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A Dynamic Contact Problem between Viscoelastic Piezoelectric Bodies with Friction and Damage

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Abstract: We consider a dynamic contact problem between two thermo-electro-viscoelastic bodies with damage and an internal state variable. The contact is bilateral and is modeled by Tresca's friction law. The damage of the materials is caused by elastic deformations. We derive a variational formulation for the model which is in the form of a system involving the displacement field, the electric potential, the internal state variable field, the temperature and the damage. Then we proved the existence of a unique weak solution to the model.

Keywords: *viscoelastic piezoelectric materials; internal state variable; damage; temperature; friction contact.*

Mathematics Subject Classification (2010): 74M10, 70K30, 70K75, 93-02.

1 Introduction

Our research paper tackles a frictional bilateral contact problem including the topic of piezoelectric, which can be explained as follows: when we apply mechanical pressure to some types of crystalline materials such as ceramics $BaTiO_3, BiFeO3$, a voltage proportional to the pressure is produced. Meanwhile, changes in shape and dimension occur if an electric field is applied to some types of crystalline materials. At present, there is a great interest in the study of piezoelectric materials for their importance in radio-electronics, electroacoustics and instrumentation. Thus, a big interest in the contact problems occurs because of the fact that parts of the equipment are in contact. So, many models have been developed to explain the interaction between the electrical and mechanical fields, see for example [2, 8] and the references therein. Frictional contact problem is a static problem of electro-elastic materials mentioned in [3] and [10], considering that the basis is

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isolated. Contact problems involving elasto-piezoelectric materials [3], viscoelastic piezoelectric materials [1] and the contact problem for electro-elastio-viscoplastic materials were studied in [7].

A mathematical investigation has been conducted for some models taking into consideration the influence of the internal damage of the material in the contact process. From the virtual power principle, general models for damage were derived in [6]. In [4], we can find the modes of mechanical damage which are derived from thermo-dynamical consideration. The ratio between the elastic moduli of the damage and damage-free materials is expressed by the function called the damage function $\zeta^\kappa = \zeta^\kappa(x,t)$ mentioned in [5,6]. In an isotropic and homogeneous elastic material, let E_Y^κ be the Young modulus of the original material and E_{eff}^κ be the current modulus, then the damage function is defined by $\zeta^\kappa = E_{eff}^\kappa / E_Y^\kappa$. This definition shows that the damage function ζ^κ is restricted to have values between zero and one; when $\zeta^\kappa = 1$, there is no damage in the material, when $\zeta^\kappa = 0$, the material is completely damaged, when $0 < \zeta^\kappa < 1$, there is partial damage and the system has a reduced load carrying capacity. The contact problem with damage has been mentioned in [9]. The differential inclusion used for the evolution of the damage field is

$$\dot{\zeta^\kappa} - \Delta\zeta^\kappa + \boldsymbol{k}^\kappa \partial\chi \boldsymbol{k}^\kappa(\zeta^\kappa) \ni \boldsymbol{S}^\kappa(\epsilon(\boldsymbol{u}^\kappa), \zeta^\kappa) \ in\ \Omega^\kappa \times [0,T], \tag{1.1}$$

where $\boldsymbol{k}^\kappa$ is a positive coefficient and $\boldsymbol{K}^\kappa$ is the set of admissible damages defined by

$$\boldsymbol{K}^\kappa = \{\zeta \in H^1(\Omega^\kappa); 0 \le \zeta \le 1.\ a.e \in \Omega^\kappa\}. \tag{1.2}$$

The paper is structured as follows. In Section 2, we present the physical setting and describe the mechanical problem. We derive a variational formulation, list the assumptions on the data, and give the variational formulation of the problem. In Section 3, we state our main existence and uniqueness result which is based on the classical result of non-linear first order evolution inequalities and equations with monotone operators and the fixed point arguments.

2 Problem Statement and Variational Formulation

The physical setting is the following. Let us consider two electro-thermovisco-elastic bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d$ ($d = 2, 3$ in applications). We put a superscript κ to indicate that the quantity is related to the domain Ω^κ. In the following, the superscript κ ranges between 1 and 2. For each domain Ω^κ, the boundary Γ^κ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^κ, Γ_2^κ and Γ_3^κ, on one hand, and in two measurable parts Γ_a^κ and Γ_b^κ, on the other hand, such that $meas\Gamma_1^\kappa > 0$, $meas\Gamma_a^\kappa > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The Ω^κ body is subject to f_0^κ forces and volume electric charges of density q_0^κ. The bodies are assumed to be clamped on $\Gamma_1^\kappa \times [0,T]$. The surface tractions f_2^κ act on $\Gamma_2^\kappa \times [0,T]$. We also assume that the electrical potential vanishes on $\Gamma_a^\kappa \times [0,T]$ and a surface electric charge of density $q_2^\kappa \times [0,T]$ is prescribed on $\Gamma_b^\kappa \times [0,T]$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The classical form of the bilateral contact with Tresca's friction and damage between two electro-thermovisco-elastic bodies with damage and an internal state variable is the following.

Problem P. For $\kappa = 1, 2$, find a displacement field $u^\kappa : \Omega^\kappa \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma^\kappa : \Omega^\kappa \times [0,T] \to \mathbb{S}^d$, an electric potential $\psi^\kappa : \Omega^\kappa \times [0,T] \to \mathbb{R}$, an electric displacement field $D^\kappa : \Omega^\kappa \times [0,T] \longrightarrow \mathbb{R}^d$, a temperature $\tau^\kappa : \Omega^\kappa \times [0,T] \longrightarrow \mathbb{R}$, a damage $\alpha^\kappa : \Omega^\kappa \times [0,T] \to \mathbb{R}$ and an internal state variable field $\beta^\kappa : \Omega^\kappa \times [0,T] \longrightarrow \mathbb{R}^m$ such that for all $t \in (0,T)$, we have

$$\begin{aligned}&\sigma^\kappa(t) = \mathcal{A}^\kappa\varepsilon(\dot{u}^\kappa(t)) + \mathcal{B}^\kappa(\varepsilon(u^\kappa(t)),\alpha^\kappa(t)) - (\mathcal{E}^\kappa)^*E(\psi^\kappa(t))\\ &+\mathcal{F}^\kappa(\beta^\kappa(t),\tau^\kappa(t)) \quad \text{in } \Omega^\kappa,\end{aligned} \tag{2.1}$$

$$D^\kappa(t) = \mathcal{E}^\kappa\varepsilon(u^\kappa(t)) + \mathcal{R}^\kappa E(\psi^\kappa(t)) + \mathcal{G}^\kappa(\beta^\kappa(t),\tau^\kappa(t)) \quad \text{in } \Omega^\kappa, \tag{2.2}$$

$$\dot{\beta}^\kappa(t) = \Theta^\kappa\big(\varepsilon(u^\kappa(t)),\alpha^\kappa(t),\beta^\kappa(t),\tau^\kappa(t)\big) \quad \text{in } \Omega^\kappa, \tag{2.3}$$

$$\dot{\tau}^\kappa(t) - \mathcal{K}_0^\kappa\Delta\tau^\kappa(t) = \Psi^\kappa\big(\varepsilon(u^\kappa(t)),\alpha^\kappa(t),\beta^\kappa(t),\tau^\kappa(t)\big) + \chi^\kappa(t) \quad \text{in } \Omega^\kappa, \tag{2.4}$$

$$\dot{\alpha}^\kappa(t) - \mathcal{K}_1^\kappa\Delta\alpha^\kappa(t) + \partial\mathbb{I}_{\mathcal{Z}^\kappa}(\alpha^\kappa(t)) \ni S^\kappa(\varepsilon(u^\kappa(t)),\alpha^\kappa(t)) \quad \text{in } \Omega^\kappa, \tag{2.5}$$

$$\operatorname{Div}\sigma^\kappa(t) + f_0^\kappa(t) = \rho^\kappa\ddot{u}^\kappa(t) \quad \text{in } \Omega^\kappa, \tag{2.6}$$

$$\operatorname{div} D^\kappa(t) = q_0^\kappa(t) \quad \text{in } \Omega^\kappa, \tag{2.7}$$

$$u^\kappa(t) = 0 \quad \text{on } \Gamma_1^\kappa, \tag{2.8}$$

$$\sigma^\kappa(t)\nu^\kappa = f_2^\kappa(t) \quad \text{on } \Gamma_2^\kappa, \tag{2.9}$$

$$\begin{cases} u_\nu^1(t) + u_\nu^2(t) = 0, \quad \sigma_\tau^1(t) = -\sigma_\tau^2(t) \equiv \sigma_\tau(t), \quad |\sigma_\tau(t)| \le g,\\ |\sigma_\tau(t)| < g \Rightarrow \dot{u}_\tau^1(t) - \dot{u}_\tau^2(t) = 0 \\ |\sigma_\tau(t)| = g \Rightarrow \exists\lambda \ge 0 \text{ such that } \sigma_\tau(t) = -\lambda\big(\dot{u}_\tau^1(t) - \dot{u}_\tau^2(t)\big),\end{cases} \quad \text{on } \Gamma_3, \tag{2.10}$$

$$\frac{\partial\alpha^\kappa(t)}{\partial\nu^\kappa} = 0 \quad \text{on } \Gamma^\kappa, \tag{2.11}$$

$$\mathcal{K}_0^\kappa\frac{\partial^\kappa\tau^\kappa(t)}{\partial\nu^\kappa} + \lambda_0^\kappa\tau^\kappa(t) = 0 \quad \text{on } \Gamma^\kappa, \tag{2.12}$$

$$\psi^\kappa(t) = 0 \quad \text{on } \Gamma_a^\kappa, \tag{2.13}$$

$$D^\kappa(t)\cdot\nu^\kappa = q_2^\kappa(t) \quad \text{on } \Gamma_b^\kappa, \tag{2.14}$$

$$u^\kappa(0) = u_0^\kappa,\ \dot{u}^\kappa(0) = v_0^\kappa,\ \alpha^\kappa(0) = \alpha_0^\kappa,\ \beta^\kappa(0) = \beta_0^\kappa,\ \tau^\kappa(0) = \tau_0^\kappa \quad \text{in } \Omega^\kappa. \tag{2.15}$$

First, equations (2.1)–(2.3) represent the electro-thermovisco-elastic constitutive law with damage and an internal state variable. The evolution of the damage field is governed by the inclusion given by the relation (2.5). Equation (2.4) represents the conservation of energy, where Ψ^κ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and χ^κ is a given volume heat source. Next, equations (2.6) and (2.7) are the equations of motion written for the stress field and of balance written for the electric displacement field, respectively, in which Div and div denote the divergence operators for tensor and vector valued functions. Conditions (2.8) and (2.9) are the displacement and traction boundary conditions, respectively. Boundary conditions (2.11), (2.12) represent, respectively, on Γ^α, a homogeneous Neumann boundary condition for the damage field and a Fourier boundary condition for the temperature, (2.13) and (2.14) represent the electric boundary conditions, and (2.15) are the initial conditions. Conditions (2.10) represent the bilateral contact condition with Tresca's friction, where $[u_\nu] = u_\nu^1 + u_\nu^2$ and $[u_\tau] = u_\tau^1 - u_\tau^2$.

