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Exact Controllability of the Reaction-Diffusion Equation under Bilinear Control

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Abstract: The goal of this paper is to study the global exact controllability of a reaction-diffusion equation in a bounded domain with Dirichlet boundary conditions. We will first consider the case of bilinear equation, then we identify a set of target states that can be exactly reached at any a priori given time. This result is then applied to prove the exact controllability of semilinear reaction-diffusion equation under distributed controls. The approach is constructive and based on linear semigroup theory and null controllability properties of linear problems.

Keywords: *exact controllability; reaction-diffusion equation; bilinear control.*

Mathematics Subject Classification (2010): 35K57, 35K58, 93C20.

1 Introduction

This paper deals with the controllability of the following semilinear reaction-diffusion equation:

$$\begin{cases} y_t = \Delta y + q(x,t)y + f(y), & \text{in } Q_T \ \ (T>0), \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with a boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. Here, $q \in L^\infty(Q_T)$ is a control function with the corresponding solution $y = y(x,t)$. The nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be a Lipschitz function such that $f(0) = 0$.

In terms of applications, the equation like (1) provides the practical description of various real problems such as chemical reactions, nuclear chain reactions, biomedical

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models etc. (see [2, 9, 10, 14, 19, 20] and the references therein). Equation (1) can be also used to describe a diffusion process with $y(x,t)$ being the concentration of a substance at the point x at time t, or a heat-transfer process, with $y(x,t)$ describing the temperature at the point x and time t (see [5] and [14], p. 17). It is shown in [2] that the equation like (1) can be also used to study the insect dispersal with constant random motion and an $(x,t)-$dependent emigration parameter. It may be also used as a model for the growth of avascular tumor [9].

The question of controllability of the bilinear reaction-diffusion equation has attracted many researchers (see e.g., [4, 5, 8] and [14]- [18]). In [4], the approximate controllability properties have been derived for the truncated bilinear version of (1) (i.e., $f = 0$) for the initial and target states y_0, y_d with finitely many changes of sign. The same question has been discussed by Fernàndez and Khapalov in [8] when the support of the bilinear control is allowed to depend on time. The exact controllability of the bilinear part of equation (1) with inhomogeneous Dirichlet conditions has been considered in [15, 17]. However, the assumptions of [15, 17] are not compatible when dealing with homogeneous Dirichlet conditions. Recently, the approximate and exact controllability have been studied for the truncated bilinear version of equation (1) under the sign condition $y_0(x)y_d(x) \geq 0,$ for almost every $x \in \Omega$ in [18]. Moreover, the partial controllability of bilinear reaction-diffusion equation has been studied in [12]. According to the maximum principle, it is not possible to steer the bilinear part of (1) from an initial state which has a constant sign to a target state that change its sign. In [13], Khapalov studied the global approximate controllability of the semilinear convection-diffusion-reaction equation by multiplicative controls while dealing with nonnegative initial and target states. In [5], Cannarsa, Floridia and Khapalov have studied the global approximate controllability properties of system (1) in the one-dimensional case for suitable classes of initial and target states that change their sign at a finite number of points. However, in the works above, the time of steering depends on the given initial and target states. In this paper, we are interested in the multiplicative controllability of the semilinear reaction diffusion system (1) at a priori given time, when the initial and target states have the same sign at almost every $x \in \Omega$ and satisfy $\ln(\frac{y^d}{y_0}) \in L^\infty(\Omega)$. We will first deal with a bilinear case ($f = 0$), then we proceed to the full equation (1). Moreover, we will see that the exact steering of the semilinear system (1) can be reduced to the controllability of its bilinear part since the nonlinear term f can be absorbed by the control in an appropriate way.

The paper is organized as follows. In the next section, we first consider the problem of exact controllability of the bilinear part of the system (1), and we will show that the steering time can be arbitrary small and uniform for all initial and reached states. Then, we apply this result to solve the problem of exact controllability of the semilinear system (1) at a priori fixed time. In the third section, we present a numerical example with simulations.

2 The Main Results

Our goal in this section is to study the exact controllability properties of the system (1) at a given time $T > 0$.

2.1 Exact controllability of the bilinear equation

Here we consider the following system:

$$\begin{cases} y_t = \Delta y + v(x,t)y, & \text{in } Q_T, \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{2}$$

From [18], one may deduce the following approximate controllability result regarding the bilinear part (2) of the system (1).

Lemma 2.1 *[18] For any initial state $y_0 \in L^2(\Omega)$, for any function $g \in W^{2,\infty}(\Omega)$ and for all $\varepsilon > 0$, there exists a time $T = T(y_0, y_d, \epsilon)$ such that the respective solution to (2) controlled with $v := \frac{g}{T}$ satisfies*

$$\|y(T) - e^g y_0\| < \varepsilon.$$

We also recall the following null-controllability of the linear heat equation.

