Volume 22, Number 5, 2022 ISSN 1562-8353

NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys

Volume 22 Number 5 2022

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NONLINEAR DYNAMICS & SYSTEMS THEORY

Volume 22, No. 5, 2022

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An International Journal of Research and Surveys

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NONLINEAR DYNAMICS AND SYSTEMS THEORY
An International Journal of Research and Surveys
Published by InforMath Publishing Group since 2001

Volume 22 Number 5 2022

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Founded by A.A. Martynyuk in 2001.
Registered in Ukraine Number: KB No 5267 / 04.07.2001.

NONLINEAR DYNAMICS AND SYSTEMS THEORY

An International Journal of Research and Surveys

Impact Factor from SCOPUS for 2021: SJR – 0.326, SNIP – 0.601 and CiteScore – 1.5

Nonlinear Dynamics and Systems Theory (ISSN 1562–8353 (Print), ISSN 1813–7385 (Online)) is an international journal published under the auspices of the S.P. Timoshenko Institute of Mechanics of National Academy of Sciences of Ukraine and Curtin University of Technology (Perth, Australia). It aims to publish high quality original scientific papers and surveys in areas of nonlinear dynamics and systems theory and their real world applications.

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Nonlinear Dynamics and Systems Theory will have 5 issues in 2023, printed in hard copy (ISSN 1562–8353) and available online (ISSN 1813–7385), by InforMath Publishing Group, Nesterov str., 3, Institute of Mechanics, Kiev, MSP 680, Ukraine, 03057. Subscription prices are available upon request from the Publisher, EBSCO Information Services (`mailto:journals@ebsco.com`), or website of the Journal: `http://e-ndst.kiev.ua`. Subscriptions are accepted on a calendar year basis. Issues are sent by airmail to all countries of the world. Claims for missing issues should be made within six months of the date of dispatch.

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Nonlinear Dynamics and Systems Theory, **22** (5) (2022) 473–488

Analysis of an Antiplane Thermo-Electro-Viscoelastic Contact Problem with Long-Term Memory

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Received: May 6, 2022; Revised: October 10, 2022

Abstract: We study a mathematical problem modeling the antiplane shear deformation of a cylinder in frictionless contact with a rigid foundation. The material is assumed to be thermo-electro-viscoelastic with long-term memory, the friction is modeled by Tresca's law and the foundation is assumed to be electrically conductive. We derive a variational formulation for the model which is in the form of a system involving the displacement field, the electric potential field and the temperature field. We prove the existence of a unique weak solution to the problem. The proof is based on the arguments of time-dependent variational inequalities, parabolic inequalities, differential equations and a fixed point theorem.

Keywords: *weak solution; variational formulation; antiplane shear deformation; thermo-electroviscoelastic material; Tresca's friction law; fixed point; variational inequality.*

Mathematics Subject Classification (2010): 74M10, 49J40, 70K70, 70K75.

1 Introduction

Anti-plane shear deformation problems arise naturally from many real world applications such as rectilinear steady flow of simple fluids [6], interface stress effects of nanostructured materials [10], structures with cracks [16], layered/composite functioning materials [15], and phase transitions in solids [17]. Considerable attention has been paid to the modelling of such kind of problems, see for instance [8] and the references therein. In particular, the review paper [8] deals with modern developments for the antiplane shear model involving linear and nonlinear solid materials, various constitutive settings and applications. Antiplane frictional contact problems are used in geophysics in order to

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describe the pre-earthquake evolution of the regions of high tectonic activity, see for instance [5] and the references therein. The mathematical analysis of models for antiplane frictional contact problems can be found in [2, 8, 18].

Currently, there is a considerable interest in frictional or frictionless contact problems involving piezoelectric materials, i.e., materials characterized by the coupling of mechanical and electrical properties. This coupling, in a piezoelectric material, leads to the appearance of electric potential when mechanical stress is present, and conversely, mechanical stress is generated when electric potential is applied. The first effect is used in mechanical sensors, and the reverse effect is used in actuators, in engineering control equipment. Piezoelectric materials for which the mechanical properties are elastic are also called electro-elastic materials and piezoelectric materials for which the mechanical properties are viscoelastic are also called electro-viscoelastic materials. General models for piezoelectric materials can be found in [3, 4, 12]. Static frictional contact problems for elastic and viscoelastic materials were studied in [11, 13, 14], under the assumption that the foundation is insulated. Contact problems with normal compliance for electro-viscoelastic materials were investigated in [9, 19]. There, variational formulations of the problems were considered and their unique solvability was proved. Antiplane problems for piezoelectric materials were considered in [18].

In paper [20], the authors have studied an antiplane contact problem for viscoelastic materials with long-term memory. This mechanical problem leads to an integro-differential variational inequality. In the present paper, we deal with an antiplane contact problem for a thermo-electro-viscoelastic cylinder, which leads to a new mathematical model, different from the one presented in [20]. The novelty of this paper consists in the fact that we model the friction by Tresca's law and the material's behavior by a thermo-viscoelastic constitutive law with long-term memory. We neglect the inertial term in the equation of motion to obtain a quasistatic approximation of the process.

Thermal effects in contact processes affect the composition and stiffness of the contacting surfaces, and cause thermal stresses in the contacting bodies. Moreover, the contacting surfaces exchange heat, and energy is lost to the surroundings. Our interest is to describe a simple physical process in which the frictional contact, viscosity and piezoelectric effects are involved, and to show that the resulting model leads to a well-posed mathematical problem. Taking into account the frictional contact between a viscous piezoelectric body and an electrically conductive foundation in the study of an antiplane problem leads to a new and interesting mathematical model which has the virtue of relative mathematical simplicity without loss of essential physical relevance.The main result we provide concerns the existence of a unique weak solution to the model. Its proof is carried out in several steps, and is based on the arguments of evolutionary variational inequalities and Banach's fixed-point theorem.

The rest of the paper is structured as follows. In Section 2, we describe the model of the frictional contact process between a thermo-electro-viscoelastic body and a conductive deformable foundation. In Section 3, we derive the variational formulation. It consists of a variational inequality for the displacement field coupled with a time-dependent variational equation for the electric potential and the heat equation for the temperature. We state our main result, the existence of a unique weak solution to the model, in Theorem 3.1. The proof of the theorem is provided in Section 4, where it is based on the arguments of evolutionary inequalities, an ordinary differential equation and a fixed-point theorem.

2 Mathematical Model

We consider a piezoelectric body $\mathcal{B}$ identified with a region in $\mathbb{R}^3$, it occupies in a fixed and undistorted reference configuration. We assume that $\mathcal{B}$ is a cylinder with generators parallel to the x_3-axis with a cross-section which is a regular region Ω in the x_1x_2-plane, $Ox_1x_2x_3$ being a Cartesian coordinate system. The cylinder is assumed to be sufficiently long so that end effects in the axial direction are negligible. Thus, $\mathcal{B} = \Omega \times \ (-\infty, +\infty)$. The cylinder is acted upon by body forces of density $\mathbf{f}_0$ and has volume free electric charges of density $\mathbf{q}_0$. It is also constrained mechanically and electrically on the boundary. To describe the boundary conditions, we denote by $\partial\Omega = \Gamma$ the boundary of Ω and we assume a partition of Γ into three open disjoint parts Γ_1, Γ_2, and Γ_3, on the one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that the one-dimensional measures of Γ_1 and Γ_a, denoted meas Γ_1 and meas Γ_a, are positive. The cylinder is clamped on $\Gamma_1 \times \ (-\infty, +\infty)$ and therefore the displacement field vanishes there. Surface tractions of density $\mathbf{f}_2$ act on $\Gamma_2 \times \ (-\infty, +\infty)$. We also assume that the electrical potential vanishes on $\Gamma a \times \ (-\infty, +\infty)$ and a surface electrical charge of density $\mathbf{q}_2$ is prescribed on $\Gamma_b \times \ (-\infty, +\infty)$. The cylinder is in contact over $\Gamma_3 \times \ (-\infty, +\infty)$ with a conductive obstacle, the so-called foundation. The contact is frictional and is modeled by Tresca's law. We are interested in the deformation of the cylinder on the time interval $[0, T]$. We assume that

$$\mathbf{f_0} = (0, 0, f_0) \quad \text{with} \quad f_0 = f_0(x_1, x_2, t) : \Omega \times [0; T] \to \mathbb{R}, \tag{1}$$

$$\mathbf{f_2} = (0, 0, f_2) \quad \text{with} \quad f_2 = f_2(x_1, x_2, t) : \Gamma_2 \times [0; T] \to \mathbb{R}, \tag{2}$$

$$q_0 = q_0 (x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{3}$$

$$q_2 = q_2 (x_1, x_2, t) : \Gamma_b \times [0, T] \to \mathbb{R}. \tag{4}$$

The forces (1), (2) and the electric charges (3), (4) are expected to give rise to deformations and to electric charges of the piezoelectric cylinder corresponding to a displacement $\mathbf{u}$ and to an electric potential field φ which are independent of x_3 and have the form

$$\mathbf{u} = (0, 0, u) \quad \text{with} \quad u = u(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}, \tag{5}$$

$$\varphi = \varphi(x_1, x_2, t) : \Omega \times [0, T] \to \mathbb{R}. \tag{6}$$

Such kind of deformation, associated to a displacement field of the form (5), is called an antiplane shear, see for instance [8] for details.

Below, the indices i and j denote components of vectors and tensors and run from 1 to 3, summation over two repeated indices is implied, and the index that follows the comma represents the partial derivative with respect to the corresponding spatial variable; also, the dot above represents the time derivative. We use S^3 for the linear space of second order symmetric tensors on $\mathbb{R}^3$ or, equivalently, the space of symmetric matrices of order 3, and "$.$", $\|.\|$ will represent the inner products and the Euclidean norms on $\mathbb{R}^3$ and S^3; we have

$$\begin{aligned} \mathbf{u}.\mathbf{v} &= u_i v_i, \quad \|\mathbf{v}\| = (\mathbf{v}.\mathbf{v})^{\frac{1}{2}} \quad \forall \mathbf{u} = (u_i), \ \mathbf{v} = (v_i) \in \mathbb{R}^3, \\ \sigma.\tau &= \sigma_{ij}\tau_{ij}, \quad \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \quad \forall \sigma = (\sigma_{ij}), \ \ \tau = (\tau_{ij}) \in S^3. \end{aligned}$$

The infinitesimal strain tensor is denoted by $\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u}))$ and the stress field by $\sigma = (\sigma_{ij})$. We also denote by $\mathbf{E}(\varphi) = (E_i(\varphi))$ the electric field and by $\mathbf{D} = (D_i)$ the electric displacement field. Here and below, in order to simplify the notation, we do not indicate the dependence of various functions on x_1, x_2, x_3 or t and we recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \quad E_i\left(\varphi\right) = -\varphi_{,i}.$$

The material's behavior is modeled by the following thermal electro-viscoelastic constitutive law with long-term memory

$$\sigma = \lambda(tr\varepsilon(\mathbf{u}))\mathbf{I} + 2\mu\varepsilon(\mathbf{u}) + 2\int_0^t \mathcal{G}(t-s)\varepsilon(\mathbf{u}(s))ds - \mathcal{E}^*\mathbf{E}(\varphi) - M_c\theta, \tag{7}$$

$$\mathbf{D} = \mathcal{E}\varepsilon(\boldsymbol{u}) + \alpha\mathbf{E}(\varphi) - \mathcal{P}\theta, \tag{8}$$

where $\lambda > 0$ and $\mu > 0$ are the Lamé coefficients, $tr(\varepsilon(\mathbf{u})) = \sum_{i=1}^{3} \varepsilon_{ii}(\mathbf{u})$, $\mathbf{I}$ is the unit tensor in $\mathbb{R}^3$, $\mathcal{G} : [0,T] \to \mathbb{R}$ is the relaxation function, $\mathcal{E}$ represents the third-order piezoelectric tensor, and $\mathcal{E}^*$ is its transpose, θ is the temperature field and $M_c := (m_{ij})$, $\mathcal{P}\,(p_i)$ are, respectively, the thermal expansion and the pyroelectric tensor which have the forms

$$M_c = \begin{pmatrix} 0 & 0 & \mathcal{M}_{c_1} \\ 0 & 0 & \mathcal{M}_{c_2} \\ \mathcal{M}_{c_1} & \mathcal{M}_{c_2} & 0 \end{pmatrix}, \qquad \mathcal{P} = \begin{pmatrix} p_1 \\ p_2 \\ 0 \end{pmatrix}.$$

We assume that $\mathcal{M}_{c_i}(x_1, x_2) : \Omega \to \mathbb{R}$, and $\quad p_i : \Omega \to \mathbb{R}$.

