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Effect of Water Scarcity in the Society: A Mathematical Model

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Abstract: Water scarcity is one of the major problems faced by all those living around the world. So, there should be a multiple way approach to be adopted to conquer the water scarcity effects in future. Keeping this in mind, we developed a mathematical model and demonstrated the effect of water scarcity through a deterministic and stochastic formats. The equilibrium point of the model is found and its stability is analyzed analytically. Numerical simulation of both the deterministic and the stochastic model is exhibited to validate our analytical findings. The attainment level of the equilibrium point is demonstrated by using the Runge-Kutta method. The comparison is also made for this equilibrium. The effect of few parameters of the model was exhibited in different figures in the numerical simulation section. Particularly the effect of the water draining rate and the rate of human population affected by water scarcity on each compartment were shown visually through plotting time vs particular compartments. Our results show the better ways for water recovery through the compartments of the model.

Keywords: *water scarcity; local stability; global stability; stochastic model.*

Mathematics Subject Classification (2010): 70Kxx; 70K42; 70K70; 93-XX; 93E03; 93C10; 93C15; 34D20; 34D23; 65C30.

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1 Introduction

Water shortages are a severe shortage in which the rates of water availability do not meet certain basic requirements specified. Water is one of the most essential natural renewable resources, and no one, neither humans nor animals, can live without it. Water comes from numerous sources, including runoff, groundwater, and surface water. The main contributor to the world growth and development are water supplies [1].

The paper concludes that there is a fixed amount of water on our planet. But so little of it is at our disposal to use. 70 percent of the earth surface is filled with 1400 million cubic kilometers of water (m km3): 2.5% is freshwater and 97.5% is saltwater, 2.5 percent is groundwater, 0.3 percent are lakes and rivers, 68.9 percent is frozen in ice caps. One-third of the population of the world currently resides in countries where the quality of the water is not adequately compromised, but by 2025, it is projected to increase by two-thirds [2].

The primary objective of this paper is to determine the scarcity of water in selected Middle East countries. For Iran, Iraq, and Saudi Arabia, the Anomaly Standardized Precipitation (WASP) index was spatially computed from 1979 to 2017. The water scarcity situation has been investigated in cities with a population of more than one million. This was done by using the methodology of the composite index to make water-related statistics more intelligible. A forecast was created for the years 2020 to 2030 to show potential improvements in the supply and demand for water in selected Middle East countries. With rising urbanization, there is a moderate to high water shortage risk for all countries at present [3]. Water shortage is a common issue in many parts of the world. Many previous water shortage evaluation strategies only considered the volume of water, and overlooked the quality of water. Moreover, the Environmental Flow Criterion (EFR) was not usually considered directly in the evaluation. In this paper, we have developed an approach to assess water scarcity by considering both water quantity and quality [4].

The formulation of a corruption control model and its analysis using the theory of differential equations are presented in paper [5]. The equilibria of the model and the stability of these equilibria are discussed in detail. Yadav, A. et al. [6] propose and evaluate mathematical models to research the dynamics of smoking activity under the influence of educational programs and also the willingness of the person to quit smoking. A nonlinear mathematical model is formulated and analyzed in paper [7] to research the relationship between the criminal population and non-criminal population by taking into account the rate of non-monotone incidence. See also [8, 9].

[10] suggested and analyzed a mathematical model using oncolytic virotherapy for cancer care. The growth of tumor cells is presumed to obey logistic growth and the interaction between tumor cells and viruses is of saturation type. Several nonlinear mathematical models are proposed and analyzed in paper [11] to study the spread of asthma due to inhaled industrial pollutants [12, 13] are also referenced.

This paper aims to illustrate the requirements to and the availability of water. As a result of growing population, rising urbanization, and rapid industrialization, combined with the need to increase agricultural production, water demand has been found to increase significantly. Water per capital supply is also slowly declining. More than 2.2 million people are expected to die every year from diseases related to polluted drinking water and poor sanitation.

As mentioned above, we have analyzed and proved that water scarcity is one of the major problems that has been proved statistically and theoretically. We are here giving a new try to prove the same by using the mathematical model.