Lemma 2.2 *[7, 11] Consider the system*

$$\begin{cases} \psi_t = \Delta\psi + b(x)\psi + \mathbf{1}_\omega u_2(x,t), & in\, \Omega \times (t_0, T), \\ \psi = 0, & on\ \partial\Omega \times (t_0, T), \\ \psi(\cdot, t_0) = \xi \in L^2(\Omega), & in\, \Omega, \end{cases} \tag{3}$$

where $0 \le t_0 < T$, $b \in L^\infty(\Omega)$ and ω is a nonempty open subset of Ω. Then there is a control $u_2 \in L^\infty(\Omega \times (t_0, T))$ such that the corresponding solution to (3) vanishes at T. Furthermore, we have

$$\|u_2\|_{L^\infty(\Omega\times(t_0,T))} \le C\, \|\xi\|_{L^2(\Omega)}, \tag{4}$$

where $C = C_{T-t_0}$ is a positive constant depending on $T - t_0$ and such that C_{T-t_0} is bounded near $t_0 \to 0^+$.

We now state the exact controllability result of the bilinear system (2).

Theorem 2.1 *Let $y_0 \in L^p(\Omega)$, ($p \ge 2$ and $p > \frac{n}{2}$) and let $y_d \in H^2(\Omega)$ such that*
i) for a.e. $x \in \Omega$, $y_0(x)y_d(x) \ge 0$ and $y_d(x) = 0 \Leftrightarrow y_0(x) = 0$,
ii) $ln(\frac{y_d}{y_0})\mathbf{1}_{E_{y_0}} \in L^\infty(\Omega)$, where $\mathbf{1}_{E_{y_0}}$ denotes the characteristic function of the set $E_{y_0} = \{x \in \Omega /\ y_0(x) \neq 0\}$,
iii) $\frac{\Delta y_d}{y_d}\mathbf{1}_{E_{y_d}} \in L^\infty(\Omega)$ and $|y_d| \ge \alpha > 0$, a.e. on some open subset $\omega \subset \Omega$.
Then for any $T > 0$, there exists a control $v \in L^\infty(Q_T)$ such that the respective solution to (2) satisfies $y(T) = y_d$, a.e. in Ω.

Proof. Let $T > 0$.
1. Approximate steering.
Let $g := ln(\frac{y_d}{y_0})\mathbf{1}_E$. It follows from the assumption (i) that $e^g y_0 = y_d$. Then, in the case where $g \in W^{2,\infty}(\Omega)$, we deduce from Lemma 2.1 that for any $\varepsilon > 0$, there exists $0 < T_1 < T$ small enough such that the corresponding solution to (2) controlled with $v_1 = \frac{g}{T}$ verifies

$$\|y(T_1) - y_d\| < \varepsilon. \tag{5}$$

Moreover, in the general case $g \in L^\infty(\Omega)$, one can construct a sequence $(g_k) \subset W^{2,\infty}(\Omega)$ which is uniformly bounded in Ω such that $g_k \to g$ in $L^2(\Omega)$, as $k \to +\infty$. We will

consider the control $v_1(x) = \dfrac{g_k}{T_1}$ for a suitably selected $k \in I\!N$ (large enough integer). Let $y(t)$ be the solution of (2) corresponding to $v_1(x)$ and to the initial state $y(0) = y_0$. Finally, let $(y_{0l}) \in L^\infty(\Omega)$ such that $y_{0l} \to y_0$ in $L^2(\Omega)$, as $l \to +\infty$.
We have the following triangular inequality:

$$\begin{aligned}\|y(T_1) - e^g y_0\| &\leq \|y(T_1) - e^{g_k} y_0\| + \|e^{g_k} y_0 - e^{g_k} y_{0l}\| + \|e^{g_k} y_{0l} - e^g y_{0l}\| + \|e^g y_{0l} - e^g y_0\| \\ &\leq \|y(T_1) - e^{g_k} y_0\| + \|e^{g_k} y_{0l} - e^g y_{0l}\| + \big(\sup_{k\in I\!N} \|e^{g_k}\|_{L^\infty(\Omega)} + e^{\|g\|_{L^\infty(\Omega)}}\big)\|y_{0l} - y_0\|.\end{aligned}$$