In the antiplane context (5), (6), when using the constitutive equations (7), (8), it follows that the stress field and the electric displacement field are given by

$$\sigma = \begin{pmatrix} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{pmatrix}, \tag{9}$$

$$\mathbf{D} = \begin{pmatrix} eu_{,1} - \alpha\varphi_{,1} - p_1\theta \\ eu_{,2} - \alpha\varphi_{,2} - p_2\theta \\ 0 \end{pmatrix}, \tag{10}$$

where α is the electric permittivity constant, e is a piezoelectric coefficient,

$$\sigma_{13} = \sigma_{31} = \mu u_{,1} + \int_0^t \mathcal{G}(t-s)u_{,1}(s)ds + e\varphi_{,1} - \mathcal{M}_{c_1}.\theta,$$

$$\sigma_{23} = \sigma_{32} = \mu u_{,2} + \int_0^t \mathcal{G}(t-s)u_{,2}(s)ds - e\varphi_{,2} - \mathcal{M}_{c_2}.\theta.$$

We assume that

$$\mathcal{E}\varepsilon = \begin{pmatrix} e\left(\varepsilon_{13} + \varepsilon_{31}\right) \\ e\left(\varepsilon_{23} + \varepsilon_{32}\right) \\ e\varepsilon_{33} \end{pmatrix} \qquad \forall \varepsilon = (\varepsilon_{ij}) \in S^3. \tag{11}$$

We also assume that the coefficients $\mathcal{G}$, μ, α, and e depend on the spatial variables x_1, x_2, but are independent of the spatial variable x_3. Since $\mathcal{E}\varepsilon.\mathbf{v} = \varepsilon.\mathcal{E}^*\mathbf{v}$ for all $\varepsilon \in S^3, \mathbf{v}$

$\in \mathbb{R}^3$, it follows from (11) that

$$\mathcal{E}^*\mathbf{v} = \begin{pmatrix} 0 & 0 & ev_1 \\ 0 & 0 & ev_2 \\ ev_1 & ev_2 & ev_3 \end{pmatrix} \quad \forall \mathbf{v} = (v_i) \in \mathbb{R}^3. \tag{12}$$

We assume that the process is mechanically quasistatic and electrically static and therefore is governed by the equilibrium equations

$$\operatorname{Div} \sigma + \mathbf{f}_0 = 0, \quad \operatorname{div} D - q_0 = 0 \ \text{ in } \mathcal{B} \times (0, T)\,,$$

where $\operatorname{Div} \sigma = (\sigma_{ij,j})$ represents the divergence of the tensor field σ. When taking into account (1), (3), (5), (6), (9), and (10), the equilibrium equations above reduce to the following scalar equations:

$$\operatorname{div}(\mu \nabla u) + \int_0^t \mathcal{G}(t-s) \ \operatorname{div}(\nabla u(s))\, ds + \ \operatorname{div}(e\nabla\varphi) - \ \operatorname{div}(\theta\, \mathcal{M}_c) + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{13}$$

$$\operatorname{div}(e\nabla u - \alpha\nabla\varphi) - \ \operatorname{div}(\theta\mathcal{P}) = q_0 \text{ in } \Omega \times (0,T) \tag{14}$$

with

$$\mathcal{M}_c = \begin{pmatrix} \mathcal{M}_{c_1} \\ \mathcal{M}_{c_2} \\ 0 \end{pmatrix}.$$

Here and below we use the notation

$$\operatorname{div} \tau = \tau_{1,1} + \tau_{1,2} \qquad \text{for} \quad \tau = (\tau_1(x_1, x_2, t), \tau_2(x_1, x_2, t))\,,$$
$$\nabla v = (v_{,1}, v_{,2})\,, \qquad \partial_\nu v = v_{,1}\nu_1 + v_{,2}\nu_2 \qquad \text{for} \qquad v = v(x_1, x_2, t)\,.$$

We now describe the boundary condition. During the process, the cylinder is clamped on $\Gamma_1 \times (-\infty, +\infty)$ and the electric potential vanishes on $\Gamma_1 \times (-\infty, +\infty)$. Thus, (5) and (6) imply that

$$u = 0 \qquad \text{on } \ \Gamma_1 \times (0,T), \tag{15}$$
$$\varphi = 0 \qquad \text{on } \ \Gamma_a \times (0,T). \tag{16}$$

Let ν denote the unit normal on $\Gamma \times (-\infty, +\infty)$. We have

$$\nu = (\nu_1, \nu_2, 0) \quad \text{with} \quad \nu_i = \nu_i(x_1, x_2) : \Gamma \to \mathbb{R},\ i = 1, 2. \tag{17}$$

For a vector $\mathbf{v}$, we denote by v_ν and $\mathbf{v}_\tau$ its normal and tangential components on the boundary, given by

$$v_\nu = \mathbf{v}.\nu \ \ , \ \ \mathbf{v}_\tau = \ \ \mathbf{v} - v_\nu \nu. \tag{18}$$

For a given stress field σ, we denote by σ_ν and σ_τ the normal and the tangential components on the boundary, that is,

$$\sigma_\nu = (\sigma\nu).\nu, \quad \sigma_\tau = \sigma\nu - \sigma_\nu \nu. \tag{19}$$

From (9), (10), and (17), we deduce that the Cauchy stress vector and the normal component of the electric diplacement field are given by

$$\begin{gathered} \sigma\nu = (0, 0, \mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s) ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu), \\ \mathbf{D}.\nu = e\partial_\nu u - \alpha\partial_\nu\varphi - \theta\mathcal{P}.\nu. \end{gathered} \tag{20}$$

Taking into account (2), (4), and (20), the traction condition on $\Gamma_2 \times (-\infty, +\infty)$ and the electric conditions on $\Gamma_b \times (-\infty, +\infty)$ are given by

$$\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu \;=\; f_2 \;\text{ on } \Gamma_2 \times (-\infty, +\infty), \quad (21)$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi \;=\; q_2 \;\text{ on } \Gamma_b \times (-\infty, +\infty). \quad (22)$$

Now, we describe the frictional contact condition and electric conditions on $\Gamma_3 \times (-\infty, +\infty)$. First, from (5) and (17), we infer that the normal displacement vanishes, $u_\nu = 0$, which shows that the contact is bilateral, that is, the contact is kept during the whole process. Using now (5) and (17)-(19), we conclude that

$$\mathbf{u}_\tau = (0, 0, u), \;\; \sigma_\tau = (0, 0, \sigma_\tau), \quad (23)$$

where

$$\sigma_\tau = (0, 0, \mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu).$$

We assume that the friction is invariant with respect to the x_3-axis and is modeled by Tresca's friction law, that is,

$$\begin{cases} |\sigma_\tau| \le g, \\ |\sigma_\tau| < g \Rightarrow \dot{\mathbf{u}}_\tau = 0, \\ |\sigma_\tau| = g \Rightarrow \exists\beta \ge 0, \text{ such that } \sigma_\tau = -\beta\dot{\mathbf{u}}_\tau, \end{cases} \quad \text{on } \Gamma_3 \times (0, T). \quad (24)$$

Here $g : \Gamma_3 \to \mathbb{R}_+$ is a given function, the friction bound, and $\dot{\mathbf{u}}_\tau$ represents the tangential velocity on the contact boundary. Using now (23), it is straightforward to see that the conditions (24) imply

$$\begin{cases} |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| \le g, \\ |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| < g \Rightarrow \dot{u}\,(t) = 0, \\ |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| = g \Rightarrow \exists\beta \ge 0, \\ \text{such that } \; \mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu = -\beta\dot{u}. \end{cases} \quad \text{on } \Gamma_3 \times (0, T). \quad (25)$$

Next, since the foundation is electrically conductive and the contact is bilateral, we assume that the normal component of the electric displacement field or the free charge is proportional to the difference between the potential on the foundation and the body's surface. Thus,

$$\mathbf{D}.\nu = k\,(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0, T),$$

where φ_F represents the electric potential of the foundation and k is the electric conductivity coefficient. We use (20) and the previous equality to obtain

$$e\partial_\nu u - \alpha\partial_\nu\varphi - \theta\mathcal{P}.\nu = k\,(\varphi - \varphi_F) \quad \text{on } \Gamma_3 \times (0, T). \quad (26)$$

Finally, we prescribe the initial displacement

$$u(0) = u_0 \qquad \text{in } \Omega, \tag{27}$$

where u_0 is the given function on Ω.

We collect the above equations and conditions to obtain the classical formulation of the antiplane problem for thermo-electro-viscoelastic materials with long-term memory, in frictional contact with a foundation.

Problem $\mathcal{P}$: Find the displacement field $u : \Omega \times (0,T) \to \mathbb{R}$, a temperature field $\theta : \Omega \times (0,T) \to \mathbb{R}_+$ and the electric potential $\varphi : \Omega \times (0,T) \to \mathbb{R}$ such that

$$\operatorname{div}(\mu\nabla u) + \int_0^t \mathcal{G}(t-s)\ \operatorname{div}(\nabla u(s))\,ds + \ \operatorname{div}(e\nabla\varphi) - \operatorname{div}(\theta\,\mathcal{M}_c) + f_0 = 0 \text{ in } \Omega\times(0,T), \tag{28}$$

$$\operatorname{div}(e\nabla u - \alpha\nabla\varphi) - \ \operatorname{div}(\theta\mathcal{P}) = q_0 \text{ in } \Omega\times(0,T), \tag{29}$$

$$\dot{\theta} - \ \operatorname{div}(K\nabla\theta) = -\mathcal{M}_c\nabla\dot{u} + h(t) \text{ in } \Omega\times(0,T), \tag{30}$$

$$u = 0 \qquad \text{on } \Gamma_1\times(0,T), \tag{31}$$

$$\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu = f_2 \ \text{ on } \Gamma_2\times(0,T), \tag{32}$$

$$\begin{cases} |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| \le g, \\ |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| < g \Rightarrow \dot{u} = 0, \\ |\mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu| = g \Rightarrow \exists\beta \ge 0, \\ \text{such that } \ \mu\partial_\nu u + \int_0^t \mathcal{G}(t-s)\partial_\nu u(s)ds + e\partial_\nu\varphi - \theta\mathcal{M}_c.\nu = -\beta\dot{u}, \end{cases} \quad on\ \Gamma_3\times(0,T), \tag{33}$$

$$\theta = 0 \qquad \text{on } \Gamma_1\cup\Gamma_2\times(0,T), \tag{34}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi = q_2 \ \text{ on } \Gamma_b\times(0,T), \tag{35}$$

$$e\partial_\nu u - \alpha\partial_\nu\varphi - \theta\mathcal{P}.\nu{=}k\,(\varphi-\varphi_F) \quad \text{on } \Gamma_3\times(0,T), \tag{36}$$

$$-\,k_{ij}\frac{\partial\theta}{\partial x_j}n_i = k_e(\theta-\theta_R) \qquad \text{on } \Gamma_3\times(0,T), \tag{37}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{38}$$

The differential equation (30) describes the evolution of the temperature field, where $K := (k_{ij})$ represents the thermal conductivity tensor, $h(t)$ is the density of volume heat sources. The associated temperature boundary condition is given by (37), where θ_R is the temperature of the foundation, and k is the heat exchange coefficient between the body and the obstacle. Finally, u_0, θ_0 represent the initial displacement and temperature, respectively.

3 Variational Formulation and Main Result

We derive now the variational formulation of Problem $\mathcal{P}$. To this end we introduce the function spaces

$$V = \left\{ v \in H^1(\Omega) \mid v = 0 \ \text{ on } \Gamma_1 \right\}, \qquad W = \left\{ \psi \in H^1(\Omega) \mid \psi = 0 \ \text{ on } \Gamma_a \right\},$$

and we assume that

$$E = \left\{ \eta \in H^1(\Omega) \mid \eta = 0 \ \text{ on } \Gamma_1 \cup \Gamma_2 \right\}.$$

Similarly, we write ζ for the trace $\gamma\zeta$ of the function $\zeta \in H^1(\Omega)$ on Γ. Since $meas\, \Gamma_1 > 0$ and $meas\, \Gamma_a > 0$, it is well known that V and W are real Hilbert spaces with the inner products

$$(u, v)_V = \int_\Omega \nabla u . \nabla v \ dx \ \ \forall\, u, v \ \in V, \ \ (\varphi, \psi)_W = \int_\Omega \nabla \varphi . \nabla \psi \ dx \ \ \forall\, \varphi, \psi \ \in W.$$

Moreover, the associated norms

$$\| \, v \, \|_V = \| \, \nabla v \, \|_{L^2(\Omega)^2} \ \ \forall\, v \in V, \ \ \| \, \psi \, \|_W = \| \, \nabla \psi \, \|_{L^2(\Omega)^2} \ \ \forall\, \psi \in W, \tag{39}$$

are equivalent on V and W, with the usual norm $\| \, . \, \|_{H^1(\Omega)}$. By Sobolev's trace theorem we deduce that there exist three positive constants $c_1 > 0$, $c_2 > 0$ and $c_3 > 0$ such that

$$\begin{gathered} \| \, v \, \|_{L^2(\Gamma_3)} \leq c_1 \, \| \, v \, \|_V \quad \forall\, v \in V, \quad \| \, \psi \, \|_{L^2(\Gamma_3)} \leq c_2 \, \| \, \psi \, \|_W \ \ \forall\, \psi \in W, \\ \| \, \eta \, \|_{L^2(\Gamma_3)} \leq c_3 \, \| \, \eta \, \|_E \quad \forall\, \eta \in E. \end{gathered} \tag{40}$$

If $(X, \| \, . \, \|_X)$ represents a real Banach space where $X = V \times W$, we denote by $C([0, T]; X)$ the space of continuous functions from $[0, T]$ to X, with the norm

$$\| \, x \, \|_{C([0;T];X)} = \max_{t \in [0,T]} \| \, x(t) \, \|_X,$$

and we use standard notations for the Lebesgue space $L^2(0, T; X)$ as well as for the Sobolev space $W^{1,2}(0, T; X)$. In particular, recall that the norm on the space $L^2(0, T; X)$ is given by the formula

$$\| \, u \, \|^2_{L^2(0,T;X)} = \int_0^T \| \, u(t) \, \|^2_X \ dt,$$

and the norm on the space $W^{1,2}(0, T; X)$ is given by the formula

$$\| \, u \, \|^2_{W^{1,2}(0,T;X)} = \int_0^T \| \, u(t) \, \|^2_X \ dt + \int_0^T \| \, \dot{u}(t) \, \|^2_X \ dt.$$

Finally, we suppress the argument X when $X = \mathbb{R}$; thus, for example, we use the notation $W^{1,2}(0, T)$ for the space $W^{1,2}(0, T; \mathbb{R})$ and the notation $\| \, . \, \|_{W^{1,2}(0,T)}$ for the norm $\| \, . \, \|_{W^{1,2}(0,T;\mathbb{R})}$.

