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# Frictional Contact Problem for Thermoviscoelastic Materials with Internal State Variable and Wear

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**Abstract:** In this paper, we study a contact problem between a deformable viscoelastic body and a rigid foundation. Thermal effects, wear and friction between surfaces are taken into account. We model the material's behavior by a nonlinear thermo-viscoelastic law with the internal state variable. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement and the heat equation for the temperature. Our proof is based on nonlinear evolution equations with monotone operators, differential equations and fixed point arguments.

**Keywords:** *thermo-viscoelastic materials; internal state variable; variational inequality of evolution; fixed point; wear.*

**Mathematics Subject Classification (2010):** 74M15, 74D10, 70K70, 70K75, 93-05, 93-10.

## 1 Introduction

During the last decades, the analysis of mathematical models in Contact Mechanics is rapidly growing. These models are suggested for different materials using different boundary conditions modelling friction, lubrication, adhesion, wear, damage, etc.

The aim of this paper is to model and establish the variational analysis of a contact problem for viscoelastic materials within the infinitesimal strain theory. The process is supposed to be subject to thermal effects, friction and wear of contacting surfaces. Mathematical models in Contact Mechanics can be found in [3, 4, 9, 11, 13].

Wear of surfaces is the degradation phenomenon of the superficial layer caused by many factors such as pressure, lubrication, friction and corrosion. Moreover, wear is a

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loss of use as a result of plastic deformations, material removal or fractures. Analysis of contact problems with wear can be found in [6, 7, 12, 16].

The constitutive laws with $k$ internal variables have been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials. Here, we consider a general model for the dynamic process of a bilateral frictional contact between a deformable body and an obstacle which results in the wear of the contacting surface. Recent models of frictional contact problems can be found in [2, 11, 14, 15]. The material obeys a viscoelastic constitutive law with thermal effects. Models taking into account thermal effects can be found in [5, 12]. We derive a variational formulation of the problem which includes a variational second order evolution inequality. We establish the existence and the uniqueness of a weak solution of the problem. The idea is to reduce the second order evolution nonlinear inequality of the system to the first order evolution inequality. After this, we use classical results on first order evolution nonlinear inequalities, a parabolic variational inequality and equations and the fixed point arguments. The novelty of this paper consists in the coupling of $k$ internal state variable, the thermal effect and wear.

The paper is structured as follows. In Section 2, we present the thermo-viscoelastic contact model with friction and provide comments on the contact boundary conditions. In Section 3, we list the assumptions on the data and derive the variational formulation. In Section 4, we present our main results on existence and uniqueness which state the unique weak solvability.

## 2 Problem Statement

The physical setting is the following. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}_d$ $(d = 2, 3)$ with a smooth $\Gamma$. The body is acted upon by body forces of density $f_0$. It is also constrained mechanically on the boundary. We consider a partition of $\Gamma$ into three disjoint measurable parts $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$, such that $meas(\Gamma_1) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. We assume that the body is fixed on $\Gamma_1$, surface traction of density $f_2$ acts on $\Gamma_2$ and a body force of density $f_0$ acts in $\Omega$. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. Then, the classical formulation of the mechanical contact problem of a thermo-visco-elastic material with an internal state variable is as follows.

**Problem P.** Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to S_d$, an internal state variable field $k : \Omega \times [0, T] \to \mathbb{R}^m$, a temperature field $\theta : \Omega \times [0, T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0, T] \to \mathbb{R}_+$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{F}(\varepsilon(u(t))) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \text{in } \Omega \times [0, T] \quad (1)$$

$$\dot{k}\,(t) = \phi\left(\sigma\,(t) - \mathcal{A}\varepsilon\left(\dot{u}\,(t)\right), \varepsilon\left(u\,(t)\right), k\,(t)\right), \quad (2)$$

$$\dot{\theta} - div\,(K_c \nabla\theta) = -M\nabla\dot{u} + q, \tag{3}$$

$$\mathrm{Div}\,\sigma + f_0 = \rho\ddot{u} \quad \text{in } \Omega \times (0,T), \tag{4}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{5}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{6}$$

$$\left\{ \begin{array}{l} \sigma_\nu = -\alpha \left|\dot{u}_\nu\right|, \ \ |\sigma_\tau| = -\mu\sigma_\nu, \\ \sigma_\tau = -\lambda\left(\dot{u}_\tau - v^*\right), \ \lambda \geq 0, \ \dot{\omega} = -kv^*\sigma_\nu, \ k > 0, \end{array} \right. \quad \text{on } \Gamma_3 \ \times [0,T], \tag{7}$$

$$-k_{ij}\frac{\partial\theta}{\partial x_i}\upsilon_j \ = k_e\,(\theta - \theta_R) \ - h_\Gamma\left(\left|\dot{u_\Gamma}\right|\right) \quad \text{on } \Gamma_3 \times (0,T), \tag{8}$$

$$\theta = 0 \quad \text{on } \Gamma_1 \cup \Gamma_2 \times (0,T), \tag{9}$$

$$u(0) = u_0, \ \dot{u}(0) = v_0, \ k(0) = k_0, \ \theta\,(0) = \theta_0 \quad \text{in } \Omega, \tag{10}$$

