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A Dynamic Contact Problem for Piezo-Thermo-Elastic-Viscoplastic Materials with Damage

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Abstract: We consider a dynamic contact problem between a piezo-thermo-elastic-viscoplastic material with damage and a rigid obstacle. The contact is frictional and bilateral, the friction is modeled by Coulomb's law with heat exchange. We employ the electro-elastic-viscoplastic with damage constitutive law for the material. The evolution of the damage is described by an inclusion of parabolic type. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and a fixed point argument.

Keywords: *viscoplastic; piezoelectric; temperature; damage; variational inequality; fixed point.*

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1 Introduction

Because of its considerable impact in everyday life and its multiple open problems, contact mechanics still remains a rich and fascinating domain of challenge. The literature devoted to various aspects of the subject is considerable, it concerns the modelling, the mathematical analysis as well as the numerical approximation of the related problems. For example, many food materials used in process engineering are elastic-viscoplastic [14] and consequently, mathematical models can be very helpful in understanding various problems related to the product development, packing, transport, shelf life testing, thermal effects, and heat transfer. It is thus important to study mathematical models that

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can be used to describe the dynamical behavior of a given elastic-viscoplastic material subjected to various highly nonlinear and even non-smooth phenomena like contact, friction, lubrication, adhesion, wear, damage, electrical and thermal effects. The uncoupled thermo-viscoplastic models were obtained in [13]. Different models have been developed to describe the interaction between the thermal and mechanical field [6]. The new papers use several types of contact for coupled materials such as thermo-mechanical, electro-mechanical and thermo-electromechanical materials. For the thermo-mechanical materials, a transmission problem in thermo-viscoplasticity is studied in [11], a thermo-viscoelastic body is considered in [5], several problems for thermo-elastic-viscoplastic materials are studied in [6–8]. For the electro-mechanical bodies, many laws of behavior are considered by many authors, see for example [1, 2, 9, 12] and references therein.

Realistically, it may be impossible to predict the electro-mechanical behaviour without thermal considerations. To achieve this, the authors have started to study a new model for thermo-electro-mechanical behaviour, see for example [4]. The aim of this paper is to study a frictionless contact problem for elastic-viscoplastic materials with piezoelectric effect, also called electro-elasto-viscoplastic materials. To this end, we consider that the material is electro-elasto-viscoplastic with an internal state variable α which may describe the damage of the system caused by elastic deformations and thermal effects. The main difficulty is that Korn's inequality cannot be applied any more. For this proposal, following the technique already developed by Duvaut and Lions [10] for Coulomb's friction models, we use the inertial term of the dynamic process to compensate the loss of coerciveness in the a priori estimates. By the change of variable, we bring the coupled second order evolution inequality into a classical first order evolution inequality. After this, we use classical results on first order evolution nonlinear inequalities, a parabolic variational inequality and equations and the fixed point arguments. Existence and uniqueness results for the boundary value problem for thermo-electro-viscoelastic materials were obtained by many authors using different functional methods. The novelty in this paper is to make the coupling of an electro-elasto-viscoplastic problem with damage and thermal effect. We employ the thermo-elastic-viscoplastic with damage constitutive law for the material. The damage of the material is caused by elastic deformations. The evolution of the damage is described by an inclusion of parabolic type. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement, a parabolic variational inequality for the damage and the heat equation for the temperature. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. A new law of behaviour for the so-called thermo-electro-elastic-viscoplastic material is given by

$$\sigma(t)=\mathcal{A}(\varepsilon(\dot{u}(t)))+\mathcal{B}(\varepsilon u(t),\alpha(t))+\int_0^t \mathcal{G}\left(\sigma(s)-\mathcal{A}(\varepsilon(\dot{u}(t))),\varepsilon\left(u\left(s\right)\right)\right)ds+\mathcal{E}^*\nabla\varphi\left(t\right)-\mathcal{M}\theta\left(t\right), \tag{1}$$

$$D\left(t\right)=\mathcal{E}\varepsilon\left(u\left(t\right)\right)-\mathcal{B}\nabla\left(\varphi\left(t\right)\right)-\mathcal{P}\theta\left(t\right), \tag{2}$$

where $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, $\mathcal{G}$, $E(\varphi)=-\nabla\varphi$, $\mathcal{E}=(e_{ijk})$, $\mathcal{M}$, $\mathcal{B}$, and $\mathcal{P}$ are the relaxation operator, electric field, piezoelectric, thermal expansion, electric permittivity and pyroelectric tensors. $\mathcal{E}^*$ is the transpose of $\mathcal{E}$.

Many types of evolution of the temperature field are given by several authors, see for example [4, 5, 8]. In this paper, we use the evolution of the temperature field obtained from the conservation of energy and define it with the following differential equation:

$$\dot{\theta}(t) - \operatorname{div}(\mathcal{K}\nabla\theta(t)) = -\mathcal{M}\nabla\dot{\mathbf{u}}(t) + q,$$

where θ is the temperature, $\mathcal{K}$ denotes the thermal conductivity tensor, $\mathcal{M}$ is the thermal expansion tensor, q is the density of volume heat sources and ψ is a nonlinear function assumed here to depend on the thermal expansion tensor and the velocity.

The differential inclusion used for the evolution of the damage field is

$$\dot{\alpha} - k_1\triangle\alpha + \partial_{\varphi}(\alpha) \ni \Phi(\varepsilon(u), \alpha), \text{ in } \Omega \times (0,T), \tag{3}$$

where $\varphi_F(\alpha)$ denotes the sub-differential of the indicator function of the set F of an admissible damage function given as follows:

$$F = \{\alpha \in H^1(\Omega) : 0 \geq \alpha \geq 1, \text{ a.e. in } \Omega\}$$

and Φ are given constitutive functions which describe the sources of the damage in the system. When $\alpha = 0$, the material is completely damaged, when $\alpha = 1$, the material is undamaged, and for $0 < \alpha < 1$, there is partial damage. The Coulomb friction is one of the useful friction laws known from the literature. This law has two basic ingredients, namely, the concept of friction threshold and its dependence on the normal stress. Various versions of the normal compliance law were recently presented in the literature [1, 2, 12]. The paper is organized as follows. In Section 2, we present the model. In Section 3, we introduce the notations, some preliminary results, a list of the assumptions on the data and we give the variational formulation of the problem. In Section 4, we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on evolutionary elliptic variational inequalities, ordinary differential equations and fixed point arguments.

2 The Model

The physical setting is the following. A thermo-electro- elastic-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with the outer Lipschitz surface Γ. This boundary is divided into three open disjoints Γ_1, Γ_2 and Γ_3, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $\operatorname{meas}(\Gamma_1) > 0$ and $\operatorname{meas}(\Gamma_a) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is subjected to the action of body forces of density f_0, a volume electric charges of density q_0 and a heat source of constant strength q.

The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface traction of density f_2 acts on $\Gamma_2 \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_b is prescribed on $\Gamma_b \times (0, T)$. Moreover, we suppose that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0, T)$. In the reference configuration, the body is in contact with an obstacle, or foundation, over the contact surface Γ_3. The contact is frictional and thermo-mechanical. The model of the contact is specified by the normal compliance and it is associated with Coulomb's law of dry friction for the mechanical contact and by an associated temperature boundary condition for the thermal contact.

The classical formulation of the mechanical problem is as follows.

Problem $\mathcal{P}$. Find the displacement field $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma}$: $\Omega \times [0, T] \to \mathbb{S}^d$, the electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$, the electric displacement field

$\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$, the temperature field $\theta : \Omega \times [0,T] \to \mathbb{R}$ and the damage field $\alpha : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t)))+\mathcal{B}(\varepsilon u(s),\alpha(t))+\int_0^t \mathcal{G}\left(\sigma(s)-\mathcal{A}(\varepsilon(\dot{u}(t))),\varepsilon\left(u\left(s\right)\right)\right) ds+\mathcal{E}^*\nabla\varphi\left(t\right)-\mathcal{M}\theta\left(t\right), \tag{4}$$

$$\mathbf{D}\left(t\right) = \mathcal{E}\epsilon\left(u\left(t\right)\right) - \mathcal{B}\nabla\left(\varphi\left(t\right)\right) - \mathcal{P}\theta\left(t\right), \tag{5}$$

$$\dot{\theta} - \operatorname{div}\left(K\nabla\theta\right) = -M\nabla\dot{u} + q \quad \text{in } \Omega \times (0,T), \tag{6}$$