In the study of Problem $\mathcal{P}$ we assume that the viscosity coefficient satisfies

$$\mathcal{G} \in W^{1,2}(0, T) \tag{41}$$

and the electric permittivity coefficient satisfies

$$\alpha \in L^\infty(\Omega) \ \text{ and there exists } \alpha^* > 0 \text{ such that } \alpha(\mathbf{x}) \geq \alpha^* \text{ a.e. } \mathbf{x} \in \Omega. \tag{42}$$

We also assume that the Lamé coefficient and the piezoelectric coefficient satisfy

$$\begin{aligned} \mu &\in L^{\infty}(\Omega) \text{ and } \mu(\mathbf{x}) > 0 \text{ a.e. } \mathbf{x} \in \Omega, && (43) \\ e &\in L^{\infty}(\Omega). && (44) \end{aligned}$$

The thermal tensor and the pyroelectric tensor satisfy

$$\mathcal{M}_c = \begin{pmatrix} \mathcal{M}_{c_1} \\ \mathcal{M}_{c_2} \\ 0 \end{pmatrix}, \quad \mathcal{M}_{c_i}(x_1, x_2) : \Omega \to \mathbb{R}, \ \mathcal{M}_{c_i} \in L^{\infty}(\Omega). \tag{45}$$

The boundary thermal data satisfy

$$h \in W^{1,2}(0,T; L^2(\Omega)), \ \ \theta_R \in W^{1,2}(0,T; L^2(\Gamma_3)), \ \ k_e \in L^{\infty}(\Omega, \mathbb{R}_+). \tag{46}$$

The thermal conductivity tensor verifies the usual symmetry and ellipticity: for some $c_k > 0$ and for all $\xi_i \in \mathbb{R}^d$,

$$K = (k_{ij}), \quad k_{ij} = k_{ji} \in L^2(\Omega), \ \forall c_k > 0, \ \xi_i \in \ \mathbb{R}^d; \quad k_{ij}\xi_i.\xi_j \le c_k \xi_i.\xi_j. \tag{47}$$

The forces, tractions, volume, and surface free charge densities have the regularity

$$\begin{aligned} f_0 &\in W^{1,2}(0,T; L^2(\Omega)), \qquad f_2 \in W^{1,2}(0,T; L^2(\Gamma_2)), && (48) \\ q_0 &\in W^{1,2}(0,T; L^2(\Omega)), \qquad q_2 \in W^{1,2}(0,T; L^2(\Gamma_b)). && (49) \end{aligned}$$

The electric conductivity coefficient and the friction bound function g satisfy the following properties:

$$\begin{aligned} k &\in L^{\infty}(\Gamma_3) \text{ and } k(\mathbf{x}) \ge 0 \text{ a.e. } \mathbf{x} \in \Gamma_3, && (50) \\ g &\in L^{\infty}(\Gamma_3) \text{ and } g(\mathbf{x}) \ge 0 \text{ a.e. } \mathbf{x} \in \Gamma_3. && (51) \end{aligned}$$

Also, we assume that the electric potential of the foundation is such that

$$\varphi_F \in W^{1,2}(0,T; L^2(\Gamma_3)). \tag{52}$$

Finally, we assume that the initial data verifies

$$u_0 \in V, \quad \theta_0 \in L^2(\Omega), \tag{53}$$

and moreover,

$$a_{\mu}(u_0, \upsilon)_V + j(\upsilon) \ge (f(0), \upsilon)_V, \ \forall \upsilon \in V. \tag{54}$$

We consider the functional $j : [0,T] \to \mathbb{R}_+$ given by

$$j(\upsilon) = \int_{\Gamma_3} g\,|\upsilon|\, da \qquad \forall \upsilon \in V, \tag{55}$$

and let $f : [0,T] \to V$ and $q : [0,T] \to W$ be defined by

$$\begin{aligned} (f(t), \upsilon)_V &= \int_{\Omega} f_0(t)\upsilon dx + \int_{\Gamma_2} f_2(t)\upsilon\, da, && (56) \\ (q(t), \psi)_W &= \int_{\Omega} q_0(t)\psi dx - \int_{\Gamma_b} q_2(t)\psi\, da + \int_{\Gamma_3} k\varphi_F(t)\,\psi da, && (57) \\ \forall \upsilon &\in V, \ \psi \in W, \forall t \in [0,T]. \end{aligned}$$

The definitions of f and q are based on Riesz's representation theorem and by (48) and (49), we infer that the integrals above are well-defined and

$$f \in W^{1,2}(0,T;V), \tag{58}$$
$$q \in W^{1,2}(0,T;W). \tag{59}$$

Next, we define the bilinear forms $a_\mu : V \times V \to \mathbb{R}$, $a_e : V \times W \to \mathbb{R}$, $a_e^* : W \times V \to \mathbb{R}$, and $a_\alpha : W \times W \to \mathbb{R}$, by the equalities

$$a_\mu(u,\upsilon) = \int_\Omega \mu\nabla u.\nabla\upsilon \; dx, \tag{60}$$
$$a_e(u,\varphi) = \int_\Omega e\nabla u.\nabla\varphi \; dx = a_e^*(\varphi,u), \tag{61}$$
$$a_\alpha(\varphi,\psi) = \int_\Omega \alpha\nabla\varphi.\nabla\psi \; dx + \int_{\Gamma_3} k\varphi\psi \; da \tag{62}$$

for all $u,\upsilon \in V,\ \varphi,\psi \in W$. Assumptions (55)–(57) imply that the integrals above are well-defined and when using (39) and (40), it follows that the forms a_μ, a_e and a_e^* are continuous; moreover, the forms a_μ and a_α are symmetric and, in addition, the form a_α is W-elliptic since

$$a_\alpha(\psi,\psi) \geq \alpha^* \|\psi\|_W^2 \quad \forall\psi \in W. \tag{63}$$

By using Green's formula, it is straightforward to derive the following variational formulation of $\mathcal{P}$. We denote by $\langle\, ,\rangle_{V'\times V}$ the duality pairing between V' and V.

Problem $\mathcal{P}_V$: Find a displacement field $u : [0;T] \to V$, an electric potential field $\varphi : [0;T] \to W$ and a temperature field $\theta : [0;T] \to E$ such that

$$\begin{aligned} &a_\mu(u(t),\upsilon - \dot u(t)) + (\textstyle\int_0^t \mathcal{G}(t-s)u(s)ds, \upsilon - \dot u(t))_V + a_e^*(\varphi(t), \upsilon - \dot u(t)) \\ &+(\mathcal{M}_c\theta(t),\upsilon - \dot u(t))_V + j(\upsilon) - j(\dot u(t)) \geq (f(t),\upsilon - \dot u(t))_V \ \forall\upsilon \in V,\ t \in (0,T), \end{aligned} \tag{64}$$

$$a_\alpha(\varphi(t),\psi) - a_e(u(t),\psi) - (\mathcal{P}\theta,\nabla\psi)_H = (q(t),\psi)_W \quad \forall\psi \in W,\ t \in [0,T], \tag{65}$$

$$\dot\theta(t) + K\theta(t) = R\dot u(t) + Q(t) \qquad \text{in } E', \tag{66}$$

$$u(0) = u_0, \qquad \theta(0) = \theta_0 \qquad \text{in } \Omega. \tag{67}$$

Here, the function $Q : [0,T] \to E'$ and the operators $K : E \to E'$, $R : V \to E'$; $\mathcal{M}_c : E \to V'$ are defined by $\forall\upsilon \in V$, $\forall\tau \in E$, $\forall\mu \in E$:

$$\langle Q(t),\mu\rangle_{E'\times E} = \int_{\Gamma_3} k_c\theta_R\mu\, ds + \int_\Omega q\mu\, dx,$$
$$\langle K\tau,\mu\rangle_{E'\times E} = \sum_{i,j=1}^d \int_\Omega k_{ij}\frac{\partial\mu}{\partial x_j}\frac{\partial\mu}{\partial x_i}\, dx + \int_{\Gamma_3} k_c\tau\mu\, ds,$$
$$\langle R\upsilon,\mu\rangle_{E'\times E} = \int_{\Gamma_3} h_\tau(|\upsilon_\tau|)\mu\, ds - \int_\Omega(\mathcal{M}_c\,\nabla\upsilon)\mu\, dx,$$
$$\langle\mathcal{M}_c\tau,\upsilon\rangle_{V'\times V} = (-\tau\mathcal{M}_c,\upsilon)_V.$$

Our main existence and uniqueness result is stated as follows.

Theorem 3.1 *Assume that (41)-(59) hold. Then there exists a unique solution* (u, θ, φ) *of problem* $\mathcal{P}_V$. *Moreover, the solution satisfies*

$$\begin{aligned} u &\in W^{1,2}(0,T;V);\ \varphi \in W^{1,2}(0,T;W), \\ \theta &\in W^{1,2}(0,T;E^{'}) \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)). \end{aligned} \tag{68}$$

An element (u, φ, θ) which solves $\mathcal{P}_V$ is called a weak solution of the mechanical problem $\mathcal{P}$. We conclude by Theorem 3.1 that the antiplane contact problem $\mathcal{P}$ has a unique weak solution, provided that (41)-(59) hold.

4 An Abstract Existence and Uniqueness Result

The proof of Theorem 3.1 is carried out in several steps that we prove in what follows. Everywhere in this section, we suppose that assumptions of Theorem 3.1 hold and we denote by $c > 0$ a generic constant, whose value may change from lines to lines.

In the first step of the proof, we introduce the set

$$\mathcal{W} = \left\{ \eta \in W^{1,2}(0,T;X) \mid \eta(0) = 0_X \right\}, \tag{69}$$

and we prove the following existence and uniqueness result.

Lemma 4.1 *For all* $\eta \in \mathcal{W}$, *there exists a unique element* $u_\eta \in W^{1,2}(0,T;X)$ *such that*

$$\begin{aligned} &a(u_\eta(t), v - \dot{u}_\eta(t)) + (\eta(t), v - \dot{u}_\eta(t))_X + j(v) - j(\dot{u}_\eta(t)) \\ &\geq (f(t), v - \dot{u}_\eta(t))_X \qquad \forall v \in X, \ \ a.e.\ t \in [0,T], \end{aligned} \tag{70}$$

$$u_\eta(0) = u_0. \tag{71}$$

Here X is a real Hilbert space endowed with the inner product $(.\,,.)_X$ and the data a is a bilinear continuous coercive and symmetric form.

Proof. We use an abstract existence and uniqueness result which may be found in [2].

In the second step, we use the displacement field u_η obtained in Lemma 4.1 and we consider the following lemma.

Lemma 4.2 *For all* $\eta \in \mathcal{W}$, *there exists a unique solution*

$$\theta_\eta \in W^{1,2}(0,T;E^{'}) \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)),\ c > 0\ \forall \eta \in L^2([0,T],V^{'}),$$

satisfying

$$\begin{cases} \dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{u}_\eta(t) + Q(t) \ \ \text{in } E^{'} \ a.e.\ t \in [0,T], \\ \theta_\eta(0) = \theta_0, \end{cases} \tag{72}$$

$$|\theta_{\eta_1} - \theta_{\eta_2}|^2_{L^2(\Omega)} \leq C \int_0^t |\dot{u}_{\eta 1}(s) - \dot{u}_{\eta 2}(s)|^2_V \, ds \quad \forall t \in [0,T], \tag{73}$$

and

$$|\dot{\theta}_{\eta_1} - \dot{\theta}_{\eta_2}|^2_{L^2(\Omega)} \leq c \int_0^t |\dot{u}_{\eta 1}(s) - \dot{u}_{\eta 2}(s)|^2_V \, ds \quad a.e. t \in [0,T]. \tag{74}$$

Proof. The existence and uniqueness result verifying (72) follows from the classical result for the first order evolution equation, applied to the Gelfand evolution triple

$$E \subset F \equiv F^{'} \subset E^{'}.$$

We verify that the operator $K : E \to E^{'}$ is linear continuous and strongly monotone, and from the expression of the operator R,

$$\upsilon_\eta \in W^{1,2}(0,T;V) \Longrightarrow R\upsilon_\eta \in W^{1,2}(0,T;F),$$

as $Q \in W^{1,2}(0,T;E)$, then $R\upsilon_\eta + Q \in W^{1,2}(0,T;E)$, we deduce (73) and (74) (see [1]).

