag{39}$$

are equivalent on V and W, with the usual norm $\| \,.\, \|_{H^1(\Omega)}$. By Sobolev's trace theorem we deduce that there exist three positive constants $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ such that

$$\begin{gathered} \| v \|_{L^2(\Gamma_3)} \leq c_1 \| v \|_V \quad \forall\, v \in V, \quad \| \psi \|_{L^2(\Gamma_3)} \leq c_2 \| \psi \|_W \quad \forall\, \psi \in W, \\ \| \eta \|_{L^2(\Gamma_3)} \leq c_3 \| \eta \|_E \quad \forall\, \eta \in E. \end{gathered} \tag{40}$$

If $(X, \| \,.\, \|_X)$ represents a real Banach space where $X = V \times W$, we denote by $C([0,T];X)$ the space of continuous functions from $[0,T]$ to X, with the norm

$$\| x \|_{C([0;T];X)} = \max_{t\in[0,T]} \| x(t) \|_X,$$

and we use standard notations for the Lebesgue space $L^2(0,T;X)$ as well as for the Sobolev space $W^{1,2}(0,T;X)$. In particular, recall that the norm on the space $L^2(0,T;X)$ is given by the formula

$$\| u \|^2_{L^2(0,T;X)} = \int_0^T \| u(t) \|^2_X \; dt,$$

and the norm on the space $W^{1,2}(0,T;X)$ is given by the formula

$$\| u \|^2_{W^{1,2}(0,T;X)} = \int_0^T \| u(t) \|^2_X \; dt + \int_0^T \| \dot{u}(t) \|^2_X \; dt.$$

Finally, we suppress the argument X when $X = \mathbb{R}$; thus, for example, we use the notation $W^{1,2}(0,T)$ for the space $W^{1,2}(0,T;\mathbb{R})$ and the notation $\| \,.\, \|_{W^{1,2}(0,T)}$ for the norm $\| \,.\, \|_{W^{1,2}(0,T;\mathbb{R})}$.

In the study of Problem $\mathcal{P}$ we assume that the viscosity coefficient satisfies

$$\mathcal{G} \in W^{1,2}(0,T) \tag{41}$$

and the electric permittivity coefficient satisfies

$$\alpha \in L^\infty(\Omega) \text{ and there exists } \alpha^* > 0 \text{ such that } \alpha(\mathbf{x}) \geq \alpha^* \text{ a.e. } \mathbf{x} \in \Omega. \tag{42}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\mu \in L^\infty(\Omega) \text{ and } \mu(\mathbf{x}) > 0 \text{ a.e. } \mathbf{x} \in \Omega, \tag{43}$$

$$e \in L^\infty(\Omega). \tag{44}$$

The thermal tensor and the pyroelectric tensor satisfy

$$\mathcal{M}_c = \begin{pmatrix} \mathcal{M}_{c_1} \\ \mathcal{M}_{c_2} \\ 0 \end{pmatrix}, \quad \mathcal{M}_{c_i}(x_1, x_2) : \Omega \to \mathbb{R}, \ \mathcal{M}_{c_i} \in L^\infty(\Omega). \tag{45}$$

The boundary thermal data satisfy

$$h \in W^{1,2}(0,T;L^2(\Omega)), \ \ \theta_R \in W^{1,2}(0,T;L^2(\Gamma_3)), \ \ k_e \in L^\infty(\Omega, \mathbb{R}_+). \tag{46}$$

The thermal conductivity tensor verifies the usual symmetry and ellipticity: for some $c_k > 0$ and for all $\xi_i \in \mathbb{R}^d$,

$$K = (k_{ij}), \quad k_{ij} = k_{ji} \in L^2(\Omega), \ \forall c_k > 0, \ \xi_i \in \ \mathbb{R}^d; \quad k_{ij}\xi_i.\xi_j \leq c_k \xi_i.\xi_j. \tag{47}$$