Now, to proceed with the variational formulation, we need the following function spaces:

$\mathbb{H}^\kappa = L^2(\Omega^\kappa)^d = \left\{u = (u_i)_{1\le i\le d};\ u_i \in L^2(\Omega^\kappa)\right\}$,
$\mathbb{H}_1^\kappa = W^{1,2}(\Omega^\kappa)^d = \left\{u = (u_i)_{1\le i\le d};\ u_i \in W^{1,2}(\Omega^\kappa)\right\}$,
$\mathcal{H}^\kappa = L^2(\Omega^\kappa)_s^{d\times d} = \left\{\sigma = (\sigma_{ij})_{1\le i,j\le d};\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega^\kappa)\right\}$,
$\mathcal{H}_1^\kappa = \left\{\sigma \in \mathcal{H}^\kappa;\ \operatorname{Div}\sigma \in \mathbb{H}^\kappa\right\}$,
$\mathbb{Y}^\kappa = L^2(\Omega^\kappa)^m = \left\{\beta = (\beta_i)_{1\le i\le m};\ \beta_i \in L^2(\Omega^\kappa)\right\}$,
$\mathcal{V}^\kappa = \left\{u \in W^{1,2}(\Omega^\kappa)^d;\ u = 0 \text{ on } \Gamma_1^\kappa\right\}$. These are real Hilbert spaces endowed with the

inner products $\langle u,v\rangle_{\mathbb{H}^\kappa}=\int_{\Omega^\kappa}u.vdx,\quad \forall u,v\in\mathbb{H}^\kappa, \langle\sigma,\theta\rangle_{\mathcal{H}^\kappa}=\int_{\Omega^\kappa}\sigma.\theta dx,\quad \forall\sigma,\theta\in\mathcal{H}^\kappa$,
$\langle u,v\rangle_{\mathbb{H}_1^\kappa}=\int_{\Omega^\kappa}u.vdx+\int_{\Omega^\kappa}\nabla u.\nabla vdx,\quad \forall u,v\in\mathbb{H}_1^\kappa$,
$\langle\sigma,\theta\rangle_{\mathcal{H}_1^\kappa}=\int_{\Omega^\kappa}\sigma.\theta dx+\int_{\Omega^\kappa}\mathrm{Div}\,\sigma.\,\mathrm{Div}\,\theta dx,\quad \forall\sigma,\theta\in\mathcal{H}^\kappa$,
$\langle\beta,k\rangle_{\mathbb{Y}^\kappa}=\int_{\Omega^\kappa}\beta.kdx,\ \ \forall\beta,k\in\mathbb{Y}, \langle u,v\rangle_{\mathcal{V}^\kappa}=(\varepsilon(u),\varepsilon(v))_{\mathcal{H}^\kappa}\ \ \forall u,v\in\mathcal{V}^\kappa$ and the associated norms $\|.\|_{\mathbb{H}^\kappa}$, $\|.\|_{\mathcal{H}^\kappa}$, $\|.\|_{\mathbb{H}_1^\kappa}$, $\|.\|_{\mathcal{H}_1^\kappa}$, $\|.\|_{\mathbb{Y}^\kappa}$ and $\|.\|_{\mathcal{V}^\kappa}$, respectively. Here and below we use the notation

$$\nabla u=(u_{i,j}),\ \varepsilon(u)=(\varepsilon_{ij}(u)),\quad \varepsilon_{ij}(u)=\frac{1}{2}(u_{i,j}+u_{j,i}),\quad \forall u\in\mathbb{H}_1^\kappa,$$
$$\mathrm{Div}\,\sigma=(\sigma_{ij,j}),\quad \forall\sigma\in\mathcal{H}_1^\kappa.$$

Completeness of the space $(\mathcal{V}^\kappa,\|.\|_{\mathcal{V}^\kappa})$ follows from the assumption $meas(\Gamma_1^\kappa>0)$, which allows the use of Korn's inequality. We denote u^κ as the trace of an element $u^\kappa\in\mathbb{H}_1^\kappa$ on Γ^κ. For every element $u^\kappa\in\mathcal{V}^\kappa$, we denote by u_ν^κ and u_τ^κ the normal and the tangential components of u on the boundary Γ^κ given by $u_\nu^\kappa=u^\kappa.\nu^\kappa$, $u_\tau^\kappa=u^\kappa-u_\nu^\kappa\nu^\kappa$. Also, for an element $\sigma^\kappa\in\mathcal{H}_1^\kappa$, we denote by $\sigma^\kappa\nu$, σ_ν^κ and σ_τ^κ the trace, the normal trace and the tangential trace of σ^κ to Γ^κ, respectively. In addition to the Sobolev trace theorem, there exists a constant $c_{tr}>0$, depending only on Ω^κ, Γ_1^κ and Γ_3 such that

$$\|u^\kappa\|_{L^2(\Gamma_3)^d}\le c_{tr}\|u^\kappa\|_{\mathcal{V}^\kappa},\quad \forall u^\kappa\in\mathcal{V}^\kappa. \tag{2.16}$$

Denote $E_0^\kappa=L^2(\Omega^\kappa)$, $E_1^\kappa=H^1(\Omega^\kappa)$, $\langle .,.\rangle_{E_0^\kappa}=\langle .,.\rangle_{L^2(\Omega^\kappa)}$, $\langle .,.\rangle_{E_1^\kappa}=\langle .,.\rangle_{H^1(\Omega^\kappa)}$, $\|.\|_{E_0^\kappa}=\|.\|_{L^2(\Omega^\kappa)}$ and $\|.\|_{E_1^\kappa}=\|.\|_{H^1(\Omega^\kappa)}$. For the electric unknowns ψ^κ and D^κ, we use the spaces

$$\mathbb{W}^\kappa=\{\psi^\kappa\in E_1^\kappa;\ \psi^\kappa=0 \text{ on } \Gamma_a^\kappa\},$$
$$\mathcal{W}^\kappa=\left\{D^\kappa=(D_i^\kappa)_{1\le i\le d};\ D_i^\kappa\in L^2(\Omega^\kappa),\ \mathrm{div}\,D^\kappa\in L^2(\Omega^\kappa)\right\}.$$

These are real Hilbert spaces with the inner products

$$\langle\psi^\kappa,\varphi^\kappa\rangle_{\mathbb{W}^\kappa}=\int_{\Omega^\kappa}\nabla\psi^\kappa.\nabla\varphi^\kappa dx,\ \ \langle D^\kappa,E^\kappa\rangle_{\mathcal{W}^\kappa}=\int_{\Omega^\kappa}D^\kappa.E^\kappa dx+\int_{\Omega^\kappa}\mathrm{div}\,D^\kappa.\,\mathrm{div}\,E^\kappa dx, \tag{2.17}$$

where $\mathrm{div}\,D^\kappa=(D_{i,i}^\kappa)$, and the associated norms are denoted by $\|.\|_{\mathbb{W}^\kappa}$ and $\|.\|_{\mathcal{W}^\kappa}$, respectively. Completeness of the space $(\mathbb{W}^\kappa,\|.\|_{\mathbb{W}^\kappa})$ is a consequence of the assumption $meas(\Gamma_a^\kappa)>0$ which allows the use of the Friedrichs-Poincaré inequality. When $\sigma^\kappa\in\mathcal{H}_1^\kappa$, $\tau^\kappa\in H^1(\Omega^\kappa)$ and $D^\kappa\in\mathcal{W}^\kappa$ are sufficiently regular functions, the following three Green's formulas hold

$$\langle\sigma^\kappa,\varepsilon(v^\kappa)\rangle_{\mathcal{H}^\kappa}+\langle\mathrm{Div}\,\sigma^\kappa,v^\kappa\rangle_{\mathbb{H}^\kappa}=\int_{\Gamma^\kappa}\sigma^\kappa\nu^\kappa.v^\kappa da,\quad \forall v^\kappa\in\mathbb{H}_1^\kappa, \tag{2.18}$$

$$\langle\Delta\tau^\kappa,\delta^\kappa\rangle_{\mathbb{H}^\kappa}+\langle\nabla\tau^\kappa,\nabla\delta^\kappa\rangle_{L^2(\Omega^\kappa)}=\int_{\Gamma^\kappa}\frac{\partial\tau^\kappa}{\partial\nu^\kappa}\delta^\kappa da,\quad \forall\delta^\kappa\in H^1(\Omega^\kappa), \tag{2.19}$$

$$(D^\kappa,\nabla\phi^\kappa)_{\mathbb{H}^\kappa}+(\mathrm{div}\,D^\kappa,\phi^\kappa)_{L^2(\Omega^\kappa)}=\int_{\Gamma^\kappa}D^\kappa\nu^\kappa\phi^\kappa da,\quad \forall\phi^\kappa\in H^1(\Omega^\kappa). \tag{2.20}$$