In the third step, we use the displacement field u_η obtained in Lemma 4.1 and θ_η obtained in Lemma 4.2 and we consider the following lemma.

Lemma 4.3 *For all $\eta \in \mathcal{W}$, there exists a unique solution $\varphi_\eta \in W^{1,2}(0,T;W)$ which satisfies*

$$a_\alpha\left(\varphi_\eta\left(t\right),\psi\right) - a_e\left(u_\eta\left(t\right),\psi\right) - \left(\mathcal{P}\theta_\eta, \nabla\psi\right)_H = \left(q\left(t\right),\psi\right)_W \;\; \forall\psi \in W,\; t \in [0,T]. \quad (75)$$

Moreover, if φ_{η_1} and $\varphi_{\eta 2}$ are the solutions of (4.7) corresponding to η_1, $\eta_2 \in C([0,T],V)$, then there exists $c > 0$ such that

$$\left\|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)\right\|_W \le c\left\|u_{\eta_1}(t) - u_{\eta_2}(t)\right\|_V \quad \forall t \in [0,T]. \quad (76)$$

Proof. Let $t \in [0,T]$. We use the properties of the bilinear form a_α and the Lax-Milgram lemma to see that there exists a unique element $\varphi_\eta(t) \in W$ which solves (75) at any moment $t \in [0,T]$. Consider now $t_1, t_2 \in [0,T]$; using (75), we get

$$\begin{aligned} a_\alpha\left(\varphi_\eta\left(t_1\right),\psi\right) - a_e\left(u_\eta\left(t_1\right),\psi\right) - \left(\mathcal{P}\theta_\eta\left(t_1\right), \nabla\psi\right)_H \\ = \left(q\left(t_1\right),\psi\right)_W \;\; \forall\psi \in W,\; t_1 \in [0,T], \end{aligned} \quad (77)$$

$$\begin{aligned} a_\alpha\left(\varphi_\eta\left(t_2\right),\psi\right) - a_e\left(u_\eta\left(t_2\right),\psi\right) - \left(\mathcal{P}\theta_\eta\left(t_2\right), \nabla\psi\right)_H \\ = \left(q\left(t_2\right),\psi\right)_W \;\; \forall\psi \in W,\; t_2 \in [0,T]. \end{aligned} \quad (78)$$

Using (77), (78) and (63), we find that

$$\begin{aligned} \alpha^*\left\|\varphi\left(t_1\right) - \varphi\left(t_2\right)\right\|_W^2 \;\; \le \;\; & (\|e\|_{L^\infty(\Omega)}\left\|u\left(t_1\right) - u\left(t_2\right)\right\|_V + \left\|q\left(t_1\right) - q\left(t_2\right)\right\|_W + \\ & \|p\|_{L^\infty(\Omega)}\left\|\theta\left(t_1\right) - \theta\left(t_2\right)\right\|_{L^2(\Omega)})\left\|\varphi\left(t_1\right) - \varphi\left(t_2\right)\right\|_W, \end{aligned}$$

and using (73) we find that

$$\begin{aligned} \alpha^*\left\|\varphi\left(t_1\right) - \varphi\left(t_2\right)\right\|_W^2 \;\; \le \;\; & (\|e\|_{L^\infty(\Omega)}\left\|u\left(t_1\right) - u\left(t_2\right)\right\|_V + \left\|q\left(t_1\right) - q\left(t_2\right)\right\|_W + \\ & \|p\|_{L^\infty(\Omega)}\left\|u\left(t_1\right) - u\left(t_2\right)\right\|_V)\left\|\varphi\left(t_1\right) - \varphi\left(t_2\right)\right\|_W. \end{aligned}$$

It follows from the previous inequality that

$$\left\|\varphi\left(t_1\right) - \varphi\left(t_2\right)\right\|_W \le c(\left\|u\left(t_1\right) - u\left(t_2\right)\right\|_V + \left\|q\left(t_1\right) - q\left(t_2\right)\right\|_W). \quad (79)$$

Then, the regularity $u_\eta \in W^{1,2}(0,T;V)$ combined with (59) and (79) imply that $\varphi_\eta \in W^{1,2}(0,T;W)$, which concludes the proof.

Now, for all $\eta \in \mathcal{W}$, we denote by u_η the solution obtained in Lemma 4.1, by θ_η the solution obtained in Lemma 4.2 and by φ_η the solution obtained in Lemma 4.3.

Step 4: In the fourth step, we consider the operator $\Lambda : \mathcal{W} \to \mathcal{W}$.

We now use Riesz's representation theorem to define the element $\Lambda\eta\,(t) \in \mathcal{W}$ by the equality

$$\begin{aligned} \langle \Lambda\eta(t), w\rangle_{\mathcal{W}} &= (\int_0^t \mathcal{G}(t-s)u_\eta(s)ds - M_c\theta_\eta, w)_V + a_e^*\left(\varphi_\eta\,(t)\,, w\right) \qquad (80) \\ \forall\eta &\in \mathcal{W},\ w \in V,\, t \in [0,T]\,. \end{aligned}$$

Clearly, for a given $\eta \in \mathcal{W}$, the function $t \to \Lambda\eta\,(t)$ belongs to $\mathcal{W}$. In this step we show that the operator $\Lambda : \mathcal{W} \to \mathcal{W}$ has a unique fixed point.

Lemma 4.4 *The operator Λ has a unique fixed point $\eta^* \in \mathcal{W}$ such that $\Lambda\eta^* = \eta^*$.*

Proof. Let $\eta_1, \eta_2 \in \mathcal{W}$ and $t \in [0,T]$. In what follows we denote by u_i, θ_i and φ_i the functions u_{η_i}, θ_{η_i} and φ_{η_i} obtained in Lemmas 4.1, 4.2 and 4.3, for i = 1, 2. Using (80) and (61), we obtain

$$\begin{aligned} &\left\|\Lambda\eta_1\,(t) - \Lambda\eta_2\,(t)\right\|_X^2 \qquad (81) \\ \leq\ & C\left(\int_0^t \left\|u_1\,(s) - u_2\,(s)\right\|_X^2\ ds + \left\|\theta_1 - \theta_2\right\|_{L^2(\Omega)}^2 + \left\|\varphi_1\,(t) - \varphi_2\,(t)\right\|_W^2\right) \forall t \in [0,T]\,. \end{aligned}$$

The constant C represents a generic positive number which may depend on $\|\theta\|_{W^{1,2}(0,T)}$, T, m_{ij} and e, and whose value may change from place to place.

Since $u_\eta \in W^{1,2}(0,T;V)$ and $\varphi_\eta \in W^{1,2}(0,T;W)$, we deduce from inequality (81) that $\Lambda_\eta \in W^{1,2}(0,T;V)$. On the other hand, (76) and arguments similar to those used in the proof of (79) yield

$$\left\|\varphi_1\,(t) - \varphi_2\,(t)\right\|_W \leq C\left\|u_1\,(t) - u_2\,(t)\right\|_V. \qquad (82)$$

Using now (73)(82) in (81), we get

$$\begin{aligned} &\left\|\Lambda\eta_1\,(t) - \Lambda\eta_2\,(t)\right\|_X^2 \\ \leq\ & C\left(\int_0^t \left\|u_1\,(s) - u_2\,(s)\right\|_X^2\ ds + \int_0^t \left\|\dot u_1\,(t) - \dot u_2\,(t)\right\|_X^2 ds + \left\|u_1\,(t) - u_2\,(t)\right\|_V^2\right). \end{aligned}$$

Using the norm on the space $W^{1,2}\,(0,T,X)$, we deduce that

$$\left\|\Lambda\eta_1\,(t) - \Lambda\eta_2\,(t)\right\|_X^2 \leq C\left\|u_1\,(s) - u_2\,(s)\right\|_X^2\ ds \qquad \forall t \in [0,T]\,. \qquad (83)$$

Taking into account (64), we have the inequalities

$$\begin{aligned} &a(u_1(t), v - \dot u_1(t)) + (\eta_1(t), v - \dot u_1(t))_X + j(v) - j(\dot u_1(t)) \\ &\geq (f(t), v - \dot u_1(t))_X \qquad \forall v \in X, \quad t \in [0,T]\,, \end{aligned}$$

and

$$\begin{aligned} &a(u_2(t), v - \dot u_2(t)) + (\eta_2(t), v - \dot u_2(t))_X + j(v) - j(\dot u_2(t)) \\ &\geq (f(t), v - \dot u_2(t))_X \qquad \forall v \in X, \quad t \in [0,T]\,, \end{aligned}$$

for all $v \in X$, $a.e.\ s \in [0,T]$. We choose $v = \dot{u}_2(s)$ in the first inequality and $v = \dot{u}_1(s)$ in the second inequality, add the result to obtain

$$\frac{1}{2}\left\|u_1(s) - u_2(s)\right\|_X^2 \leq -\left(\eta_1(s) - \eta_2(s), \dot{u}_1(s) - \dot{u}_2(s)\right)_X \quad \text{a.e. } s \in [0,T].$$

Let $t \in [0,T]$. Integrating the previous inequality from 0 to t using (68), we obtain

$$\begin{array}{c} \frac{1}{2}\left\|u_1(t) - u_2(t)\right\|_X^2 \leq -\left(\eta_1(t) - \eta_2(t), u_1(t) - u_2(t)\right)_X \\ \int_0^t \left(\dot{\eta}_1(s) - \dot{\eta}_2(s), u_1(s) - u_2(s)\right)_X\ ds. \end{array}$$

We deduce that

$$\begin{array}{c} C\left\|u_1(t) - u_2(t)\right\|_X^2 \leq \left\|\eta_1(t) - \eta_2(t)\right\|_X \left\|u_1(t) - u_2(t)\right\|_X \\ + \int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X \left\|u_1(s) - u_2(s)\right\|_X\ ds. \end{array}$$

Using Young's inequality, we get

$$\begin{array}{c} \left\|u_1(t) - u_2(t)\right\|_X^2 \leq C(\left\|\eta_1(t) - \eta_2(t)\right\|_X^2 + \int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds \\ + \int_0^t \left\|u_1(s) - u_2(s)\right\|_X^2\ ds). \end{array} \tag{84}$$

On the other hand, as

$$\eta_1(t) - \eta_2(t) = \int_0^t \dot{\eta}_1(s) - \dot{\eta}_2(s)\, ds,$$

we can obtain

$$\left\|\eta_1(t) - \eta_2(t)\right\|_X^2 \leq C \int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds. \tag{85}$$

Using now (85) in (84), we have

$$\left\|u_1(t) - u_2(t)\right\|_X^2 \leq C(\int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds + \int_0^t \left\|u_1(s) - u_2(s)\right\|_X^2\ ds).$$

Taking into account Gronwall's inequality, we deduce

$$\left\|u_1(t) - u_2(t)\right\|_X^2 \leq C \int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds. \tag{86}$$

From (83), (86), we obtain

$$\left\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\right\|_X^2 \leq C \int_0^t \left\|\dot{\eta}_1(s) - \dot{\eta}_2(s)\right\|_X^2\ ds.$$

Iterating the last inequality m times, we infer

$$\left\|\Lambda^m\eta_1(t) - \Lambda^m\eta_2(t)\right\|_X^2 \leq C^m \int_0^t \int_0^{s_1} \ldots\ldots \int_0^{s_{m-1}} \left\|\dot{\eta}_1(s_m) - \dot{\eta}_2(s_m)\right\|_X^2\ ds_m \ldots\ldots ds_1,$$

where Λ^m denotes the power of operator Λ. The last inequality gives

$$\left\|\Lambda^m\eta_1(t) - \Lambda^m\eta_2(t)\right\|_{W^{1.2}(0,T;X)}^2 \leq \frac{C^m T^m}{m!} \left\|\eta_1(t) - \eta_2(t)\right\|_{W^{1.2}(0,T;X)}^2,$$

which implies that for m sufficiently large, the power Λ^m of Λ is a contraction in the Banach space, since

$$\lim_{m\to\infty}\frac{C^m T^m}{m!}=0,$$

it follows now from Banach's fixed-point theorem that there exists a unique element $\eta^*\in\mathcal{W}$ such that $\Lambda^m\eta^*=\eta^*$. Moreover, since

$$\Lambda^m\left(\Lambda\eta^*\right)=\Lambda\left(\Lambda^m\eta^*\right)=\Lambda\eta^*,$$

we deduce that $\Lambda\eta^*$ is also a fixed point of the operator Λ^m. By the uniqueness of the fixed point, we conclude that $\Lambda\eta^*=\eta^*$, which shows that η^* is a fixed point, we conclude that $\Lambda\eta^*=\eta^*$. *Step 5:* In the fifth and last step of our demonstration, we have now all the ingredients to provide the proof of Theorem 3.1.