$$\omega(0) = \omega_0 \text{ on } \Gamma_3. \tag{11}$$

First, (1) represents the thermal viscoelastic constitutive law with long-term memory, $\theta$ represents the temperature, $M := (m_{ij})$ represents the thermal expansion tensor. We denote by $\varepsilon(u)$ (respectively, by $\mathcal{A}$, $\mathcal{F}$, $\mathcal{B}$) the linearized strain tensor (respectively, the viscosity nonlinear tensor, the elasticity operator, the relaxation function), $\phi$ is also a nonlinear constitutive function which depends on $k$. There is a variety of choices for the internal state variables, for reference in the field, see [8,10]. Equation (3) describes the evolution of the temperature field, where $K_c := (k_{ij})$ represents the thermal conductivity tensor, $q$ is the density of volume heat sources. (4) represents the equation of motion, where $\rho$ represents the mass density; we mention that $Div$ is the divergence operator. (5) - (6) are the displacement and the traction boundary condition, respectively. (7) describes the frictional bilateral contact with wear described above on the potential contact surface. (8) represents the associated temperature boundary condition on $\Gamma_3$, where $\theta_R$ is the temperature of the foundation, and $k_e$ is the exchange coefficient between the body and the obstacle. The equation (9) means that the temperature vanishes on $\Gamma_1 \cup \Gamma_2 \times (0,T)$. In (10), $u_0$ is the initial displacement, $v_0$ is the initial velocity, $k_0$ is the initial internal state variable and $\theta_0$ is the initial temperature. In (11), $\omega_0$ is the initial wear.

## 3 Variational Formulation and Preliminaries

For a weak formulation of the problem, first, we introduce some notations. The indices $i$, $j$, $k$, $l$ range from 1 to $d$ and summation over the repeated indices is implied. The index that follows the comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i.j} = \frac{\partial u_i}{\partial x_j}$. We also use the following notations:

$$H = \mathbb{L}^2(\Omega)^d, \mathcal{H} = \{\sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji} \in \mathbb{L}^2(\Omega)\},$$
$$H_1 = \{u = (u_i)/\varepsilon(u) \in \mathcal{H}\}, \mathcal{H}_1 = \{\sigma \in \mathcal{H}/Div\sigma \in H\}.$$

The operators of deformation $\varepsilon$ and divergence $Div$ are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathcal{H}, H_1$, and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$(u,v)_H = \int_\Omega u_i v_i dx, \forall u,v \in H, (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij}dx, \forall \sigma,\tau \in \mathcal{H},$
$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \forall u,v \in H_1, (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H,$
$\sigma,\tau \in \mathcal{H}_1.$

We denote by $|.|_H$ (respectively, by $|.|_{\mathcal{H}}$ , $|.|_{H_1}$, and $|.|_{\mathcal{H}_1}$) the associated norm on the space $H$ ( respectively, $\mathcal{H}$, $H_1$, and $\mathcal{H}_1$).

The following Green's formula holds:

$$(\sigma,\varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}(\sigma),v)_H = \int_\Gamma \sigma\nu\cdot v da \qquad \forall v \in H^1(\Omega)^d,$$

and for the displacement field, we need the closed subspace of $H_1$ defined by

$$V = \{v \in H_1(\Omega) : v = 0 \text{ on } \Gamma_1\}.$$

The set of admissible internal state variables is given by

$$Y = \{\alpha = (\alpha_i)/\alpha_i \in L^2(\Omega), 1 \le i \le m\}.$$

Let us define

$$E = \{\eta \in H_1(\Omega) : \eta = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds, i.e., there exists a positive constant $C_k$, which depends only on $\Omega$, $\Gamma_1$, such that

$$|\varepsilon(v)|_{\mathcal{H}} \ge C_k |v|_{H_1(\Omega)^d}, \quad \forall v \in V.$$

On the space $V$, we consider the inner product and the associated norm given by

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad |v|_V = |\varepsilon(v)|_{\mathcal{H}} \quad \forall u,v \in V. \tag{12}$$

It follows that $|.|_{H_1}$ and $|.|_V$ are equivalent norms on $V$ . Therefore $(V, |.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and Korn's inequality, there exists a positive constant $C_0$ which depends only on $\Omega$, $\Gamma_1$ and $\Gamma_3$ such that

$$|v|_{L^2(\Gamma_3)^d} \le C_0 |v|_V \quad \forall v \in V. \tag{13}$$

In the study of the mechanical problem (1)−(11), we make the following assumptions that the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{a) } \exists\, L_{\mathcal{A}} > 0 : |\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)| \le L_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|, \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d,\ p.p.\ x \in \Omega, \\ \text{b) } \exists\, m_{\mathcal{G}} > 0 : (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2), \varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}} |\varepsilon_1 - \varepsilon_2|^2, \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \\ \text{c) The mapping } x \to \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega,\ \forall \varepsilon \in \mathbb{S}^d, \\ \text{d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x},0) \in \mathcal{H}. \end{cases} \tag{14}$$