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}_0 = \rho\ddot{u} \ \text{ in } \Omega \times (0,T), \tag{7}$$

$$\dot{\alpha} - K\triangle\alpha + \partial\varphi_K(\alpha) \ni \Phi(\varepsilon(u) - \alpha) \text{ in } \Omega \times (0,T), \tag{8}$$

$$\operatorname{div}\mathbf{D} - q_0 = 0 \ \text{ in } \Omega \times (0,T), \tag{9}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{10}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{11}$$

$$\sigma_\tau = p_r\left(u_\nu - h\right) on\ \Gamma_3 \times (0,T), \tag{12}$$

$$\begin{cases} \|\sigma_\tau\| \le \mu p\|R\sigma_\nu\|, \\ \|\sigma_\tau\| < \mu p\|R\sigma_\nu\| \Longrightarrow \dot{\mathbf{u}}_\tau = 0, \\ \|\sigma_\tau\| = \mu p\|R\sigma_\nu\| \Longrightarrow \exists\ \lambda > 0 : \quad \sigma_\tau = -\lambda\ \dot{\mathbf{u}}_\tau \text{ on } \ \Gamma_3 \times (0,T), \end{cases} \tag{13}$$

$$- K_{ij}\frac{\partial\theta}{\partial v}\nu_j = K_e(\theta - \theta_R) - h_\tau(|\dot{u}_\tau|) \quad \text{on } \Gamma_3 \times [0,T], \tag{14}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \text{ on } \Gamma \times (0,T), \tag{15}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = \mathbf{0} \ \text{ on } \ \Gamma_3 \times (0,T), \tag{16}$$

$$\theta = 0 \ \text{ on } \quad (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{17}$$

$$\varphi = 0 \quad \text{ on } \quad \Gamma_a \times (0,T), \tag{18}$$

$$\mathbf{D}\cdot\,\boldsymbol{\nu} = q_b \ \text{ on } \ \Gamma_b \times (0,T), \tag{19}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = \psi\left(u_\nu - h\right)\phi_L\left(\varphi - \varphi_0\right) \quad \text{ on } \Gamma_3 \times (0,T), \tag{20}$$

$$u(0) = u_0,\ \dot{u}(0) = v_0,\ \alpha(0) = \alpha_0 \text{ and } \theta(0) = \theta_0 \text{ in } \Omega. \tag{21}$$

We now describe problem (4)-(21) and provide the explanation of the equations and the boundary conditions. Equations (4) and (5) represent the thermo-electro-elastic-viscoplastic constitutive law, the evolution of the temperature field is governed by a differential equation given by the relation (6), assumed to be a rather general function of the strains. Next equations (20) and (9) are the steady equations for the stress and electric-displacement field, conditions (10) and (11) are the displacement and traction boundary conditions. Equation (17) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0,T)$ which implies that there is only an electro-mechanical effect on $(\Gamma_1 \cup \Gamma_2)$.

Next, (18) and (19) represent the electric boundary conditions for the electrical potential on Γ_a, and the electric charges on Γ_b, respectively. We use (19) as the electrical contact condition on Γ_3 which represents a regularized condition. Equation (20) represents the initial displacement field and the initial damage field, where u_0 is the initial displacement, and θ_0 is the initial temperature.

We turn to the contact conditions (12)-(14) and describe the frictional thermo-mechanical contact on the potential contact surface Γ_3. The relation (12) describes a normal compliance conditions with Coulomb's law. The equation (14) represents an associated temperature boundary condition on the contact surface. The equation (16) shows that there are no electric charges on the contact surface. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L \quad \text{if} \quad s < -L, \\ -s \quad \text{if} \quad -L \leq s \leq 0, \\ 0 \quad \text{if} \quad s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator R_τ defined below, is motivated by mathematical arguments but it is not restrictive from a physical point of view since no restriction on the size of the parameter L is made in what follows, where $u^1_\tau - u^2_\tau$ stands for the jump of the displacements in the tangential direction. R_ν is the truncation operator given by

$$R_\nu(s) = \begin{cases} v \quad \text{if} \quad |v| \leq L, \\ L\frac{v}{|v|} \quad \text{if} \quad |v| > L. \end{cases}$$

3 Variational Formulation

In order to obtain the variational formulation of the Problem $\mathcal{P}$, we use the following notations and preliminaries

3.1 Notations and preliminaries.

In this short section, we recall some preliminary material and notations. For more details, we refer the reader to [7, 10]. The indices i, j, k and l run from 1 to d and summation over repeated indices is implied. An index that follows the comma represents the partial derivative with respect to the corresponding component of the spatial variable. We also use the following notations:

$$H = L^2(\Omega)^d = \{\ \mathbf{u} = (u_i) : u_i \in L^2(\Omega)\}, \quad \mathcal{H} = \{\ \sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H^1(\Omega)^d = \{\ \mathbf{u} = (v_i) \in H :\ \varepsilon(\mathbf{u}) \in \mathcal{H}\ \}, \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H}\text{:}\ \ \operatorname{Div}\sigma \in H\}.$$

The operators of deformation ε and Div are defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = (u_{i,j} + u_{j,i})/2, \quad \operatorname{Div}\sigma = (\sigma_{ij,j}) \cdot$$

The associated norms on spaces H, $H^1(\Omega)^d$, $\mathcal{H}$, and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$, and $\|\cdot\|_{\mathcal{H}_1}$ respectively. Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and $\gamma : H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $v \in H^1(\Omega)^d$, we also use the notation v to denote the trace γv of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on Γ . Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to represent its positive part, that is, $r_+ = \max(0, r)$. To obtain the variational formulation of the problem (4)-(21), we introduce, for the bonding field, the sets

$$W = \left\{\phi \in H^1(\Omega)^d : \phi = 0 \text{ on } \ \Gamma_a\right\}, \quad \mathcal{W} = \left\{D = (D_i) :\ D_i \in L^2(\Omega), \operatorname{div}D\ \in L^2(\Omega)\right\}.$$

On the spaces V, W, $\mathcal{W}$, we define the following inner products:

$$(\mathrm{u}\cdot\mathrm{v})_V = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \forall u, v \in V, \tag{22}$$

$$(\varphi, \phi)_W = (\nabla_\varphi, \nabla_\phi)_{\mathcal{W}}, \forall \varphi, \phi \in W, \tag{23}$$

$$(w, z)_E = (\nabla_w, \nabla_z)_{\mathcal{H}}, \forall w, z \in E, \tag{24}$$

where $E = \left\{\gamma \in H^1(\Omega) : \gamma = 0 \text{ a.e. on } \Gamma_1 \cup \Gamma_2 \right\}$.

Therefore, the spaces $(V, (\cdot,\cdot)_V)$, $(W\ ,(\cdot,\cdot)_W)$ and $(E, (\cdot,\cdot)_E)$ are real Hilbert spaces.

3.2 Assumptions on the data

We now list the assumptions on the problem's data.

The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l}
\text{(a) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\
\|\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\| \leqslant L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\| \quad \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{a.e. } x \in \Omega\ , \\
\text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\
(\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geqslant m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2, \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega\ , \\
\text{(c) The mapping } x \longrightarrow \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega\ , \forall\ \varepsilon \in \mathbb{S}^d, \\
\text{(d) The mapping } x \longrightarrow \mathcal{A}(x,\varepsilon) \text{ belongs to } \mathcal{H}.
\end{array}\right. \tag{25}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l}
\text{(a) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\
\|\mathcal{B}(x,\varepsilon_1) - \mathcal{B}(x,\varepsilon_2)\| \leqslant L_{\mathcal{B}} \|\varepsilon_1 - \varepsilon_2\| \quad \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega, \\
\text{(b) The mapping } x \longrightarrow \mathcal{B}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega\ , \forall\ \varepsilon \in \mathbb{S}^d, \\
\text{(c) The mapping } x \longrightarrow \mathcal{B}(x,0) \text{ belongs to } \mathcal{H}.
\end{array}\right. \tag{26}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\left\{\begin{array}{l}
\text{(a) There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\
\|\mathcal{G}(x,\sigma_1,\varepsilon_1) - \mathcal{G}(x,\sigma_2,\varepsilon_2)\| \leqslant L_{\mathcal{G}} (\|\sigma_1 - \sigma_2\| + \|\varepsilon_1 - \varepsilon_2\|)\ , \\
\text{for all } \sigma_1,\ \sigma_2,\ \varepsilon_1,\ \varepsilon_2 \in \mathbb{S}^d \text{ , a.e. } x \in \Omega, \\
\text{(b) The mapping } x \longrightarrow \mathcal{G}(x,\sigma,\varepsilon) \text{ is Lebesgue measurable on } \Omega\ , \forall\ \varepsilon \in \mathbb{S}^d, \\
\text{for any } \varepsilon, \sigma \in \mathbb{S}^d, \\
\text{(c) The mapping } x \longrightarrow \mathcal{G}(x,0,0) \in \mathcal{H}.
\end{array}\right. \tag{27}$$

The piezoelectric operator $\mathcal{E}$: $\Omega \times \mathbb{S}^d \longrightarrow \mathbb{R}^d$ satisfies

$$\left\{\begin{array}{l}
\text{(a) } \mathcal{E}(x,\tau) = (e_{ijk}\ , \tau_{jk})\ , \ \forall \tau = (\tau_{jk}) \in \mathbb{S}^d, \text{ a.e. } x \text{ in } \Omega\ , \\
\text{(b) } e_{ijk} = e_{ikj} \in L^\infty(\Omega)\ , \ 1 \leqslant i,j,k \leqslant d.
\end{array}\right. \tag{28}$$

The thermal expansion operator $\mathcal{M}$:$\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\left\{\begin{array}{l}
\text{(a) There exists a constant } L_{\mathcal{M}} > 0 \text{ such that} \\
\|\mathcal{M}(x,\theta_1) - \mathcal{M}(x,\theta_2)\| \leqslant L_{\mathcal{M}} \|\theta_1 - \theta_2\| \quad \forall\ \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega\ , \\
\text{(b) The mapping } x \longrightarrow \mathcal{M}(x,\theta) \text{ is Lebesgue measurable on } \Omega, \forall\ \theta \in \mathbb{R}, \\
\text{(c) The mapping } x \longrightarrow \mathcal{M}(x,0) \in \mathcal{H}.
\end{array}\right. \tag{29}$$