Existence. Let $\eta^*\in W^{1.2}\left(0,T;V\right)$ be the fixed point of the operator Λ, and let u_{η^*}, θ_{η^*} and φ_{η^*} be the solutions defined in Lemmas 4.1, 4.2 and 4.3, respectively, for $\eta=\eta^*$. It follows from (80) that

$$\langle\eta^*(t),w\rangle_V=(\int_0^t\mathcal{G}(t-s)u_{\eta^*}(s)ds-M_c\theta_\eta,w))_V+a_e^*\left(\varphi_{\eta^*}\left(t\right),w\right)\ \ \forall w\in V,\, t\in\left[0,T\right],\tag{87}$$

and, therefore, (64), (66), and (76) imply that $(u_{\eta^*},\theta_{\eta^*},\varphi_{\eta^*})$ is a solution of problem $\mathcal{P}_V$. Regularity (68) of the solution follows from Lemmas 4.1, 4.2 and 4.3.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Λ. It can also be obtained by using arguments similar to those used in [20] and [9].

5 Conclusion

This work models the phenomenon of contact with friction between a cylindrical body and a foundation. These contact phenomena abound in industry and in everyday life, so they play an important role in the behavior of mechanical structures.

The envisaged mechanical model is an antiplane one. We recall that the antiplane shear deformation is the expected deformation of a very long cylinder loaded in the direction of its generators. In such a model, the displacement vectorial field is parallel to the generators of the cylinder and it is independent of the axial coordinate. Due to their simplicity in the writing of the equations without loss of physical relevance, antiplane models have enjoyed special attention in recent years. The antiplane models appear in the technical literature in engineering, describing the functioning of various mechanisms, and in geophysics, focusing on the deformation of the tectonic plates, and in particular, on earthquakes.

The novelty of the result obtained is the coupling of an electro-viscoelastic problem and a thermal effect.

The problem is formulated as a coupled system of evolutionary variational inequality for the displacement field with a time-dependent variational equation for the electric potential field and the heat equation for the temperature. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem.

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Nonlinear Dynamics and Systems Theory, **22** (5) (2022) 489–502

Controllability of Dynamic Equations with Memory

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Received: May 26, 2022 Revised: November 11, 2022

Abstract: In this work, we consider a control system governed by a dynamic equation with memory. We obtain conditions under which the system is approximately controllable and approximately controllable on free time. In order to do this, we use a technique developed by Bashirov et al. [4–6], where we can avoid fixed point theorems. But first of all, we prove the existence and uniqueness of solutions of the system and after that, we prove the prolongation of solutions under some additional condition. Finally, we present several examples to illustrate the applicability of our results.

Keywords: *controllability; semilinear dynamic equations; memory; time scales.*

Mathematics Subject Classification (2010): 93C10, 93C23, 34N05, 34K42.

1 Introduction

Control theory addresses how a system can be modified through feedback, in particular, how an arbitrary initial state can be directed either exactly or approximately close to a given final state using a control in a set of admissible controls. In the last decades, control theory of dynamic equations on time scales has attracted the attention of several researches, because this is a powerful tool that allows to study from a unified point of view controllability of continuous systems, discrete systems, systems in which the time variable can vary both continuously and discretely, as well as other types of time variables. Among the works made, we can cite Bartosiewicz [1] who explored linear positive control systems, Bartosiewicz and Pawłuszewicz [2, 3] reviewed linear systems, Janglajew and Pawłuszewicz [15] analyzed constrained local controllability of linear dynamic systems,

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Bohner and Wintz [8] studied controllability and observability of linear systems, Grow and Wintz [13] proved existence and uniqueness of solutions to a bilinear state system with locally essentially bounded coefficients on an unbounded time scale. Approximate and exact controllability of semilinear systems on time scales was studied by Duque, Leiva and Uzcátegui in [10,11], Malik and Kumar in [18] established exact controllability for time-varying neutral differential equations with impulses. More works can be seen in [9,17,19] and references therein.

In this regard, in this paper, we will consider a control system governed by the dynamic equation with memory

$$\begin{cases} z^{\Delta}(t) = - A(t)z^{\sigma}(t) + B(t)u(t) + a\displaystyle\int_{t_0}^{t} M(t,s)g(s,z_{\tau}(s))\Delta s \\ \qquad\qquad + bf(t,z(t),u(t)), \quad t \geq t_0 \geq 0, \\ z(t) = \phi(t), \quad t \in [\tau(t_0), t_0]_{\mathbb{T}}, \end{cases} \tag{1}$$

where $z(t) \in \mathbb{R}^n$ is the state function, $z_{\tau}(t) = z(\tau(t))$, and $\tau : \mathbb{T} \to \mathbb{T}$ is the delay function which is increasing and unbounded on $\mathbb{T}$ such that $\tau(t) \leq t$ for $t \in \mathbb{T}$ (see [12]). $A \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$, $B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times m})$, the control $u \in L^2_{\Delta}(\mathbb{T}, \mathbb{R}^m)$, $M : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ is a function that is locally essentially bounded on $\mathbb{T}\times\mathbb{T}$, the functions $f : \mathbb{T}\times\mathbb{R}^n\times\mathbb{R}^m \to \mathbb{R}^n$, $g : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$ are rd-continuous and there exist rd-continuous functions $L_f, L_g : \mathbb{T} \to \mathbb{R}^+$ such that

C1) $\|f(t,z,u) - f(t,\tilde{z},\tilde{u})\| \leq L_f(t)(\|z-\tilde{z}\| + \|u-\tilde{u}\|)$, with $f(t,0,0) = 0$,

C2) $\|g(t,z) - g(t,\tilde{z})\| \leq L_g(t)\,\|z-\tilde{z}\|$, with $g(t,0) = 0$.

The function ϕ lies in the space $C_{\mathrm{rd}}([\tau(t_0), t_0]_{\mathbb{T}}, \mathbb{R}^n)$, which is a Banach space endowed with the norm

$$\|\phi\|_0 = \sup\{\|\phi(t)\| : t \in [\tau(t_0), t_0]_{\mathbb{T}}\}.$$

In this paper, we suppose that the time scale $\mathbb{T}$ satisfies $-\infty < \tau(t_0) < \sup\mathbb{T} = \infty$.

The main goal of this work is to study controllability of system (1). Specifically, we shall show that under certain conditions, controllability of the associated linear system implies controllability of the semilinear dynamic equation with memory. In order to prove this assertion, we impose some conditions on the nonlinear terms presented in the system, and then apply a direct approach developed by A. E. Bashirov et al. (see [4–6]) to avoid fixed point theorems, and approximate controllability is achieved. But before that, we prove existence, uniqueness and continuation of solutions of the system. Finally, we consider some examples in which our results can be applied.

2 Preliminaries

Before studying system (1), we give a brief introduction to the calculus on time scales, especially to clarify notations and definitions, which will help for a better understanding of the reader. For more details about time scales theory, we recommend the excellent monograph [7].

Time scales theory was introduced by Stefan Hilger (see [14]). We define a time scale as any arbitrary nonempty closed subset of $\mathbb{R}$, this set is denoted by $\mathbb{T}$. For every $t \in \mathbb{T}$, the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined, respectively, as

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{and} \quad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}.$$

A point $t \in \mathbb{T}$ is said to be right-dense if $\sigma(t) = t$ and $t < \sup \mathbb{T}$, right-scattered if $\sigma(t) > t$, left-dense if $\rho(t) = t$ and $t > \inf \mathbb{T}$, left-scattered if $\rho(t) < t$, isolated if $\rho(t) < t < \sigma(t)$. The function $\mu : \mathbb{T} \to [0, \infty)$ defined by $\mu(t) := \sigma(t) - t$ is known as the graininess function. It is assumed that $\mathbb{T}$ has the topology inherited from standard topology on the real numbers. The time scale interval $[a, b]_{\mathbb{T}}$ is defined by $[a, b]_{\mathbb{T}} = \{t \in \mathbb{T} : a \leq t \leq b\}$, with $a, b \in \mathbb{T}$, and similarly we define open intervals and open neighborhoods.

Definition 2.1 (See [7]) A function $f : \mathbb{T} \to \mathbb{R}^n$ is said to be right-dense continuous, or just rd-continuous, if f is continuous at every right-dense point $t \in \mathbb{T}$ and $\lim_{s \to t^-} f(s)$ exists (finite) for every left-dense point $t \in \mathbb{T}$. The class of all rd-continuous functions $f : \mathbb{T} \to \mathbb{R}^n$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$. We define $f^\sigma : \mathbb{T} \to \mathbb{R}^n$ by $f^\sigma = f \circ \sigma$. We define the set $\mathbb{T}^\kappa$ by $\mathbb{T}^\kappa = \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$ if $\mathbb{T}$ has a left-scattered maximum, and $\mathbb{T}^\kappa = \mathbb{T}$ otherwise.

Definition 2.2 (See [7]) A function $f : \mathbb{T} \to \mathbb{R}^n$ is called delta differentiable (or simply Δ-differentiable) at $t \in \mathbb{T}^\kappa$ provided there exists $f^\Delta(t)$ with the property that given $\varepsilon > 0$, there is a neighborhood $U = (t - \delta, t + \delta)_{\mathbb{T}}$ for some $\delta > 0$ such that

$$\left\| f^\sigma(t) - f(s) - f^\Delta(t)(\sigma(t) - s) \right\| \leq \varepsilon \left| \sigma(t) - s \right| \text{ for all } s \in U.$$

In this case, $f^\Delta(t)$ is called the Δ-derivative of f at t.

If f is Δ-differentiable at $t \in \mathbb{T}^\kappa$, then it is easy to show that (see [7, Theorem 1.16])

$$f^\Delta(t) = \begin{cases} \dfrac{f^\sigma(t) - f(t)}{\sigma(t) - t} & \text{if } \sigma(t) > t, \\[2ex] \lim\limits_{s \to t} \dfrac{f(t) - f(s)}{t - s} & \text{if } \sigma(t) = t. \end{cases}$$

Definition 2.3 (See [7]) A function $F : \mathbb{T} \to \mathbb{R}^n$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}^n$ if $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}^\kappa$. The Cauchy integral is defined by

$$\int_s^t f(\tau)\Delta\tau = F(t) - F(s), \quad t, s \in \mathbb{T},$$

where F is an antiderivative of f.

A function $p : \mathbb{T} \to \mathbb{R}$ is said to be regressive if $1 + \mu(t)p(t) \neq 0$, $t \in \mathbb{T}$, and positively regressive if $1 + \mu(t)p(t) > 0$, $t \in \mathbb{T}$. We will denote by $\mathcal{R}$ the set of all regressive and rd-continuous functions, and by $\mathcal{R}^+$ the set of all positively regressive and rd-continuous functions.

Definition 2.4 [See [7]] If $p \in \mathcal{R}$, then the generalized exponential function is defined by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right),$$

where

$$\xi_\mu(z) := \begin{cases} \frac{\mathrm{Log}(1+\mu z)}{\mu} & \text{if } \mu > 0, \\ z & \text{if } \mu = 0, \end{cases}$$

where $z \in \mathbb{C}_\mu := \{z \in \mathbb{C} : z \neq 1/\mu\}$ and $\mathrm{Log} z = \log|z| + i \arg z$, $-\pi < \arg z \leq \pi$.

Definition 2.5 (See [7]) Let A be an $n \times n$ matrix-valued function on $\mathbb{T}$. We say that A is rd-continuous on $\mathbb{T}$ if each entry of A is rd-continuous on $\mathbb{T}$, and the class of all such rd-continuous $n \times n$ matrix-valued functions on $\mathbb{T}$ is denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{n\times n})$. A is called regressive (with respect to $\mathbb{T}$) provided $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^\kappa$, and the class of all such regressive and rd-continuous functions is denoted by $\mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$.

Let $t_0 \in \mathbb{T}$ and A be an $n \times n$ regressive matrix-valued function defined on $\mathbb{T}$. Then, the unique solution of the initial value problem

$$X^\Delta = A(t)X, \quad X(t_0) = I,$$

is called the matrix exponential function, denoted by $e_A(t, t_0)$, and satisfies the properties

a) $e_0(t, s) \equiv I$ and $e_A(t, t) \equiv I$,

b) $e_A(t, s)e_A(s, r) = e_A(t, r)$,

c) $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t, s)$,

d) $e_A(t, s) = e_A^{-1}(s, t) = e^*_{\ominus A^*}(s, t)$,

e) $e_A(t, s)e_B(t, s) = e_{A\oplus B}(t, s)$ if $A(t)$ and $B(t)$ commute,

where for $A, B \in \mathcal{R}(\mathbb{T}, \mathbb{R}^{n\times n})$,

$$A \oplus B = A + B + \mu AB \quad \text{and} \quad \ominus A = -(I + \mu A)^{-1}A.$$

3 Existence and Uniqueness

In this section, we show existence and uniqueness of solutions for system (1). The next theorem is a consequence of straightforward computation.