In order to simplify the notations, we define the spaces

$$\mathcal{V}=\{u=(u^1,u^2)\in\mathcal{V}^1\times\mathcal{V}^2;\ \ u_\nu^1+u_\nu^2=0 \text{ on } \Gamma_3\},$$
$$\mathbb{H}=\mathbb{H}^1\times\mathbb{H}^2,\ \ \mathbb{H}_1=\mathbb{H}_1^1\times\mathbb{H}_1^2,\ \ \mathcal{H}=\mathcal{H}^1\times\mathcal{H}^2,\ \ \mathcal{H}_1=\mathcal{H}_1^1\times\mathcal{H}_1^2,\ \ \mathbb{Y}=\mathbb{Y}^1\times\mathbb{Y}^2,$$
$$E_0=E_0^1\times E_0^2,\ \ E_1=E_1^1\times E_1^2,\ \ \mathbb{W}=\mathbb{W}^1\times\mathbb{W}^2,\ \ \mathcal{W}=\mathcal{W}^1\times\mathcal{W}^2.$$

The spaces $\mathcal{V}$, $\mathbb{H}$, $\mathcal{H}$, $\mathbb{Y}$, E_0, E_1, $\mathbb{W}$ and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $\langle .,. \rangle_{\mathcal{V}}$, $\langle .,. \rangle_{\mathbb{H}}$, $\langle .,. \rangle_{\mathcal{H}}$, $\langle .,. \rangle_{\mathbb{Y}}$, $\langle .,. \rangle_{E_0}$, $\langle .,. \rangle_{E_1}$, $\langle .,. \rangle_{\mathbb{W}}$, and $\langle .,. \rangle_{\mathcal{W}}$. The associate norms will be denoted by $\|.\|_{\mathcal{V}}$, $\|.\|_{\mathbb{H}}$, $\|.\|_{\mathcal{H}}$, $\|.\|_{\mathbb{Y}}$, $\|.\|_{E_0}$, $\|.\|_{E_1}$, $\|.\|_{\mathbb{W}}$, and $\|.\|_{\mathcal{W}}$, respectively.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $p \in [1,+\infty]$, $k \in [1,+\infty[$. We denote by $\mathcal{C}(0,T;X)$ and $\mathcal{C}^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\pi\|_{\mathcal{C}(0,T;X)} = \max_{t\in[0,T]} \|\pi(t)\|_X, \quad \|\pi\|_{\mathcal{C}^1(0,T;X)} = \max_{t\in[0,T]} \|\pi(t)\|_X + \max_{\pi\in[0,T]} \|\dot{\pi}(t)\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable. Moreover, if X_1 and X_2 are real Hilbert spaces, then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $\langle .,. \rangle_{X_1\times X_2}$.

We now list assumptions on the data. Assume the operators $\mathcal{A}^\kappa$, $\mathcal{B}^\kappa$, $\mathcal{F}^\kappa$, $\mathcal{G}^\kappa$, $\mathcal{R}^\kappa$, Θ^κ, Ψ^κ, S^κ, and $\mathcal{E}^\kappa$ satisfy the following conditions $\big(L_{\mathcal{A}^\kappa}$, $m_{\mathcal{A}^\kappa}$, $L_{\mathcal{B}^\kappa}$, $L_{\mathcal{F}^\kappa}$, $L_{\mathcal{G}^\kappa}$, $m_{\mathcal{R}^\kappa}$, L_{Θ^κ}, L_{Ψ^κ} and L_{S^κ} being positive constants$\big)$ for $\kappa = 1,2$:

H(1): (a) $\mathcal{A}^\kappa : \Omega^\kappa \times \mathbb{S}^d \to \mathbb{S}^d$;
(b) $|\mathcal{A}^\kappa(x,\varepsilon_1) - \mathcal{A}^\kappa(x,\varepsilon_2)| \leq L_{\mathcal{A}^\kappa}|\varepsilon_1 - \varepsilon_2|,\ \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega^\kappa$;
(c) $(\mathcal{A}^\kappa(x,\varepsilon_1) - \mathcal{A}^\kappa(x,\varepsilon_2)).(\varepsilon_1 - \varepsilon_2) \geq m_{\mathcal{A}^\kappa}|\varepsilon_1 - \varepsilon_2|^2, \quad \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d$ a.e $x \in \Omega^\kappa$;
(d) $\mathcal{A}^\kappa(.,\varepsilon)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d$;
(e) $\mathcal{A}^\kappa(.,0)$ belongs to $\mathcal{H}^\kappa$.

H(2): (a) $\mathcal{B}^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$;
(b) $|\mathcal{B}^\kappa(x,\varepsilon_1,r_1) - \mathcal{B}^\kappa(x,\varepsilon_2,r_2)| \leq L_{\mathcal{B}^\kappa}\big(|\varepsilon_1 - \varepsilon_2| + |r_1 - r_2|\big)$;
$\forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, r_1, r_1 \in \mathbb{R}$, a.e. $x \in \Omega^\kappa$;
(c) $\mathcal{B}^\kappa(.,\varepsilon,r)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d$, $r \in \mathbb{R}$;
(d) $\mathcal{B}^\kappa(.,0,0)$ belongs to $\mathcal{H}^\kappa$.

H(3): (a) $\mathcal{F}^\kappa : \Omega^\kappa \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{S}^d$;
(b) $|\mathcal{F}^\kappa(x,k_1,r_1) - \mathcal{F}^\kappa(x,k_2,r_2)| \leq L_{\mathcal{F}^\kappa}\big(|k_1 - k_2| + |r_1 - r_2|\big)$;
$\forall k_1,k_2 \in \mathbb{R}^m, r_1, r_1 \in \mathbb{R}, \quad$ a.e. $x \in \Omega^\kappa$;
(c) $\mathcal{F}^\kappa(.,k,r)$ is measurable on Ω^κ, for all $k \in \mathbb{R}^m$, $r \in \mathbb{R}$;
(d) $\mathcal{F}^\kappa(.,0,0)$ belongs to $\mathcal{H}^\kappa$. H(4): (a) $\mathcal{G}^\kappa : \Omega^\kappa \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^d$;
(b) $|\mathcal{G}^\kappa(x,k_1,r_1) - \mathcal{G}^\kappa(x,k_2,r_2)| \leq L_{\mathcal{G}^\kappa}\big(|k_1 - k_2| + |r_1 - r_2|\big)$;
$\forall k_1,k_2 \in \mathbb{R}^m, r_1, r_1 \in \mathbb{R}, \quad$ a.e. $x \in \Omega^\kappa$.
(c) $\mathcal{G}^\kappa(.,k,r)$ is measurable on Ω^κ, for all $k \in \mathbb{R}^m$, $r \in \mathbb{R}$;
(d) $\mathcal{G}^\kappa(.,0,0)$ belongs to $\mathcal{H}^\kappa$.

H(5): (a) $\mathcal{R}^\kappa : \Omega^\kappa \times \mathbb{R}^d \to \mathbb{R}^d$;
(b) $\mathcal{R}^\kappa = (r^\kappa_{ij}),\ r^\kappa_{ij} = r^\kappa_{ji} \in L^\infty(\Omega^\kappa), \quad 1 \leq i,j \leq d$;
(c) $\mathcal{R}^\ell v.v \geq m_{\mathcal{R}^\kappa}|v|^2, \quad \forall v \in \mathbb{R}^d$, a.e. $x \in \Omega^\kappa$.

H(6): (a) $\Theta^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$;
(b) $|\Theta^\kappa(x,\varepsilon_1,r_1,k_1,d_1) - \Theta^\kappa(x,\varepsilon_2,r_2,k_2,d_2)| \leq$;
$L_{\Theta^\kappa}\big(|\varepsilon_1 - \varepsilon_2| + |r_1 - r_2| + |k_1 - k_2| + |d_1 - d_2|\big)$;
$\forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, k_1,k_2 \in \mathbb{R}^m, r_1,r_2,d_1,d_2 \in \mathbb{R}, \quad a.e.\ x \in \Omega^\kappa$;
(c) $\Theta^\kappa(.,\varepsilon,r,k,d)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d, k \in \mathbb{R}^m, r,d \in \mathbb{R}$;
(d) $\Theta^\kappa(.,0,0,0,0)$ belongs to $L^2(\Omega^\kappa)$.

H(7): (a) $\Psi^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$;
(b) $|\Psi^\kappa(x, \varepsilon_1, r_1, k_1, d_1) - \Psi^\kappa(x, \varepsilon_2, r_2, k_2, d_2)| \leq$;
$L_{\Psi^\kappa}\big(|\varepsilon_1 - \varepsilon_2| + |r_1 - r_2| + |k_1 - k_2| + |d_1 - d_2|\big)$;
$\forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, k_1, k_2 \in \mathbb{R}^m, r_1, r_2, d_1, d_2 \in \mathbb{R}$, *a.e.* $x \in \Omega^\kappa$;
(c) $\Psi^\kappa(., \varepsilon, r, k, d)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d, k \in \mathbb{R}^m, r, d \in \mathbb{R}$;
(d) $\Psi^\kappa(., 0, 0, 0, 0)$ belongs to $L^2(\Omega^\kappa)$.

H(8): (a) $S^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$;
(b) $|S^\kappa(x, \varepsilon_1, r_1) - S^\kappa(x, \varepsilon_2, r_2)| \leq L_{S^\kappa}\big(|\varepsilon_1 - \varepsilon_2| + |r_1 - r_2|\big)$;
$\forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \forall r_1, r_2 \in \mathbb{R}$ a.e. $x \in \Omega^\kappa$;
(c) $S^\kappa(., \varepsilon, r)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d,\ r \in \mathbb{R}$;
(d) $S^\kappa(., 0, 0)$ belongs to $L^2(\Omega^\kappa)$.