The forces, tractions, volume, and surface free charge densities have the regularity

$$f_0 \in W^{1,2}(0,T;L^2(\Omega)), \qquad f_2 \in W^{1,2}(0,T;L^2(\Gamma_2)), \tag{48}$$

$$q_0 \in W^{1,2}(0,T;L^2(\Omega)), \qquad q_2 \in W^{1,2}(0,T;L^2(\Gamma_b)). \tag{49}$$

The electric conductivity coefficient and the friction bound function g satisfy the following properties:

$$k \in L^\infty(\Gamma_3) \ \text{ and } \ k(\mathbf{x}) \geq 0 \ \text{ a.e. } \mathbf{x} \in \Gamma_3, \tag{50}$$

$$g \in L^\infty(\Gamma_3) \ \text{ and } \ g(\mathbf{x}) \geq 0 \ \text{ a.e. } \mathbf{x} \in \Gamma_3. \tag{51}$$

Also, we assume that the electric potential of the foundation is such that

$$\varphi_F \in W^{1,2}(0,T;L^2(\Gamma_3)). \tag{52}$$

Finally, we assume that the initial data verifies

$$u_0 \in V, \quad \theta_0 \in L^2(\Omega), \tag{53}$$

and moreover,

$$a_\mu(u_0, \upsilon)_V + j(\upsilon) \geq (f(0), \upsilon)_V, \ \forall \upsilon \in V. \tag{54}$$

We consider the functional $j : [0,T] \to \mathbb{R}_+$ given by

$$j(\upsilon) = \int_{\Gamma_3} g\,|\upsilon|\,da \qquad \forall \upsilon \in V, \tag{55}$$

and let $f : [0,T] \to V$ and $q : [0,T] \to W$ be defined by

$$(f(t), \upsilon)_V = \int_\Omega f_0(t)\upsilon dx + \int_{\Gamma_2} f_2(t)\upsilon\,da, \tag{56}$$

$$(q(t), \psi)_W = \int_\Omega q_0(t)\psi dx - \int_{\Gamma_b} q_2(t)\psi\,da + \int_{\Gamma_3} k\varphi_F(t)\,\psi da, \tag{57}$$

$$\forall \upsilon \in V, \ \psi \in W, \forall t \in [0,T].$$

The definitions of f and q are based on Riesz's representation theorem and by (48) and (49), we infer that the integrals above are well-defined and

$$f \in W^{1,2}(0,T;V), \tag{58}$$
$$q \in W^{1,2}(0,T;W). \tag{59}$$

Next, we define the bilinear forms $a_\mu : V \times V \to \mathbb{R}$, $a_e : V \times W \to \mathbb{R}$, $a_e^* : W \times V \to \mathbb{R}$, and $a_\alpha : W \times W \to \mathbb{R}$, by the equalities

$$a_\mu(u,\upsilon) = \int_\Omega \mu \nabla u.\nabla \upsilon \, dx, \tag{60}$$
$$a_e(u,\varphi) = \int_\Omega e \nabla u.\nabla \varphi \, dx = a_e^*(\varphi, u), \tag{61}$$
$$a_\alpha(\varphi,\psi) = \int_\Omega \alpha \nabla \varphi.\nabla \psi \, dx + \int_{\Gamma_3} k\varphi\psi \, da \tag{62}$$

for all $u, \upsilon \in V$, $\varphi, \psi \in W$. Assumptions (55)–(57) imply that the integrals above are well-defined and when using (39) and (40), it follows that the forms a_μ, a_e and a_e^* are continuous; moreover, the forms a_μ and a_α are symmetric and, in addition, the form a_α is W-elliptic since

$$a_\alpha(\psi,\psi) \geq \alpha^* \|\psi\|_W^2 \quad \forall \psi \in W. \tag{63}$$

By using Green's formula, it is straightforward to derive the following variational formulation of $\mathcal{P}$. We denote by $\langle\, , \rangle_{V' \times V}$ the duality pairing between V' and V.