Using the principle of an ordinary differential equation, we analyze our model and record comprehensive results of numerical simulations to support the analytical results. First, our model is expanded to the model of stochastic differential equations. The outcomes of deterministic and stochastic models were also compared. The remainder of this paper is structured as follows, Section 2 explains the model and the presence of equilibria and illustrates local stability, global equilibrium stability. Section 3 addresses the remaining stochastic model. Section 4 displays the effects of simulation for deterministic and stochastic models. Our results are summarized in Section 5 as a conclusion.

2 The Model and Analysis

We proposed and analyzed a non linear model for water scarcity by dividing into four different compartments [14], namely, the total usage of water (W), the human (H), water scarcity (W_s), water recover (W_r). All variables are time t functions. The transfer diagram of the model is described in Figure 1. The mathematical model is suggested as follows, in view of the above considerations:

$$\begin{aligned} \frac{dW}{dt} &= \Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r, \\ \frac{dH}{dt} &= \alpha_2 WH - \beta H - \mu H - \mu_1 H, \\ \frac{dW_s}{dt} &= \alpha_1 W + \beta H - \delta_1 W_s, \\ \frac{dW_r}{dt} &= \delta_1 W_s - \delta_2 W_r. \end{aligned} \tag{1}$$

In Table 1, the parameters used in model (1) are defined.

Table 1: Description of parameters.

Parameter	Description
Λ	Recruitment rate
α_1	Water draining rate
α_2	The rate of consumption of water by a human
δ_1	The recovery rate of water resource
δ_2	The rate at which water becomes normal level water
β	Rate of human population affected by water scarcity
μ	Natural death rate
μ_1	Death rate due to water scarcity

2.1 Existence of equilibria

Our model's equilibrium is calculated by setting the right-hand side of the model to zero [15]. The system has the following equilibria, namely, the endemic equilibrium (EE)

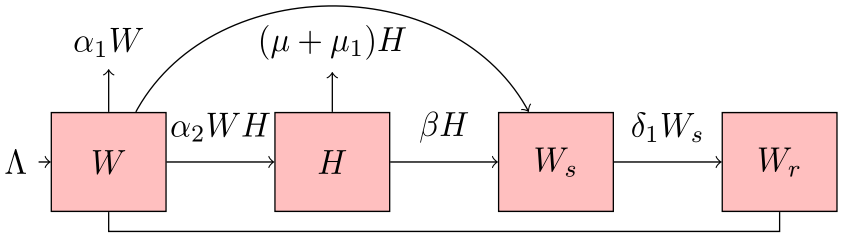

Figure 1: Transfer Diagram of the Model.

$E^*\,(W^*, H^*, W_s^*, W_r^*)$, where

$$W^* = \frac{k_1}{\alpha_2}, \tag{2}$$

$$H^* = \frac{\Lambda}{k_1-\beta}, \tag{3}$$

$$W_s^* = \frac{\Lambda\alpha_2\beta - \alpha_1 k_1\beta + \alpha_1 k_1^2}{\delta_1\alpha_2(k_1-\beta)}, \tag{4}$$

$$W_r^* = \frac{\Lambda\alpha_2\beta - \alpha_1 k_1\beta + \alpha_1 k_1^2}{\delta_2\alpha_2(k_1-\beta)}, \tag{5}$$

where $\quad k_1 = \beta + \mu + \mu_1$.

2.2 Stability analysis

The system's variational matrix is given by

$$M = \begin{pmatrix} -(\alpha_1+\alpha_2 H) & -\alpha_2 W & 0 & \delta_2 \\ \alpha_2 H & \alpha_2 W - k_1 & 0 & 0 \\ \alpha_1 & \beta & -\delta_1 & 0 \\ 0 & 0 & \delta_1 & -\delta_2 \end{pmatrix}.$$

2.2.1 Stability analysis of EE point

The variation matrix M* corresponding to the point E^* of the endemic equilibrium, is given by

$$M^* = \begin{pmatrix} n_{11} & n_{12} & 0 & n_{14} \\ n_{21} & n_{22} & 0 & 0 \\ n_{31} & n_{32} & n_{33} & 0 \\ 0 & 0 & n_{43} & n_{44} \end{pmatrix},$$

where

$n_{11} = -(\alpha_1+\alpha_2 H), \quad n_{12} = -\alpha_2 W, \quad n_{14} = \delta_2$
$n_{21} = \alpha_2 H, \quad n_{22} = \alpha_2 W - k_1$
$n_{31} = \alpha_1, \quad n_{32} = \beta, \quad n_{33} = -\delta, n_{43} = \delta_1, \quad n_{44} = -\delta_2.$