The elasticity operator $\mathcal{F} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} \text{a) There exists a constant } L_F > 0 \text{ such that} \\ \quad |\mathcal{F}(x,\varepsilon_1) - \mathcal{F}(x,\varepsilon_2)| \le L_F (|\varepsilon_1 - \varepsilon_2|) \\ \quad \forall \varepsilon_1,\varepsilon_2 \in S_d, \quad \text{a.e. } x \in \Omega. \\ \text{b) The mapping } x \to \mathcal{F}(x,\varepsilon) \text{ is Lebesgue measurable} \\ \quad \text{on } \Omega, \text{ for any } \varepsilon \in S_d. \\ \text{c) The mapping } x \mapsto \mathcal{F}(x,0) \text{ is in } \mathcal{H}. \end{cases} \tag{15}$$

The relaxation function $\mathcal{B} : [0,T] \times \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\left\{ \begin{array}{l} \text{a) } \mathcal{B}_{ijkh} \in W^{1.\infty}(0,T;\mathbb{L}^\infty(\Omega)), \\ \text{b) } \mathcal{B}(t)\,\sigma \cdot \tau = \sigma \cdot \mathcal{B}(t)\,\tau, \forall \sigma, \tau \in \mathbb{S}^d,\ p.p.t \in [0,T]\,,\ p.p.\text{on } \Omega. \end{array} \right. \tag{16}$$

The function $\phi : \Omega \times S_d \times S_d \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$\left\{ \begin{array}{l} \text{a) There exists a constant } L_\phi > 0 \text{ such that} \\ \quad |\phi(x,\sigma_1,\xi_1,k_1) - \phi(x,\sigma_2,\xi_2,k_2)\ | \leq\ L_\phi\left(|\sigma_1-\sigma_2| + |\xi_1-\xi_2| + |k_1-k_2|\right), \\ \quad \forall \sigma_1, \sigma_{2,}\varepsilon_1, \varepsilon_2 \in S_d \ \text{ and } k_1,k_2 \in \mathbb{R}^m,\ \text{a.e. } x \in \Omega. \\ \text{b) For any } \sigma, \varepsilon \in S_d \text{ and } k \in \mathbb{R}^m,\quad x \to \phi(x,\sigma,\varepsilon,k) \text{ is Lebesgue measurable on } \Omega. \\ \text{c) The mapping } \mathbf{x} \mapsto \phi(x,0,0,0) \text{ is in } L^2(\Omega)^m. \end{array} \right. \tag{17}$$

The function $h_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\left\{ \begin{array}{l} \quad \text{a ) There exists a constant } L_\tau > 0 \text{ such that} \\ \ |h_\tau(x,r_1) - h_\tau(x,r_2)| \leq L_h\,|r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}_+\ ,\ \text{a.e. } x \in \Gamma_3. \\ \text{b) } x \mapsto p_\tau(.,0) \text{ is Lebesgue measurable on } \Gamma_3,\ \forall r \in \mathbb{R}_+. \end{array} \right. \tag{18}$$

For the temperature, we use the following Green's formula:

$$\int_\Omega \dot{\theta}\tau dx - \int_\Omega div\left(K_c \nabla\theta\right) = \int_\Omega -\left(M_e \nabla \dot{u}\right)\tau dx + \int_\Omega q\tau dx \quad \forall \tau \in E. \tag{19}$$

The mass density satisfies

$$\rho \in L^\infty(\Omega),\ \text{there exists } \rho^* > 0 \text{ such that} \quad \rho \geq \rho^* \text{ a.e. } x \in \Omega. \tag{20}$$

We also suppose that the forces, the tractions, the volume, the surface free charges densities and the functions $\alpha$ and $\mu$ have the regularity

$$\left\{ \begin{array}{l} f_0 \in L^2(0,T;H), \quad f_2 \in L^2(0,T;L^2(\Gamma_2)^d), \\ \ \alpha \in \mathbb{L}^\infty(\Gamma_3) \quad \alpha(x) \geq \alpha^* > 0, \quad p.p.\text{on } \Gamma_3, \\ \ \mu \in \mathbb{L}^\infty(\Gamma_3), \quad \mu(x) > 0, \quad p.p.\text{on } \Gamma_3, \end{array} \right. \tag{21}$$

$$q \in W^{1,2}(0,T;L^2(\Omega)),\ \theta_R \in W^{1,2}(0,T;L^2(\Gamma_3)),\ k_e \in L^\infty(\Omega,\mathbb{R}_+)\ , \tag{22}$$

$$\left\{ \begin{array}{l} K_c\ = (k_{ij}), (k_{ij} = k_{ji} \in L^\infty(\Omega), \\ \forall c_k \geq 0, \zeta_i \in \mathbb{R}^d, k_{ij}\zeta_i\zeta_j \geq c_k\zeta_i\zeta_j, \end{array} \right. \tag{23}$$

$$M = (m_{ij}), m_{ij} = m_{ji} \in L^\infty(\Omega). \tag{24}$$

The initial data satisfy

$$u_0 \in V,\ v_0 \in H,\ \theta_0 \in E,\ k_0 \in Y,\ \omega_0 \in L^\infty(\Gamma_3). \tag{25}$$