Problem $\mathcal{P}_V$: Find a displacement field $u : [0;T] \to V$, an electric potential field $\varphi : [0;T] \to W$ and a temperature field $\theta : [0;T] \to E$ such that

$$\begin{aligned} &a_\mu(u(t), \upsilon - \dot{u}(t)) + (\textstyle\int_0^t \mathcal{G}(t-s)u(s)ds, \upsilon - \dot{u}(t))_V + a_e^*(\varphi(t), \upsilon - \dot{u}(t)) \\ &+(\mathcal{M}_c\theta(t), \upsilon - \dot{u}(t))_V + j(\upsilon) - j(\dot{u}(t)) \geq (f(t), \upsilon - \dot{u}(t))_V \ \forall \upsilon \in V, \, t \in (0,T), \end{aligned} \tag{64}$$

$$a_\alpha(\varphi(t), \psi) - a_e(u(t), \psi) - (\mathcal{P}\theta, \nabla\psi)_H = (q(t), \psi)_W \quad \forall \psi \in W, \ t \in [0,T], \tag{65}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t) \qquad \text{in } E', \tag{66}$$

$$u(0) = u_0, \qquad \theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{67}$$

Here, the function $Q : [0,T] \to E'$ and the operators $K : E \to E'$, $R : V \to E'$; $\mathcal{M}_c : E \to V'$ are defined by $\forall \upsilon \in V$, $\forall \tau \in E$, $\forall \mu \in E$:

$$\langle Q(t), \mu \rangle_{E' \times E} = \int_{\Gamma_3} k_c \theta_R \mu \, ds + \int_\Omega q\mu \, dx,$$
$$\langle K\tau, \mu \rangle_{E' \times E} = \sum_{i,j=1}^d \int_\Omega k_{ij} \frac{\partial \mu}{\partial x_j} \frac{\partial \mu}{\partial x_i} \, dx + \int_{\Gamma_3} k_c \tau \mu \, ds,$$
$$\langle R\upsilon, \mu \rangle_{E' \times E} = \int_{\Gamma_3} h_\tau(|\upsilon_\tau|)\mu \, ds - \int_\Omega (\mathcal{M}_c \nabla \upsilon)\mu \, dx,$$
$$\langle \mathcal{M}_c\tau, \upsilon \rangle_{V' \times V} = (-\tau \mathcal{M}_c, \upsilon)_V.$$

Our main existence and uniqueness result is stated as follows.

Theorem 3.1 *Assume that (41)-(59) hold. Then there exists a unique solution (u, θ, φ) of problem $\mathcal{P}_V$. Moreover, the solution satisfies*

$$\begin{aligned} u &\in W^{1,2}(0,T;V);\ \varphi \in W^{1,2}(0,T;W), \\ \theta &\in W^{1,2}(0,T;E') \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)). \end{aligned} \tag{68}$$

An element (u, φ, θ) which solves $\mathcal{P}_V$ is called a weak solution of the mechanical problem $\mathcal{P}$. We conclude by Theorem 3.1 that the antiplane contact problem $\mathcal{P}$ has a unique weak solution, provided that (41)-(59) hold.

4 An Abstract Existence and Uniqueness Result

The proof of Theorem 3.1 is carried out in several steps that we prove in what follows. Everywhere in this section, we suppose that assumptions of Theorem 3.1 hold and we denote by $c > 0$ a generic constant, whose value may change from lines to lines.

In the first step of the proof, we introduce the set

$$\mathcal{W} = \left\{\eta \in W^{1,2}(0,T;X) \mid \eta(0) = 0_X\right\}, \tag{69}$$

and we prove the following existence and uniqueness result.

Lemma 4.1 *For all $\eta \in \mathcal{W}$, there exists a unique element $u_\eta \in W^{1,2}(0,T;X)$ such that*

$$\begin{aligned} &a(u_\eta(t), v - \dot{u}_\eta(t)) + (\eta(t), v - \dot{u}_\eta(t))_X + j(v) - j(\dot{u}_\eta(t)) \\ &\geq (f(t), v - \dot{u}_\eta(t))_X \qquad \forall v \in X,\ \ a.e.\ t \in [0,T], \end{aligned} \tag{70}$$

$$u_\eta(0) = u_0. \tag{71}$$

Here X is a real Hilbert space endowed with the inner product $(.,.)_X$ and the data a is a bilinear continuous coercive and symmetric form.