The tangential function satisfies:

$$\begin{cases} h_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ verifies:} \\ (a) : \exists \;\; L_\tau > 0 \text{ s.t. } \; |\, h_\tau(x, r_1 - h_\tau(x, r_2)\,| \leq L^\tau \,|\, r_1 - r_2 \,|, \\ \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3. \\ (b) : \text{The mapping } \mathbf{x} \mapsto h_\tau(x, r) \text{ belongs to } L^2(\Gamma_3). \end{cases} \tag{30}$$

The electric permittivity operator $B = (Bij) : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a)} \;\; B\,(x, E) = (Bij\,(x)\,E_j) \;\forall\; E = (E_i) \in \mathbb{R}^d \;\;, \text{ a.e. } x \in \Omega, \\ \text{(b) } Bij = Bji \in \;\; L^\infty\,(\Omega)\,, \quad 1 \;\leqslant i, j \;\leqslant d, \\ \text{(c) There exists a constant } \;M_{\mathcal{B}} > 0 \;\text{ such that } BE.E \geqslant M_{\mathcal{B}}\,|E|^2\,, \\ \;\forall\; E = (E_i) \in \mathbb{R}^d \;\;, \text{ a.e. in } \Omega. \end{cases} \tag{31}$$

The thermal conductivity operator $\mathcal{K} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } \;L_{\mathcal{K}} > 0 \text{ such that} \\ \;\; \|\mathcal{K}\,(x, r_1) - \mathcal{K}\,(x, r_2)\| \leqslant L_{\mathcal{K}}\,\|r_1 - r_2\| \quad \text{ for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega, \\ \text{(b) } m_{ij} = m_{ji} \in \;\; L^\infty\,(\Omega)\,, \quad 1 \;\leqslant i, j \;\leqslant d, \\ \text{(c) The mapping } x \longrightarrow S\,(x, 0, 0) \text{ belongs to } L^2\,(\Omega)\,. \end{cases} \tag{32}$$

The damage source function $\Phi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_\Phi > 0 \text{ such that} \\ |\Phi(x, \eta_1, \omega_1, \beta_1) - \Phi(x, \eta_2, \omega_2, \beta_2)| \leq L_\Phi(|\eta_1 - \eta_2| + |\omega_1 - \omega_2| + |\beta_1 - \beta_2|) \\ \text{for all } \eta_1, \; \eta_2, \; \omega_1, \; \omega_2 \in \mathbb{S}^d, \; \beta_1, \beta_2 \in \mathbb{R}, \; x \in \Omega, \\ \text{(b) The mapping } x \longrightarrow \Phi(x, \eta, \omega, \beta) \text{ is Lebesgue measurable on } \Omega, \\ \text{for any } \eta, \; \omega \in \mathbb{S}^d \text{ and for all } \beta \in \mathbb{R}, \\ \text{(c) The mapping } x \longrightarrow \Phi(x, 0, 0, 0) \text{ belongs to } \mathbb{L}^2(\Omega). \end{cases} \tag{33}$$

The function $\Psi : \varepsilon \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_\Psi > 0 \text{ such that} \\ |\Psi(x, \sigma_1, \varepsilon_1, \theta_1, \xi_1) - \Psi(x, \sigma_2, \varepsilon_2, \theta_2, \xi_2)| \leq L_\Psi(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| \\ \;+|\theta_1 - \theta_2| + |\xi_1 - \xi_2|), \text{ for all } \sigma_1, \; \sigma_2, \; \varepsilon_1, \; \varepsilon_2 \in \mathbb{S}_n, \; \theta_1, \theta_2, \xi_1, \xi_2 \in \mathbb{R}, \; x \in \Omega, \\ \text{(b) The mapping } x \longrightarrow \Psi(x, \sigma, \varepsilon, \theta, \xi) \text{ is Lebesgue measurable on } \Omega, \\ \text{for all } \sigma, \; \varepsilon \in \mathbb{S}_n \text{ and for all } \theta, \xi \in \mathbb{R}, \\ \text{(c) The mapping } x \longrightarrow \Psi(x, 0, 0, 0) \text{ belongs to } \mathbb{L}^2(\varepsilon). \end{cases} \tag{34}$$

We also suppose that the body forces and surface tractions have the regularity

$$f_0 \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)), f_2 \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)), \rho \in \mathbb{L}^\infty(\Omega), \tag{35}$$

$$q_0 \in C\left(0, T, L^2\,(\Omega)\right), \qquad q_2 \in C\left(0, T, L^2\,(\Gamma_b)\right), \tag{36}$$

$$q_2\,(t) = 0 \text{ on } \Gamma_3, \forall t \in [0, T]\,. \tag{37}$$

The functions g and μ have the following properties:

$$g \in L^2\,(\Gamma_3)\,, \quad g\,(x) \geqslant 0, \;\; a.e. \text{ on } \;\Gamma_3, \tag{38}$$

$$\mu \in L^\infty\,(\Gamma_3)\,, \quad \mu\,(x) > 0, \quad a.e. \text{ on } \Gamma_3, \tag{39}$$

here μ is the coefficient of friction. The initial displacement field satisfies

$$u_0 \in V, \tag{40}$$

and the initial temperature field satisfies

$$\theta_0 \in E, \quad \theta_F \in L^2\left(0, T, L^2\left(\Gamma_3\right)\right), k_e \in L^\infty\left(\Omega, \mathbb{R}_+\right), q_{th} \in L^2\left(0, T, E^{'}\right). \tag{41}$$

Using the above notation and Green's formulas, we obtain the variational formulation of the mechanical problem (4)-(21) for all functions $v \in V,\ w \in W_{th},\ \phi \in W_e$ and a.e. $t \in (0, T)$, given as follows.

Problem $\mathcal{PV}$. Find the displacement $\mathbf{u} : [0, T] \to V$, the stress $\sigma : [0, T] \to \mathcal{H}_1$, and an electric potential $\varphi : [0, T] \to W$, the electric displacement $D : [0, T] \longrightarrow H$ and the temperature $\theta : [0, T] \longrightarrow V$, and the damage $\alpha : [0, T] \longrightarrow E_1$ such that

$$\begin{aligned}\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{B}(u(t), \alpha(t)) + \int_0^t \mathcal{G}\left(\sigma(s) - \mathcal{A}(\varepsilon(\dot{u}(t))), \varepsilon\left(u\left(s\right)\right)\right) ds \\ + \mathcal{E}^* \nabla \varphi\left(t\right) - \mathcal{M}\theta\left(t\right),\end{aligned} \tag{42}$$

$$(\ddot{u}(t), v - \dot{u}(t))_{V' \times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}\ (\mathbf{v}(\mathbf{t}) - \dot{\mathbf{u}}(t))_{\mathcal{H}} + j(\mathbf{v}(t)) - j(\dot{\mathbf{u}}(t)) \geqslant (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \tag{43}$$

$$(\dot{\alpha}(t), \zeta - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \zeta - \alpha(t)) \geq (\Phi(\varepsilon(u(t))), \alpha(t), \zeta - \alpha(t))_{L^2(\Omega)}, \tag{44}$$

for all $\alpha(t) \in F$, $\zeta \in F$ and $t \in [0, T]$.

$$D\left(t\right) = \mathcal{E}\varepsilon\left(u\left(t\right)\right) - \mathcal{B}\nabla\left(\varphi\left(t\right)\right) - \mathcal{P}\theta\left(t\right), \tag{45}$$

$$\left(D\left(t\right), \nabla_\phi\right)_H = -\left(q_e\left(t\right), \phi\right)_W + \left(h\left(u\left(t\right), \varphi\right), \phi\right)_W, \ \forall \varphi \in W, \quad t \in [0, T], \tag{46}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t) \quad \text{ on } E^{'}, \tag{47}$$

$$u(0) = u_0,\ \dot{u}(0) = v_0,\ \alpha(0) = \alpha_0 \text{ and } \theta(0) = \theta_0 \text{ on } \Omega. \tag{48}$$

Here, the function $Q : [0, T] \to E'$ and the operators $K : E \to E',\ R : V \to\ E';\ M : E \to V'$ are defined by $\forall v \in V$, $\forall \tau \in E$, $\forall \eta \in E$:

$$\begin{aligned}\langle Q(t), \eta\rangle_{E' \times E} &= \int_{\Gamma_3} k_e \theta_R \eta\, ds + \int_\Omega q\eta\, dx, \\ \langle K\tau, \eta\rangle_{E' \times E} &= \sum_{i,j=1}^d \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \eta}{\partial x_i}\, dx + \int_{\Gamma_3} k_e \tau\eta\, ds, \\ \langle Rv, \eta\rangle_{E' \times E} &= \int_{\Gamma_3} h_\tau(|v_\tau|)\eta\, ds - \int_\Omega (M_e \nabla v)\eta\, dx, \\ \langle M\tau, v\rangle_{V' \times V} &= (-\tau M_e, \varepsilon(v))_{\mathcal{H}},\end{aligned}$$

where $J_\varepsilon : V \times V \to \mathbb{R}$, $f : [0; T] \to V$, $q_e : [0; T] \to W$ and $\gamma : V \times W \to W$ are respectively defined by

$$J\varepsilon\ (N, v) = \int_{\Gamma_3} \mu p \left|R \times N_\nu\right| \sqrt{|v_\tau|^2 + \varepsilon^2} da, \quad \forall v \in V, \quad \forall \varepsilon > 0, \tag{49}$$