H(9): (a) $\mathcal{E}^\kappa : \Omega^\kappa \times \mathbb{S}^d \to \mathbb{R}^d$;
(b) $\mathcal{E}^\kappa = (e^\kappa_{ijk}),\ e^\kappa_{ijk} = e^\kappa_{ikj} \in L^\infty(\Omega^\kappa),\ 1 \leq i, j, k \leq d$;
(c) $\mathcal{E}^\kappa \varepsilon.\upsilon = \varepsilon.(\mathcal{E}^\kappa)^* \upsilon,\ \ \forall \varepsilon \in \mathbb{S}^d, \upsilon \in \mathbb{R}^d$.

We suppose that the mass density, the forces, the traction densities and the foundation's temperatures satisfy

H(10): (a) $\rho^\kappa \in L^\infty(\Omega^\kappa)$, $\exists \rho_0 > 0$; $\rho^\kappa(x) \geq \rho_0$ a.e. $x \in \Omega^\kappa$;
(b) $f^\kappa_0 \in L^2(0, T; L^2(\Omega^\kappa)^d)$, $f^\kappa_2 \in L^2(0, T; L^2(\Gamma^\kappa_2)^d)$;
(c) $q^\kappa_0 \in \mathcal{C}(0, T; L^2(\Omega^\kappa))$, $q^\kappa_2 \in \mathcal{C}(0, T; L^2(\Gamma^\kappa_b))$;
(d) $\chi^\kappa \in L^2(0, T; L^2(\Omega^\kappa))$.

The energy coefficient, microcrack diffusion coefficient and the friction yield limit g satisfy

H(11): $\mathcal{K}^\kappa_0,\ \mathcal{K}^\kappa_1 > 0$, $\ g \in L^\infty(\Gamma_3),\ g \geq 0$, a.e. on Γ_3.

Finally, we assume that the initial values satisfy the regularity

H(12): $\beta^\kappa_0 \in \mathbb{Y}^\kappa$, $u^\kappa_0 \in \mathcal{V}^\kappa$, $v^\kappa_0 \in \mathbb{H}^\kappa$, $\alpha^\kappa_0 \in \mathcal{Z}^\kappa$, $\tau^\kappa_0 \in E^\kappa_1$.

We will use a modified inner product on $\mathbb{H}$, given by

$$\langle\!\langle u, v\rangle\!\rangle_{\mathbb{H}} = \sum_{\kappa=1}^{2} \langle \rho^\kappa u^\kappa, v^\kappa \rangle_{\mathbb{H}^\kappa}, \quad \forall u, v \in \mathbb{H} \tag{2.21}$$

and let $|\!|\!|.|\!|\!|_{\mathbb{H}}$ be the associated norm. It follows from assumption H(8)(a) that $|\!|\!|.|\!|\!|_{\mathbb{H}}$ and $\|.\|_{\mathbb{H}}$ are equivalent norms on $\mathbb{H}$, and the inclusion mapping of $(\mathcal{V}, \|.\|_{\mathcal{V}})$ into $(\mathbb{H}, |\!|\!|.|\!|\!|_{\mathbb{H}})$ is continuous and dense. We denote by $\mathcal{V}'$ the dual of $\mathcal{V}$. Identify $\mathbb{H}$ with its own dual. Then

$$\langle u, v\rangle_{\mathcal{V}' \times \mathcal{V}} = \langle\!\langle u, v\rangle\!\rangle_{\mathbb{H}}, \quad \forall u \in \mathbb{H}, v \in \mathcal{V}. \tag{2.22}$$

We define five mappings $F : [0, T] \to \mathcal{V}'$, $Q : [0, T] \to \mathbb{W}$, $a_0 : E_1 \times E_1 \to \mathbb{R}$, a_1 :

$E_1 \times E_1 \to \mathbb{R}$ and $J : \mathcal{V} \to \mathbb{R}$, respectively, by

$$\langle F(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} = \sum_{\kappa=1}^{2}\int_{\Omega^\kappa} f_0^\kappa(t)\cdot v^\kappa dx + \sum_{\kappa=1}^{2}\int_{\Gamma_2^\kappa} f_2^\kappa(t)\cdot v^\kappa da \quad \forall v \in \mathcal{V}, \tag{2.23}$$

$$\langle Q(t), \zeta\rangle_{\mathbb{W}} = \sum_{\kappa=1}^{2}\int_{\Omega^\kappa} q_0^\kappa(t)\zeta^\kappa dx - \sum_{\kappa=1}^{2}\int_{\Gamma_b^\kappa} q_2^\kappa(t)\zeta^\kappa da \quad \forall \zeta \in \mathbb{W}, \tag{2.24}$$

$$a_0(\xi,\zeta) = \sum_{\kappa=1}^{2}\mathcal{K}_0^\kappa\int_{\Omega^\kappa}\nabla\xi^\kappa.\nabla\zeta^\kappa dx + \sum_{\kappa=1}^{2}\lambda_0^\kappa\int_{\Gamma^\kappa}\xi^\kappa\zeta^\kappa da, \tag{2.25}$$

$$a_1(\xi,\zeta) = \sum_{\kappa=1}^{2}\mathcal{K}_1^\kappa\int_{\Omega^\kappa}\nabla\xi^\kappa.\nabla\zeta^\kappa dx, \tag{2.26}$$

$$J(u) = \int_{\Gamma_3} g\left|u_\tau^1 - u_\tau^2\right| da. \tag{2.27}$$

We note that conditions H(10)(b) and H(10)(c) imply

$$F \in L^2(0,T;\mathcal{V}'), \quad Q \in \mathcal{C}(0,T;\mathbb{W}). \tag{2.28}$$

We now turn to deriving a variational formulation of the mechanical problem P. To that end we assume that $\{u^\kappa, \sigma^\kappa, \psi^\kappa, D^\kappa, \tau^\kappa, \alpha^\kappa, \beta^\kappa\}$ with $\kappa = 1,2$ are sufficiently smooth functions satisfying (2.1)–(2.15) and let $w = (w^1, w^2) \in \mathcal{V}$ and $t \in [0,T]$. First, we use Green's formula (2.18) and by (2.6) (2.8), (2.9) and (2.21)–(2.23), we find

$$\begin{aligned}\langle \ddot{u}(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum_{\kappa=1}^{2}\langle\sigma^\kappa, \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} &= \langle F(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &+ \sum_{\kappa=1}^{2}\int_{\Gamma_3}\sigma^\kappa(t)\nu^\kappa.(w^\kappa - \dot{u}^\kappa(t))da.\end{aligned} \tag{2.29}$$

Using now (2.9) and definition of $\mathcal{V}$, we achieve

$$\sum_{\kappa=1}^{2}\sigma^\kappa(t)\nu^\kappa.(w^\kappa - \dot{u}^\kappa(t)) = \sigma_\tau(t).\big((w_\tau^1 - w_\tau^2) - (\dot{u}_\tau^1(t) - \dot{u}_\tau^2(t))\big)$$

and use the frictional contact conditions (2.9) and the definition (2.27) to obtain

$$\sum_{\kappa=1}^{2}\int_{\Gamma_3}\sigma^\kappa(t)\nu^\kappa.(w^\ell - \dot{u}^\kappa(t))da \geq -J(w) + J(\dot{u}(t)). \tag{2.30}$$

Finally, we combine (2.1), (2.29) and (2.30) to deduce that

$$\begin{aligned}&\langle \ddot{u}(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum_{\kappa=1}^{2}\langle\mathcal{A}^\kappa\varepsilon(\dot{u}^\kappa) + \mathcal{B}^\kappa(\varepsilon(u^\kappa), \alpha^\kappa), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} \\ &+ \sum_{\kappa=1}^{2}\langle(\mathcal{E}^\kappa)^*\nabla\psi^\kappa, \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} + \sum_{\kappa=1}^{2}\langle\mathcal{F}^\kappa(\beta^\kappa, \tau^\kappa), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} \\ &+ J(w) - J(\dot{u}(t)) \geq \langle F(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}}.\end{aligned} \tag{2.31}$$

Similarly, let $\phi = (\phi^1, \phi^2) \in \mathbb{W}$ and $t \in [0,T]$, from (2.2), (2.7), (2.14), (2.19) and (2.24), we deduce that

$$\sum_{\kappa=1}^{2} \big\langle \mathcal{R}^\kappa \nabla\psi^\kappa(t) - \mathcal{E}^\kappa \varepsilon(u^\kappa(t)) - \mathcal{G}^\kappa(\beta^\kappa(t), \tau^\kappa(t)), \nabla\phi^\kappa \big\rangle_{\mathbb{H}^\kappa} = \langle Q(t), \phi\rangle_{\mathbb{W}}. \tag{2.32}$$

On the other hand, let $\xi = (\xi^1, \xi^2) \in \mathcal{Z}$ and $t \in [0,T]$. Then, using (2.5), we have

$$\begin{aligned}&\sum_{\kappa=1}^{2} \langle \dot\alpha^\kappa(t), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\kappa)} - \sum_{\kappa=1}^{2} \langle \mathcal{K}_1^\kappa \triangle \alpha^\kappa(t), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\kappa)}\\ &\geq \sum_{\kappa=1}^{2} \langle S^\kappa(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\kappa)}.\end{aligned}$$

Combining this inequality with (2.11), (2.19) and (2.26), we obtain

$$\begin{aligned}&\sum_{\kappa=1}^{2} \langle \dot\alpha^\kappa(t), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\kappa)} + a_1(\alpha(t), \xi - \alpha(t))\\ &\geq \sum_{\kappa=1}^{2} \langle S^\kappa(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\kappa)}.\end{aligned} \tag{2.33}$$