Let $L \in I\!N$ be such that

$$\big(\sup_{k\in I\!N} \|e^{g_k}\|_{L^\infty(\Omega)} + e^{\|g\|_{L^\infty(\Omega)}}\big)\|y_{0L} - y_0\| < \frac{\epsilon}{3},$$

and for such value of L, we consider K such that

$$\|e^{g_K} - e^g\| \|y_{0L}\|_{L^\infty(\Omega)} < \frac{\epsilon}{3}.$$

Finally, for this value of K, it comes from Lemma 2.1 that there exists $T > 0$ such that

$$\|y(T_1) - e^{g_K} y_0\| < \frac{\epsilon}{3}.$$

We conclude that

$$\|y(T_1) - e^g y_0\| < \epsilon.$$

Hence, since $e^g y_0 = y_d$, it comes that (5) holds for some $0 < T_1 < T$.
2. Exact steering.
Let us consider the following system:

$$\begin{cases} y_t = \Delta y + v(x,t)y, & \text{in } \Omega \times (T_1, T), \\ y(0,t) = 0, & \text{on } \partial\Omega \times (T_1, T), \\ y(T_1) = y(T_1^-), & \text{in } \Omega\cdot \end{cases} \tag{6}$$

Let $z = y - y_d$, where y satisfies (6). Thus z satisfies

$$\begin{cases} z_t = \Delta z + v(x,t)(z + y_d) + \Delta y_d, & \text{in } \Omega \times (T_1, T), \\ z(0,t) = 0, & \text{on } \partial\Omega \times (T_1, T), \\ z(T_1) = y(T_1^-) - y_d, & \text{in } \Omega. \end{cases} \tag{7}$$

In order to prove Theorem 2.1, it is sufficient to prove that (7) is exact null controllable. Let $T_2 \in (T_1, T)$ be close to T_1, so we can assume in the sequel that $0 < T_2 - T_1 < 1$. Then consider the following time-independent control in (T_1, T_2) :

$$v_2(x) = -\frac{\Delta y_d}{y_d} \mathbf{1}_{E_{y_d}}, \ a.e., \text{ in } \Omega.$$

From the definition of v_2, we have $v_2 y_d + \Delta y_d = 0$, a.e. in Ω. Thus the system (7) can be reduced to the following one:

$$\begin{cases} z_t = \Delta z + v_2(x)z, & \text{in } \Omega \times (T_1, T_2), \\ z = 0, & \text{on } \partial\Omega \times (T_1, T_2), \\ z(T_1) = y(T_1^-) - y_d, & \text{in } \Omega, \end{cases} \tag{8}$$

whose solution is given by

$$z(t) = S(t-T_1)z(T_1) + \int_{T_1}^{t} S(t-s)v_2(x)z(s)ds, \ \forall t \in [T_1, T_2]. \tag{9}$$

Then, since $S(t)$ is a contraction semigroup,

$$\|z(t)\| \leq \|z(T_1)\| + \|v_2\|_{L^\infty(\Omega)} \int_{T_1}^{t} \|z(s)\|ds$$

for all $t \in [T_1, T_2]$. Gronwall's inequality gives

$$\|z(t)\| \leq C\|z(T_1)\|, \ \forall t \in [T_1, T_2], \ C > 0. \tag{10}$$

Moreover, we know that $S(t)(L^p(\Omega)) \subset L^\infty(\Omega)$, and for all $\xi \in L^p(\Omega)$, we have

$$\|S(t)\xi\|_{L^\infty(\Omega)} \leq C\, t^{-\frac{n}{2p}} \|\xi\|_{L^p(\Omega)}, \forall t > 0, \tag{11}$$

where the constant C is independent of ξ. We also have $S(t)(L^\infty(\Omega)) \subset L^\infty(\Omega)$, and for all $\xi \in L^\infty(\Omega)$, we have (see [6], p. 44)

$$\|S(t)\xi\|_{L^\infty(\Omega)} \leq \|\xi\|_{L^\infty(\Omega)}, \forall t \geq 0.$$

Using the smooth effect of the heat semigroup $S(t)$, we can take the mild solution $z(t)$ in the space of continuous function equipped with the supremum norm. Then, by taking the $L^\infty-$norm in (9) and using (11), we get

$$\|z(t)\|_{L^\infty(\Omega)} \leq C(t-T_1)^{-\frac{n}{2p}} \|z(T_1)\|_{L^1(\Omega)} + C\|v_2\|_{L^\infty(\Omega)} \int_{T_1}^{t} (t-s)^{-\frac{n}{2p}} \|z(s)\|_{L^1(\Omega)} ds$$

for all $t \in [T_1, T_2]$, and for some constant $C > 0$ which is independent of $\eta := T_2 - T_1$. Then, when using (10), it comes

$$\|z(t)\|_{L^\infty(\Omega)} \leq C(t-T_1)^{-\frac{n}{2p}} \|z(T_1)\|_{L^1(\Omega)} + C\|v_2\|_{L^\infty(\Omega)} \|z(T_1)\| \int_{T_1}^{t} (t-s)^{-\frac{n}{2p}} ds$$

for all $t \in [T_1, T_2]$, and in particular,

$$\|z(T_2)\|_{L^\infty(\Omega)} \leq C\eta^{-\frac{n}{2p}} \|z(T_1)\|,$$

where C is a positive constant which is independent of $\eta \in (0, 1)$. Thus (5) implies

$$\|z(T_2)\|_{L^\infty(\Omega)} \leq C\eta^{-\frac{n}{2p}} \epsilon \tag{12}$$

for some constant $C > 0$ which is independent of η.