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v}\, da. \tag{50}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$

$$a(\alpha, \zeta) = \kappa \int_{\Omega} \nabla\alpha \cdot \nabla\zeta \, dx, \tag{51}$$

$$(q_e(t), \phi)_W = \int_{\Omega} q_0(t)\phi dx - \int_{\Gamma_b} q_2(t)\phi \, da, \tag{52}$$

$$(\gamma (u, \varphi), \phi)_W = \int_{\Gamma_3} \psi (u_\nu - h) \phi_L(\varphi - \varphi_0)\phi \, da \tag{53}$$

for all u, $v \in V$, θ, $w \in W$, $\phi \in W$ and $t \in [0; T]$. We note that the definitions of f and q_e are based on the Riesz representation theorem. Moreover, the conditions (35) and (36) imply that

$$f \in C(0, T, V), \quad q_e \in C(0, T, We). \tag{54}$$

We denote by $\|.\|_V$, $\|.\|_H$ and $.\,\|.\|_{V'}$ the norms on the spaces V, H and V', respectively, and we use $(.,.)_{V' \times V}$ for the duality pairing between V' and V. Note that if $f \in H$, then

$$(f, v)_{V' \times V} = (f, v)_H, \forall v \in H. \tag{55}$$

The existence of the unique solution of problem $\mathcal{P}_V$ is stated and proved in the next section.

4 Existence and Uniqueness of the Solution

Our main existence and uniqueness result is the following.

Theorem 4.1 *Assume that (25)-(41) hold. Then, if $N_\psi < \frac{m_\beta}{a_0^2}$, there exists a unique solution $\{u, \sigma, \theta, \varphi, D\}$ to problem $\mathcal{P}_V$ satisfying*

$$u \in W^{1,2}(0; T; V) \cap C^1(0; T; V) \cap W^{2,2}(0; T; V'), \sigma \in C(0; T; \mathcal{H}), \tag{56}$$

$$\varphi \in C(0; T; W), D \in C(0; T; \mathcal{W}), \tag{57}$$

$$\theta \in W^{1,2}(0; T; E') \cap L^2(0; T; E) \cap C(0; T; L^2(\Omega)). \tag{58}$$

$$\alpha \in W^{1,2}\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^1(\Omega)\right). \tag{59}$$

Functions u, σ, θ, φ, D, θ and α, which satisfy (42)-(48), are called the weak solution to the contact problem $\mathcal{P}$. We conclude that, under the assumptions (25)-(40) and if $N_\psi < \frac{m_\beta}{a_0^2}$ is satisfied, the mechanical problem (4)-(21) has a unique weak solution satisfying (56)-(58).

The proof of Theorem 4.1 is carried out in several steps. It is based on the results of evolutionary variational inequalities, ordinary differential equations and fixed point arguments.

In the first step, we let $\eta \in \mathbb{L}^2(0, T; V)$ be given and consider the following variational inequality.

Problem $\mathcal{PV}u_\eta$. Find a displacement field $u_\eta : [0; T] \to V$ such that $\forall\, t \in [0, T]$,

$$\begin{aligned}(\ddot{u}(t), v - \dot{u}(t))_{V' \times V} + (\mathcal{A}\varepsilon\left(\dot{u_\eta}(t)\right), \varepsilon(v - \dot{u_\eta}(t)))_{\mathcal{H}} + j(u_\eta(t), v) - j(u_\eta(t), \dot{u}_\eta(t)) \\ + \left(\eta(t), v - \dot{u_\eta}(t)\right)_V \geq (f(t), v - \dot{u_\eta}(t))_V,\end{aligned} \tag{60}$$

$u_\eta(0) = u_0, \dot{u}_\eta(0) = v_0$ for all u_η, $v \in V$. In the study of the problem $\mathcal{PV}u_\eta$, we have the following result.

Lemma 4.1 *$\mathcal{PV}u_\eta$ has a unique solution satisfying the regularity expressed in* (56)*:*

$$u_\eta(t) = u_0 + \int_0^t v_{\eta g_\eta}(s)ds \quad \forall t \in [0, T].$$

We define the operator $A : V \to V'$ by

$$(Av, w)_{V' \times V} = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{H}}, \quad \forall v, w \in V. \tag{61}$$

We consider the following variational inequality.
Problem $\mathcal{PV}v_\eta$. Find a displacement field $v_\eta : [0; T] \times \Omega \to V$ such that $\forall\, t \in [0, T]$.

$$\begin{aligned}&(\dot{v}_{N\eta}(t), w - v_{N\eta}(t))_{V' \times V} + (Av_{N\eta}(t)), w - v_{N\eta}(t))_{V' \times V} + j(N, w)\\ &-j(N, v_{N\eta}(t)) \geq (f_\eta(t), w - v_{N\eta}(t))_{V' \times V}, \ \forall w \in V,\end{aligned} \tag{62}$$

$$v_{N\eta}(0) = v_o. \tag{63}$$

In the study of **Problem** $\mathcal{PV}v_\eta$, we have the following result.

Lemma 4.2 *For all $N \in \mathbb{L}^2(0, T, \mathcal{H}_1)$ and $\eta \in \mathbb{L}^2(0, T, V')$, the* **Problem** *$\mathcal{PV}v_\eta$ has a unique solution with the regularity $v_{N\eta} \in C(0, T, H) \cap \mathbb{L}^2(0, T, V) \cap W^{1,2}(0, T, V')$.*

Proof. We begin by the step of regularization we defined, for all $\varepsilon > 0$,

$$\dot{J}\varepsilon\ (N, v) = \int_{\Gamma_3} \mu p \left|R \times N_\nu\right| \sqrt{|v_\tau|^2 + \varepsilon^2} da, \quad \forall v \in V, \quad \forall \varepsilon > 0.$$

After some algebra, for all $\varepsilon > 0$, $\dot{J}_\varepsilon$ is C^1 convex on V, and its Frechet derivative satisfies

$$\forall c > 0, \ \forall w \in V \quad \left|\dot{J}'_\varepsilon(N, w)\right|_{V'} \leq C|N|_{\mathbb{L}^2(\Gamma_3)}. \tag{64}$$

From (25) and the monotonicity of $\dot{J}'_\varepsilon$, it follows from the classical first order evolution equation that

$$\forall \varepsilon > 0, \quad v^\varepsilon_{N\eta} \in \mathbb{L}^2(0, T, V) \cap C(0, T, H) \text{ and } \dot{v}^\varepsilon_{N\eta} \in \mathbb{L}^2(0, T, V')$$

such that

$$\left\{\begin{array}{l}\dot{v}^\varepsilon_{N\eta}(t) + Av^\varepsilon_{N\eta} + j'_\varepsilon(N, v^\varepsilon_{N\eta}) = f_\eta(t) \text{ in } V', \text{ a.e. } t \in [0, T],\\ v^\varepsilon_{N\eta}(0) = v_0.\end{array}\right. \tag{65}$$

Therefore, $v^\varepsilon_{N\eta} \in \mathbb{L}^2(0, T; V) \cap W^{1,2}(0, T; V')$, we obtain

$$\left\{\begin{array}{l}\left(\dot{v}^\varepsilon_{N\eta}(t), w - v^\varepsilon_{N\eta}\right)_{V' \times V} + \left(Av^\varepsilon_{N\eta}(t), w - v^\varepsilon_{N\eta}\right)_{V' \times V} + j_\varepsilon(N, w)\\ -j_\varepsilon(N, v^\varepsilon_{N\eta}(t)) \geq \left(f_\eta(t), w - v^\varepsilon_{N\eta}(t)\right)_{V' \times V} \forall w \in V, \text{ a.e. } t \in [0, T].\end{array}\right. \tag{66}$$

Using (25) and the monotony of j'_ε, we deduce that

$$\exists C > 0, \ \forall t \in [0, T] : \left|v^\varepsilon_{N\eta}(t)\right| \leq C \int_0^T \left|v^\varepsilon_{N\eta}(t)\right|^2_V dt \leq C \int_0^T \left|\dot{v}^\varepsilon_{N\eta}(t)\right|^2_{V'} dt \leq C,$$

using a subsequence to find that

$$\begin{cases} v^{\varepsilon}_{N\eta} \rightharpoonup v_{N\eta} \text{ weakly in } \mathbb{L}^2(0,T;V) \text{ and weakly in } \mathbb{L}^\infty(0,T;H), \\ \dot{v}^{\varepsilon}_{N\eta} \rightharpoonup \dot{v}_{N\eta} \text{ star weakly in } \mathbb{L}^2(0,T;V'). \end{cases} \tag{67}$$