We will use a modified inner product on the Hilbert space, given by

$$((u,v))_H = (\rho u, v)_H \quad \forall\ u, v \in H, \tag{26}$$

and we let $\|.\|_H$ be the associated norm given by

$$\|v\|_H = (\rho v, v)^{\frac{1}{2}} \quad \forall v \in H. \tag{27}$$

It follows from assumption (20) that $\| \, . \, \|_H$ and $|\cdot|_H$ are equivalent norms on $H$, and also the inclusion mapping of $(V, |\cdot|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by $V'$ the dual space of $V$. Identifying $H$ with its own dual, we can write the Gelfand triple

$$V \subset H \subset V'.$$

We use the notation $(.,.)_{V'\times V}$ to represent the duality pairing between $V'$ and $V$ , recall that

$$(u,v)_{V'\times V} = ((u,v))_H \quad \forall u \in H, \forall v \in V. \tag{28}$$

Let $f: \ [0,T] \to V'$ be the function defined by

$$(f(t),v)_{V'\times V} = \int_\Omega f_0(t).v dx + \int_{\Gamma_2} f_2(t).v da \ \ \forall \ \mathbf{v} \in V. \tag{29}$$

Next, we denote by $j : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$

$$j(u,v) = \int_{\Gamma_3} \alpha \, |u_\nu| \, (\mu \, |v_\tau - v^*|) \, da \; . \tag{30}$$

Let $\varphi : V \times V \to \mathbb{R}$ be the function defined by

$$\varphi(u,v) = \int_{\Gamma_3} \alpha \, |u_\nu| \, |v_\nu| \, da, \forall v \in V. \tag{31}$$

Let us introduce the operator $A : V \to V'$

$$(Au,v)_{V'\times V} = (\mathcal{A}(\varepsilon(u)), \varepsilon(v))_{\mathcal{H}}$$

for all $u, v \in V$ and $t \in [0,T]$. Note that

$$f \in L^2(0,T;V'). \tag{32}$$

Using standard arguments based on Green's formulas we can derive the following variational formulation of problem P.

**Problem PV.** Find a displacement field $u : [0,T] \to V$, a stress field $\sigma : [0,T] \to \mathcal{H}$, an internal state variable field $k : [0,T] \to Y$, a temperature field $\theta : \Omega \times [0,T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0,T] \to \mathbb{R}_+$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{F}(\varepsilon(u(t))) + \int\limits_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \text{in } \Omega \times [0,T] \,, \tag{33}$$

$$\dot{k}\,(t) = \phi\left(\sigma\,(t) - \mathcal{A}\varepsilon\left(\dot{u}\,(t)\right), \varepsilon\,(u\,(t))\,, k\,(t)\right), \tag{34}$$

$$\begin{gathered}(\ddot{u}(t), w - \dot{u}(t))_{V'\times V} + \ (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} + j(\dot{u}, w) - j(\dot{u}, \dot{u}(t)) + \varphi(\dot{u}, w) - \varphi(\dot{u}, \dot{u}(t)) \\ \geq \big(f(t), w - \dot{u}(t)\big)\,, \ \forall u, w \in V,\end{gathered} \tag{35}$$

$$\dot{\theta}\,(t) + K\theta\,(t) = R\dot{u}\,(t) + Q\,(t) \quad t \in (0,T), \tag{36}$$

$$\dot{\omega} = -kv^*\sigma_\nu, \tag{37}$$

$$u(0) = u_0, \ \dot{u}(0) = v_0, \ k(0) = k_0, \ \theta\,(0) = \theta_0, \tag{38}$$

where $Q : [0,T] \to E', K : E \to E', R : V \to E'$ are given by

$$(Q(t), \mu)_{E' \times E} = \int_{\Gamma_3} k_e \theta_R(t) \mu da + \int_{\Omega} q(t) \mu dx, \tag{39}$$

$$(K\tau, \mu)_{E' \times E} = \sum_{i,j=1}^{d} \int_{\Omega} k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \mu da, \tag{40}$$

$$(Rv, \mu)_{E' \times E} = \int_{\Gamma_3} h_\tau(|v_\tau|) \mu da - \int_{\Omega} (M \nabla v)\, dx \tag{41}$$

for all $v \in V, \mu, \tau \in E$.

The proof of the existence and uniqueness of solution to problem PV will be given in the next section.

## 4 Existence and Uniqueness Result

Now, we propose our existence and uniqueness result.

**Theorem 4.1** *Let the assumptions (14)–(25) hold. Then the problem has a unique solution* $\{u, \sigma, k, \omega, \theta\}$ *satisfying*

$$u \in C^1(0,T;H) \cap W^{1.2}(0,T;V) \cap W^{2.2}(0,T;V') \tag{42}$$

$$\sigma \in L^2(0,T;\mathcal{H}), \quad Div\sigma \in L^2(0,T;V'), \tag{43}$$

$$k \in W^{1,2}(0,T;Y), \tag{44}$$

$$\omega \in C^1(0,T;\mathbb{L}^2(\Gamma_3)), \tag{45}$$

$$\theta \in W^{1,2}(0,T;E') \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)). \tag{46}$$

*We conclude that under the assumptions (14)–(25), the mechanical problem (1)-(11) has a unique weak solution with the regularity (42)-(46).*

The proof of this theorem will be carried out in several steps.

**The first step**: let $g \in L^2(0,T;V)$ and $\eta = (\eta^1, \eta^2) \in L^2(0,T;V' \times Y)$ be given, and prove that there exists a unique solution $u_{g\eta}$ of the following intermediate problem.