The bi-quadratic equation is

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$\begin{aligned}
a_1 &= -(n_{11} + n_{22} + n_{33} + n_{44}),\\
a_2 &= n_{11}n_{22} + n_{22}n_{33} + n_{33}n_{44} + n_{11}n_{33} + n_{11}n_{44} + n_{22}n_{44} - n_{12}n_{21},\\
a_3 &= -n_{11}n_{22}n_{33} - n_{11}n_{22}n_{44} - n_{11}n_{33}n_{44} - n_{22}n_{33}n_{44} + n_{12}n_{21}n_{33}\\
&+ n_{12}n_{21}n_{44} - n_{14}n_{43}n_{31},\\
a_4 &= n_{11}n_{22}n_{33}n_{44} + n_{14}n_{22}n_{31}n_{43} - n_{14}n_{21}n_{32}n_{43} - n_{12}n_{21}n_{33}n_{44}.
\end{aligned}$$

E^* will be locally asymptotically stable by using the Routh-Hurwitz criteria if the following conditions are satisfied: $a_1 > 0$, $a_3 > 0$, $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$, $a_3 > 0$.

If two other inequalities referred to above are satisfied, E^* is locally asymptotically stable [16].

2.2.2 Global stability of endemic equilibrium

In order to analyze the global stability of the endemic equilibrium E^*, we adopt the approach developed by Korobeinikov [8] and successfully applied in [9]. E^* exists for all $x, y, z, u > \epsilon$, for some $\epsilon > 0$.

Let $k_1y = [\beta+\mu+\mu_1]y = g(x, y, z, u)$ be positive and monotonic functions in $\mathbb{R}^4_+$ (for more details, see [8, 9]).

$$\begin{aligned}
V(x,y,z,u) &= x - \int_\epsilon^x \frac{g(x^*,y^*,z^*,u^*)}{g(\eta,y^*,z^*,u^*)}\,d\eta + y - \int_\epsilon^y \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,\eta,z^*,u^*)}\,d\eta\\
&+z - \int_\epsilon^z \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,y^*,\eta,u^*)}\,d\eta + u - \int_\epsilon^w \frac{g(x^*,y^*,z^*,u^*)}{h(x^*,y^*,z^*,\eta)}\,d\eta. \qquad (6)
\end{aligned}$$

If $g(x, y, z, u)$ is monotonic with respect to its variables, then the state E is the only extreme and the global minimum of this function. So, obviously,

$$\begin{aligned}
&\frac{\partial V}{\partial x} = 1 - \frac{g(x^*,y^*,z^*,u^*)}{g(x,y^*,z^*,u^*)}, \frac{\partial V}{\partial y} = 1 - \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,y,z^*,u^*)},\\
&\frac{\partial V}{\partial z} = 1 - \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,y^*,z,u^*)}, \frac{\partial V}{\partial u} = 1 - \frac{g(x^*,y^*,z^*,u^*)}{g(x^*,y^*,z^*,u)}. \qquad (7)
\end{aligned}$$

The functions $g(x, y, z, u)$ and $g(x, y, z, u)$ grow monotonically, then have only one stationary point. Further, since

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} &= \frac{g(x^*,y^*,z^*,u^*)}{[g(x,y^*,z^*,u^*)]^2}.\frac{g(x,y^*,z^*,u^*)}{\partial x},\\
\frac{\partial^2 V}{\partial y^2} &= \frac{g(x^*,y^*,z^*,u^*)}{[g(x^*,y,z^*,u^*)]^2}.\frac{g(x^*,y,z^*,u^*)}{\partial y},\\
\frac{\partial^2 V}{\partial z^2} &= \frac{g(x^*,y^*,z^*,u^*)}{[g(x^*,y^*,z,u^*)]^2}.\frac{g(x^*,y^*,z,u^*)}{\partial z},
\end{aligned}$$