It follows that

$$v_{N\eta} \in C(0,T;H) \text{ and } v^{\varepsilon}_{N\eta}(t) \rightharpoonup v_{N\eta}(t) \text{ weakly in } H,\ \forall t \in [0,T]. \tag{68}$$

Integrating (66), we have $\forall w \in \mathbb{L}^2(0,T;V)$,

$$\int_0^T (\dot{v}^{\varepsilon}_{N\eta}(t), w)_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{N\eta}(t), w)_{V'\times V} dt + \int_0^T j_\varepsilon(N,w) dt \geq \int_0^T (f_\eta(t), w(t))_{V'\times V} dt,$$

then we have

$$\begin{array}{l} \int_0^T (\dot{v}^{\varepsilon}_{N\eta}(t), w)_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{N\eta}(t), w)_{V'\times V} dt + \int_0^T j_\varepsilon(N,w) dt \\ \geq \int_0^T (\dot{v}^{\varepsilon}_{N\eta}(t), v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{N\eta}(t), v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \\ \int_0^T j_\varepsilon(N, v^{\varepsilon}_{N\eta}(t)) dt + \int_0^T (f_\eta(t), w(t) - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt \\ \geq \frac{1}{2} \left| v^{\varepsilon}_{N\eta}(t) \right|_H^2 - \frac{1}{2} \left| v^{\varepsilon}_{N\eta}(0) \right|_H^2 + \int_0^T (Av^{\varepsilon}_{N\eta}(t), v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \\ \int_0^T j_\varepsilon(v^{\varepsilon}_{N\eta}(t)) dt + \int_0^T (f_\eta(t), w(t) - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt. \end{array}$$

From (67) and (68) we obtain that for all $w \in \mathbb{L}^2(0,T;V)$,

$$\begin{array}{l} \int_0^T (\dot{v}^{\varepsilon}_{N\eta}(t), w - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{N\eta}(t), w - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \\ \int_0^T (j(N,w) - j(N, v_{N\eta}))\, dt \geq \int_0^T (f_\eta(t), w(t) - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt. \end{array}$$

The previous inequality implies (see [10]) that

$$\begin{array}{l} \left(\dot{v}^{\varepsilon}_{N\eta}(t), w - v^{\varepsilon}_{N\eta}\right)_{V'\times V} + \left(Av^{\varepsilon}_{N\eta}(t), w - v^{\varepsilon}_{N\eta}\right)_{V'\times V} + j_\varepsilon(N,w) \\ -j_\varepsilon(N, v^{\varepsilon}_{N\eta}(t)) \geq \left(f_\eta(t), w - v^{\varepsilon}_{N\eta}(t)\right)_{V'\times V} \forall w \in V,\ t \in [0,T]. \end{array}$$

We conclude that **Problem** $\mathcal{PV}v_\eta$ has at least a solution $v_{N\eta} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \cap W^{1,2}(0,T;V')$ and $\dot{v}_{N\eta} \in \mathbb{L}^2(0,T;V')$. For the uniqueness, let $v^1_{N\eta}$, $v^2_{N\eta}$ be two solutions of **Problem** $\mathcal{PV}v_\eta$, we use (62) to obtain for a.e. $t \in [0,T]$,

$$(\dot{v}^2_{N\eta}(t) - \dot{v}^1_{N\eta}(t), v^2_{N\eta}(t) - v^1_{N\eta}(t)) - (Av^2_{N\eta}(t) - Av^1_{N\eta}(t), v^2_{N\eta}(t) - v^1_{N\eta}(t)) \leq 0.$$

Integrating the previous inequality, using (25) and (61), we find

$$\frac{1}{2}\left|v^2_{N\eta}(t) - v^1_{N\eta}(0)\right|_V^2 + m_{\mathcal{A}} \int_0^T \left|v^2_{N\eta}(s) - v^1_{N\eta}(s)\right|_V^2 ds \leq 0, \quad \forall t \in [0,T],$$

which implies $v^1_{N\eta} = v^2_{N\eta}$. Let us consider now $u_{N\eta} : [0,T] \to V$ is the function defined by

$$u_{N\eta}(t) = \int_0^T v_{N\eta}(s) ds + u_0, \quad \forall t \in [0,T]. \tag{69}$$

In the study of **Problem** $\mathcal{PV}u_\eta$, we have the following result.

Lemma 4.3 **Problem** $\mathcal{PV}u_\eta$ *has a unique solution satisfying the regularity expressed in* (56)*.*

Proof. The proof of Lemma 4.3 is a consequence of Lemma 4.2 together with (69). In the second step, let $\eta \in C(0,T;V)$, we use the displacement field u_η obtained in Lemma 4.1 and consider the following variational problem.
Problem $\mathcal{PV}\varphi_\eta$. Find an electrical potential $\varphi_\eta : [0;T] \to W$ such that $\forall t \in [0,T]$,

$$(B\ \nabla\varphi_\eta\,(t)\,, \nabla_\phi) - (\mathcal{E}\ \varepsilon\,(u_\eta\,(t))\,, \nabla\phi)_H + (\gamma\,(u_\eta\,(t)\,, \varphi_\eta\,(t))\,, \phi)_W = (q_e\,(t)\,, \phi)\,w.\ \forall\ \phi \in W. \tag{70}$$

We have the following result.

Lemma 4.4 *$\mathcal{PV}\varphi_\eta$ has a unique solution φ_η which satisfies the regularity expressed in (57). Moreover, if φ_{η_1} and φ_{η_2} are solutions of (70) corresponding to $\eta_1, \eta_2 \in \mathbb{L}^2(0,T;V)$, then there exists $C > 0$ such that*

$$|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)|_W \le C\,|u_{\eta_1}\,(t) - u_{\eta_2}\,(t)|_V\,, \forall t \in [0,T]\,. \tag{71}$$

Proof. The same result for this Lemma 4.4 is given in [12]. In the third step, we let $\lambda \in \mathbb{L}^2\left(0,T;\mathbb{L}^2(\Omega)\right)$ be given and consider the following variational problem for the temperature field.
Problem $\mathcal{PV}\theta_\lambda$. Find a temperature field $\theta_\lambda : [0,T] \longrightarrow E$ such that

$$\begin{cases} \dot\theta_\lambda(t) + K\theta_\lambda(t) = R\dot u_\eta(t) + Q(t) \text{ in } E' \text{ a.e. } t \in [0,T], \\ \theta_\lambda(0) = \theta_0, \end{cases} \tag{72}$$

for all $\theta_\lambda, w \in E$, a.e. $t \in (0,T)$. For the Problem $\mathcal{PV}\theta_\lambda$ we have the following result.

Lemma 4.5 *$\mathcal{PV}\theta_\lambda$ has a unique solution such that*

$$\theta_\lambda \in L^2(0,T;E) \cap \mathcal{C}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;E'). \tag{73}$$

Moreover, $\exists C > 0$ such that $\forall \lambda_1, \lambda_2 \in L^2(0,T;V')$,

$$\|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} \le C \int_0^T \|\lambda_1(s) - \lambda_2(s)\|^2_{E'}\,ds, \quad \forall t \in [0,T]. \tag{74}$$

Proof. The result follows from the classical first order evolution equation given in [3]. Here the Gelfand triple is given by

$$E \subset L^2\,(\Omega) = (L^2\,(\Omega))' \subset E'.$$

The operator K is linear continuous and coercive. By Korn's inequality, we have

$$|K(u)|_{\mathcal{H}} \ge C|u|_{H_1}, \text{ for all } u \in V,$$

with C being a strictly positive constant defined only on Ω and Γ_1. Therefore

$$(K\tau, \tau)_{E'\times E} \ge C|\tau|^2_E. \tag{75}$$

In the fourth step, we let $\mu \in \mathbb{L}^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.
Problem $\mathcal{PV}\alpha_\mu$. Find the damage field $\alpha_\mu : [0,T] \longrightarrow H^1(\Omega)$ such that $\alpha_\mu \in F$ and

$$(\dot\alpha_\mu(t), \zeta - \alpha_\mu(t))_{\mathbb{L}^2(\Omega)} + a\,(\alpha_\mu(t), \zeta - \alpha_\mu(t)) \ \ge (\mathcal{S}(\varepsilon(u_\mu(t)), \alpha_\mu(t)), \zeta - \alpha_\mu(t))_{\mathbb{L}^2(\Omega)}\,, \tag{76}$$

$$\alpha_\mu(0) = \alpha_0, \tag{77}$$

for all $\alpha(t) \in F$, $\zeta \in F$ and $t \in [0,T]$. Note that if $f \in H$, then

$$(f,v)_{V' \times V} = (f,v)_H\,, \quad \forall v \in H.$$

Theorem 4.2 *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed and convex set of V. Assume that $a(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$ is a continuous and symmetric form such that for some constants $\zeta > 0$ and c_0,*

$$a(v,v) = c_0\|v\|_H^2 \geq \zeta\|v\|_V^2, \quad \forall v \in H.$$

Then, for every $u_0 \in K$ and $f \in \mathbb{L}^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H) \cap \mathbb{L}^2(0,T,V)$ such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$ and for almost all $t \in [0,T]$,

$$(\dot{u}(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_H, \quad \forall v \in K.$$

We apply this theorem to **Problem** $\mathcal{PV}\alpha_\mu$.