**Problem PV$_{g\eta}$.** Find the displacement field $u_{g\eta} : [0,T] \to V$ such that for a.e. $t \in (0,T)$,

$$\begin{cases} (\ddot{u}_{g\eta}(t), w - \dot{u}_{g\eta}(t))_{V' \times V} + \ (\mathcal{A}\varepsilon(\dot{u}_{g\eta}(t)), \varepsilon(w - \dot{u_{g\eta}}(t)))_{\mathcal{H}} + \\ \left(\eta^1, w - \dot{u_{g\eta}}(t)\right)_{V' \times V} + j(g, w) - j(g, \dot{u_{g\eta}}(t)) \geq \left(f_{g\eta}(t), w - \dot{u}_{g\eta}(t)\right), \ \forall w \in V, \end{cases} \tag{47}$$

$$u_{g\eta}(0) = u_0, \quad \dot{u}_{g\eta}(0) = v_0. \tag{48}$$

We define $f_{g\eta}(t) \in V$ for $a.e.t \in [0.T]$ by

$$(f_{g\eta}(t), w)_{V' \times V} = \left(f(t) - \eta^1(t), w\right)_{V' \times V}, \quad \forall w \in V. \tag{49}$$

From (29), we deduce that

$$f_\eta \in L^2(0,T;V'). \tag{50}$$

Let now $u_\eta : [0.T] \to V$ be the function defined by

$$u_\eta\,(t) = \int_0^t v_\eta\,(s)\,ds + u_0, \forall t \in [0,T]\,. \tag{51}$$

Concerning Problem $\mathrm{PV}_{g\eta}$, we have the following result.

**Lemma 4.1** *There exists a unique solution to problem $PV_{g\eta}$ with the regularity.*

$$v_\eta \in L^2(0,T;V) \text{ and } \dot{v}_\eta \in L^2(0,T;V'). \tag{52}$$

***Proof.*** The proof by nonlinear first order evolution inequalities is given in [9].

**The second step**: we use the displacement $u_{g\eta}$ to consider the following variational problem.

Let us consider now the operator $\Lambda_\eta\,(g) : \mathbb{L}^2(0,T;V) \to \mathbb{L}^2(0,T;V)$ defined by

$$\Lambda_\eta\,(g) = v_{g\eta}\;. \tag{53}$$

We have the following lemma.

**Lemma 4.2** *The operator $\Lambda_\eta$ has a unique fixed point $g_\eta^* \in \mathbb{L}^2(0,T;V)$.*

***Proof.*** Let $g_1, g_2 \in \mathbb{L}^2(0,T;V)$ and let $\eta = \left(\eta^1,\eta^2\right) \in L^2\,(0,T;V' \times Y)$. Using similar arguments as in (47), (51), we find

$$\begin{aligned}&(\dot{v}_1\,(t) - \dot{v}_2\,(t)\,, v_1\,(t) - v_2\,(t)) + (\mathcal{A}\varepsilon\,(v_1\,(t)) - \mathcal{A}\varepsilon\,(v_2\,(t))\,, \varepsilon\,(v_1\,(t)) - \varepsilon\,(v_2\,(t)))+\\ &+j(g_1, v_1\,(t)) - j(g_1, v_2\,(t)) - j(g_2, v_1\,(t)) + j(g_2, v_2\,(t)) \le 0.\end{aligned} \tag{54}$$

From the definition of the functional $j$ given by (30), we have

$$\begin{aligned}&j(g_1, v_2\,(t)) - j(g_1, v_1\,(t)) - j(g_2, v_2\,(t)) + j(g_2, v_1\,(t)) = \textstyle\int_{\Gamma_3} (\alpha\,|g_{1\nu}| - \alpha\,|g_{2\nu}|)\\ &(\mu\,|v_{1\tau} - v^*| - \mu\,|v_{2\tau} - v^*|)\,da.\end{aligned} \tag{55}$$

From (13), (21) we find

$$j(g_1, v_2\,(t)) - j(g_1, v_1\,(t)) - j(g_2, v_2\,(t)) + j(g_2, v_1\,(t)) \le C\,|g_1 - g_2|_V\,|v_1 - v_2|_V\,. \tag{56}$$

Integrating the inequality (54) with respect to time, using the initial conditions $v_2\,(0) = v_1\,(0) = v_0$, using (14) , (56) and the inequality

$$2ab \le \frac{C}{m_{\mathcal{A}}}a^2 + \frac{m_{\mathcal{A}}}{C}b^2,$$

we find

$$|v_2\,(t) - v_1\,(t)|_V^2 \le C\int_0^t |g_2\,(s) - g_1\,(s)|_V^2\,ds. \tag{57}$$

From (53) and (57), we find that

$$|\Lambda_\eta g_2\,(t) - \Lambda_\eta g_1\,(t)|_V^2 \le C\int_0^t |g_2\,(s) - g_1\,(s)|_V^2\,ds.$$

Reiterating this inequality $m$ times, we obtain

$$\left|\Lambda_\eta^m g_2(t) - \Lambda_\eta^m g_1(t)\right|_{\mathbb{L}^2(0,T;V)} \leq \frac{C^m T^m}{m!} \left|g_2(t) - g_1\right|_{\mathbb{L}^2(0,T;V)}. \tag{58}$$

Since $\lim\limits_{m\to+\infty} \frac{C^m T^m}{m!} = 0$, it follows that exists a positive integer $m$ such that $\frac{C^m T^m}{m!} < 1$ and, therefore, (58) shows that $\Lambda_\eta^m$ is a contraction on the Banach space $\mathbb{L}^2(0,T;V)$. Thus, from Banach's fixed point theorem, the operator $\Lambda_\eta$ has a unique fixed point $g_\eta^* \in \mathbb{L}^2(0,T;V)$.