$$\frac{\partial^2 V}{\partial u^2} = \frac{g(x^*, y^*, z^*, u^*)}{[g(x^*, y^*, z^*, u)]^2} \cdot \frac{g(x^*, y^*, z^*, u)}{\partial u}$$

are non negative, $g(x, y, z, u)$ and $h(x, y, z, u)$ have minimum. That is,

$$V(x, y, z, u) \geq V(x^*, y^*, z^*, u^*)$$

and hence, V is a Lyapunov function, and its derivative is given by

$$\begin{aligned}
\frac{dV}{dt} &= x' - x' \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)} + y' - y' \frac{h(x^*, y^*, z^*, u^*)}{g(x^*, y, z^*, u^*)} + z' - z' \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y^*, z, u^*)} + \\
&\quad u' - u' \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u)} \\
&= \alpha_1 x^* (1 - \frac{x}{x^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)}\right) - \delta_2 u^* (1 - \frac{u}{u^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)}\right) \\
&\quad + k_1 y^* (1 - \frac{y}{y^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y, z^*, u^*)}\right) - \alpha_1 x^* (1 - \frac{x}{x^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y^*, z, u^*)}\right) \\
&\quad - \beta y^* (1 - \frac{y}{y^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y^*, z, u^*)}\right) + \delta_1 z^* (1 - \frac{z}{z^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y^*, z, u^*)}\right) \\
&\quad - \delta_1 z^* (1 - \frac{z}{z^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y^*, z^*, u)}\right) + \delta_2 u^* (1 - \frac{u}{u^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y^*, z^*, u)}\right) \\
&\quad + g(x^*, y^*, z, u) \left(1 - \frac{g(x, y, z, u)}{g(x^*, y^*, z, u)}\right) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)}\right) \\
&\quad - g(x^*, y^*, z, u) \left(1 - \frac{g(x, y, z, u)}{g(x^*, y^*, z, u)}\right) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y, z^*, u^*)}\right). \qquad (8)
\end{aligned}$$

It is noted here that $g(x^*, y^*, z^*, u^*) = h(x^*, y^*, z, u)$ is explicitly given as g and h in terms of x, y, z and u.

Since $E > 0$, the function $g(x, y, z, u)$ is concave with respect to y, z and u and

$$\frac{\partial^2 g(x, y, z, u)}{\partial y^2} \leq 0, \quad \frac{\partial^2 g(x, y, z, u)}{\partial z^2} \leq 0,$$

then $\dfrac{dV}{dt} \leq 0$ for all $x, y, z, u > 0$. Also, the monotonicity of $g(x, y, z, u)$ with respect to x, y, z and u ensures that

$$\begin{array}{c}
(1 - \frac{x}{x^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)}\right) \leq 0, (1 - \frac{y}{y^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y, z^*, u^*)}\right) \leq 0, \\
(1 - \frac{z}{z^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y^*, z, u^*)}\right) \leq 0, (1 - \frac{u}{u^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y^*, z^*, u)}\right) \leq 0
\end{array} \qquad (9)$$

holds for all $x, y, z, u > 0$. Thus, we establish the following result.

Theorem 2.1 *The endemic equilibrium E^* of model (1) is globally asymptotically stable whenever conditions outlined in Eq.* (9) *are satisfied [17].*

3 Stochastic Model

We are expanding our deterministic model to stochastic systems here, as stochastic models are more able to capture random variations of the biological dynamics of the problem. The derivation of an SDE model is based on the method developed by Yuan et al. [18]. Let $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))^T$ be a continuous random variable for $(W(t), H(t), W_s(t), W_r(t))^T$ and T denote the transpose of a matrix.