Lemma 4.6 *There exists a unique solution α_μ to the auxiliary problem $\mathcal{PV}\alpha_\mu$ such that*

$$\alpha_\mu \in W^{1,2}(0,T;\mathbb{L}^2(\Omega)) \cap \mathbb{L}^2(0,T;H^1(\Omega)). \tag{78}$$

The above lemma follows from the standard result for parabolic variational inequalities.

Proof. The inclusion mapping of $\left(H^1(\Omega), \|\cdot\|_{H^1(\Omega)}\right)$ into $(\mathbb{L}^2(\Omega), \|\cdot\|_{\mathbb{L}^2(\Omega)})$ is continuous and its range is dense. We denote by $\left(H^1(\Omega)\right)'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $\mathbb{L}^2\Omega$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset \mathbb{L}^2(\Omega) \subset \left(H^1(\Omega)\right)'.$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))' \times H^1(\Omega)}$ to represent the duality pairing between $\left(H^1(\Omega)\right)'$ and $H^1(\Omega)$, we have

$$(\alpha,\beta)_{(H^1(\Omega))' \times H^1(\Omega)} = (\alpha,\beta)_{\mathbb{L}^2(\Omega)}, \ \ \forall \alpha \in \mathbb{L}^2(\Omega), \ \beta \in H^1(\Omega),$$

and we note that F is a closed convex set in $H^1(\Omega)$. Then we use the definition of the bilinear form a given by (51), and the fact that $\alpha_\mu \in F$.

Problem $\mathcal{PV}\sigma_{\eta,\lambda,\mu}$. Find a stress field $\sigma_{\eta\lambda\mu} : [0,T] \longrightarrow \mathcal{H}$,

$$\sigma_{\eta\lambda\mu}(t) = \mathcal{B}(\varepsilon\left(u_\eta(t)\right), \alpha_\mu(t)\left(v\right))_{\mathcal{H}} + \int_0^t \mathcal{G}\left(\sigma(s), \varepsilon\left(u_\eta\left(s\right)\right)\right) ds - \mathcal{M}\theta_\lambda(t), \ \forall t \in [0,T]. \tag{79}$$

In the study of problem $\mathcal{PV}\sigma_{\eta\lambda\mu}$, we have the following result.

Lemma 4.7 *There exists a unique solution of problem $\mathcal{PV}\sigma_{\eta\lambda\mu}$, which satisfies* (56). *Moreover, if u_{η_i}, θ_{λ_i}, α_{μ_i} and $\sigma_{\eta_i,\lambda_i,\mu_i}$ represent the solution of problems $\mathcal{PV}u_{\eta_i}$, $\mathcal{PV}\theta_{\lambda_i}$, $\mathcal{PV}\alpha_{\mu_i}$ and $\mathcal{PV}\sigma_{\eta_i,\lambda_i,\mu_i}$, respectively, for $i = 1,2$, then there exists $c > 0$ such that*

$$\begin{aligned}
&\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}^2} \leq C(\|u_{\eta_1(t) - u_{\eta_2}(t)}\|_V^2 \\
&\quad + \int_0^t (\|u_{\eta_1(s) - u_{\eta_2}(s)}\|_V^2 + \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|_V^2 + \|\alpha_{\mu_1}(s) - \alpha_{\mu_2}(s)\|_V^2)\, ds).
\end{aligned} \tag{80}$$

Proof. Let $\Pi_{\eta,\lambda;\mu} : \mathbb{L}^2(0,T,\mathcal{H}) \longrightarrow \mathbb{L}^2(0,T;\mathcal{H})$ be the mapping given by

$$\Pi_{\eta,\lambda,\mu}\sigma(t) = \mathcal{B}(\varepsilon(u_\eta(t)), \alpha_\mu(t)) + \int_0^t \mathcal{G}(\sigma(s), \varepsilon(u_\eta(s)))\,ds - \mathcal{M}\theta_\lambda(t). \tag{81}$$

Let $\sigma_i \in \mathbb{L}^2(0,T;\mathcal{H}) : i = 1,2$ and $t_1 \in [0,T]$. Using hypothesis (27) and Hölder's inequality, we find

$$\|\Pi_{\eta,\lambda,\mu}\sigma_1(t) - \Pi_{\eta,\lambda,\mu}\sigma_2(t)\|^2_{\mathcal{H}} \leq L^2_{\mathcal{G}} T \int_0^t \|\sigma_1(s) - \sigma_2(s)\|^2_{\mathcal{H}}\,ds.$$

It follows from this inequality that for m large enough, a power $\Pi^m_{\eta,\lambda,\mu}$ of the mapping $\Pi_{\eta,\lambda,\mu}$ is a contraction of the Banach space $\mathbb{L}^2(0,T;\mathcal{H})$, and therefore there exists a unique element $\sigma_{\eta,\lambda,\mu} \in \mathbb{L}^2(0,T;\mathcal{H})$ such that $\Pi_{\eta,\lambda,\mu}\sigma_{\eta,\lambda,\mu} = \sigma_{\eta,\lambda,\mu}$. Moreover, $\sigma_{\eta,\lambda,\mu}$ is the unique solution of the problem $\mathcal{PV}\sigma_{\eta\lambda\mu}$. If u_{η_i}, θ_{λ_i} α_{μ_i} and $\sigma_{\eta_i,\lambda_i,\mu_i}$ represent the solution of the problems $\mathcal{PV}u_{\eta_i}$, $\mathcal{PV}\theta_{\lambda_i}$, $\mathcal{PV}\alpha_{\mu_i}$ and $\mathcal{PV}\sigma_{\eta_i\lambda_i\mu_i}$, respectively, for $i = 1,2$, then we use (4), (25), (26) and Young's inequality to obtain

$$\begin{aligned}
&\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}^2} \leq C(\|u_{\eta_1(t)-u_{\eta_2}(t)}\|^2_V \\
&\quad+ \int_0^t (\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}^2} + \|u_{\eta_1(s)-u_{\eta_2}(s)}\|^2_V + \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|^2_V \\
&\quad+ \|\alpha_{\mu_1}(s) - \alpha_{\mu_2}(s)\|^2_V)\,ds).
\end{aligned}$$

This permits us to obtain, using Gronwall's lemma, the inequality (80). Finally, we consider the operator Λ such that

$$\Lambda(\eta,\lambda,\mu)(t) = (\Lambda^1(\eta,\lambda,\mu)(t), \Lambda^2(\eta,\lambda,\mu)(t), \Lambda^3(\eta,\lambda,\mu)(t)), \tag{82}$$

where Λ^1, Λ^2 and Λ^3 are defined by

$$\begin{aligned}
&(\Lambda^1(\eta(t),\lambda(t),\mu(t),v(t))_{V'\times V} = \mathcal{B}(\varepsilon\left(u_\eta(t)\right), \varepsilon\left(v(t)\right))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\eta\left(t\right), \varepsilon\left(v(t)\right))_{\mathcal{H}} \\
&\quad+ \dot{J}_\varepsilon(u_\eta(t), v(t)) + \left(\int_0^t \mathcal{G}\left(\sigma_{\eta,\lambda,\mu}(s), \varepsilon\left(u_\eta(s)\right)\right)\,ds - \mathcal{M}\theta_\lambda(t), \varepsilon(v(t))\right)_{\mathcal{H}}, \quad \forall v \in V,
\end{aligned} \tag{83}$$

$$\Lambda^2(\eta(t),\lambda(t),\mu(t),v(t) = \Psi(\sigma_{\eta,\lambda,\mu}(t), \varepsilon(u_\eta(t)), \theta_\lambda(t))) \tag{84}$$

and

$$\Lambda^3(\eta(t),\lambda(t),\mu(t),v(t)) = \Phi(\sigma_{\eta,\lambda,\mu}(t), \varepsilon(u_\eta(t)), \alpha_\mu(t))). \tag{85}$$

Here, for $\eta \in \mathbb{L}^2(0,T;V)$, $\lambda \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Omega))$ and $\mu \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Omega))$, u_η, ϕ_η, θ_λ, α_μ and $\sigma_{\eta,\lambda,\mu}$ represent the displacement field, the potential electric field, the temperature, the damage field and the stress field obtained in Lemmas 4.1, 4.4, 4.5, 4.6 and 4.7. We have the following result.