Let us now consider the control $v(x,t) = v_2(x) + v_3(x,t)$ on $[T_2, T]$, $v_3 \in L^\infty(\Omega \times (T_2, T))$ (with $v_3(t) = 0, \ t \in (T_1, T_2)$). When using this control, the system (7) becomes

$$\begin{cases} z_t = \Delta z + v_2(x)z + v_3(x,t)(z + y_d), & \text{in } \Omega \times (T_2, T), \\ z = 0, & \text{on } \partial\Omega \times (T_2, T), \\ z(T_2) = y(T_2^-) - y_d, & \text{in } \Omega. \end{cases} \tag{13}$$

Let us consider the following linear system:

$$\begin{cases} \psi_t = \Delta\psi + v_2(x)\psi + \mathbf{1}_\omega u_1(x,t), & \text{in } \Omega \times (T_2, T), \\ \psi(0,t) = 0, & \text{on } \partial\Omega \times (T_2, T), \\ \psi(T_2) = z(T_2), & \text{in } \Omega. \end{cases} \tag{14}$$

From Lemma 2.2, there exists a control $u_1 \in L^\infty(\Omega \times (T_2, T))$ such that the corresponding solution to (14) satisfies $\psi(.,T) = 0$. Furthermore, the steering control u_1 satisfies the estimate

$$\|u_1\|_{L^\infty(\Omega\times(T_2,T))} \leq C\|\psi(T_2)\|_{L^2(\Omega)} \tag{15}$$

for some positive constant C which is independent of T_2. In other words, for some positive constant C (depending though on $T - T_2$). Moreover, since T_2 is small enough, we can (according to the Lemma 2.2) choose C only dependent on T.
The solution of (14) is given, for all $t \in [T_2, T]$, by

$$\psi(t) = S(t - T_2)\psi(T_2) + \int_{T_2}^{t} S(t-s)(v_2(x)\psi(s) + u_1(\cdot, s))ds. \tag{16}$$

Since $\psi(T_2) \in L^\infty(\Omega)$, we have (see [6], p. 44)

$$\|S(t)\psi(T_2)\|_{L^\infty(\Omega)} \leq \|\psi(T_2)\|_{L^\infty(\Omega)}.$$

Since $\psi(T_2) \in L^p(\Omega)$ and $u_1 \in L^\infty(\Omega \times (T_2, T))$, we can see that $\psi(t) \in L^p(\Omega),\ T_2 \leq t \leq T$.

Moreover, from (11), we have

$$\|S(t-s)\psi(s)\|_{L^\infty(\Omega)} \leq C(t-s)^{-\frac{n}{2p}}\|\psi(s)\|_{L^p(\Omega)},\ 0 \leq s < t,$$

and

$$\|S(t-s)\mathbf{1}_\omega u_1(\cdot,s)\|_{L^\infty(\Omega)} \leq C(t-s)^{-\frac{n}{2p}}\|u_1(\cdot,s)\|_{L^p(\Omega)},\ 0 \leq s < t\cdot$$

Thus from (16), we have $\psi(t) \in L^\infty(\Omega)$ for all $t \in (T_2, T]$, and

$$\|\psi(t)\|_{L^\infty(\Omega)} \leq \|\psi(T_2)\|_{L^\infty(\Omega)} + C\|u_1\|_{L^\infty(\Omega\times(T_2,T))} + C\int_{T_2}^{t} \|\psi(s)\|_{L^\infty(\Omega)}$$

for some C which is independent of η.

Gronwall's inequality yields, via (12) and (15),

$$\|\psi(t)\|_{L^\infty(\Omega)} \leq C_*\eta^{-\frac{n}{2p}}\varepsilon,\ t \in [T_2, T],$$

for η small enough and for some constant $C_* > 0$ which is independent of η.
Since $|y_d| \geq \alpha > 0$ a.e. in ω, we can choose ε and η small enough such that $\eta > (\frac{C_*\epsilon}{\alpha})^{\frac{2p}{n}}$. Hence

$$|\psi(x,t) + y_d| > 0, \text{ a.e } \omega \times (T_2, T).$$

This enables us to define a control v_3 in $\Omega \times (T_2, T)$ through the following relation:

$$v_3(x,t)(\psi(x,t) + y_d) = u_1(x,t).$$

Since $u_1 \in L^\infty(\Omega \times (T_2, T))$, it follows that $v_3 \in L^\infty(\Omega \times (T_2, T))$.
Using the control $v(x,t) = v_2(x) + v_3(x,t),\ t \in (T_2, T)$ in the system (13), leads to the following one:

$$\begin{cases} z_t = \Delta z + v_2(x)z + \dfrac{u_1(x,t)}{\psi(x,t) + y_d}(z + y_d), & \text{in } \Omega \times (T_2, T), \\ z(0,t) = 0, & \text{on } \partial\Omega \times (T_2, T), \\ z(T_2) = y(T_2^-) - y_d, & \text{in } \Omega, \end{cases} \tag{17}$$

which admits ψ as a solution. Hence, by uniqueness, we have $z = \psi$ a.e in $\Omega \times (T_2, T)$.
Finally, returning to initial system (2), the control is then defined by

$$v(x,t) = \begin{cases} v_1(x), & \text{in } (0, T_1), \\ v_2(x), & \text{in } (T_1, T_2), \\ v_2(x) + \dfrac{u_1(x,t)}{\psi(x,t) + y_d}, & \text{in } (T_2, T), \end{cases}$$

so that $v \in L^\infty(Q_T)$ and $y(T) = y_d$.

Remark 2.1 The result of Theorem 2.1 improves the results from the literature in terms of the steering time which is here independent of the initial and target states (see for instance [4, 18]).

2.2 Exact controllability of the semilinear system

Presently, the system (1) is considered. The next theorem introduces significant differences with respect to the literature in terms of the proof techniques. Indeed, the method used in [5] consists of shifting the points of sign change by making use of a finite sequence of initial-value pure diffusion problems. In [18], a static control was used to study the approximate controllability of the truncated part of (1), and the equation at hand becomes linear so that one can apply the linear semigroup theory. In the context of equation (1), the central idea of our method is to try to select the bilinear control in such a way that the corresponding trajectory of (1) can be approximated by the bilinear term $v(x,t)y(t)$ on a small interval of steering $[0, T]$. In other words, the effect of the pure diffusion (i.e. $v = 0$ and $f = 0$) as well as the one of the nonlinearity becomes negligible as $T \to 0^+$.

Our exact controllability result for semilinear case is as follows.

Theorem 2.2 *Let $T > 0$. If y_0 and y_d satisfy the assumptions of Theorem 2.1, then there exists a control $q(\cdot,\cdot) \in L^\infty(Q_T)$ for which the respective solution to (1) is such that $y(T) = y_d$.*

Proof. The idea consists in looking for a control that makes the system (1) equivalent to its bilinear part (2) so that one may apply the results of the previous section.
Let us observe that (at least formally) the system (1) can be written as follows:

$$\begin{cases} y_t = \Delta y + (q(x,t) + \frac{f(y)}{y}\mathbf{1}_{E_y})y, & \text{in } Q_T, \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{18}$$

where $E_y = \{(x,t) \in Q_T : y(x,t) \neq 0\}$. This leads us to consider the following bilinear system:

$$\begin{cases} \varphi_t = \Delta\varphi + v(x,t)\varphi, & \text{in } Q_T, \\ \varphi(0,t) = 0, & \text{on } \Sigma_T, \\ \varphi(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{19}$$

According to Theorem 2.1, there exists $v \in L^\infty(Q_T)$ for which the solution of the system (19) is such that $\varphi(T) = y_d$.

From the assumptions on f, we deduce that $|f(y(x))| \leq L|y(x)|$ for a.e $x \in \Omega$, where L is a Lipschitz constant of f. Thus we have $\dfrac{f(\varphi)}{\varphi}\mathbf{1}_{E_\varphi} \in L^\infty(\Omega)$, where $E_\varphi = \{(x,t) : \varphi(x,t) \neq 0\}$.

Consider the following bilinear system:

$$\begin{cases} y_t = \Delta y + \big(v(x,t) - \frac{f(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}\big)y + f(y), & \text{in } Q_T, \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{20}$$

and let us set $q(x,t) = v(x,t) - \dfrac{f(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}$ in (20), where φ is the solution of (19) corresponding to the steering control v. It is apparent that φ is a solution of (20). Hence, by uniqueness, we have that $y = \varphi$ is the unique solution of (20) corresponding to the control $q(x,t) = v(x,t) - \dfrac{f(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}$. Then the controllability result of the theorem follows from Theorem 2.1.

Remark 2.2 1. In the case where $f(0) \neq 0$, we can use the control $q(x,t) = v(x,t) - \dfrac{f(\varphi) - f(0)}{\varphi}\mathbf{1}_{E_\varphi}$.