Let $\Delta X = X(t+\Delta t) - X(t) = (\Delta X_1, \Delta X_2, \Delta X_3, \Delta X_4)^T$ denote the random vector for the change in random variables during the time interval Δt. Here, we'll write transition maps that define all possible changes in the SDE model between states. Based on our ODE model system (1), here we see that within a small time interval Δt, there are 9 possible changes between states. Changes in the state and their probabilities are discussed in Table 2. In the case, the state change ΔX is denoted by $\Delta X = (-1, 1, 0, 0)$. The probability of this change is determined by

Prob $(\Delta X_1, \Delta X_2, \Delta X_3, \Delta X_4) = (-1, 1, 0, 0) | (X_1, X_2, X_3, X_4) = P_3 = \alpha_2 X_1 X_2 + o(\Delta t)$ by neglecting terms higher than o(Δ t), the following expectation change $E(\Delta X)$ and its covariance matrix $V(\Delta X)$ associated with ΔX, can be identified. The expectation of ΔX is

$$E(\Delta X) = \sum_{i=1}^{8} P_i (\Delta X)_i \Delta t = \begin{pmatrix} \Lambda - \alpha_1 X_1 - \alpha_2 X_1 X_2 + \delta_2 X_4 \\ \alpha_2 X_1 X_2 - \beta X_2 - \mu X_2 - \mu_1 X_2 \\ \alpha_1 X_1 + \beta X_2 - \delta_1 X_3 \\ \delta_1 X_3 - \delta_2 X_4 \end{pmatrix} \Delta t$$

$$= f(X_1, X_2, X_3, X_4)\Delta \text{t}.$$

Table 2: Possible changes of states and their probabilities.

Possible stage change	Probability of state changes
$(\Delta x)_1 = (1, 0, 0, 0)^T$	$P_1 = \Lambda \Delta t + o(\Delta t)$
$(\Delta x)_2 = (-1, 0, 1, 0)^T$	$P_2 = \alpha_1 X_1 \Delta t + o(\Delta t)$
$(\Delta x)_3 = (-1, 1, 0, 0)^T$	$P_3 = \alpha_2 X_1 X_2 \Delta t + o(\Delta t)$
$(\Delta x)_4 = (1, 0, 0, -1)^T$	$P_4 = \delta_2 X_4 \Delta t + o(\Delta t)$
$(\Delta x)_5 = (0, -1, 1, 0)^T$	$P_5 = \beta X_2 \Delta t + o(\Delta t)$
$(\Delta x)_6 = (0, -1, 0, 0)^T$	$P_6 = \mu X_2 \Delta t + o(\Delta t)$
$(\Delta x)_7 = (0, -1, 0, 0)^T$	$P_7 = \mu_1 X_2 \Delta t + o(\Delta t)$
$(\Delta x)_8 = (0, 0, -1, 1)^T$	$P_8 = \delta_1 X_3 \Delta t + o(\Delta t)$
$(\Delta x)_9 = (0, 0, 0, 0)^T$	$P_9 = (1 - \sum_{i=1}^{8} P_i) + o(\Delta t)$

It can be noted here that the expectation vector and also the function f are in the same form as those of the ODE system (1).

Since the covariance matrix $V(\Delta X) = E((\Delta X)(\Delta X)^T) - E(\Delta X)(E(\Delta X)^T)$ and $E((\Delta X)(\Delta X)^T) = f(X)(f(X)^T)\Delta t$, it can be approximated with the diffusion matrix Ω times Δt by neglecting the term of $(\Delta t)^2$ so that $V(\Delta X) \approx E((\Delta X)(\Delta X)^T)$. That is,

$$E((\Delta X)(\Delta X)^T) = \sum_{i=1}^{8} P_i(\Delta X)_i(\Delta X)_i^T \Delta t = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & 0 \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & 0 & V_{43} & V_{44} \end{pmatrix}.\Delta t = \Omega.\Delta t,$$

where each component of the diffusion matrix of 4×4 is symmetric, positive-definite, and can be obtained by

$$V_{11} = P_1 + P_2 + P_3 + P_4 = \Lambda + \alpha_1 X_1 + \alpha_2 X_1 X_2 + \delta_2 X_4,$$
$$V_{22} = P_3 + P_5 + P_6 + P_7 = \alpha_2 X_1 X_2 + \beta X_2 + \mu X_2 + \mu_1 X_2,$$
$$V_{33} = P_2 + P_5 + P_8 = \alpha_1 X_1 + \beta X_2 + \delta_1 X_3,$$
$$V_{44} = P_4 + P_8 = \delta_2 X_4 + \delta_1 X_3,$$
$$V_{12} = V_{21} = -P_3 = -\alpha_2 X_1 X_2,$$
$$V_{34} = V_{43} = -P_4 = -\delta_1 X_3,$$
$$V_{14} = V_{41} = -P_4 = -\delta_2 X_4,$$
$$V_{13} = V_{31} = -P_2 = -\alpha_1 X_1,$$
$$V_{23} = V_{32} = -P_5 = -\beta X_2.$$