Lemma 4.8 *The operator Λ has a unique fixed point $(\eta^*,\lambda^*,\mu^*) \in \mathbb{L}^2(0,T;V \times \mathbb{L}^2(\Omega)) \times \mathbb{L}^2(\Omega))$.*

Proof. We show for a positive integer m, the mapping Λ^m is a contraction on $\mathbb{L}^2(0,T;V \times \mathbb{L}^2(\Omega)) \times \mathbb{L}^2(\Omega))$. To this end, we suppose that (η_1,λ_1,μ_1) and (η_2,λ_2,μ_2)

are two functions in $\mathbb{L}^2(0,T;V\times\mathbb{L}^2(\Omega))\times\mathbb{L}^2(\Omega))$ and denote $u_{\eta_i}=u_i$, $\dot u_{\eta_i}=v_i$, $\varphi_{\eta_i}=\varphi_i$, $\theta_{\lambda_i}=\theta_i$, $\alpha_{\mu_i}=\alpha_i$ and $\sigma_{\eta_i,\lambda_i,\mu_i}=\sigma_i$ for $i=1,2$. We have

$$\begin{aligned}\|\Lambda^1(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^1(\eta_2,\lambda_2,\mu_2)(t)\|^2_{V'} &\le C\|R_\nu(u_\nu(t)-u_{2\nu}(t))\|^2_{\mathbb{L}^2(\Gamma_3)}\\ &+C\|R_\tau(u_\tau(t)-u_{2\tau}(t))\|^2_{\mathbb{L}^2(\Gamma_3)}+\|\mathcal{B}\varepsilon(u_1(t))-\mathcal{B}\varepsilon(u_2(t))\|^2_{\mathcal{H}}\\ &+\|\varepsilon^*\nabla\varphi_1(t)-\varepsilon^*\nabla\varphi_2(t)\|^2_{\mathcal{H}}+C\|\alpha_1(t)-\alpha_2(t)\|^2_{\mathbb{L}^2(\Omega)}\\ &+\int_0^t\|\mathcal{G}(\sigma_1(s),\varepsilon(u_2(s)))-\mathcal{G}(\sigma_2(s),\varepsilon(u_2(s)))\|^2_{\mathcal{H}}\,ds\\ &+C\|\theta_1(t)-\theta_2(t)\|^2_{\mathbb{L}^2\Omega}.\end{aligned}\tag{86}$$

Therefore, from (26), (27), (28) and the definition of R_ν, R_τ, we obtain

$$\begin{aligned}\|\Lambda^1(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^1(\eta_2,\lambda_2,\mu_2)(t)\|^2_{V'} &\le C(\|u_1(t)-u_2(t)\|^2_V\\ &+\int_0^t\|u_1(s)-u_2(s)\|^2_V\,ds+\int_0^t\|\sigma_1(s)-\sigma_2(s)\|^2_{\mathcal{H}}\,ds\\ &+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_E\,ds+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_F\,ds\\ &+\|\varphi_1(t)-\varphi_2(t)\|^2_W).\end{aligned}\tag{87}$$

We use estimate (81) to obtain

$$\begin{aligned}\|\Lambda^1(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^1(\eta_2,\lambda_2,\mu_2)(t)\|^2_{V'} &\le C(\|u_1(t)-u_2(t)\|^2_V\\ &+\int_0^t\|u_1(s)-u_2(s)\|^2_V\,ds+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_E\,ds\\ &+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_F\,ds+\|\varphi_1(t)-\varphi_2(t)\|^2_W).\end{aligned}\tag{88}$$

Recall that above u_{η_ν} and u_{η_τ} denote the normal and the tangential component of the function u_η, respectively. By similar arguments, from the function Φ and the definition of Λ^2, it follows that

$$\begin{aligned}\|\Lambda^2(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^2(\eta_2,\lambda_2,\mu_2)(t)\|^2_E &\le C(\|u_1(t)-u_2(t)\|^2_V\\ &+\int_0^t\|\sigma_1(s)-\sigma_2(s)\|^2_V\,ds+\|\theta_1(s)-\theta_2(s)\|^2_E)\\ &\le C(\|u_1(t)-u_2(t)\|^2_V+\|\theta_1(s)-\theta_2(s)\|^2_E\\ &+\int_0^t\|u_1(s)-u_2(s)\|^2_V\,ds+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_E\,ds).\end{aligned}\tag{89}$$

On the other hand, by (33), (80) and the definition of Λ^3, we get

$$\begin{aligned}\|\Lambda^3(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^3(\eta_2,\lambda_2,\mu_2)(t)\|^2_F &\le C(\|u_1(t)-u_2(t)\|^2_V\\ &+\|\varphi_1(t)-\varphi_2(t)\|^2_W+\int_0^t\|u_1(s)-u_2(s)\|^2_V\,ds\\ &+\|\alpha_1(s)-\alpha_2(s)\|^2_F+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_F\,ds).\end{aligned}\tag{90}$$

Also, since

$$u_i(t) = \int_0^t v_i(s)\,ds + u_0, \quad t \in [0,T], \tag{91}$$

we have

$$u_i(t) = \|u_1(t) - u_2(t)\|_V^2 \leq \int_0^t \|v_i(s)\|_V^2\,ds + u_0, \tag{92}$$

which implies

$$\|u_1(t) - u_2(t)\|_V^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2\,ds \leq C \int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds. \tag{93}$$

Therefore

$$\begin{aligned} \|\Lambda(\eta_1,\lambda_1,\mu_1)(t) - \Lambda(\eta_2,\lambda_2,\mu_2)(t)\|_{V'\times E\times F}^2 &\leq C(\|u_1(t) - u_2(t)\|_V^2 \\ &+ \int_0^t \|u_1(s) - u_2(s)\|_V^2\,ds + \|\alpha_1(s) - \alpha_2(s)\|_F^2 \\ &+ \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_F^2\,ds + \|\theta_1(s) - \theta_2(s)\|_E^2 \\ &+ \int_0^t \|\theta_1(s) - \theta_2(s)\|_E^2\,ds + \|\varphi_1(t) - \varphi_2(t)\|_W^2). \end{aligned} \tag{94}$$

Moreover, from (60), we obtain

$$\begin{aligned} (\dot v_1 - \dot v_2, v_1 - v_2)_{V'\times V} &= (\mathcal{A}\varepsilon(v_1) - \mathcal{A}\varepsilon(v_2), \varepsilon(v_2 - v_1))_{V'\times V} \\ &+ (\eta_1 - \eta_2, v_1 - v_2)_{V'\times V} \leq 0. \end{aligned} \tag{95}$$

We integrate this equality with respect to time, use the initial conditions, $v_1(0) = v_2(0) = v_0$, (27) and (61) to find

$$m_{\mathcal{A}} \int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds \leq C \int_0^t \|\eta_1(s) - \eta_2(s)\|_V \|v_1(s) - v_2(s)\|_V\,ds \tag{96}$$

for all $t \in [0,T]$. Then, using the inequality $2ab \leq \frac{a^2}{m_{\mathcal{A}}} + m_{\mathcal{A}} b^2$, we obtain

$$\int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds \leq C \int_0^t \|\eta_1(s) - \eta_2(s)\|_V\,ds, \forall t \in [0,T]. \tag{97}$$

Since $u_1(0) = u_2(0) = u_0$, we have

$$\|u_1(s) - u_2(s)\|_V^2 \leq C \int_0^T \|v_1(s) - v_2(s)\|_V\,ds, \tag{98}$$

and from (74), we have

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leq C \int_0^T \|\lambda_1(s) - \lambda_2(s)\|_{E'}^2 ds, \quad \forall t \in [0,T], \tag{99}$$

and from (71), we have

$$\|\varphi_1(t) - \varphi_2(t)\|_W^2 \leq C\|u_1(t) - u_2(t)\|_V^2, \quad \forall t \in [0,T]. \tag{100}$$

We substitute (93) and (100) in (94) to obtain

$$\|\Lambda(\eta_1,\lambda_1,\mu_1)(t)-\Lambda(\eta_2,\lambda_2,\mu_2)(t)\|^2_{V'\times E\times F}\le C(\int_0^t\|v_1(s)-v_2(s)\|^2_V\,ds + \|\theta_1(s)-\theta_2(s)\|^2_E+\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_E\,ds + \|\alpha_1(s)-\alpha_2(s)\|^2_F+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|^2_F\,ds), \tag{101}$$

on the other hand, from (76), we deduce that

$$(\dot\alpha_1-\dot\alpha_2,\alpha_1-\alpha_2)_{F'\times F}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\le(\mu_1-\mu_2,\alpha_1-\alpha_2)_F,\ \text{a.e. } t\in[0,T]. \tag{102}$$

Integrating the previous inequality with respect to time, using the initial conditions $\alpha_1(0)=\alpha_2(0)=\alpha_0$ and inequality $a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\ge 0$, we find

$$\frac{1}{2}\|\alpha_1(s)-\alpha_2(s)\|^2_F\le\int_0^t(\mu_1(s)-\mu_2(s),\alpha_1(s)-\alpha_2(s))_F\,ds. \tag{103}$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\alpha_1(s)-\alpha_2(s)\|^2_F\le C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_F\,ds,\quad\forall t\in[0,T]. \tag{104}$$

We substitute (97), (99) and (104) in (101) to obtain

$$\|\Lambda(\eta_1,\lambda_1,\mu_1)(t)-\Lambda(\eta_2,\lambda_2,\mu_2)(t)\|^2_{V'\times E\times F}\le C\int_0^t\|((\eta_1,\lambda_1,\mu_1)(s) - (\eta_2,\lambda_2,\mu_2)(s))\|^2_{V'\times E\times F}\,ds. \tag{105}$$

Reintegrating this inequality n times, we obtain

$$\|\Lambda^n(\eta_1,\lambda_1,\mu_1)-\Lambda^n(\eta_2,\lambda_2,\mu_2)\|^2_{\mathbb{L}^2(0,T;V'\times E\times F)}\le\frac{C^nT^n}{n!}\|((\eta_1,\lambda_1,\mu_1) - (\eta_2,\lambda_2,\mu_2))\|^2_{\mathbb{L}^2(V'\times E\times F)}, \tag{106}$$

thus, for n sufficiently large, Λ^n is a contraction on the Banach space $\mathbb{L}^2(0,T;V'\times E\times F)$ and so Λ has a unique fixed point. Now, we have all ingredients to prove Theorem 4.1.