Theorem 3.1 *Consider a control $u \in L^2_\Delta(\mathbb{T}, \mathbb{R}^n)$. Then z is a solution of system* (1) *if and only if z satisfies the integral equation*

$$z(t) = \begin{cases} \phi(t), \quad t \in [\tau(t_0), t_0]_\mathbb{T}, \\ e_{\ominus A}(t, t_0)\phi(t_0) + \displaystyle\int_{t_0}^t e_{\ominus A}(t, s)B(s)u(s)\Delta s \\ \qquad +a \displaystyle\int_{t_0}^t e_{\ominus A}(t, s) \left[\int_{t_0}^s M(s, \xi)g(\xi, z_\tau(\xi))\Delta\xi\right] \Delta s \\ \qquad +b \displaystyle\int_{t_0}^t e_{\ominus A}(t, s)f(s, z(s), u(s))\Delta s, \quad t \geq t_0. \end{cases} \tag{2}$$

For fixed $\eta > t_0$, we denote

$$M_e = \sup\{\|e_{\ominus A}(t, s)\| : t, s \in [t_0, \eta]_\mathbb{T}\}, \quad M = \sup\{\|M(t, s)\| : t, s \in [t_0, \eta]_\mathbb{T}\},$$
$$\bar{L}_f = \sup\{L_f(t) : t \in [t_0, \eta]_\mathbb{T}\}, \quad \bar{L}_g = \sup\{L_g(t) : t \in [t_0, \eta]_\mathbb{T}\}.$$

Theorem 3.2 *Suppose there exists $\eta > t_0$ such that*

$$M_e \left(|a|\, M\bar{L}_g\eta + |b|\, \bar{L}_f\right) \eta < 1. \tag{3}$$

Then, for any $\phi \in C_{\mathrm{rd}}([\tau(t_0), t_0]_\mathbb{T}, \mathbb{R}^n)$ and $u \in L^2_\Delta(\mathbb{T}, \mathbb{R}^m)$, system (1) *has a unique solution through (t_0, ϕ) defined on $[\tau(t_0), \eta]_\mathbb{T}$.*

Proof. Let $\eta > t_0$ be such that (3) holds and consider $\phi \in C_{\rm rd}([\tau(t_0), t_0]_{\mathbb{T}}, \mathbb{R}^n)$ and $u \in L^2_\Delta([t_0,\eta]_{\mathbb{T}}, \mathbb{R}^m)$. Now, finding a solution of system (1) through (t_0, ϕ) is equivalent to solving the integral equation (2). In order to do this, we consider the function space

$$C_{{\rm rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n) = \{z \in C_{\rm rd}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n) : z(t) = \phi(t) \text{ for } t \in [\tau(t_0), t_0]_{\mathbb{T}}\},$$

which is a Banach space endowed with the norm $\|z\|_* = \sup\{\|z(t)\| : t \in [\tau(t_0), \eta]_{\mathbb{T}}\}$, and we show that the operator

$$\mathcal{T} : C_{{\rm rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n) \longrightarrow C_{{\rm rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n)$$

defined by

$$(\mathcal{T}z)(t) = \begin{cases} \phi(t), \quad t \in [\tau(t_0), t_0]_{\mathbb{T}}, \\ e_{\ominus A}(t,t_0)\phi(t_0) + \displaystyle\int_{t_0}^t e_{\ominus A}(t,s)B(s)u(s)\Delta s \\ \qquad + a\displaystyle\int_{t_0}^t e_{\ominus A}(t,s)\left[\int_{t_0}^s M(s,\xi)g(\xi, z_\tau(\xi))\Delta\xi\right]\Delta s \\ \qquad + b\displaystyle\int_{t_0}^t e_{\ominus A}(t,s)f(s,z(s),u(s))\Delta s, \quad t \in [t_0, \eta]_{\mathbb{T}} \end{cases} \tag{4}$$

has a unique fixed point. Indeed, if $t \in [\tau(t_0), t_0]_{\mathbb{T}}$, then $(\mathcal{T}z)(t) = \phi(t) = z(t)$. If $t \in [t_0, \eta]_{\mathbb{T}}$, then for $z, \tilde{z} \in C_{{\rm rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n)$ with $z \neq \tilde{z}$, we have

$$\begin{aligned}
&\|(\mathcal{T}z)(t) - (\mathcal{T}\tilde{z})(t)\| \\
&\le |a| \int_{t_0}^t \|e_{\ominus A}(t,s)\| \left[\int_{t_0}^s \|M(s,\xi)\| \, \|g(\xi, z_\tau(\xi)) - g(\xi, \tilde{z}_\tau(\xi))\| \, \Delta\xi\right] \Delta s \\
&\quad + |b| \int_{t_0}^t \|e_{\ominus A}(t,s)\| \, \|f(s,z(s),u(s)) - f(s,\tilde{z}(s),u(s))\| \, \Delta s \\
&\le |a| \int_{t_0}^t \|e_{\ominus A}(t,s)\| \left[\int_{t_0}^s M L_g(\xi) \, \|z(\tau(\xi)) - \tilde{z}(\tau(\xi))\| \, \Delta\xi\right] \Delta s \\
&\quad + |b| \int_{t_0}^t \|e_{\ominus A}(t,s)\| \, L_f(s) \, \|z(s) - \tilde{z}(s)\| \, \Delta s \\
&\le |a| \int_{t_0}^t M_e \left[\int_{t_0}^s M\bar{L}_g \, \|z - \tilde{z}\|_* \, \Delta\xi\right] \Delta s + |b| \int_{t_0}^t M_e \bar{L}_f \, \|z - \tilde{z}\|_* \, \Delta s \\
&\le |a| \int_{t_0}^t M_e M \bar{L}_g \eta \, \|z - \tilde{z}\|_* \, \Delta s + |b| \, M_e \bar{L}_f \eta \, \|z - \tilde{z}\|_* \\
&\le M_e \left(|a| \, M\bar{L}_g \eta + |b| \, \bar{L}_f\right) \eta \, \|z - \tilde{z}\|_* .
\end{aligned}$$

Therefore, using (3), we have

$$\|\mathcal{T}z - \mathcal{T}\tilde{z}\|_* \le M_e \left(|a| \, M\bar{L}_g\eta + |b| \, \bar{L}_f\right) \eta \, \|z - \tilde{z}\|_* < \|z - \tilde{z}\|_*,$$

so that $\mathcal{T}$ satisfies all assumptions of the Banach contraction theorem, and therefore, $\mathcal{T}$ has only one fixed point in the space $C_{{\rm rd}_\phi}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^n)$, which is the solution of problem (1).

Definition 3.1 We shall say that $[\tau(t_0), \eta)_{\mathbb{T}}$ is the maximal interval of existence of the solution z of system (1) if there is no solution of (1) on $[\tau(t_0), \eta^*)_{\mathbb{T}}$ with $\eta^* > \eta$.

Theorem 3.3 *If z is a solution of system* (1) *on $[\tau(t_0),\eta)_{\mathbb{T}}$ and η is maximal, then either $\eta=\infty$ or $z(t)$ is not bounded on any neighborhood of η.*

Proof. Suppose that $\eta<\infty$ and there is a neighborhood U of η such that $\|z(t)\|\le R$ for $t\in U$. In this case, we can suppose that $\|z(t)\|\le R$ for all $t\in[\tau(t_0),\eta)_{\mathbb{T}}$. If η is left-dense, then there is an increasing sequence $\{\eta_k\}_{k\ge1}$ such that $\lim\limits_{k\to\infty}\eta_k=\eta$ and $\lim\limits_{k\to\infty}z(\eta_k)=z^*$ for some $z^*\in\mathbb{R}^n$. We shall see that $\lim\limits_{t\to\eta^-}z(t)=z^*$.

Let $\varepsilon>0$ be small enough. Since $\lim\limits_{k\to\infty}\eta_k=\eta$, we can take $\eta_{_N}\in(\eta-\varepsilon,\eta)_{\mathbb{T}}$ such that $\|z(\eta_{_N})-z^*\|<\varepsilon$. For $t\in(\eta-\varepsilon,\eta)_{\mathbb{T}}$ with $t>\eta_{_N}$, we have

$$\|z(t)-z^*\|\le\|z(t)-z(\eta_{_N})\|+\|z(\eta_{_N})-z^*\|.$$

Now,

$$\begin{aligned}
\|z(t)-z(\eta_{_N})\| &\le \|e_{\ominus A}(t,t_0)-e_{\ominus A}(\eta_{_N},t_0)\|\,\|\phi(t_0)\| \\
&\quad+\int_{t_0}^{\eta_N}\|e_{\ominus A}(t,s)-e_{\ominus A}(\eta_{_N},s)\|\,\|B(s)\|\,\|u(s)\|\,\Delta s \\
&\quad+|a|\int_{t_0}^{\eta_N}\|e_{\ominus A}(t,s)-e_{\ominus A}(\eta_{_N},s)\|\left[\int_{t_0}^{s}\|M(s,\xi)\|\,\|g(\xi,z_\tau(\xi))\|\,\Delta\xi\right]\Delta s \\
&\quad+|b|\int_{t_0}^{\eta_N}\|e_{\ominus A}(t,s)-e_{\ominus A}(\eta_{_N},s)\|\,\|f(s,z(s),u(s))\|\,\Delta s \\
&\quad+\int_{\eta_N}^{t}\|e_{\ominus A}(t,s)\|\,\|B(s)\|\,\|u(s)\|\,\Delta s \\
&\quad+|a|\int_{\eta_N}^{t}\|e_{\ominus A}(t,s)\|\left[\int_{t_0}^{s}\|M(s,\xi)\|\,\|g(\xi,z_\tau(\xi))\|\,\Delta\xi\right]\Delta s \\
&\quad+|b|\int_{\eta_N}^{t}\|e_{\ominus A}(t,s)\|\,\|f(s,z(s),u(s))\|\,\Delta s \\
&\le \|e_{\ominus A}(t,t_0)-e_{\ominus A}(\eta_{_N},t_0)\|\,\|\phi(t_0)\| \\
&\quad+\int_{t_0}^{\eta_N}\|e_{\ominus A}(t,s)-e_{\ominus A}(\eta_{_N},s)\|\,\|B(s)\|\,\|u(s)\|\,\Delta s \\
&\quad+|a|\int_{t_0}^{\eta_N}\|e_{\ominus A}(t,s)-e_{\ominus A}(\eta_{_N},s)\|\left[\int_{t_0}^{s}M\bar{L}_g\,\|z(\tau(\xi))\|\,\Delta\xi\right]\Delta s \\
&\quad+|b|\int_{t_0}^{\eta_N}\|e_{\ominus A}(t,s)-e_{\ominus A}(\eta_{_N},s)\|\,\bar{L}_f(\|z(s)\|+\|u(s)\|)\Delta s \\
&\quad+\int_{\eta_N}^{t}\|e_{\ominus A}(t,s)\|\,\|B(s)\|\,\|u(s)\|\,\Delta s \\
&\quad+|a|\int_{\eta_N}^{t}\|e_{\ominus A}(t,s)\|\left[\int_{t_0}^{s}M\bar{L}_g\,\|z(\tau(\xi))\|\,\Delta\xi\right]\Delta s \\
&\quad+|b|\int_{\eta_N}^{t}\|e_{\ominus A}(t,s)\|\,\bar{L}_f(\|z(s)\|+\|u(s)\|)\Delta s \\
&\le \|e_{\ominus A}(t,t_0)-e_{\ominus A}(\eta_{_N},t_0)\|\,\|\phi(t_0)\| \\
&\quad+\int_{t_0}^{\eta}\|e_{\ominus A}(t,s)-e_{\ominus A}(\eta_{_N},s)\|\,\|B(s)\|\,\|u(s)\|\,\Delta s
\end{aligned}$$

$$
\begin{aligned}
&+ |a| \int_{t_0}^{\eta} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_{_N},s)\| \, M\bar{L}_g Rs\Delta s \\
&+ |b| \int_{t_0}^{\eta} \|e_{\ominus A}(t,s) - e_{\ominus A}(\eta_{_N},s)\| \, \bar{L}_f(R + \|u(s)\|)\Delta s \\
&+ \int_{\eta_{_N}}^{\eta} M_e \|B(s)\| \, \|u(s)\| \, \Delta s + |a| \int_{\eta_{_N}}^{\eta} M_e M\bar{L}_g Rs\Delta s \\
&+ |b| \int_{\eta_{_N}}^{\eta} M_e \bar{L}_f(R + \|u(s)\|)\Delta s.
\end{aligned}
$$

Hence, we get that, if $\eta_{_N} \to \eta$, then $\|z(t) - z(\eta_N)\| \to 0$, so $\lim\limits_{t\to\eta^-} z(t) = z^*$, and therefore, $z(t)$ can be continued beyond of η, contradicting our assumption.

If η is left-scattered, then $\rho(\eta) \in (t_0, \eta)_{\mathbb{T}}$ so that the solution z exists also at η, namely, by putting

$$
\begin{aligned}
z(\eta) =& [I + \mu(\rho(\eta))A(\rho(\eta))]^{-1} \Big\{ z(\rho(\eta)) + \mu(\rho(\eta))B(\rho(\eta))u(\rho(\eta)) \\
&+ a\mu(\rho(\eta)) \int_{t_0}^{\rho(\eta)} M(\rho(\eta), s)g(s, z_\tau(s))\Delta s + b\mu(\rho(\eta))f(\rho(\eta), z(\rho(\eta)), u(\rho(\eta))) \Big\},
\end{aligned}
$$

which is a contradiction.