**Lemma 4.3** *Now, define $k_\eta \in W^{1,2}(0,T;Y)$ by*

$$k_\eta(t) = k_0 + \int_0^t \eta^2(s)\, ds. \tag{59}$$

*Then there exists $C > 0$ such that*

$$|k_1(s) - k_2(s)|_Y^2 \leq C \int_0^t \left|\eta_1^2(s) - \eta_2^2(s)\right|_{Y'}^2 ds. \tag{60}$$

In the third step, we use the displacement field $u_\eta$ obtained in Lemma 4.1 and $k_\eta$ defined in (59) to consider the following variational problem for the temperature field.

**Problem PV$_\theta$.** Find $\theta_\eta : [0,T] \to E'$ satisfying for a.e. $t \in (0,T)$,

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{u}_\eta(t) + Q(t) \quad t \in (0,T), \text{ in } E', \tag{61}$$

$$\theta_\eta(0) = \theta_0. \tag{62}$$

**Lemma 4.4** *Problem $\boldsymbol{PV}_\theta$ has a unique solution*

$$\theta_\eta \in W^{1,2}(0;T;E') \cap L^2(0;T;E) \cap C(0;T;L^2(\Omega)), \quad \forall \eta \in L^2(0,T;V'),$$

*satisfying*

$$|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)|_{L^2(\Omega)}^2 \leq C \int_0^t |\upsilon_1(s) - \upsilon_2(s)|_V^2 ds \qquad \forall\, t \in (0,T). \tag{63}$$

***Proof.*** The existence and uniqueness result verifying (61) follows from the classical result on the first order evolution equation, applied to the Gelfand evolution triple

$$E \subset F \equiv F' \subset E'.$$

We verify that the operator $K$ is linear continuous and strongly monotone. Now from the expression of the operator $R$, $v_\eta \in W^{1,2}(0,T;V) \Rightarrow Rv_\eta \in W^{1,2}(0,T;F)$, as $Q \in W^{1,2}(0,T;E)$, then $Rv_\eta + Q \in W^{1,2}(0,T;E)$, we deduce (63), (See [1]).

Finally, as a consequence of these results, and using the properties of $\mathcal{F}$, $\mathcal{E}$, $\mathcal{G}$, $\phi$, and $j$ for $t \in [0,T]$, we consider the element

$$\Lambda\eta(t) = \left(\Lambda^1\eta(t), \Lambda^2\eta(t)\right) \in V' \times Y, \tag{64}$$

$$\begin{aligned}\left(\Lambda^1(\eta), w\right)_{V'\times V} &= (\mathcal{F}(\varepsilon(u_\eta(t))), w)_V + \\ &+ (\int_0^t \mathcal{B}(t-s)\varepsilon(u_\eta(s))ds, w)_V - (\theta_\eta(t)\mathcal{M}, \varepsilon(w))_{\mathcal{H}} + \varphi(\dot{u}, w)\ \forall w \in V,\end{aligned} \tag{65}$$

$$\Lambda^2\eta(t) = \phi\left(\sigma_\eta(t), \varepsilon\left(u_\eta(t)\right), k_\eta(t)\right). \tag{66}$$

Here, for every $\eta \in L^2(0,T;V' \times Y)$, $u_\eta$, $\theta_\eta$ represent the displacement field and the temperature field obtained in Lemmas 4.1, 4.4, respectively, and $k_\eta$ is the internal state variable given by (59). We have the following result.

**Lemma 4.5** *The operator $\Lambda$ has a unique fixed point $\eta^* \in L^2(0,T;V' \times Y)$.*

***Proof.*** Let $\eta_1$, $\eta_2 \in L^2(0,T;V' \times Y)$. Write for $i = 1.2, u_{\eta i} = u_i$, $\dot{u}_{\eta i} = v_{\eta i} = v_i$, $\sigma_{\eta i} = \sigma_i$, $k_{\eta i} = k_i$, $\theta_{\eta i} = \theta_i$. Using (12), (15), (16), (24), (31), we have

$$\begin{aligned}&\left|\Lambda^1\eta_1(t) - \Lambda^1\eta_2(t)\right|^2_{V'} \leq C(|u_1(t) - u_2(t)|^2_V + \textstyle\int_0^t |u_1(s) - u_2(s)|^2_V \ ds + \\ &|\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega} + |v_1(t) - v_2(t)|^2_V).\end{aligned} \tag{67}$$

By similar arguments, from (66), (33) and (17), it follows that

$$\left|\Lambda^2\eta_1(t) - \Lambda^2\eta_2(t)\right|^2_Y \leq C(|\sigma_1(t) - \sigma_2(t)|^2_{\mathcal{H}} + |u_1(t) - u_2(t)|^2_V + |k_1(t) - k_2(t)|^2_Y). \tag{68}$$