For the temperature, let $\delta = (\delta^1, \delta^2) \in E_1$ and $t \in [0,T]$. Using (2.4), (2.12) and (2.19), we have

$$\begin{aligned}&\sum_{\kappa=1}^{2} \Big\langle \Psi^\ell\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t), \beta^\kappa(t), \tau^\kappa(t)\big) + \chi^\kappa(t), \delta^\kappa \Big\rangle_{L^2(\Omega^\kappa)}\\ &= \sum_{\kappa=1}^{2} \langle \dot\tau^\kappa(t), \delta^\kappa\rangle_{L^2(\Omega^\kappa)} - \sum_{\kappa=1}^{2} \int_{\Omega^\kappa} \mathcal{K}_0^\kappa \Delta\tau^\kappa(t)\delta^\kappa dx\\ &= \sum_{\kappa=1}^{2} \langle \dot\tau^\kappa(t), \delta^\kappa\rangle_{L^2(\Omega^\kappa)} + \sum_{\kappa=1}^{2} \int_{\Omega^\kappa} \mathcal{K}_0^\kappa \nabla\tau^\kappa(t)\nabla\delta^\kappa dx + \sum_{\kappa=1}^{2} \int_{\Gamma^\kappa} \lambda_0^\kappa \tau^\kappa(t)\delta^\kappa da.\end{aligned}$$

We use now (2.25) in the previous equality to obtain

$$a_0(\tau(t), \delta) = \sum_{\kappa=1}^{2} \Big\langle \Psi^\ell\big(\varepsilon(u^\kappa), \alpha^\kappa, \beta^\kappa, \tau^\kappa\big)(t), \delta^\kappa \Big\rangle_{E_0^\kappa} - \sum_{\kappa=1}^{2} \langle \dot\tau^\kappa(t) - \chi^\kappa(t), \delta^\kappa\rangle_{E_0^\kappa}. \tag{2.34}$$

We now gather the constitutive law (2.3), the initial condition (2.15), inequalities (2.31), (2.33), and equalities (2.32), (2.34) to obtain the following weak formulation of the piezoelectric contact problem P.

Problem PV. Find $u = (u^1, u^2) : [0,T] \to \mathcal{V}$, $\psi = (\psi^1, \psi^2) : [0,T] \to \mathbb{W}$, $\tau = (\tau^1, \tau^2) : [0,T] \to E_1$, $\alpha = (\alpha^1, \alpha^2) : [0,T] \to E_1$ and $\beta = (\beta^1, \beta^2) : [0,T] \to \mathbb{Y}$

such that for a.e.$t \in (0, T)$,

$$\dot{\beta}^\kappa(t) = \Theta^\kappa\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t), \beta^\kappa(t), \tau^\kappa(t)\big) \quad \text{in } \Omega^\kappa, \kappa = 1, 2, \tag{2.35}$$

$$\left.\begin{array}{l} \langle \ddot{u}(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum\limits_{\kappa=1}^{2} \langle \mathcal{A}^\kappa \varepsilon(\dot{u}^\kappa(t)) + \mathcal{B}^\kappa(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} \\ + \sum\limits_{\kappa=1}^{2} \langle (\mathcal{E}^\kappa)^* \nabla\psi^\kappa(t), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} + \sum\limits_{\kappa=1}^{2} \langle \mathcal{F}^\kappa(\beta^\kappa(t), \tau^\kappa(t)), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} \\ + J(w) - J(\dot{u}(t)) \geq \langle F(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} \quad \forall w \in \mathcal{V}, \end{array}\right\} \tag{2.36}$$

$$\begin{array}{l} \sum\limits_{\kappa=1}^{2} \left\langle \mathcal{R}^\kappa \nabla\psi^\kappa(t) - \mathcal{E}^\kappa \varepsilon(u^\kappa(t)) - \mathcal{G}^\kappa(\beta^\kappa(t), \tau^\kappa(t)), \nabla\phi^\kappa \right\rangle_{\mathbb{H}^\kappa} = \langle Q(t), \phi\rangle_{\mathbb{W}}, \\ \forall \phi \in \mathbb{W}, \end{array} \tag{2.37}$$

$$\left.\begin{array}{l} \alpha(t) \in \mathcal{Z}, \quad \sum\limits_{\kappa=1}^{2} \langle \dot{\alpha}^\kappa(t), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\kappa)} + a(\alpha(t), \xi - \alpha(t)) \\ \geq \sum\limits_{\kappa=1}^{2} \langle S^\kappa\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)\big), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\ell)} \quad \forall \xi \in \mathcal{Z}, \end{array}\right\} \tag{2.38}$$

$$\left.\begin{array}{l} a_0(\tau(t), \delta) = \sum_{\kappa=1}^{2} \left\langle \Psi^\kappa\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t), \beta^\kappa(t), \tau^\kappa(t)\big), \delta^\kappa \right\rangle_{E_0^\kappa} \\ - \sum_{\kappa=1}^{2} \langle \dot{\tau}^\kappa(t) - \chi^\kappa(t), \delta^\kappa\rangle_{E_0^\kappa} \quad \forall \delta \in E_1, \end{array}\right\} \tag{2.39}$$

$$\begin{array}{l} u(0) = (u_0^1, u_0^2), \; \dot{u}(0) = (v_0^1, v_0^2), \; \alpha(0) = (\alpha_0^1, \alpha_0^2), \; \beta(0) = (\beta_0^1, \beta_0^2), \\ \tau(0) = (\tau_0^1, \tau_0^2). \end{array} \tag{2.40}$$

The existence of a unique solution to Problem **PV** will be presented in the next section.

3 Main Existence and Uniqueness Result

Now, we propose our existence and uniqueness result.

Theorem 3.1 *Under the assumptions H(1)–H(12), there exists a unique solution* $\{u, \psi, \tau, \alpha, \beta\}$ *to problem PV. Moreover, the solution satisfies*

$$u \in W^{1,2}(0, T; \mathcal{V}) \cap \mathcal{C}^1(0, T; \mathbb{H}) \cap W^{2,2}(0, T; \mathcal{V}'), \tag{3.1}$$

$$\psi \in \mathcal{C}(0, T; \mathbb{W}), \tag{3.2}$$

$$\tau \in W^{1,2}(0, T; E_0) \cap L^2(0, T; E_1), \tag{3.3}$$

$$\alpha \in W^{1,2}(0, T; \mathbb{Y}), \tag{3.4}$$

$$\beta \in W^{1,2}(0, T; E_0) \cap L^2(0, T; E_1). \tag{3.5}$$

The functions $\{\sigma, D, u, \psi, \tau, \alpha, \beta\}$, which satisfy (2.1), (2.2) and (2.35)–(2.40) , are called the weak solution of the thermo-piezoelectric contact Problem P. We conclude by Theorem 3.1 that, under the assumptions H(1)–H(12), the mechanical problem (2.1)–(2.15) has a unique weak solution $\{\sigma, D, u, \psi, \tau, \alpha, \beta\}$. To precuse the regularity of the weak solution, we note that the constitutive relation (2.1)–(2.2), the assumptions H(1)–H(5), H(9) and the regularities (3.1)–(3.3) show that $\sigma \in \mathcal{C}(0, T; \mathcal{H})$ and $D \in \mathcal{C}(0, T; \mathbb{H})$. We test (2.36) with $v^\kappa \in \mathcal{C}_0^\infty(\Omega^\kappa; \mathbb{R}^d)$ and $v^{3-\kappa} = 0$. Then we take $\phi^\kappa \in \mathcal{C}_0^\infty(\Omega^\kappa)$ and $\phi^{3-\kappa} = 0$ in (2.37) to obtain that

$$\operatorname{Div} \sigma^\kappa(t) + f_0^\kappa(t) = \rho^\kappa \ddot{u}^\kappa(t), \quad \operatorname{div} D^\kappa(t) = q_0^\kappa(t),$$

almost everywhere in Ω^κ for a.e. $t \in (0,T)$ and $\kappa = 1,2$. Next, we use assumptions H(10) to deduce that $\operatorname{Div}\sigma^\kappa \in L^2(0,T;\mathbb{H}^\kappa)$, $\operatorname{div} D^\kappa \in \mathcal{C}(0,T;E_0^\kappa)$, $\kappa = 1,2$, which shows that

$$\sigma \in L^2(0,T;\mathcal{H}_1), \quad D \in \mathcal{C}(0,T;\mathcal{W}). \tag{3.6}$$

We conclude that the weak solution $\{\sigma, D, u, \psi, \tau, \alpha, \beta\}$ of the thermo-piezoelectric contact Problem P has the regularity (3.1)–(3.6).

The proof of Theorem 3.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 3.1 hold, and let a $\eta = (\eta^1, \eta^2) \in L^2(0,T;\mathcal{V}')$ be given. In the first step, we consider the following variational problem.

Problem P_{u_η}. Find $u_\eta = (u_\eta^1, u_\eta^2) : [0,T] \to \mathcal{V}$ such that for a.e. $t \in (0,T)$,

$$\left.\begin{array}{l} \langle \ddot{u}_\eta(t), w - \dot{u}_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum\limits_{\kappa=1}^{2} \left\langle \mathcal{A}^\kappa \varepsilon(\dot{u}_\eta^\kappa(t)), \varepsilon(w^\kappa - \dot{u}_\eta^\kappa(t))\right\rangle_{\mathcal{H}^\kappa} \\ +J(w) - J(\dot{u}_\eta(t)) \geq \langle F(t) - \eta(t), w - \dot{u}_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}}, \ \forall w \in \mathcal{V}, \\ u_\eta(0) = (u_0^1, u_0^2), \quad \dot{u}_\eta(0) = (v_0^1, v_0^2). \end{array}\right\} \tag{3.7}$$

We define the mappings $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ and $F_\eta : [0,T] \longrightarrow \mathcal{V}'$, respectively, by

$$\begin{array}{l} \langle \mathcal{A}u, v\rangle_{\mathcal{V}'\times\mathcal{V}} = \sum\limits_{\kappa=1}^{2} \langle \mathcal{A}^\kappa \varepsilon(u^\kappa), \varepsilon(v^\kappa)\rangle_{\mathcal{H}^\kappa}, \quad \forall u, v \in \mathcal{V}, \\ \langle F_\eta(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} = \langle F(t) - \eta(t), v\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad \forall t \in [0,T], \ v \in \mathcal{V}. \end{array} \tag{3.8}$$

Use velocities $v_\eta^\kappa = \dot{u}_\eta^\kappa$ with $\kappa = 1,2$. So, Problem P_{u_η} has been rewritten.