2. The result of Theorem 2.2 extends the results of approximate multiplicative controllability of semilinear systems established in [5] to the case of several dimensions. Moreover, the result of Theorem 2.2 also holds for a nonlinearity $f = f(t,x,y,\nabla y)$ which is globally Lipschitz in y uniformly w.r.t the other parameters (see [13]).

The next result provides a set of states that can be reached with additive controls through the following semilinear system:

$$\begin{cases} y_t = \Delta y + f(y) + u(x,t), & \text{in } Q_T, \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{21}$$

Corollary 2.1 *Let assumptions of Theorem 2.1 hold. Then for any $T > 0$, there exists a control $u \in L^2(0,T;L^2(\Omega))$ for which the respective solution to (21) satisfies $y(T) = y_d$.*

Proof. It suffices to take $u(x,t) = q(x,t)y(x,t)$, where q is the steering control of (1) from y_0 to y_d at T and y is the corresponding solution of (1).

3 Simulation

In this section, we investigate the exact controllability for the one dimensional version of system (1). Note that the approximate controllability of such models has been considered in the bilinear and semilinear context in [4, 5] (see also [14] for different interpretations of these equations).

Let us consider the system (1) with $f(x) = \sin(x)$, $x \in \mathbb{R}$. This function constitutes a prototype of (non trivial) smooth Lipschitz nonlinearities that vanish at the origin, which is widely used for illustrative and numerical examples (see for instance [1,3,16]). As initial and final data, let us take $y_0 = 10^{-2}(x+10^{-2})(1.01-x)$ and $y_d = y_0 e^x$ in $\Omega = (0,1)$.

Thus, we have $a(x) = \frac{1}{T}\ln\frac{y_d}{y_0} = \frac{x}{T}$ and $g(x) = -\dfrac{\Delta y_d}{y_d} = -1 + \dfrac{4x}{(x+10^{-2})(1.01-x)}$.

According to Theorem 2.2, we deduce that for every $T > 0$, there are positive real numbers T_1 and T_2 which are small enough and for which the control

$$q(x,t) = \begin{cases} \dfrac{x}{T_1} - \dfrac{\sin(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}, & (0,T_1), \\ -1 + \dfrac{4x}{(x+10^{-2})(1.01-x)} - \dfrac{\sin(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}, & (T_1,T_2), \\ -1 + \dfrac{4x}{(x+10^{-2})(1.01-x)} + \dfrac{u(x,t)}{\psi(x,t)+y_d} - \dfrac{\sin(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}, & (T_2,T), \end{cases}$$

($E_\varphi = \{(x,t) : \varphi(x,t) \neq 0\}$) achieves the exact steering of system (1) from y_0 to y_d at T, where φ solves (19) with

$$v(x,t) = \begin{cases} \frac{x}{T_1}, & (0,T_1), \\ -1 + \frac{4x}{(x+10^{-2})(1.01-x)}, & (T_1,T_2), \\ -1 + \frac{4x}{(x+10^{-2})(1.01-x)} + \frac{u(x,t)}{\psi(x,t)+y_d}, & (T_2,T), \end{cases}$$

and $u(x,t)$ is the control of null-controllability of the linear system

$$\begin{cases} \psi_t = \Delta\psi - \psi + u(x,t), & \text{in } \Omega \times (T_2,T), \\ \psi(0,t) = 0, & \text{on } \partial\Omega \times (T_2,T), \\ \psi(T_2) = y(T_2) - y_d, & \text{in } \Omega, \end{cases}$$

and ψ is the corresponding solution.

Here we consider a globally distributed control $u(x,t)$, which can be taken time-independent (see [14], p.57)

$$u(x,t) = -\sum_{k=1}^{\infty} \frac{(\pi^2k^2+1)e^{-(T-T_2)(\pi^2k^2+1)}}{e^{-(T-T_2)(k^2\pi^2+1)} - 1}\Big(\int_0^\pi (y(\xi,T_2) - y_d(\xi))\varphi_k(\xi)d\xi\Big)\varphi_k(x), \quad (22)$$

where $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, $k \geq 1$, are the eigenfunctions of A associated to the eigenvalues $\lambda_k = -k^2\pi^2$.

Now, note that system (1) with control $q(x,t)$ and system (19) with control $v(x,t)$ have the same state and it suffices to simulate the latter. We will give simulations for $T = 1$, $T_1 = 0.01$ and $T_2 = 0.02$, and we will follow the three steps given below.

Step 1. Approximate steering: Solve system (1), controlled on the time interval $(0,T_1)$, by $v(x,t) = v_1(x) = \frac{1}{T_1}\ln(\frac{y_d}{y_0}) = 100x$ to get $y(x,T_1)$.