Taking into account that

$$\sigma_i(t) = \mathcal{A}(\varepsilon(\dot{u}_i(t))) + \eta_i^1(t), \ \forall t \in [0,T], \tag{69}$$

by (14), and using (69), we find

$$|\sigma_1(t) - \sigma_2(t)|^2_{\mathcal{H}_1} \leq C\left(|v_1(t) - v_2(t)|^2_V + \left|\eta_1^1(t) - \eta_2^1(t)\right|^2_{V'}\right). \tag{70}$$

So

$$\begin{aligned}\left|\Lambda^2\eta_1(t) - \Lambda^2\eta_2(t)\right|^2_Y \quad &\leq \quad C(|v_1(t) - v_2(t)|^2_V + \left|\eta_1^1 - \eta_2^1\right|^2_{V'} + |u_1(t) - u_2(t)|^2_V \\ &\qquad + |k_1(t) - k_2(t)|^2_Y).\end{aligned} \tag{71}$$

Consequently,

$$\begin{aligned}|\Lambda\eta_1(t) - \Lambda\eta_2(t)|^2_{V'\times Y} &\leq C(|u_1(t) - u_2(t)|^2_V + |k_1(t) - k_2(t)|^2_Y + \left|\eta_1^1(t) - \eta_2^1(t)\right|^2_{V'} \\ &+ |\theta_1(t) - \theta_2(t)|^2_{L^2(\Omega} + |v_1(t) - v_2(t)|^2_V + \textstyle\int_0^t |u_1(s) - u_2(s)|^2_V \, ds.\end{aligned} \tag{72}$$

Since $u_1$ and $u_2$ have the same initial value, we get

$$|u_1(t) - u_2(t)|^2_V \leq C\int_0^t |v_1(s) - v_2(s)|^2_V \, ds.$$

From this inequality, (72) and (63), we obtain

$$\begin{aligned}&|\Lambda\eta_1(t) - \Lambda\eta_2(t)|^2_{V'\times Y} \leq C(\textstyle\int_0^t |v_1(s) - v_2(s)|^2_V \, ds + |v_1(t) - v_2(t)|^2_V + \\ &|k_1(t) - k_2(t)|^2_Y + \left|\eta_1^1(t) - \eta_2^1(t)\right|^2_{V'}), \forall t \in [0,T].\end{aligned}$$

Moreover, from (54), we obtain

$$\begin{aligned}&\big(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)\big) + (\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))) + \\ &+ (\eta_1(t) - \eta_2(t), v_1(t) - v_2(t)) \leq j(v_1(t), v_2(t)) - j(v_1(t), v_1(t)) \\ &-j(v_2(t), v_2(t)) + j(v_2(t), v_1(t)).\end{aligned} \tag{73}$$

From the definition of the functional $j$ given by (30), and using (13),(23), we get

$$j(v_1(t), v_2(t)) - j(v_1(t), v_1(t)) - j(v_2(t), v_2(t)) + j(v_2(t), v_1(t)) \leq C\,|v_1(t) - v_2(t)|_V^2 . \tag{74}$$

Integrating the inequality(73) with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (14),(74) and using the Cauchy-Schwartz inequality and the inequality

$$2ab \leq m_{\mathcal{A}} a^2 + \frac{1}{m_{\mathcal{A}}} b^2,$$

we find

$$\int_0^t |v_1(s) - v_2(s)|_V^2\, ds \leq C \int_0^t \left|\eta_1^1(s) - \eta_2^1(s)\right|_{V'}^2 ds. \tag{75}$$

It follows now from (59),(63) and (75) that

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{V'\times Y}^2 \leq C \int_0^t |\eta_1(s) - \eta_2(s)|_{V'\times Y'}^2\, ds.$$

Reiterating the previous inequality $n$ times, we find that

$$|\Lambda^n\eta_1 - \Lambda^n\eta_2|_{L^2(0,T;\, V'\times Y)}^2 \leq \tfrac{C^n T^n}{n!} \int_0^t |\eta_1(s) - \eta_2(s)|_{V'\times Y}^2\, ds.$$

This inequality shows that for $n$ large enough, the operator $\Lambda^n$ is a contraction on the Banach space $L^2(0,T;\ V'\times Y)$, and so $\Lambda$ has a unique fixed point. Next, we consider the operator $\mathcal{L} : C(0,T;\mathbb{L}^2(\Gamma_3)) \to C(0,T;\mathbb{L}^2(\Gamma_3))$ defined by

$$\mathcal{L}\omega(t) = -kv^* \int_0^t \sigma_\nu(s)\, ds, \forall t \in [0,T]. \tag{76}$$

**Lemma 4.6** *The operator $\mathcal{L} : C(0,T;\mathbb{L}^2(\Gamma_3)) \to C(0,T;\mathbb{L}^2(\Gamma_3))$ has a unique point element $\omega^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ such that $\mathcal{L}\omega^* = \omega^*$.*

***Proof.*** Using $\omega_1$, $\omega_2 \in C(0,T;\mathbb{L}^2(\Gamma_3))$, we have

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 \leq kv^* \int_0^t |\sigma_1(s) - \sigma_2(s)|^2\, ds.$$