Problem P_{v_η}. Find $v_\eta = (v_\eta^1, v_\eta^2) : [0,T] \to \mathcal{V}$ such that for a.e. $t \in (0,T)$,

$$\left.\begin{array}{l} \langle \dot{v}_\eta(t), w - v_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \langle \mathcal{A}v_\eta(t), w - v_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + J(w) - J(v_\eta(t)) \\ \geq \langle F_\eta(t), w - v_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad \forall w \in \mathcal{V}. \\ v_\eta(0) = (v_0^1, v_0^2). \end{array}\right\} \tag{3.9}$$

Lemma 3.1 *Assume that H(1) and H(11) hold, then the mappings $\mathcal{A}$ and J defined, respectively, by (3.8) and (2.27) satisfy*

$$\left\{\begin{array}{l} (a) \ \ \mathcal{A} : \mathcal{V} \to \mathcal{V}' \ \textit{is semi-continuous and strongly monotonous}, \\ (b) \ \ \exists\, C_{\mathcal{A}}^1 \geq 0, \ \exists C_{\mathcal{A}}^2 \geq 0 \ \textit{such that} \ \ \|\mathcal{A}u\|_{\mathcal{V}'} \leq C_{\mathcal{A}}^1 \|u\|_{\mathcal{V}} + C_{\mathcal{A}}^2, \quad \forall u \in \mathcal{V}, \\ (c) \ \ \textit{for all sequence } (u_k) \textit{ and } u \textit{ in } L^2(0,T;\mathcal{V}) \textit{ such that } \ u_k \rightharpoonup u \textit{ weakly in } L^2(0,T;\mathcal{V}), \\ \qquad \mathcal{A}u_k \rightharpoonup \mathcal{A}u \textit{ weakly in } L^2(0,T;\mathcal{V}') \\ \qquad \textit{and} \quad \lim_{\mathrm{k}\to+\infty} \ \inf \int_0^T \langle \mathcal{A}u_k(s), u_k(s)\rangle_{\mathcal{V}'\times\mathcal{V}} ds \geq \int_0^T \langle \mathcal{A}u(s), u(s)\rangle_{\mathcal{V}'\times\mathcal{V}} ds \end{array}\right.$$

$$\left\{\begin{array}{l} (a') \ J : \mathcal{V} \to \mathbb{R} \textit{ is a convex and lower semi-continuous functional.} \\ \qquad \textit{There exists a sequence of } \mathcal{C}^1 \textit{ convex functions } (J_k) : \mathcal{V} \to \mathbb{R} \\ (b') \ \ \exists\, C_g \geq 0 \quad \textit{such that} \quad \|J_k'(u)\|_{\mathcal{V}'} \leq C_g, \quad \forall k \in \mathbb{N}, \quad \forall u \in \mathcal{V}, \\ (c') \ \lim_{\mathrm{k}\to+\infty} \int_0^T J_k(u(s)) ds = \int_0^T J(u(s)) ds, \quad \forall u \in L^2(0,T;\mathcal{V}), \\ (d') \ \ \textit{There exists a sequence } (u_k) \textit{ and } u \textit{ in } L^2(0,T;\mathcal{V}) \textit{ such that} \\ \qquad u_k \rightharpoonup u \textit{ weakly in } L^2(0,T;\mathcal{V}), \ \textit{then} \lim_{\mathrm{k}\to+\infty} \ \inf \int_0^T J_k(u_k(s)) ds \geq \int_0^T J(u(s)) ds, \end{array}\right.$$

where $J_k'(u)$ is the Fréchet derivative of J_k at u.

Proof. From the definition (3.8) and assumption H(1), we can verify that $\mathcal{A}$ satisfies the conditions (a)-(b), and applying the Lebesgue theorem, we deduce the condition (c). On the other hand, by using the continuous embedding $\mathcal{V} \hookrightarrow L^2(\Gamma_3)^d$, we find that J is convex and continuous. To approximate the function J, we use the following functional $J_k : \mathcal{V} \to \mathbb{R}$ defined by

$$J_k(u) = \int_{\Gamma_3} g\sqrt{\left|u_\tau^1 - u_\tau^2\right|^2 + k^{-1}}\, da, \quad \forall u = (u^1, u^2) \in \mathcal{V},\ \forall k \in \mathbb{N}^*.$$

We verify that the Fréchet derivative of J_k at $u = (u^1, u^2)$ is given by

$$\langle J_k'(u), h\rangle_{\mathcal{V}'\times\mathcal{V}} = \int_{\Gamma_3} g\frac{(u_\tau^1 - u_\tau^2, h_\tau^1 - h_\tau^2)_{\mathbb{R}^d}}{\sqrt{\left|u_\tau^1 - u_\tau^2\right|^2 + k^{-1}}} da, \quad \forall h = (h^1, h^2) \in \mathcal{V}. \tag{3.10}$$

Then J_k is of class $\mathcal{C}^1$. Direct algebraic computations show that for all $a \geq 0$, $b \geq 0$ such that $a + b = 1$, and for all reals x and y, $k \geq 1$,

$$\sqrt{(ax + by)^2 + k^{-1}} \leq a\sqrt{x^2 + k^{-1}} + b\sqrt{y^2 + k^{-1}}.$$

Then J_k is convex for all $k \in \mathbb{N}^*$. From (3.10), it follows that

$$\exists c \geq 0,\ \forall u \in \mathcal{V},\ \|J_k'(u)\|_{\mathcal{V}'} \leq c\|g\|_{L^\infty(\Gamma_3)},$$

therefore (b') is satisfied. From the definition of J_k, we have $\lim_{k\to+\infty} J_k(u) = J(u)$ and as J_k is continuous on $\mathcal{V}$, applying the Lebesgue theorem, we deduce the property (c'). Finally, (d') is a consequence of the fact that

$$\forall u \in \mathcal{V},\ \forall k \in \mathbb{N}^*, \quad J_k(u) \geq J(u),$$

which finishes the proof.

Lemma 3.2 *Problem P_{v_η} has a unique solution v_η which satisfies*

$$v_\eta \in \mathcal{C}(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap W^{1,2}(0, T; \mathcal{V}').$$

The proof of Lemma 3.2 is found in [9, p.48].

Let now $u_\eta = (u_\eta^1, u_\eta^2) : [0, T] \to \mathcal{V}$ be the function defined by

$$u_\eta^\kappa(t) = \int_0^t v_\eta^\kappa(s) ds + u_0^\kappa, \quad \forall t \in [0, T], \quad \kappa = 1, 2. \tag{3.11}$$

In the study of Problem P_{u_η}, we have the following result.

Lemma 3.3 *P_{u_η} has a unique solution satisfying the regularity expressed in (3.1).*

Proof. The proof of Lemma 3.3 is a consequence of Lemma 3.2 and the relation (3.11). In the second step, let $\pi = (\pi^1, \pi^2) \in L^2(0, T; E_0)$ and consider the auxiliary problem.

Problem P_{τ_π}. Find $\tau_\pi = (\tau_\pi^1, \tau_\pi^2) : [0, T] \to E_0$ such that for a.e. $t \in (0; T)$,

$$\sum_{\kappa=1}^2 \langle \dot{\tau}_\pi^\kappa(t) - \pi^\kappa(t) - \chi^\kappa(t), \delta^\kappa\rangle_{E_0^\kappa} + a_0(\tau_\pi(t), \delta) = 0,\ \forall \delta \in E_0, \tag{3.12}$$

$$\tau_\pi(0) = (\tau_0^1, \tau_0^2). \tag{3.13}$$

Lemma 3.4 *There exists a unique solution τ_π to the auxiliary problem P_{τ_π} satisfying* (3.3).

Proof. The proof of Lemma 3.4 is a consequence of the Poincaré-Friedrichs inequality and the definitions (2.25) of the operator $a_0(.,.)$.

In the third step, let $\mu = (\mu^1, \mu^2) \in L^2(0, T, \mathbb{Y})$ be given, and define $\beta_\mu = (\beta^1_\mu, \beta^2_\mu) \in W^{1,2}(0, T, \mathbb{Y})$ by

$$\beta^\kappa_\mu(t) = \beta^\kappa_0 + \int_0^t \mu^\kappa(s)ds, \quad \kappa = 1, 2. \tag{3.14}$$

We use $u_\eta = (u^1_\eta, u^2_\eta)$ obtained in Lemma 3.3 and $\tau_\pi = (\tau^1_\pi, \tau^2_\pi)$ obtained in Lemma 3.4 to construct the following variational problem.