A matrix D square root of the symmetric, positive-definite diffusion matrix Ω is such that $K = \Omega^{1/2}$. Use an equivalent matrix K such that $\Omega = KK^T$, where K has the dimension of a 4×7 matrix.

$$K = \begin{pmatrix} \sqrt{\Lambda} & -\sqrt{\alpha_1 X_1} & -\sqrt{\alpha_2 X_1 X_2} & \sqrt{\delta_2 X_4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\alpha_2 X_1 X_2} & 0 & -\sqrt{\beta X_2} & -\sqrt{(\mu+\mu_1)X_2} & 0 \\ 0 & \sqrt{\alpha_1 X_1} & 0 & 0 & \sqrt{\beta X_2} & 0 & -\sqrt{\delta_1 X_3} \\ 0 & 0 & 0 & -\sqrt{\delta_2 X_4} & 0 & 0 & \sqrt{\delta_1 X_3} \end{pmatrix}.$$

Then, the Ito stochastic differential model has the following form:

$$dX(t) = f(X_1, X_2, X_3, X_4)dt + K.dW(t)$$

with the initial condition $X(0) = (X_1(0), X_2(0), X_3(0), X_4(0))^T$ and a Wiener process, $W(t) = (W_1(t), W_2(t), W_3(t), W_4(t), W_5(t), W_6(t), W_7(t))^T$. We get the stochastic differential equation model as follows:

$$\begin{aligned} dW &= [\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r]dt + \sqrt{\Lambda}dW_1 - \sqrt{\alpha_1 W}dW_2 - \sqrt{\alpha_2 WH}dW_3 + \sqrt{\delta_2 W_r}dW_4, \\ dH &= [\alpha_2 WH - \beta H - \mu H - \mu_1 H]dt + \sqrt{\alpha_2 WH}dW_3 - \sqrt{\beta H}dW_5 - \sqrt{(\mu+\mu_1)H}dW_6, \\ dW_s &= [\alpha_1 W + \beta H - \delta W_s]dt + \sqrt{\alpha_1 W}dW_2 + \sqrt{\beta H}dW_5 - \sqrt{\delta_1 W_s}dW_7, \\ dW_r &= [\delta_1 W_s - \delta_2 W_r]dt - \sqrt{\delta_2 W_r}dW_4 + \sqrt{\delta_1 W_s}dW_7. \end{aligned} \tag{10}$$

4 Numerical Simulation

Here, we simulate both deterministic and stochastic models for the following set of parameters: $\Lambda = 200$, $\alpha_1 = 0.02$, $\alpha_2 = 0.04$, $\mu = 0.0143$, $\mu_1 = 0.08$, $\beta = 0.093$, $\delta_1 = 0.02$, $\delta_2 = 0.0001$.

The system (1) is simulated for various sets of parameters satisfying the condition of local and globally asymptotic stability of equilibrium E^*. For both deterministic and stochastic models, the simulation results are shown in Fig. 2. The stochastic model (SDE model) is simulated by the method of Euler-Maruyama, and Fig. 2 plots the mean of the 100 runs. Here, the results of the stochastic model seem better than those of the deterministic model as the curve corresponding to scarcity lies below the one that corresponds to the deterministic model $\Lambda = 100$, $\alpha_1 = 0.00002$, $\alpha_2 = 0.004$, $\mu = 0.0143$,

$\mu_1 = 0.08$, $\beta = 0.093$, $\delta_1 = 0.02$, $\delta_2 = 0.9$. The system (1) is simulated for different sets of parameters satisfying the condition of local and globally asymptotic stability of equilibrium E^* (see Fig. 3).

Figs. 4 – 7 demonstrate the impact of various parameters on the equilibrium level of water scarcity and recovery.