Proof. We use an abstract existence and uniqueness result which may be found in [2].

In the second step, we use the displacement field u_η obtained in Lemma 4.1 and we consider the following lemma.

Lemma 4.2 *For all $\eta \in \mathcal{W}$, there exists a unique solution*

$$\theta_\eta \in W^{1,2}(0,T;E') \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)),\ c > 0\ \forall \eta \in L^2([0,T],V'),$$

satisfying

$$\begin{cases} \dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{u}_\eta(t) + Q(t) \quad in\ E'\ \ a.e.\ t \in [0,T], \\ \theta_\eta(0) = \theta_0, \end{cases} \tag{72}$$

$$|\theta_{\eta_1} - \theta_{\eta_2}|^2_{L^2(\Omega)} \leq C \int_0^t |\dot{u}_{\eta 1}(s) - \dot{u}_{\eta 2}(s)|^2_V\, ds \qquad \forall t \in [0,T], \tag{73}$$

and

$$|\dot{\theta}_{\eta_1} - \dot{\theta}_{\eta_2}|^2_{L^2(\Omega)} \leq c \int_0^t |\dot{u}_{\eta 1}(s) - \dot{u}_{\eta 2}(s)|^2_V\, ds \qquad a.e.t \in [0,T]. \tag{74}$$

Proof. The existence and uniqueness result verifying (72) follows from the classical result for the first order evolution equation, applied to the Gelfand evolution triple

$$E \subset F \equiv F^{'} \subset E^{'}.$$

We verify that the operator $K : E \to E^{'}$ is linear continuous and strongly monotone, and from the expression of the operator R,

$$\upsilon_\eta \in W^{1,2}(0,T;V) \Longrightarrow R\upsilon_\eta \in W^{1,2}(0,T;F),$$

as $Q \in W^{1,2}(0,T;E)$, then $R\upsilon_\eta + Q \in W^{1,2}(0,T;E)$, we deduce (73) and (74) (see [1]).

In the third step, we use the displacement field u_η obtained in Lemma 4.1 and θ_η obtained in Lemma 4.2 and we consider the following lemma.

Lemma 4.3 *For all* $\eta \in \mathcal{W}$ *, there exists a unique solution* $\varphi_\eta \in W^{1,2}(0,T;W)$ *which satisfies*

$$a_\alpha(\varphi_\eta(t), \psi) - a_e(u_\eta(t), \psi) - (\mathcal{P}\theta_\eta, \nabla\psi)_H = (q(t), \psi)_W \ \ \forall\psi \in W, \ t \in [0,T]. \quad (75)$$

Moreover, if φ_{η_1} *and* $\varphi_{\eta 2}$ *are the solutions of (4.7) corresponding to* $\eta_1, \eta_2 \in C([0,T], V)$, *then there exists* $c > 0$ *such that*

$$\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\|_W \le c\,\|u_{\eta_1}(t) - u_{\eta_2}(t)\|_V \quad \forall t \in [0,T]. \quad (76)$$

Proof. Let $t \in [0,T]$. We use the properties of the bilinear form a_α and the Lax-Milgram lemma to see that there exists a unique element $\varphi_\eta(t) \in W$ which solves (75) at any moment $t \in [0,T]$. Consider now $t_1, t_2 \in [0,T]$; using (75), we get

$$\begin{aligned} a_\alpha(\varphi_\eta(t_1), \psi) - a_e(u_\eta(t_1), \psi) - (\,\mathcal{P}\theta_\eta(t_1), \nabla\psi)_H \\ = (q(t_1), \psi)_W \ \ \forall\psi \in W, \ t_1 \in [0,T], \end{aligned} \quad (77)$$

$$\begin{aligned} a_\alpha(\varphi_\eta(t_2), \psi) - a_e(u_\eta(t_2), \psi) - (\,\mathcal{P}\theta_\eta(t_2), \nabla\psi)_H \\ = (q(t_2), \psi)_W \ \ \forall\psi \in W, \ t_2 \in [0,T]. \end{aligned} \quad (78)$$