Problem $P_{\psi_{\eta\pi\mu}}$. Find $\psi_{\eta\pi\mu} = (\psi^1_{\eta\pi\mu}, \psi^2_{\eta\pi\mu}) : [0, T] \to \mathbb{W}$ such that for a.e. $t \in (0, T)$,

$$\begin{aligned}&\textstyle\sum_{\kappa=1}^2 \left\langle \mathcal{R}^\kappa \nabla \psi^\kappa_{\eta\pi\mu}(t), \nabla\phi^\kappa \right\rangle_{\mathbb{H}^\kappa} - \sum_{\kappa=1}^2 \left\langle \mathcal{E}^\kappa \varepsilon(u^\kappa_\eta(t)) + \mathcal{G}^\kappa(\beta^\kappa_\mu(t), \tau^\kappa_\pi(t)), \nabla\phi^\kappa \right\rangle_{\mathbb{H}^\kappa}\\ &= (Q(t), \phi)_{\mathbb{W}} \quad \forall\phi \in \mathbb{W}.\end{aligned} \tag{3.15}$$

We have the following result.

Lemma 3.5 *Problem $P_{\psi_{\eta\pi\mu}}$ has a unique solution $\psi_{\eta\pi\mu} = (\psi^1_{\eta\pi\mu}, \psi^2_{\eta\pi\mu})$ which satisfies the regularity* (3.2).

Proof. We define a bilinear form $b(.,.) : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ by

$$b(\psi, \phi) = \sum_{\kappa=1}^2 \langle \mathcal{R}^\kappa \nabla\psi^\kappa, \nabla\phi^\kappa \rangle_{\mathbb{H}^\kappa}, \quad \forall\psi, \phi \in \mathbb{W}. \tag{3.16}$$

We use H(5) and (3.16) to show that the bilinear form $b(.,.)$ is continuous, symmetric and coercive on $\mathbb{W}$, moreover, using (2.24) and the Riesz representation theorem, we may define an element $Q_{\eta\pi\mu} : [0, T] \to \mathbb{W}$ such that

$$\begin{aligned}&\langle Q_{\eta\pi\mu}(t), \phi\rangle_{\mathbb{W}} = \langle Q(t), \phi\rangle_{\mathbb{W}} + \textstyle\sum_{\kappa=1}^2 \left\langle \mathcal{E}^\kappa \varepsilon(u^\kappa_\eta(t)) + \mathcal{G}^\kappa(\beta^\kappa_\mu(t), \tau^\kappa_\pi(t)), \nabla\phi^\kappa \right\rangle_{\mathbb{H}^\kappa}\\ &\quad \forall\phi \in \mathbb{W},\ t \in (0, T).\end{aligned}$$

We apply the Lax-Milgram theorem to deduce that there exists a unique element $\psi_{\eta\pi\mu}(t) = (\psi^1_{\eta\pi\mu}(t), \psi^2_{\eta\pi\mu}(t)) \in \mathbb{W}$ such that

$$b(\psi_{\eta\pi\mu}(t), \phi) = \langle Q_{\eta\pi\mu}(t), \phi\rangle_{\mathbb{W}} \quad \forall\phi \in \mathbb{W}. \tag{3.17}$$

We conclude that $\psi_{\eta\pi\mu}$ is a solution of Problem $P_{\psi_{\eta\pi\mu}}$. Let $t_1, t_2 \in [0, T]$, it follows from (3.15) that

$$\begin{aligned}\|\psi_{\eta\pi\mu}(t_1) - \quad &\psi_{\eta\pi\mu}(t_2)\|_{\mathbb{W}} \le C\big(\|u_\eta(t_1) - u_\eta(t_2)\|_{\mathcal{V}} + \|\beta_\mu(t_1) - \beta_\mu(t_2)\|_{\mathbb{Y}}\\ &+\|\tau_\pi(t_1) - \tau_\pi(t_2)\|_{E_0} + \|Q(t_1) - Q(t_2)\|_{\mathbb{W}}\big).\end{aligned} \tag{3.18}$$

Due to (2.28), (3.2), (3.3) and $\beta_\mu \in W^{1,2}(0, T; \mathbb{Y})$, inequality (3.18) implies that $\psi_{\eta\pi\mu} \in \mathcal{C}(0, T; \mathbb{W})$. In the fourth step, let $\theta = (\theta^1, \theta^1) \in L^2(0.T; E_0)$ be given and consider the following initial-value problem.

Problem P_{α_θ}. *Find* $\alpha_\theta = (\alpha_\theta^1, \alpha_\theta^2) : [0,T] \to E_1$ such that for a.e. $t \in (0,T)$,

$$\alpha_\theta(t) \in \mathcal{Z}, \quad \sum_{\kappa=1}^{2} \langle \dot\alpha_\theta^\kappa(t) - \theta^\kappa(t), \mu^\kappa - \alpha_\theta^\kappa(t) \rangle_{L^2(\Omega^\kappa)} + a_1(\alpha_\theta(t), \mu - \alpha_\theta(t)) \geq 0, \quad \forall \mu \in \mathcal{Z}. \tag{3.19}$$

In the study of Problem P_{α_θ}, we have the following result.

Lemma 3.6 *The problem P_{α_θ} has a unique solution $\alpha_\theta = (\alpha_\theta^1, \alpha_\theta^2)$ which satisfies the regularity (3.5).*

Proof. We use a standard result for parabolic variational inequalities [9, p.47]. Finally, we now pass to the final step of the proof of Theorem 3.1 in which we use a fixed point argument. To this end, we consider the mapping

$$\Sigma : L^2(0,T; \mathcal{V}' \times \mathbb{Y} \times E_0 \times E_0) \to L^2(0,T; \mathcal{V}' \times \mathbb{Y} \times E_0 \times E_0)$$

defined by

$$\Sigma(\eta,\mu,\pi,\theta) = \big(\Sigma_1(\eta,\mu,\pi,\theta), \Sigma_2(\eta,\mu,\pi,\theta), \Sigma_3(\eta,\mu,\pi,\theta), \Sigma_4(\eta,\mu,\pi,\theta)\big) \tag{3.20}$$

with

$$\langle \Sigma_1(\eta,\mu,\pi,\theta)(t), v \rangle_{\mathcal{V}'\times\mathcal{V}} = \sum_{\kappa=1}^{2} \Big\langle \mathcal{B}^\kappa(\varepsilon(u_\eta^\kappa(t)), \alpha_\theta^\kappa(t)) + \big(\mathcal{E}^\kappa\big)^* \nabla \psi_{\eta\pi\mu}^\kappa(t),\ \varepsilon(v^\kappa) \Big\rangle_{\mathcal{H}^\kappa}$$

$$+ \sum_{\kappa=1}^{2} \big\langle \mathcal{F}^\kappa(\beta_\mu^\kappa(t), \tau_\pi^\kappa(t)),\ \varepsilon(v^\kappa) \big\rangle_{\mathcal{H}^\kappa}, \quad \forall v \in \mathcal{V}, \tag{3.21}$$

$$\Sigma_2(\eta,\mu,\pi,\theta)(t) = \Big(\Theta^1\big(\varepsilon(u_\eta^1(t)), \alpha_\theta^1(t), \beta_\mu^1(t), \tau_\pi^1(t)\big)\ ,\ \Theta^2\big(\varepsilon(u_\eta^2(t)), \alpha_\theta^2(t), \beta_\mu^2(t), \tau_\pi^2(t)\big)\Big), \tag{3.22}$$

$$\Sigma_3(\eta,\mu,\pi,\theta)(t) = \Big(\Psi^1\big(\varepsilon(u_\eta^1(t)), \alpha_\theta^1(t), \beta_\mu^1(t), \tau_\pi^1(t)\big)\ ,\ \Psi^2\big(\varepsilon(u_\eta^2(t)), \alpha_\theta^2(t), \beta_\mu^2(t), \tau_\pi^2(t)\big)\Big), \tag{3.23}$$

$$\Sigma_4(\eta,\mu,\pi,\theta)(t) = \Big(S^1(\varepsilon(u_\eta^1(t)), \alpha_\theta^1(t))\ ,\ S^2(\varepsilon(u_\eta^2(t)), \alpha_\theta^2(t))\Big). \tag{3.24}$$

We have the following result.

Lemma 3.7 *The operator Σ has a unique fixed point $(\eta_*, \mu_*, \pi_*, \theta_*) \in L^2(0,T; \mathcal{V}' \times \mathbb{Y} \times E_0 \times E_0)$.*

Proof. Let $(\eta_1,\mu_1,\pi_1,\theta_1)$, $(\eta_2,\mu_2,\pi_2,\theta_2)$ in $L^2(0,T; \mathcal{V}' \times \mathbb{Y} \times E_0 \times E_0)$ and let $t \in [0,T]$. For simplicity, we use the notation $u_i = u_{\eta_i}$, $v_i = \dot u_{\eta_i}$, $\psi_i = \psi_{\eta_i\pi_i\mu_i}$, $\beta_i = \beta_{\mu_i}$, $\tau_i = \tau_{\pi_i}$ and $\alpha_i = \alpha_{\theta_i}$ for $i = 1,2$. From the definition (3.20)–(3.24) combined with the assumptions H(2), H(3) and H(6)–H(9), we conclude that there is $C > 0$ such that

$$\big\|\Sigma\big(\eta_1,\mu_1,\pi_1,\theta_1\big)(t) - \Sigma\big(\eta_2,\mu_2,\pi_2,\theta_2\big)(t)\big\|_{\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0}^2 \leq C\big(\big\|u_1(t) - u_2(t)\big\|_{\mathcal{V}}^2$$

$$+ \big\|\psi_1(t) - \psi_2(t)\big\|_{\mathbb{W}}^2 + \big\|\beta_1(t) - \beta_2(t)\big\|_{\mathbb{Y}}^2 + \big\|\tau_1(t) - \tau_2(t)\big\|_{E_0}^2 + \big\|\alpha_1(t) - \alpha_2(t)\big\|_{E_0}^2\big). \tag{3.25}$$

Moreover, from (3.11), we have

$$\|u_1(t) - u_2(t)\|_{\mathcal{V}} \le \int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{V}} ds, \quad \forall t \in [0, T]. \tag{3.26}$$