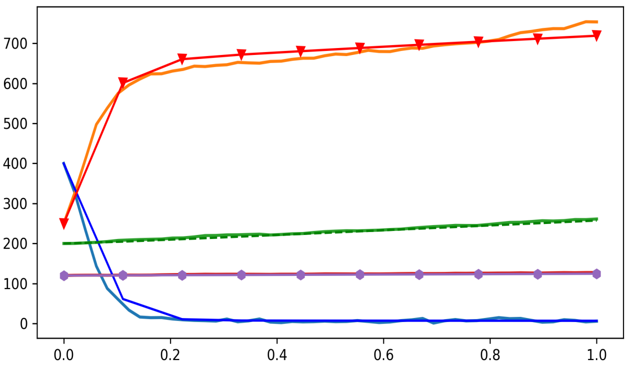

Figure 2: Variation of all compartments of the model showing the effect of stochastic and deterministic models.

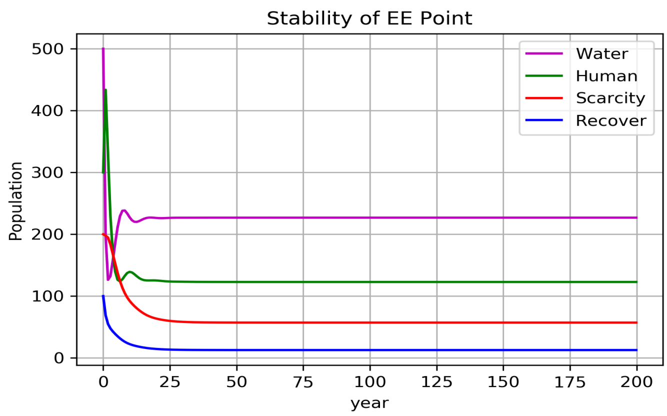

Figure 3: Variation of all compartments of the model showing the stability.

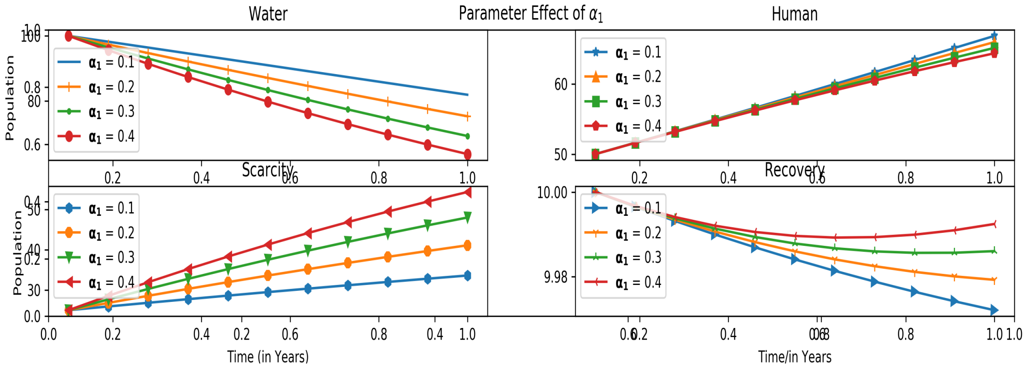

Figure 4: Effect of α_1 on the variation of all compartments of the model.

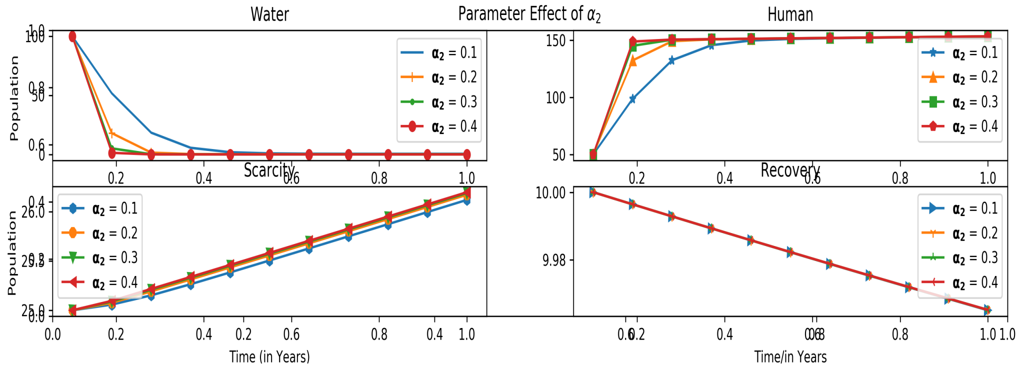

Figure 5: Effect of α_2 on the variation of all compartments of the model.