Using (77), (78) and (63), we find that

$$\begin{aligned} \alpha^* \|\varphi(t_1) - \varphi(t_2)\|_W^2 \ \le \ & (\|e\|_{L^\infty(\Omega)} \|u(t_1) - u(t_2)\|_V + \|q(t_1) - q(t_2)\|_W + \\ & \|p\|_{L^\infty(\Omega)} \|\theta(t_1) - \theta(t_2)\|_{L^2(\Omega)}\ ) \|\varphi(t_1) - \varphi(t_2)\|_W, \end{aligned}$$

and using (73) we find that

$$\begin{aligned} \alpha^* \|\varphi(t_1) - \varphi(t_2)\|_W^2 \ \le \ & (\|e\|_{L^\infty(\Omega)} \|u(t_1) - u(t_2)\|_V + \|q(t_1) - q(t_2)\|_W + \\ & \|p\|_{L^\infty(\Omega)} \|u(t_1) - u(t_2)\|_V\ ) \|\varphi(t_1) - \varphi(t_2)\|_W. \end{aligned}$$

It follows from the previous inequality that

$$\|\varphi(t_1) - \varphi(t_2)\|_W \le c(\|u(t_1) - u(t_2)\|_V + \|q(t_1) - q(t_2)\|_W). \quad (79)$$

Then, the regularity $u_\eta \in W^{1,2}(0,T;V)$ combined with (59) and (79) imply that $\varphi_\eta \in W^{1,2}(0,T;W)$, which concludes the proof.

Now, for all $\eta \in \mathcal{W}$, we denote by u_η the solution obtained in Lemma 4.1, by θ_η the solution obtained in Lemma 4.2 and by φ_η the solution obtained in Lemma 4.3.

Step 4: In the fourth step, we consider the operator $\Lambda : \mathcal{W} \to \mathcal{W}$.

We now use Riesz's representation theorem to define the element $\Lambda\eta\,(t) \in \mathcal{W}$ by the equality

$$\begin{aligned} \langle \Lambda\eta(t), w\rangle_{\mathcal{W}} &= (\int_0^t \mathcal{G}(t-s)u_\eta(s)ds - M_c\theta_\eta, w)_V + a_e^*\left(\varphi_\eta\left(t\right), w\right) \\ \forall\eta &\in \mathcal{W},\ w \in V,\ t \in [0,T]\,. \end{aligned} \tag{80}$$

Clearly, for a given $\eta \in \mathcal{W}$, the function $t \to \Lambda\eta\,(t)$ belongs to $\mathcal{W}$. In this step we show that the operator $\Lambda : \mathcal{W} \to \mathcal{W}$ has a unique fixed point.

Lemma 4.4 *The operator Λ has a unique fixed point $\eta^* \in \mathcal{W}$ such that $\Lambda\eta^* = \eta^*$.*

Proof. Let $\eta_1, \eta_2 \in \mathcal{W}$ and $t \in [0,T]$. In what follows we denote by u_i, θ_i and φ_i the functions u_{η_i}, θ_{η_i} and φ_{η_i} obtained in Lemmas 4.1, 4.2 and 4.3, for i = 1, 2. Using (80) and (61), we obtain

$$\begin{aligned} &\left\|\Lambda\eta_1\left(t\right) - \Lambda\eta_2\left(t\right)\right\|_X^2 \\ \leq\ & C\left(\int_0^t \left\|u_1\left(s\right) - u_2\left(s\right)\right\|_X^2\ ds + \left\|\theta_1 - \theta_2\right\|_{L^2(\Omega)}^2 + \left\|\varphi_1\left(t\right) - \varphi_2\left(t\right)\right\|_W^2\right) \forall t \in [0,T]\,. \end{aligned} \tag{81}$$

The constant C represents a generic positive number which may depend on $\|\theta\|_{W^{1,2}(0,T)}$, T, m_{ij} and e, and whose value may change from place to place.