Proof. (of Theorem 4.1). Let $(\eta^*,\lambda^*,\mu^*)\in\mathbb{L}^2(0,T;V'\times\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega))$ be the fixed point of Λ defined by (82), (83), (84) and (85) and

$$u_*=u_{\eta^*},\quad \varphi_*=\varphi_{\eta^*},\quad \theta_*=\theta_{\eta^*}\quad\text{and}\quad \alpha_*=\alpha_{\eta^*}. \tag{107}$$

Let $\sigma_*:[0,T]\longrightarrow\mathcal{H}$ be the function defined by

$$\sigma_*=\mathcal{A}\varepsilon(\dot u_*)+\varepsilon^*\nabla\varphi_*+\sigma_{\eta^*,\lambda^*,\mu^*}. \tag{108}$$

We prove that $\{u_*,\sigma_*,\varphi_*,\theta_*,\alpha_*\}$ satisfies (42), (48) and the regularities (56)-(58). Indeed, we write (60) and use (107) to find

$$(\ddot u_*(t),v)_{V'\times V}+(\mathcal{A}\varepsilon(\dot u_*(t)),\varepsilon(v))_{\mathcal{H}}+J_\varepsilon(\dot u_*(t),v) + (\eta^*(t),v)_{V'\times V}\ge(f(t),v)_{V'\times V},\quad\forall v\in V\ \text{a.e.},\ t\in 0,T, \tag{109}$$

we use equalities $\Lambda^1(\eta^*, \lambda^*, \mu^*) = \mu^*$, $\Lambda^2(\eta^*, \lambda^*, \mu^*) = \lambda^*$ and $\Lambda^3(\eta^*, \lambda^*, \mu^*) = \eta^*$, it follows that

$$\begin{aligned}(\eta_*(t), v)_{V'\times V} &= (\mathcal{B}\varepsilon(u_*(t)), \varepsilon(v))_{\mathcal{H}} + (\varepsilon^*\nabla\varphi_*(t), \varepsilon(v))_{\mathcal{H}} \\ &+ \left(\int_0^t \mathcal{G}(\sigma_{\eta^*,\lambda^*,\mu^*}(s), \varepsilon(u_*(s), \alpha_*(s)))\, ds - \mathcal{M}\theta_*(t), \varepsilon(v)\right)_{\mathcal{H}} \\ &+ J_\varepsilon(u_*(t), v(t)),\end{aligned} \tag{110}$$

$$\lambda_*(t) = \Phi(\sigma_{\eta^*,\lambda^*,\mu^*}(t), \varepsilon(u_*(t)), \theta_*(t)), \tag{111}$$

$$\mu_*(t) = \Psi(\sigma_{\eta^*,\lambda^*,\mu^*}(t), \varepsilon(u_*(t)), \alpha_*(t)). \tag{112}$$

We now substitute (110) in (109) to obtain

$$\begin{aligned}&(\ddot{u}_*(t), v)_{V'\times V} + (\mathcal{A}\varepsilon(\dot{u}_*(t), \varepsilon(v))_{\mathcal{H}} + (\mathcal{B}\varepsilon(u_*)(t), \varepsilon(v), \alpha_*(t))_{\mathcal{H}} + (\varepsilon^*\nabla\varphi_*, \varepsilon(v))_{\mathcal{H}} \\ &+ \left(\int_0^t \mathcal{G}(\sigma_{\eta^*,\lambda^*,\mu^*}(s), \varepsilon(u_*(s)))\, ds - \mathcal{M}\theta_*(t), \varepsilon(v)\right)_{\mathcal{H}} \\ &+ J_\varepsilon(u_*(t), v) \geq (f(t), \dot{v})_{V'\times V}, \quad \forall v \in V.\end{aligned} \tag{113}$$

It follows from Lemma 4.7 and (108) that $\sigma_* \in \mathbb{L}^2(0, T; \mathcal{H})$ and (43) implies that

$$\operatorname{div} \sigma_* + f_0(t) = \rho\ddot{u}_*(t), \text{ a.e., } t \in [0, T].$$

We write (72) for $\lambda = \lambda^*$ to find that (74) is satisfied, also write (76) for $\mu = \mu^*$ to find that (76) is satisfied , we consider now (60) for $\eta = \eta^*$ to find that (60) is satisfied. Next, the regularities (56)-(59) follow from Lemmas 4.1, 4.2, 4.4, 4.5, 4.6 and the regularity (56) follows from Lemma 4.7, the uniqueness part of Theorem 4.1 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (82)-(85) and thus follows the unique solvability of the problems $\mathcal{PV}u_\eta$, $\mathcal{PV}\varphi_\eta$, $\mathcal{PV}\theta_\lambda$, $\mathcal{PV}\alpha_\mu$ and $\mathcal{PV}\sigma_{\eta,\lambda,\mu}$, which completes the proof.

5 Conclusion

As a conclusion, we can say that our model, which describes the contact problem with damage and thermal effect for an electro-elasto-viscoplastic problem, based on thermodynamics is developed to describe the self-heating and stress-strain behavior of thermoplastic polymers under tensile loading. The constitutive model considers temperature-dependent elasticity, nonlinear viscoplastic flow and damage evolution. The literature devoted to various aspects of the subject is considerable, it concerns the modelling and the mathematical analysis of the related problems. For example, many food materials used in process engineering are elastic-viscoplastic, mathematical models can be very helpful in understanding various problems related to the product development, packing, transport, shelf life testing, thermal effects, and heat transfer. It is thus important to study mathematical models that can be used to describe the dynamical behavior of a given elastic-viscoplastic material subjected to various highly nonlinear and even non-smooth phenomena like contact, friction, lubrication, adhesion, wear, damage, electrical and thermal effects. Thermal effects in contact processes affect the composition and stiffness of the contacting surfaces, and cause thermal stresses in the contacting bodies. Moreover, the contacting surfaces exchange heat and energy is lost to the surroundings.

References

[1] A. Aissaoui and N. Hemici. A frictional contact problem with damage and adhesion for an electro elastic-viscoplastic body. *Electronic Journal of Differential Equations* **11** (2014) 1–19.

[2] A. Aissaoui and N. Hemici. Bilateral contact problem with adhesion and damage. *Electronic Journal of Qualitative Theory of Differential Equations* **18** (2014) 1–16.

[3] V. Barbu. *Optimal Control of Variational Inequalities*, Pitman, Boston, 1984.

[4] H. Benaissa, E. Essou and R. Fakhar. Existence results for unilateral contact problem with friction of thermo-electro-elasticity. *Applied Mathematics and Mechanics* **36** (2015) 911–926.

[5] E. Bonetti, G. Bonfanti and R. Rossi. Modeling via the internal energy balance and analysis of adhesive contact with friction in thermoviscoelasticity *Nonlinear Analysis: Real World Applications* **22** (2015) 473–507.

[6] I. Boukaroura and S. Djabi. Analysis of a quasistatic contact problem with wear and damage for thermo-viscoelastic materials. *Malaya Journal of Matematik* **6** (2018) 299–309.

[7] H. Brézis. Equations et inequations nonlinéaires dans les espaces vectoriels en dualité. *Annales de l'Institut Fourier* **18** (1968) 115–175.

[8] A. Djabi, A. Merouani and A. Aissaoui. A frictional contact problem with wear involving elastic-viscoplastic materials with damage and thermal efects. *Electronic Journal of Qualitative Theory of Differential Equations* **27** (2015) 1–18.

[9] A. Djabi and A. Merouani. Bilateral contact problem with friction and wear for an electro elastic-viscoplastic materials with damage. *Taiwanese Journal of Mathematics* **19** (2015) 1161–1182.

[10] G. Duvaut and J.-L. Lions. *Les Inequations en Mecanique et en Physique*. Dunod, Paris, 1972.

[11] A. Merouani and F. Messelmi. Quasi-static transmission problem in thermo-viscoplasticity. *International Journal of Open Problems in Computer Science and Mathematics* **6** (2013) 29–46.

[12] M. Selmani. A Frictional Contact Problem Involving Piezoelectric Materials with Long Memory. *Mediterranean Journal of Mathematics* **12** (2015) 1177–1197.

[13] A. Merouani and A. Djabi. A monotony method in quasistatic processes for viscoplastic materials. *Stud. Univ. Bobeş-Bolyai Math.* **8** (2) (2013) 263–278.

[14] S. Migŏrski, A. Ochal and M. Sofonea. Analysis of a dynamic elastic-viscoplastic contact problem with friction. *Discrete Contin. Dyn. Syst. Ser. B* **10** (4) (2008) 887-–902.

[15] L. Maiza, T. Hadj Ammar and M. Said Ameur. A Dynamic Contact Problem for Elasto-Viscoplastic Piezoelectric Materials with Normal Compliance, Normal Damped Response and Damage. *Nonlinear Dynamics and Systems Theory* **21** (3) (2021) 280—302.

[16] P. G. Ciarlet. *Mathematical Elasticity, Vol. I: Three Dimensional Elasticity.* Elsevier, North-Holland, 1988.