Step 2. Computation of the additive control $u(x,t)$: Solve (19) on the time interval $(0,T_2)$, by taking the control

$$v(x,t) = v_1(x,t) = \begin{cases} 100x, & (0,T_1), \\ -1 + \dfrac{4x}{(x+10^{-2})(1.01-x)}, & (T_1,T_2). \end{cases}$$

This gives $y(\xi, T_2)$, which enables us to compute the control $u(x,t)$ using the formula (22).

Step 3. Exact steering: Consider the solution ψ:

$$\psi(x,t) = e^{T_2 - t} S(t - T_2) y(x, T_2) + \int_{T_2}^{t} e^{-(t-s)} S(t-s) u(x,s) ds$$

of the equation

$$\psi_t = \Delta\psi - \psi + u(x,t),\ t \in (T_2, T),$$

with $\psi(T_2^+) = y(T_2^-)$ as the initial state. Then, we use the relation $y(x,t) = \psi(x,t) + y^d,\ t \in (T_2, T)$ to get $y(x,T) = y_d$. Below are the figures corresponding to the exact steering with the error function.

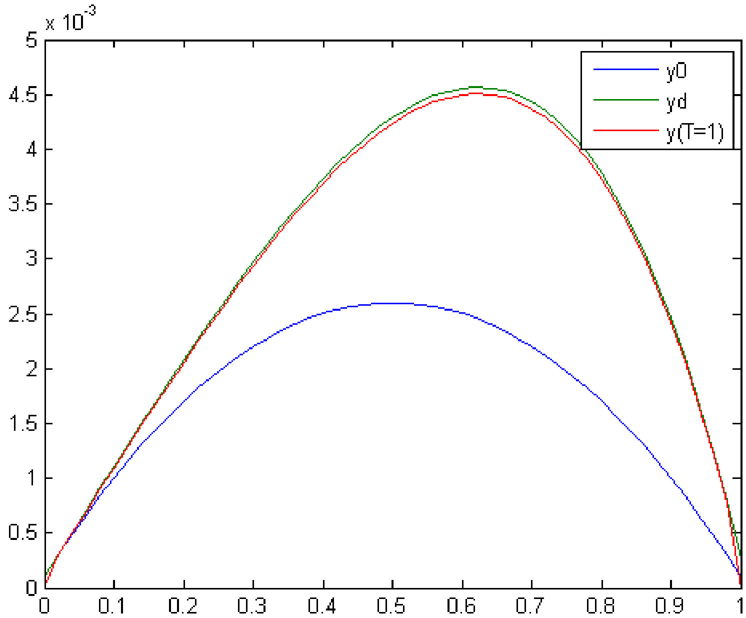

Figure 1: The evolution of the state at T.

• Figure 1 reflects the exact steering and shows that the trajectory $y(t)$ becomes very close to the desirable state at time T.

• Figure 2 describes the variation of the error function defined by $E(t) = \|y(t) - y_d\|, t \in [0,T]$, and shows that $E(t)$ tends to zero when t becomes close to the time of steering T.

4 Conclusion

In the present paper, the multiplicative controllability of a semilinear reaction-diffusion equation is considered in several space dimensions. The approach is constructive and consists in using a set of three controls applied subsequently in time. First, a static

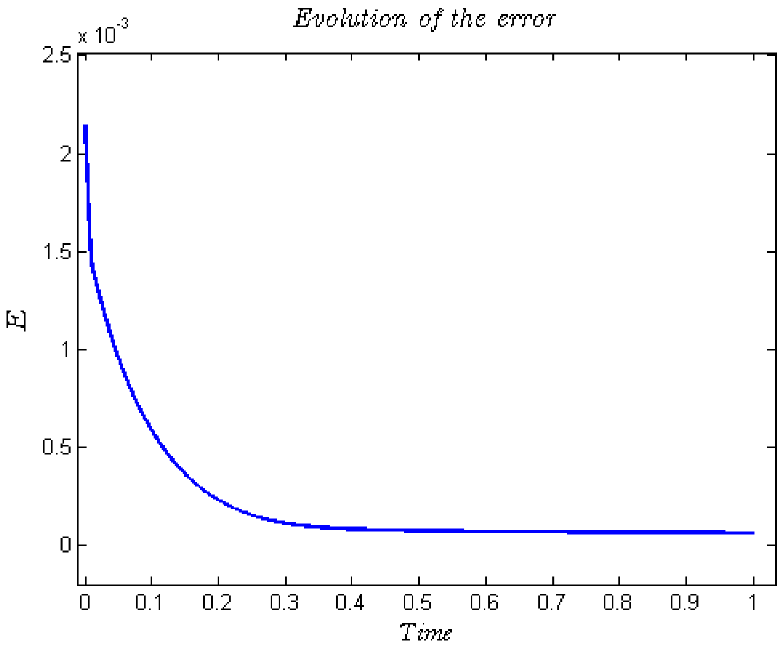

Figure 2: The variation of the error.