Theorem 3.4 *If there exists Δ-differentiable $\varphi : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}^+$ such that*

$$
\|g(t,z)\| \leq \varphi^{\Delta}(t), \tag{5}
$$

then the solution of system (1) *is defined on $[\tau(t_0), \infty)_{\mathbb{T}}$.*

Proof. Suppose that $z(t)$ is defined on $[\tau(t_0), \eta)_{\mathbb{T}}$ with $\eta < \infty$. Then, for $t \in (t_0, \eta)_{\mathbb{T}}$, we have

$$
\begin{aligned}
\|z(t)\| \leq & \|e_{\ominus A}(t,t_0)\| \, \|\phi(t_0)\| + \int_{t_0}^{t} \|e_{\ominus A}(t,s)\| \, \|B(s)\| \, \|u(s)\| \, \Delta s \\
&+ |a| \int_{t_0}^{t} \|e_{\ominus A}(t,s)\| \left[\int_{t_0}^{s} \|M(s,\xi)\| \, \|g(\xi, z_\tau(\xi))\| \, \Delta\xi \right] \Delta s \\
&+ |b| \int_{t_0}^{t} \|e_{\ominus A}(t,s)\| \, \|f(s, z(s), u(s))\| \, \Delta s \\
\leq & M_e \|\phi(t_0)\| + \int_{t_0}^{t} M_e \|B(s)\| \, \|u(s)\| \, \Delta s + |a| \int_{t_0}^{t} M_e \left[\int_{t_0}^{s} M\varphi^{\Delta}(\xi)\Delta\xi \right] \Delta s \\
&+ |b| \int_{t_0}^{t} ML_f(s)(\|z(s)\| + \|u(s)\|)\Delta s \\
\leq & M_e \|\phi(t_0)\| + \int_{t_0}^{\eta} \left(M_e \|B(s)\| + |b| \, M\bar{L}_f \right) \|u(s)\| \, \Delta s + |a| \int_{t_0}^{\eta} M_e M\varphi(s)\Delta s \\
&+ |b| \int_{t_0}^{t} M\bar{L}_f \, \|z(s)\| \, \Delta s.
\end{aligned}
$$

By using Gronwall's inequality (see [7, Corollary 6.8]), we obtain

$$\begin{aligned}\|z(t)\| \leq &\Big[M_e \|\phi(t_0)\| + \int_{t_0}^{\eta} \big(M_e \|B(s)\| + |b|\, \bar{L}_f M\big) \|u(s)\| \,\Delta s \\ &\quad + |a| \int_{t_0}^{\eta} M_e M \varphi(s) \Delta s\Big] e_{|b|\bar{L}_f M}(t, t_0) \\ \leq &\Big[M_e \|\phi(t_0)\| + \int_{t_0}^{\eta} \big(M_e \|B(s)\| + |b|\, \bar{L}_f M\big) \|u(s)\| \,\Delta s \\ &\quad + |a| \int_{t_0}^{\eta} M_e M \varphi(s) \Delta s\Big] e_{|b|\bar{L}_f M}(\eta, t_0).\end{aligned}$$

This implies that $\|z(t)\|$ stays bounded in any neighborhood of η. So, from Theorem 3.3, we get $\eta = \infty$. This completes the proof.

4 Controllability of the Linear Equation

In order to study controllability of system (1), in this section, we shall present some characterization of controllability of a linear system associated to (1), namely,

$$\begin{cases} z^{\Delta}(t) = -A(t) z^{\sigma}(t) + B(t) u(t), \quad t \in [\delta, \eta]_{\mathbb{T}}, \\ z(\delta) = z^0. \end{cases} \tag{6}$$

The results presented in this section can be seen in [11], of course, with obvious modifications.

Note that, for all $z^0 \in \mathbb{R}^n$ and $u \in L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m)$, the initial value problem (6) admits only one solution, which is given by

$$z(t) = e_{\ominus A}(t, \delta) z^0 + \int_{\delta}^{t} e_{\ominus A}(t, s) B(s) u(s) \Delta s. \tag{7}$$

Definition 4.1 We say that (6) is controllable on $[\delta, \eta]_{\mathbb{T}}$ if for every $z^0, z^1 \in \mathbb{R}^n$, there exists $u \in L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ such that the solution z of (6) corresponding to u satisfies $z(\eta) = z^1$.

Definition 4.2 For the linear system (6), we define the following concepts:

1) The controllability operator $\mathcal{B}^{\eta} : L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m) \to \mathbb{R}^n$ is defined by

$$\mathcal{B}^{\eta} u = \int_{\delta}^{\eta} e_{\ominus A}(\eta, s) B(s) u(s) \Delta s. \tag{8}$$

2) The Gramian map is defined by $\mathcal{L}_{\mathcal{B}^{\eta}} = \mathcal{B}^{\eta} \mathcal{B}^{\eta *}$.

Proposition 4.1 *The adjoint $\mathcal{B}^{\eta *} : \mathbb{R}^n \to L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ of the operator $\mathcal{B}^{\eta}$ is given by*

$$(\mathcal{B}^{\eta *} z)(t) = B^*(t) e^*_{\ominus A}(\eta, t) z$$

and

$$\mathcal{L}_{\mathcal{B}^{\eta}} z = \int_{\delta}^{\eta} e_{\ominus A}(\eta, s) B(s) B^*(s) e^*_{\ominus A}(\eta, s) z \Delta s.$$

Theorem 4.1 *System* (6) *is controllable on* $[\delta, \eta]_{\mathbb{T}}$ *if and only if one of the following statements holds:*

1) $\mathrm{Range}(\mathcal{B}^{\eta}) = \mathbb{R}^n$,

2) $\langle L_{\mathcal{B}^{\eta}} z, z\rangle > 0$ *for every* $z \in \mathbb{R}^n \setminus \{0\}$,

3) there exists $\gamma > 0$ *such that* $\|\mathcal{B}^{\eta *} z\|_{L^2_{\Delta}} \geq \gamma \|z\|$ *for every* $z \in \mathbb{R}^n$,

4) $\mathcal{L}_{\mathcal{B}^{\eta}}$ *is invertible. Moreover,* $\mathcal{G}_{\eta} = \mathcal{B}^{\eta *}\mathcal{L}^{-1}_{\mathcal{B}^{\eta}}$ *is a right inverse of* $\mathcal{B}^{\eta}$, *and the control* $u \in L^2_{\Delta}([\delta, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ *steering the system from the initial state* z^{δ} *to a final state* z^1 *is given by*
$$u = \mathcal{B}^{\eta *}\mathcal{L}^{-1}_{\mathcal{B}^{\eta}}\left(z^1 - e_{\ominus A}(\eta, \delta) z^0\right). \tag{9}$$

5 Approximate Controllability of the Nonlinear System

Definition 5.1 (Approximate Controllability) System (1) is said to be approximately controllable on $[t_0, \eta]_{\mathbb{T}}$ if for every $\phi \in C_{\mathrm{rd}}([\tau(t_0), t_0]_{\mathbb{T}}, \mathbb{R}^m)$, $z^1 \in \mathbb{R}^n$ and $\varepsilon > 0$, there exists a control $u \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ such that the solution z of (1) corresponding to u satisfies
$$z(t_0) = \phi(t_0) \quad \text{and} \quad \left\|z(t) - z^1\right\| < \varepsilon.$$

Theorem 5.1 *Suppose the system* (1) *is defined on* $[t_0, \eta]_{\mathbb{T}}$, *where* η *is such that* (3) *is satisfied. Assume that*

i) η *is left-dense,*

ii) there exists Δ*-differentiable* $\varphi : [t_0, \eta]_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $\|g(t, z)\| \leq \varphi^{\Delta}(t)$ *for all* $t \in [t_0, \eta]_{\mathbb{T}}$,

iii) there exists rd-continuous $\psi : [t_0, \eta]_{\mathbb{T}} \to \mathbb{R}^+$ *such that* $\|f(t, z, u)\| \leq \psi(t)$ *for all* $t \in [t_0, \eta]_{\mathbb{T}}$.

If the linear system (6) *is controllable on* $[\delta, \eta]_{\mathbb{T}}$, *with* $t_0 \leq \delta < \eta$, *then system* (1) *is approximately controllable on* $[t_0, \eta]_{\mathbb{T}}$.

Proof. Given $\phi \in C_{\mathrm{rd}}([\tau(t_0), t_0]_{\mathbb{T}}, \mathbb{R}^n)$, a final state z^1 and $\varepsilon > 0$, we want to find a control $u^{\varepsilon} \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$ steering the solution of system (1) to an ε-neighborhood of z^1 at time η. Indeed, let $\varepsilon > 0$ and consider a control $u \in L^2_{\Delta}([t_0, \eta]_{\mathbb{T}}, \mathbb{R}^m)$, arbitrary but fixed, and the corresponding solution $z(t) = z(t, t_0, \phi, u)$ of system (1). Since η is left-dense, there exists $\delta_{\varepsilon} \in (t_0, \eta)_{\mathbb{T}}$ such that
$$\eta - \delta_{\varepsilon} < \frac{\varepsilon}{M_e(|a|\, M\bar{\varphi} + |b|\, \bar{\psi})},$$
where $\bar{\varphi} = \sup\{\varphi(t) : t \in [t_0, \eta]_{\mathbb{T}}\}$ and $\bar{\psi} = \sup\{\psi(t) : t \in [t_0, \eta]_{\mathbb{T}}\}$. We define the control $u^{\varepsilon} \in L^2_{\Delta}([\tau(t_0), \eta]_{\mathbb{T}}, \mathbb{R}^m)$ by
$$u^{\varepsilon}(t) = \begin{cases} u(t) & \text{if } t \in [t_0, \delta_{\varepsilon}]_{\mathbb{T}}, \\ \tilde{u}(t) & \text{if } t \in (\delta_{\varepsilon}, \eta]_{\mathbb{T}}, \end{cases} \tag{10}$$
where
$$\tilde{u}(t) = B^*(t) e^*_{\ominus A}(\eta, t)\mathcal{L}^{-1}_{\mathcal{B}^{\eta}}\left(z^1 - e_{\ominus A}(\eta, \delta_{\varepsilon}) z(\delta_{\varepsilon})\right)$$

is the control steering the solution of system (6) from the initial state $z(\delta_\varepsilon)$ to the final state z^1 on $[\delta_\varepsilon, \eta]_{\mathbb{T}}$. The corresponding solution $z^{\delta_\varepsilon}(\cdot) = z^{\delta_\varepsilon}(\cdot, t_0, \phi, u^\varepsilon)$ of problem (1) at time η can be expressed by

$$\begin{aligned}
z^{\delta_\varepsilon}(\eta) =& e_{\ominus A}(\eta, t_0)\phi(t_0) + \int_{t_0}^{\eta} e_{\ominus A}(\eta, s)B(s)u^\varepsilon(s)\Delta s \\
&+ a\int_{t_0}^{\eta} e_{\ominus A}(\eta, s)\left[\int_{t_0}^{s} M(s,\xi)g(\xi, z_\tau^{\delta_\varepsilon}(\xi))\Delta\xi\right]\Delta s \\
&+ b\int_{t_0}^{\eta} e_{\ominus A}(\eta, s)f(s, z^{\delta_\varepsilon}(s)u_\alpha^\varepsilon(s))\Delta s \\
=& e_{\ominus A}(\eta, \delta_\varepsilon)\Bigg\{e_{\ominus A}(\delta_\varepsilon, t_0)\phi(t_0) + \int_{t_0}^{\delta_\varepsilon} e_{\ominus A}(\delta_\varepsilon, s)B(s)u(s)\Delta s \\
&+ a\int_{t_0}^{\delta_\varepsilon} e_{\ominus A}(\delta_\varepsilon, s)\left[\int_{t_0}^{s} M(s,\xi)g(\xi, z_\tau(\xi))\Delta\xi\right]\Delta s \\
&+ b\int_{t_0}^{\delta_\varepsilon} e_{\ominus A}(\delta_\varepsilon, s)f(s, z(s), u(s))\Delta s\Bigg\} \\
&+ \int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta, s)B(s)\tilde{u}(s)\Delta s + a\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta, s)\left[\int_{t_0}^{s} M(s,\xi)g(\xi, z_\tau^{\delta_\varepsilon}(\xi))\Delta\xi\right]\Delta s \\
&+ b\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta, s)f(s, z^{\delta_\varepsilon}(s), \tilde{u}(s))\Delta s \\
=& e_{\ominus A}(\eta, \delta_\varepsilon)z(\delta_\varepsilon) + \int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta, s)B(s)\tilde{u}(s)\Delta s \\
&+ a\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta, s)\left[\int_{t_0}^{s} M(s,\xi)g(\xi, z_\tau^{\delta_\varepsilon}(\xi))\Delta\xi\right]\Delta s \\
&+ b\int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta, s)f(s, z^{\delta_\varepsilon}(s), \tilde{u}(s))\Delta s.
\end{aligned}$$

On the other hand, the corresponding solution $y(\cdot) = y(\cdot, \delta_\varepsilon, y(\delta_\varepsilon), \tilde{u})$ of initial value problem (6) at time $t = \eta$ is given by

$$y(\eta) = e_{\ominus A}(\eta, \delta_\varepsilon)y(\delta_\varepsilon) + \int_{\delta_\varepsilon}^{\eta} e_{\ominus A}(\eta, s)B(s)\tilde{u}(s)\Delta s.$$

Since the linear system (6) is controllable on $[\delta_\varepsilon, \eta]_{\mathbb{T}}$, we have that $y(\eta) = z^1$. Taking $y(\delta_\varepsilon) = z(\delta_\varepsilon)$, we get

$$\begin{aligned}
\left\|z^{\delta_\varepsilon}(\eta) - z^1\right\| \le& |a|\int_{\delta_\varepsilon}^{\eta} \|e_{\ominus A}(\eta, s)\|\left[\int_{t_0}^{s} \|M(s,\xi)\|\left\|g(\xi, z_\tau^{\delta_\varepsilon}(\xi))\right\|\Delta\xi\right]\Delta s \\
&+ |b|\int_{\delta_\varepsilon}^{\eta} \|e_{\ominus A}(\eta, s)\|\left\|f(s, z^{\delta_\varepsilon}(s), \tilde{u}(s))\right\|\Delta s \\
\le& |a|\int_{\delta}^{\eta} M_e\left[M\int_{t_0}^{s} \varphi^\Delta(\xi)\Delta\xi\right]\Delta s + |b|\int_{\delta_e}^{\eta} M_e\psi(s)\Delta s \\
\le& |a|\int_{\delta_\varepsilon}^{\eta} M_e M\varphi(s)\Delta s + |b|\int_{\delta_\varepsilon}^{\eta} M_e\psi(s)\Delta s
\end{aligned}$$

$$\le M_e \left(|a|\, M\bar{\varphi} + |b|\, \bar{\psi}\right)(\eta - \delta_\varepsilon) < \varepsilon.$$

So we get that system (1) is approximately controllable.