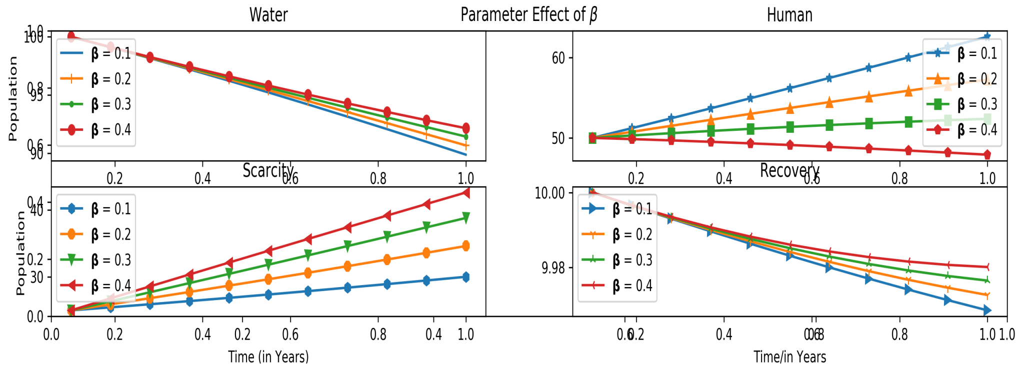

Figure 6: Effect of β on the variation of all compartments of the model.

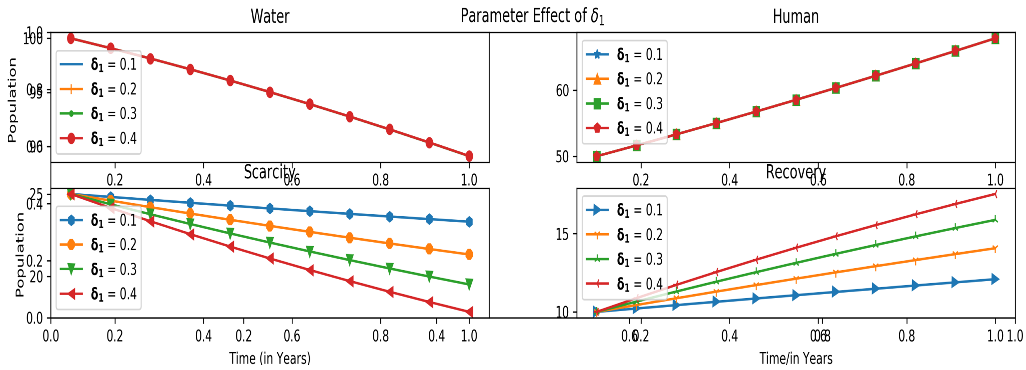

Figure 7: Effect of δ_1 on the variation of all compartments of the model.

5 Result of Discussion and Conclusion

In this paper, a deterministic mathematical model on water resource-related water scarcity problems was proposed and analyzed. We calculate the equilibrium of the proposed model and analyze in detail the local stability and global stability of endemic equilibria.

Further, we extended the deterministic model to a stochastic model and compared numerical simulation results of both models. The resuls of the stochastic model showed that the water scarcity decreased comparatively to the deterministic model. The impact of various parameters on the equilibrium point of water scarcity and recovery is demonstrated. As a society, we have a social responsibility to reduce the scarcity of water. Therefore, we have developed a model of possible strategies to predict better results. Simulations using this model showed the effectiveness of progressing from human to water scarcity.

When the value β (the rate of human population affected by water scarcity) increases in time, the stable point is differed in all compartment (see Fig. 6). Figs. 4 and 5 depict if the values α_1 and α_2 increase or decrease, there is no major difference in all compartments. Fig. 7 depicts if the parameter δ_1 (the rate of water recovery) is increasing in time, the water scarcity is decreased and the recovery is increased.

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