Since $u_\eta \in W^{1,2}(0,T;V)$ and $\varphi_\eta \in W^{1,2}(0,T;W)$, we deduce from inequality (81) that $\Lambda_\eta \in W^{1,2}(0,T;V)$. On the other hand, (76) and arguments similar to those used in the proof of (79) yield

$$\left\|\varphi_1\left(t\right) - \varphi_2\left(t\right)\right\|_W \leq C\left\|u_1\left(t\right) - u_2\left(t\right)\right\|_V. \tag{82}$$

Using now (73)(82) in (81), we get

$$\begin{aligned} &\left\|\Lambda\eta_1\left(t\right) - \Lambda\eta_2\left(t\right)\right\|_X^2 \\ \leq\ & C\left(\int_0^t \left\|u_1\left(s\right) - u_2\left(s\right)\right\|_X^2\ ds + \int_0^t \left\|\dot{u}_1\left(t\right) - \dot{u}_2\left(t\right)\right\|_X^2 ds + \left\|u_1\left(t\right) - u_2\left(t\right)\right\|_V^2\right). \end{aligned}$$

Using the norm on the space $W^{1,2}\left(0,T,X\right)$, we deduce that

$$\left\|\Lambda\eta_1\left(t\right) - \Lambda\eta_2\left(t\right)\right\|_X^2 \leq C\left\|u_1\left(s\right) - u_2\left(s\right)\right\|_X^2\ ds \qquad \forall t \in [0,T]\,. \tag{83}$$

Taking into account (64), we have the inequalities

$$\begin{aligned} &a(u_1(t), v - \dot{u}_1(t)) + (\eta_1(t), v - \dot{u}_1(t))_X + j(v) - j(\dot{u}_1(t)) \\ &\geq (f(t), v - \dot{u}_1(t))_X \qquad \forall v \in X, \quad t \in [0,T]\,, \end{aligned}$$

and

$$\begin{aligned} &a(u_2(t), v - \dot{u}_2(t)) + (\eta_2(t), v - \dot{u}_2(t))_X + j(v) - j(\dot{u}_2(t)) \\ &\geq (f(t), v - \dot{u}_2(t))_X \qquad \forall v \in X, \quad t \in [0,T]\,, \end{aligned}$$

for all $v \in X$, $a.e.\ s \in [0,T]$. We choose $v = \dot{u}_2(s)$ in the first inequality and $v = \dot{u}_1(s)$ in the second inequality, add the result to obtain

$$\frac{1}{2}\left\|u_1(s) - u_2(s)\right\|_X^2 \leq -\left(\eta_1(s) - \eta_2(s), \dot{u}_1(s) - \dot{u}_2(s)\right)_X \quad \text{a.e. } s \in [0,T].$$

Let $t \in [0,T]$. Integrating the previous inequality from 0 to t using (68), we obtain

$$\begin{gathered}\tfrac{1}{2}\left\|u_1(t) - u_2(t)\right\|_X^2 \leq -\left(\eta_1(t) - \eta_2(t), u_1(t) - u_2(t)\right)_X \\ \textstyle\int_0^t \left(\dot{\eta}_1(s) - \dot{\eta}_2(s), u_1(s) - u_2(s)\right)_X\ ds.\end{gathered}$$

We deduce that

$$\begin{gathered}C\left\|u_1(t) - u_2(t)\right\|_X^2 \leq \left\|\eta_1(t) - \eta_2(t)\right\|_X \left\|u_1(t) - u_2(t)\right\|_X \\ + \textstyle\int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X \left\|u_1(s) - u_2(s)\right\|_X\ ds.\end{gathered}$$

Using Young's inequality, we get

$$\begin{gathered}\left\|u_1(t) - u_2(t)\right\|_X^2 \leq C(\left\|\eta_1(t) - \eta_2(t)\right\|_X^2 + \textstyle\int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds \\ + \textstyle\int_0^t \left\|u_1(s) - u_2(s)\right\|_X^2\ ds).\end{gathered} \tag{84}$$

On the other hand, as

$$\eta_1(t) - \eta_2(t) = \int_0^t \dot{\eta}_1(s) - \dot{\eta}_2(s)\, ds,$$

we can obtain

$$\left\|\eta_1(t) - \eta_2(t)\right\|_X^2 \leq C\int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds. \tag{85}$$