control is used to achieve the approximate steering in L^2 at a small time T_1. Then, a second static control is used in a small time interval $[T_1, T_2]$ so that the approximate steering becomes in L^∞ sense. Finally, in the remaining time interval $[T_2, T]$, we exploit a $(x, t)-$dependent control law that ensures the zero controllability of an appropriate linear system (with an additive control) to guarantee the exact steering. Furthermore, the considered methods allow us to achieve the approximate and exact steering (for a given couple of the initial and desirable states) at arbitrary small control time which can be fixed in advance.

References

[1] D. Aimene, K. Laoubi and D. Seba. On approximate controllability of impulsive fractional semilinear systems with deviated argument in Hilbert spaces. *Nonlinear Dynamics and Systems Theory* **20** (2020) 465–478.

[2] H. T. Banks, P. M. Kareiva and L. Zia, Analyzing field studies of insect dispersal using two-dimensional transport equations. *Environmental Entomology* **17** (5) (1988) 815–820.

[3] Burton, T. A., and Purnaras, I. K. Fixed Point Regions, Unified Construction of Fixed Point Mappings for Integral, Quadratic, and Fractional Equations. *Nonlinear Dynamics and Systems Theory* **20** (4) (2020) 347.

[4] P. Cannarsa and A. Khapalov, Multiplicative controllability for reaction-diffusion equation with target states admitting finitely many changes of sign. *Discrete and continuous dynamical systems, Series B* **14** (2010) 1293–1311.

[5] P. Cannarsa, G. Floridia and A. Y. Khapalov, Multiplicative controllability for semilinear reaction-diffusion equations with finitely many changes of sign. *Journal de Mathématiques Pures et Appliquées* 2017 **108** (4) 425–458.

[6] T. Cazenave and A. Haraux, An introduction to semilinear evolution equations. *Oxford Lecture Series in Mathematics and its Applications 13, Oxford University Press, Oxford* 1998.

[7] E. Fernàndez-Cara and E. Zuazua. Null and approximate controllability for weakly blowing-up semilinear heat equations, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **17** (2000) 583–616.

[8] L. A. Fernàndez and A. Y. Khapalov. Controllability properties for the one-dimensional Heat equation under multiplicative or nonnegative additive controls with local mobile support. *ESAIM: Control Optim. Calc. Var.* **18**(4) (2012) 1207–1224.

[9] S. C. Ferreira, Jr., M. L. Martins and M. J. Vilela. Reaction-diffusion model for the growth of avascular tumor. *Phys. Rev. E 65*, 021907, Published 23 (2002).

[10] A. Friedman. PDE problems arising in mathematical biology. *Networks and Heterogeneous Media* **7** (4) (2012).

[11] A. V. Fursikov, and O. Yu. Imanuvilov. Controllability of Evolution Equations. Lecture Notes 34 *Seoul National University, Korea*. 1996.

[12] M. Jidou Khayar and M. Ouzahra. Partial controllability of the bilinear reaction-diffusion equation. *International Journal of Dynamics and Control* (2018) 1–8.

[13] A. Y. Khapalov. Controllability of the semilinear parabolic equation governed by a multiplicative control in the reaction term: A qualitative approach. *SIAM J. Control. Optim.* **41** (2003) 1886–1900.

[14] A. Y. Khapalov. Controllability of partial differential equations governed by multiplicative controls. *A Lecture Notes in Mathematics* 1995 Springer-Verlag, Berlin 2010.

[15] P. Lin, Zhong Cheng Zhou and Hang Gao. Exact controllability of the parabolic system with bilinear control. *Applied Mathematics Letters* **19** (2006) 568–575.

[16] H. Najib, R. El Ayadi, Y. Ouakrim and M. Ouzahra. Exact controllability of a semilinear reaction-diffusion equation governed by a bilinear control. *Applicable Analysis* (2021) 1–25.

[17] M. Ouzahra, A. Tsouli and A. Boutoulout. Exact controllability of the heat equation with bilinear control. *Mathematical methods in the applied sciences* **38** (18) (2015) 5074–5084.

[18] M. Ouzahra. Approximate and exact controllability of a reaction-diffusion equation governed by bilinear control. *European Journal of Control* **32** (2016) 32–38.

[19] T. Roose, S.J. Chapman and P.K. Maini. Mathematical models of avascular tumor growth. *SIAM Rev.* **49** (2) (2007) 179–208.

[20] S. Salsa. Partial differential equations in action: from modelling to theory. *Springer-Verlag Italia*, Milano, 2008.