Substituting $\eta = \eta_1$, $w = v_2$ and $\eta = \eta_2$, $w = v_1$ in (3.7), we find

$$\langle \dot{v}_1 - \dot{v}_2, v_1 - v_2\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum_{\kappa=1}^{2} \langle \mathcal{A}^\kappa \varepsilon(v_1^\kappa) - \mathcal{A}^\kappa \varepsilon(v_2^\kappa), \varepsilon(v_1^\kappa - v_2^\kappa)\rangle_{\mathcal{H}^\kappa} + \langle \eta_1 - \eta_2, v_1 - v_2\rangle_{\mathcal{V}'\times\mathcal{V}} \le 0.$$

We integrate this inequality with respect to time, use the initial conditions $v_1(0) = v_2(0) = (v_0^1, v_0^2)$, the assumption H(1)(c) and the inequality $\langle \dot{v}_1 - \dot{v}_2, v_1 - v_2\rangle_{\mathcal{V}'\times\mathcal{V}} \ge 0$ to find that

$$\min(m_{\mathcal{A}^1}, m_{\mathcal{A}^2}) \int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{V}}^2 ds \le - \int_0^t \langle \eta_1(s) - \eta_2(s), v_1(s) - v_2(s)\rangle_{\mathcal{V}'\times\mathcal{V}} ds.$$

Then, using the inequality $2ab \le \frac{a^2}{\epsilon} + \epsilon b^2$, we obtain

$$\int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{V}}^2 ds \le C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{V}'}^2 ds, \tag{3.27}$$

where C is a positive constant that may change from line to line.
From (3.26) and (3.27), we deduce

$$\|u_1(t) - u_2(t)\|_{\mathcal{V}}^2 \le C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{V}'}^2 ds. \tag{3.28}$$

The definition (3.14) yields

$$\|\beta_1(t) - \beta_2(t)\|_{\mathbb{Y}}^2 \le \int_0^t \|\mu_1(s) - \mu_2(s)\|_{\mathbb{Y}}^2 ds. \tag{3.29}$$

On the other hand, from (3.12), we can write

$$\begin{aligned} &\left\langle \dot{\tau}_1(t) - \dot{\tau}_2(t), \tau_1(t) - \tau_2(t)\right\rangle_{E_0} + a_0(\tau_1(t) - \tau_2(t), \tau_1(t) - \tau_2(t)) \\ &= \left\langle \pi_1(t) - \pi_2(t), \tau_1(t) - \tau_2(t)\right\rangle_{E_0} \quad a.e. t \in (0, T). \end{aligned}$$

We integrate this equality with respect to time, and use the initial conditions $\tau_1(0) = \tau_2(0) = (\tau_0^1, \tau_0^2)$ and inequality $a_0(\tau_1 - \tau_2, \tau_1 - \tau_2) \ge 0$ to find

$$\frac{1}{2}\|\tau_1(t) - \tau_2(t)\|_{E_0}^2 \le \int_0^t \|\pi_1(s) - \pi_2(s)\|_{E_0} . \|\tau_1(s) - \tau_2(s)\|_{E_0} ds.$$

Then, using the inequality $2ab \le a^2 + b^2$, we obtain

$$\|\tau_1(t) - \tau_2(t)\|_{E_0}^2 \le \int_0^t \|\pi_1(s) - \pi_2(s)\|_{E_0}^2 ds + \int_0^t \|\tau_1(s) - \tau_2(s)\|_{E_0}^2 ds$$

and, by using Gronwall's inequality, we obtain

$$\|\tau_1(t) - \tau_2(t)\|_{E_0}^2 \le C \int_0^t \|\pi_1(s) - \pi_2(s)\|_{E_0}^2 ds \quad \text{a.e. } t \in (0, T). \tag{3.30}$$

Also, (3.15) and the arguments similar to those used in the proof of (3.18) yield

$$\|\psi_1(t)-\psi_2(t)\|_{\mathbb{W}} \le C\big(\|u_1(t)-u_2(t)\|_{\mathcal{V}}+\|\beta_1(t)-\beta_2(t)\|_{\mathbb{Y}}+\|\tau_1(t)-\tau_2(t)\|_{E_0}\big) \ \ a.e.\ t\in(0,T). \tag{3.31}$$

Furthermore, by substituting $\theta=\theta_1,\ \mu=\alpha_1$ and $\theta=\theta_2,\ \mu=\alpha_2$ in (3.19) and subtracting the two inequalities obtained, we find

$$\begin{aligned}&(\dot\alpha_1(t)-\dot\alpha_2(t),\alpha_1(t)-\alpha_2(t))_{E_0}+a_1(\alpha_1(t)-\alpha_2(t),\alpha_1(t)-\alpha_2(t))\\ &\qquad\le(\theta_1(t)-\theta_2(t),\alpha_1(t)-\alpha_2(t))_{E_0},\quad \text{a.e. } t\in(0,T).\end{aligned}$$

We integrate the previous inequality and applying the inequality of Hölder and Young with Gronwall's lemma, we deduce that

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{E_0}\le C\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{E_0}ds \quad \text{a.e. } t\in(0,T). \tag{3.32}$$

We substitute (3.28)–(3.32) in (3.25), we obtain

$$\begin{aligned}&\big\|\Sigma\big(\eta_1,\mu_1,\pi_1,\theta_1\big)(t)-\Sigma\big(\eta_2,\mu_2,\pi_2,\theta_2\big)(t)\big\|^2_{\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0}\le\\ &C\textstyle\int_0^t\big\|(\eta_1,\mu_1,\pi_1,\theta_1\big)(s)-\big(\eta_2,\mu_2,\pi_2,\theta_2\big)(s)\big\|^2_{\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0}ds \quad \text{a.e. } t\in(0,T).\end{aligned}$$

Reiterating this inequality n times leads to

$$\begin{aligned}&\big\|\Sigma^n\big(\eta_1,\mu_1,\pi_1,\theta_1\big)-\Sigma^n\big(\eta_2,\mu_2,\pi_2,\theta_2\big)\big\|^2_{L^2(0,T;\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0)}\le\\ &\tfrac{C^nT^n}{n!}\big\|(\eta_1,\mu_1,\pi_1,\theta_1\big)-\big(\eta_2,\mu_2,\pi_2,\theta_2\big)\big\|^2_{L^2(0,T;\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0)}.\end{aligned}$$

Thus, for n sufficiently large, Σ^n is a contraction on the Banach space $L^2(0,T;\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0)$, and so Σ has a unique fixed point.

Now, we have all the ingredients to prove Theorem 3.1.

Proof. Let $(\eta_*,\mu_*,\pi_*,\theta_*\big)\in L^2(0,T;\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0)$ be the fixed point Σ defined by (3.20)–(3.24) and denote

$$u_*=u_{\eta_*},\quad \tau_*=\tau_{\pi_*},\quad (\dot u^\kappa_*(t)),\varepsilon(w^\kappa-\psi_*=\psi_{\eta_*\pi_*\mu_*},\quad \alpha_*=\alpha_{\theta_*},\quad \beta_*=\beta_{\mu_*}. \tag{3.33}$$

We prove $\{u_*,\psi_*,\tau_*,\alpha*,\beta_*\}$ satisfies (2.35)–(2.40) and the regularities (3.1)–(3.5). Indeed, we write (3.7) for $\eta=\eta_*$ and use (3.33) to find

$$\begin{aligned}&\langle\ddot u_*(t),w-\dot u_*(t)\rangle_{\mathcal{V}'\times\mathcal{V}}+\sum_{\kappa=1}^{2}\langle\mathcal{A}^\kappa\varepsilon\dot u^\kappa_*(t))\rangle_{\mathcal{H}^\kappa}+J(w)-J(\dot u_*(t))\\ &+\langle\eta_*(t),w-\dot u_*(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\ge\langle F(t),w-\dot u_*(t)\rangle_{\mathcal{V}'\times\mathcal{V}},\quad \forall w\in\mathcal{V},\ a.e.\ t\in(0,T).\end{aligned} \tag{3.34}$$

Equation $\Sigma_1\big(\eta_*,\mu_*,\pi_*,\theta_*\big)=\eta_*$ combined with (3.21) shows that $\ for\ \ a.e.t\in(0,T)$,

$$\langle\eta_*(t),v\rangle_{\mathcal{V}'\times\mathcal{V}}=\sum_{\kappa=1}^{2}\Big\langle\mathcal{B}^\kappa(\varepsilon(u^\kappa_*(t)),\alpha^\kappa_*(t))+\mathcal{F}^\kappa(\beta^\kappa_*(t),\tau^\kappa_*(t))+\big(\mathcal{E}^\kappa\big)^*\nabla\psi^\kappa_*(t),\ \varepsilon(v^\kappa)\Big\rangle_{\mathcal{H}^\kappa}, \tag{3.35}$$

We substitute (3.35) in (3.34) and use (3.33) to see that (2.36) is satisfied. From $\Sigma_2\big(\eta_*,\mu_*,\pi_*,\theta_*\big)=\mu_*$ and (3.14), we see that (2.35) is satisfied. We write now (3.15) for

$(\eta, \pi, \mu) = (\eta_*, \pi_*, \mu_*)$ and use (3.33) to find (2.37). The equalities $\Sigma_3\big(\eta_*, \mu_*, \pi_*, \theta_*\big) = \pi_*$ and $\Sigma_4\big(\eta_*, \mu_*, \pi_*, \theta_*\big) = \theta_*$, combined with (3.12), (3.19) show that (2.38)–(2.39) are satisfied. Next, (2.40) and the regularity (3.1)–(3.5) follow from Lemmas 3.1, 3.4, 3.5 and 3.6 and the relation (3.14), which concludes the existence part of Theorem 3.1. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Σ defined by (3.20)–(3.24) combined with the unique solvability of Problems P_{u_η}, P_{τ_π}, $P_{\varphi_{\eta\pi\mu}}$ and P_{α_θ}.

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