Using now (85) in (84), we have

$$\left\|u_1(t) - u_2(t)\right\|_X^2 \leq C(\int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds + \int_0^t \left\|u_1(s) - u_2(s)\right\|_X^2\ ds).$$

Taking into account Gronwall's inequality, we deduce

$$\left\|u_1(t) - u_2(t)\right\|_X^2 \leq C\int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds. \tag{86}$$

From (83), (86), we obtain

$$\left\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\right\|_X^2 \leq C\int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds.$$

Iterating the last inequality m times, we infer

$$\left\|\Lambda^m\eta_1(t) - \Lambda^m\eta_2(t)\right\|_X^2 \leq C^m \int_0^t \int_0^{s_1} \ldots\ldots \int_0^{s_{m-1}} \left\|\dot{\eta}_1(s_m) - \dot{\eta}_2(s_m)\right\|_X^2\ ds_m \ldots\ldots ds_1,$$

where Λ^m denotes the power of operator Λ. The last inequality gives

$$\left\|\Lambda^m\eta_1(t) - \Lambda^m\eta_2(t)\right\|_{W^{1.2}(0,T;X)}^2 \leq \frac{C^m T^m}{m!}\left\|\eta_1(t) - \eta_2(t)\right\|_{W^{1.2}(0,T;X)}^2,$$

which implies that for m sufficiently large, the power Λ^m of Λ is a contraction in the Banach space, since

$$\lim_{m\to\infty} \frac{C^m T^m}{m!} = 0,$$

it follows now from Banach's fixed-point theorem that there exists a unique element $\eta^* \in \mathcal{W}$ such that $\Lambda^m \eta^* = \eta^*$. Moreover, since

$$\Lambda^m (\Lambda\eta^*) = \Lambda (\Lambda^m \eta^*) = \Lambda\eta^*,$$

we deduce that $\Lambda\eta^*$ is also a fixed point of the operator Λ^m. By the uniqueness of the fixed point, we conclude that $\Lambda\eta^* = \eta^*$, which shows that η^* is a fixed point, we conclude that $\Lambda\eta^* = \eta^*$. *Step 5:* In the fifth and last step of our demonstration, we have now all the ingredients to provide the proof of Theorem 3.1.

Existence. Let $\eta^* \in W^{1.2}(0, T; V)$ be the fixed point of the operator Λ, and let u_{η^*}, θ_{η^*} and φ_{η^*} be the solutions defined in Lemmas 4.1, 4.2 and 4.3, respectively, for $\eta = \eta^*$. It follows from (80) that

$$\langle \eta^*(t), w\rangle_V = (\int_0^t \mathcal{G}(t-s) u_{\eta^*}(s) ds - M_c\theta_\eta, w))_V + a_e^* (\varphi_{\eta^*}(t), w) \ \ \forall w \in V,\ t \in [0, T], \tag{87}$$

and, therefore, (64), (66), and (76) imply that $(u_{\eta^*}, \theta_{\eta^*}, \varphi_{\eta^*})$ is a solution of problem $\mathcal{P}_V$. Regularity (68) of the solution follows from Lemmas 4.1, 4.2 and 4.3.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ. It can also be obtained by using arguments similar to those used in [20] and [9].

5 Conclusion

This work models the phenomenon of contact with friction between a cylindrical body and a foundation. These contact phenomena abound in industry and in everyday life, so they play an important role in the behavior of mechanical structures.

The envisaged mechanical model is an antiplane one. We recall that the antiplane shear deformation is the expected deformation of a very long cylinder loaded in the direction of its generators. In such a model, the displacement vectorial field is parallel to the generators of the cylinder and it is independent of the axial coordinate. Due to their simplicity in the writing of the equations without loss of physical relevance, antiplane models have enjoyed special attention in recent years. The antiplane models appear in the technical literature in engineering, describing the functioning of various mechanisms, and in geophysics, focusing on the deformation of the tectonic plates, and in particular, on earthquakes.

The novelty of the result obtained is the coupling of an electro-viscoelastic problem and a thermal effect.