From (12) and using (14)-(16), we find

$$\begin{aligned}&|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}_1}^2 \leq C(|u_1(t) - u_2(t)|_V^2 + |v_1(t) - v_2(t)|_V^2 + \\ &\textstyle\int_0^t |u_1(s) - u_2(s)|_V^2\, ds + |\theta_1(t) - \theta_2(t)|_{H^1(\Omega)}^2).\end{aligned} \tag{77}$$

Using (63), we obtain

$$\begin{aligned}&|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}_1}^2 \leq C(\textstyle\int_0^t |u_1(s) - u_2(s)|_V^2\, ds + |u_1(t) - u_2(t)|_V^2 + \\ &|v_1(t) - v_2(t)|_V^2 + \textstyle\int_0^t |v_1(s) - v_2(s)|_V^2\, ds.\end{aligned} \tag{78}$$

From (51), we have

$$\begin{gathered}\textstyle\int_0^t |u_1(s) - u_2(s)|_V^2\, ds + |u_1(t) - u_2(t)|_V^2 + |v_1(t) - v_2(t)|_V^2 \leq \\ C \textstyle\int_0^t |v_1(s) - v_2(s)|_V^2\, ds.\end{gathered}$$

So

$$\begin{aligned}&\int_0^t |u_1(s)-u_2(s)|_V^2\,ds+|u_1(t)-u_2(t)|_V^2+|v_1(t)-v_2(t)|_V^2\le\\&C(\int_0^t |v_1(s)-v_2(s)|_V^2\,ds+|\omega_1(t)-\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}).\end{aligned}\tag{79}$$

By Gronwall's inequality, we find

$$\int_0^t |u_1(s)-u_2(s)|_V^2\,ds+|u_1(t)-u_2(t)|_V^2+|v_1(t)-v_2(t)|_V^2\le C\,|\omega_1(t)-\omega_2(t)|^2_{\mathbb{L}^2(\Gamma_3)}.$$

So, we have

$$|\sigma_1(t)-\sigma_2(t)|^2_{\mathcal{H}_1}\le C\int_0^t |\omega_1(s)-\omega_2(s)|^2_{\mathbb{L}^2(\Gamma_3)}\,ds.\tag{80}$$

Using (80), we find

$$|\mathcal{L}\omega_1(t)-\mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}\le C\int_0^t |\omega_1(s)-\omega_2(s)|_{\mathbb{L}^2(\Gamma_3)}\,ds.$$

Reiterating the previous inequality $p$ times, we find that

$$|\mathcal{L}\omega_1(t)-\mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}\le\frac{(Ct)^p}{p!}\,|\omega_1(t)-\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}.$$

This inequality shows that for $p$ large enough, the operator $\mathcal{L}^p$ is a contraction on the Banach space $C(0,T;\mathbb{L}^2(\Gamma_3))$, and so $\mathcal{L}$ has a unique fixed point $\omega^*\in C(0,T;\mathbb{L}^2(\Gamma_3))$.

Now we have all the ingredients to prove Theorem 4.1.

**Existence.** Let $g^*=g^*_{\eta^*}$ be the fixed point of $\Lambda_{\eta^*}$ defined by Lemma 4.2, let $\eta_*=\left(\eta^1_*,\eta^2_*\right)\in L^2\,(0,T;V'\times Y)$ be the fixed point of $\Lambda$ defined by (65) and (66), $k_{\eta^*}(t)=k_0+\int_0^t\eta^2_*(s)\,ds$, and let $\omega^*\in C(0,T;\mathbb{L}^2(\Gamma_3))$ be the fixed point $\mathcal{L}$ defined by (76) and let $(u_{\eta_*},\theta_{\eta_*})$ be the solution to Problems $\mathbf{PV}_{g\eta}$, $\mathbf{PV}_\theta$ for $\eta=\eta_*$, that is, $u=u_{\eta_*}, k=k_{\eta_*}$, $\theta=\theta_{\eta_*}$, and

$$\sigma(t)=\mathcal{A}(\varepsilon(\dot u(t)))+\mathcal{F}(\varepsilon(u(t)))+\int_0^t\mathcal{B}(t-s)\varepsilon(u(s))ds-\theta(t)\mathcal{M}.$$

It results from (65) and (66), for $\Lambda^1(\eta_*)=\eta^1$ and $\Lambda^2(\eta_*)=\eta^2$, that $(u,\sigma,k,\theta,\omega)$ is a solution of Problem **PV**. The regularities (42)-(46) follow from Lemmas 4.1, 4.3, 4.4 and 4.6.

**Uniqueness.** The uniqueness of the solution follows from the uniqueness of the fixed point of the operators $\Lambda_\eta$, $\Lambda$ and $\mathcal{L}$.

## 5 Concluding Remark

Scientific research and recent papers in mechanics are articulated around two main components, one devoted to the laws of behavior and the other devoted to the boundary conditions imposed on the body.

The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables on the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials. Our model is obtained by combining the thermoviscoelastic constitutive law with a long memory term, wear, friction and the internal state variable $k$. The model is developed to describe the self-heating and stress-strain behavior of thermoviscoelastic polymers under tensile loading when the rate of deformation depends on the internal variable $k$.

Mathematically, the idea is to reduce the second order nonlinear evolution inequality of the system to the first order evolution inequality. After this, we use classical results on first order evolution nonlinear inequalities, parabolic inequalities, differential equations and fixed point arguments.

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