The problem is formulated as a coupled system of evolutionary variational inequality for the displacement field with a time-dependent variational equation for the electric potential field and the heat equation for the temperature. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem.

References

[1] S. Adly and O. Chau. On some dynamic thermal non clamped contact problems. *Mathematical Programming*, Ser. B (2013) 5–26.

[2] F. Andreu, J. M. Mazón and M. Sofonea. Entropy solutions in the study of antiplane shear deformations for elastic solids. *Math. Models Methods Appl. Sci.* **10** (2000) 99–126.

[3] O. Baiz and H. Benaissa. Analysis of Dynamic Frictional Contact Problem for Electro-Elastic Materials. *Nonlinear Dynamics and Systems Theory* **21** (2) (2021) 150–165.

[4] R. C. Batra and J. S. Yang. Saint-Venant's principle in linear piezoelectricity. *J. Elasticity* **38** (1995) 209–218.

[5] M. Campillo, C. Dascalu and I. R. Ionescu. Instability of a periodic system of faults. *Geophys. J. Int.* **159** (2004) 212–222.

[6] R. L. Fosdick and J. Serrin. Rectilinear steady flow of simple fluids. *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences.* **332** (1590) (1973) 311–333.

[7] T.V. Hoarau-Mantel and A. Matei. Analysis of a viscoelastic antiplane contact problem with slip-dependent friction. *Int. J. Appl. Math. Comput. Sci.* **12** (2002) 51–58.

[8] C. O. Horgan and K. L. Miller. Antiplane shear deformations for homogeneous and inhomogeneous anisotropic linearly elastic solids. *J. Appl. Mech.* **61** (1994) 23–29.

[9] Z. Lerguet, M. Shillor and M. Sofonea. A frictional contact problem for an electro-viscoelastic body. *Electron. J. Differ. Equ. (electronic only) 2007* (2007) 16 pages.

[10] J. Luo and X. Wang. On the anti-plane shear of an elliptic nano inhomogeneity. *European Journal of Mechanics A/Solids* **28** (2009) 926–934.

[11] F. Maceri and P. Bisegna. The unilateral frictionless contact of a piezoelectric body with a rigid support. *Math. Comput. Modelling* **28** (1998) 19–28.

[12] L. Maiza, T. Hadj Ammar and M. Said Ameur. A Dynamic Contact Problem for Elasto-Viscoplastic Piezoelectric Materials with Normal Compliance, Normal Damped Response and Damage. *Nonlinear Dynamics and Systems Theory* **21** (3) (2021) 280–302.

[13] S. Migórski, A. Ochal and M. Sofonea. Weak solvability of antiplane frictional contact problems for elastic cylinders. *Nonlinear Anal., Real World Appl.* **11** (2010) 172–183.

[14] S. Migórski, A. Ochal and M. Sofonea. Analysis of a piezoelectric contact problem with subdifferential boundary condition. *Proc. R. Soc. Edinb., Sect. A, Math.* **144** (2014) 1007–1025.

[15] F. Narita and Y. Shindo. Layered piezoelectric medium with interface crack under anti-plane shear. *Theoretical and Applied Fracture Mechancis* **30** (2) (1998) 119–126.

[16] G. H. Paulino, M. T. A. Saif and S. Mukherjee. A finite elastic body with a curved crack loaded in anti-plane shear International. *Journal of Solids and Structures* **30** (8) (1993) 1015–1037.

[17] S. A. Silling. Consequences of the Maxwell relation for anti-plane shear deformations of an elastic solid. *Journal of elasticity* **19** (3) (1988) 241–284.

[18] M. Sofonea, M. Dalah and A. Ayadi. Analysis of an antiplane electro-elastic contact problem. *Adv. Math. Sci. Appl.* **17** (2007) 385–400.

[19] M. Sofonea and El. H. Essoufi. Quasistatic frictional contact of a viscoelastic piezoelectric body. *Adv. Math. Sci. Appl.* **14** (2004) 613–631.

[20] M. Sofonea, C. P. Niculescu and A.Matei. An antiplane contact problem for viscoelastic materials with long-term memory. *Math. Model. Anal.* **11** (2006) 213–228.