6 Approximate Controllability on Free Time

In this section, we prove the approximate controllability on free time of the system

$$\begin{cases} z^{\Delta}(t) = -A(t)z^{\sigma}(t) + B(t)u(t) + bf(t, z(t), u(t)), & t \ge t_0 \ge 0, \\ z(t_0) = z^0, \end{cases} \tag{11}$$

which is the system (1) without memory (i.e., taking $a \equiv 0$).

Definition 6.1 (Approximate Controllability on Free Time) System (11) is said to be approximately controllable on free time if for every $z^0, z^1 \in \mathbb{R}^n$, and $\varepsilon > 0$, there exist $\eta \in \mathbb{T}$ and $u \in L^2_\Delta([t_0, \eta]_\mathbb{T}, \mathbb{R}^m)$ such that the corresponding solution of (1) satisfies

$$\left\| z(\eta) - z^1 \right\| < \varepsilon.$$

Theorem 6.1 *Suppose that*

i) There exists $M_e > 0$ such that $\|e_{\ominus A}(t, s)\| \le M_e$ for all $t, s \in \mathbb{T}$,

ii) there exists rd-continuous $\psi : [t_0, \infty)_\mathbb{T} \to \mathbb{R}^+$ such that

$$\|f(t, z, u)\| \le \psi(t) \quad \textit{with} \quad \int_{t_0}^{\infty} \psi(s)\Delta s < \infty.$$

If the linear system (6) is controllable on each interval $[\delta, \eta]_\mathbb{T}$, then the system (11) is approximately controllable on free time.

Proof. For $\varepsilon > 0$, $z_0 \in \mathbb{R}^n$ and a final state z^1, we want to find $\eta > t_0$ and a control $u^\varepsilon \in L^2_\Delta([t_0, \eta]_\mathbb{T}, \mathbb{R}^m)$ steering the solution of system (11) to an ε-neighborhood of z^1 at time η. Since $\displaystyle\int_{t_0}^{\infty} \psi(s)\Delta s < \infty$, we can choose $\delta_\varepsilon, \eta \in \mathbb{T}$ big enough with $t_0 < \delta_\varepsilon < \eta$ such that

$$\int_{\delta_\varepsilon}^{\eta} \psi(s)\Delta s < \frac{\varepsilon}{|b|\, M_e}.$$

Now, defining $u^\varepsilon \in L^2_\Delta([t_0, \eta]_\mathbb{T}, \mathbb{R}^m)$ as in (10) and proceeding similarly as in the proof of Theorem 5.1, we have

$$\left\| z^{\delta_\varepsilon}(\eta) - z^1 \right\| \le |b| \int_{\delta_\varepsilon}^{\eta} \|e_{\ominus A}(\eta, s)\| \left\| f(s, z^{\delta_\varepsilon}(s), \tilde{u}(s)) \right\| \Delta s < \varepsilon.$$

So we get that system (11) is approximately controllable on free time.

7 Examples

Example 7.1 Let us consider the time scale $\mathbb{T} = \mathbb{P}_{1,1} = \bigcup_{k=0}^{\infty}[2k, 2k+1]$ and the control system

$$\begin{cases} z^{\Delta}(t) = -z^{\sigma}(t) + 2u(t) + \frac{1}{100}\displaystyle\int_1^t e_{\ominus 1}(t,s)\sin(s)\sin(z(s/5))\Delta s \\ \qquad\qquad + \frac{1}{20}\cos(t)\sin(z(t)+u(t)), \quad t \in [1,5]_{\mathbb{T}}, \\ z(t) = \phi(t), \quad t \in [\frac{1}{5}, 1]_{\mathbb{T}}, \end{cases} \tag{12}$$

where $t_0 = 1$, $\tau(t) = \frac{t}{5}$, $M(t,s) = e_{\ominus 1}(t,s)$, $g(t,z) = \sin(t)\sin(z)$, $f(t,z,u) = \cos(t)\sin(z(t)+u(t))$, $A(t) = 1$, $B(t) = 2$ and $e_{\ominus A}(t,s) = e_{\ominus 1}(t,s)$. Since

$$\|g(t,z) - g(t,\tilde{z})\| \le |\sin(t)|\,\|z-\tilde{z}\|, \quad g(t,0) = 0,$$
$$\|f(t,z,u) - f(t,\tilde{z},\tilde{u})\| \le |\cos(t)|\,(\|z-\tilde{z}\| + \|u-\tilde{u}\|), \quad f(t,0,0) = 0,$$

and $M_e\left(|a|\,M\bar{L}_g\eta + |b|\,\bar{L}_f\right)\eta < \frac{1}{2}$, Theorem 3.2 ensures existence and uniqueness of solutions for problem (12) on $[\frac{1}{5}, 5]_{\mathbb{T}}$. On the other hand,

$$\|g(t,z)\| \le \varphi^{\Delta}(t) \quad \text{for all} \quad t \in [1,5]_{\mathbb{T}} \quad \text{with} \quad \varphi(t) = t,$$
$$\|f(t,z,u)\| \le \psi(t) \quad \text{for all} \quad t \in [1,5]_{\mathbb{T}} \quad \text{with} \quad \psi(t) = 1.$$

Furthermore, $\mathcal{L}_{\mathcal{B}^5} = 4\displaystyle\int_{\delta_\varepsilon}^{5} e_{\ominus(1\oplus 1)}(5,s)\Delta s > 0$, so this operator is invertible, and hence the linear system

$$\begin{cases} z^{\Delta}(t) = -z^{\sigma}(t) + 2u(t), \quad t \in [\delta_\varepsilon, 5]_{\mathbb{T}}, \\ z(\delta_\varepsilon) = z^0, \end{cases}$$

is controllable and, since $\eta = 5$ is left-dense, by Theorem 5.1, system (12) is approximately controllable on $[1,5]_{\mathbb{T}}$.

Example 7.2 Let us consider the time scale $\mathbb{T} = \{3^n : n \in \mathbb{N}_0\}$ and the control system

$$\begin{cases} z^{\Delta}(t) = -2z^{\sigma}(t) + u(t) + \dfrac{1}{3t^2}\left(\tanh(z(t)) + \dfrac{u(t)}{1+u^2(t)}\right), \quad t > 1, \\ z(1) = z_0, \end{cases} \tag{13}$$

where $f(t,z,u) = \frac{1}{3t^2}\left(\tanh(z) + \frac{u}{1+u^2}\right)$, $A(t) = 2$, $B(t) = 1$ and $e_{\ominus A}(t,s) = e_{\ominus 2}(t,s)$. It is easy to see that the solution of (13) is defined on $[1,\infty)_{\mathbb{T}}$. On the other hand, we have

$$\|f(t,z,u)\| \le \frac{1}{3t^2}\left\|\tanh(z) + \frac{u}{1+u^2}\right\| \le \psi(t) \quad \text{with} \quad \psi(t) = \frac{2}{3t^2} \quad \text{and} \quad \int_1^{\infty}\frac{\Delta t}{t^2} < \infty.$$

For $\eta > \delta_\varepsilon$, the linear system

$$\begin{cases} z^{\Delta}(t) = -2z^{\sigma}(t) + u(t), \quad t \in [\delta_\varepsilon, \eta]_{\mathbb{T}}, \\ z(\delta_\varepsilon) = z^0, \end{cases}$$

is controllable since the operator $\mathcal{L}_{\mathcal{B}^\eta} = \displaystyle\int_{\delta_\varepsilon}^{\eta} e_{\ominus(2\oplus 2)}(\eta,s)\Delta s$ is invertible. Hence, by Theorem 6.1, we have that system (13) is approximately controllable on free time.

8 Conclusion and Final Remark

In this paper, we study a control system governed by a dynamic equation with memory on time scales. Specifically, first of all, we prove existence and uniqueness of solutions, then under an additional condition, and by applying Gronwall's inequality on time scales, we prove the prolongation of solutions. After that, we prove approximate controllability of the system assuming that the associated linear control problem on time scales is exactly controllable on $[\delta,\eta]_{\mathbb{T}}$, for any $\delta\in(t_0,\eta)_T$ with η being a left-dense point. In the case where the time scale does not have left-dense points, we consider the system without memory and we prove, under additional conditions, controllability on free time, i.e., we prove the existence of a time η such that the system (1) is approximately controllable. For difference equations, approximate controllability on free time was introduced by Uzcategui and Leiva in [16]. Finally, two examples show that our results are feasible. Of course, this work can be extended to evolution equations with memory on time scales in infinite-dimensional Banach spaces using strongly continuous semigroups on time scales approach.

Acknowledgements

The authors are thankful to the referee for carefully reading the manuscript and for suggesting several essential improvements.

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Nonlinear Dynamics and Systems Theory, **22** (5) (2022) 503–521

A Dynamic Contact Problem for Piezo-Thermo-Elastic-Viscoplastic Materials with Damage

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Received: June 5, 2022; Revised: November 16, 2022

Abstract: We consider a dynamic contact problem between a piezo-thermo-elastic-viscoplastic material with damage and a rigid obstacle. The contact is frictional and bilateral, the friction is modeled by Coulomb's law with heat exchange. We employ the electro-elastic-viscoplastic with damage constitutive law for the material. The evolution of the damage is described by an inclusion of parabolic type. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. The proof is based on a classical existence and uniqueness result on parabolic inequalities, differential equations and a fixed point argument.

Keywords: *viscoplastic; piezoelectric; temperature; damage; variational inequality; fixed point.*

Mathematics Subject Classification (2010): 74M10, 74M15, 74F05, 49J40, 74R05, 74C10, 70K70, 70K75.

1 Introduction

Because of its considerable impact in everyday life and its multiple open problems, contact mechanics still remains a rich and fascinating domain of challenge. The literature devoted to various aspects of the subject is considerable, it concerns the modelling, the mathematical analysis as well as the numerical approximation of the related problems. For example, many food materials used in process engineering are elastic-viscoplastic [14] and consequently, mathematical models can be very helpful in understanding various problems related to the product development, packing, transport, shelf life testing, thermal effects, and heat transfer. It is thus important to study mathematical models that

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can be used to describe the dynamical behavior of a given elastic-viscoplastic material subjected to various highly nonlinear and even non-smooth phenomena like contact, friction, lubrication, adhesion, wear, damage, electrical and thermal effects. The uncoupled thermo-viscoplastic models were obtained in [13]. Different models have been developed to describe the interaction between the thermal and mechanical field [6]. The new papers use several types of contact for coupled materials such as thermo-mechanical, electro-mechanical and thermo-electromechanical materials. For the thermo-mechanical materials, a transmission problem in thermo-viscoplasticity is studied in [11], a thermo-viscoelastic body is considered in [5], several problems for thermo-elastic-viscoplastic materials are studied in [6–8]. For the electro-mechanical bodies, many laws of behavior are considered by many authors, see for example [1, 2, 9, 12] and references therein.

Realistically, it may be impossible to predict the electro-mechanical behaviour without thermal considerations. To achieve this, the authors have started to study a new model for thermo-electro-mechanical behaviour, see for example [4]. The aim of this paper is to study a frictionless contact problem for elastic-viscoplastic materials with piezoelectric effect, also called electro-elasto-viscoplastic materials. To this end, we consider that the material is electro-elasto-viscoplastic with an internal state variable α which may describe the damage of the system caused by elastic deformations and thermal effects. The main difficulty is that Korn's inequality cannot be applied any more. For this proposal, following the technique already developed by Duvaut and Lions [10] for Coulomb's friction models, we use the inertial term of the dynamic process to compensate the loss of coerciveness in the a priori estimates. By the change of variable, we bring the coupled second order evolution inequality into a classical first order evolution inequality. After this, we use classical results on first order evolution nonlinear inequalities, a parabolic variational inequality and equations and the fixed point arguments. Existence and uniqueness results for the boundary value problem for thermo-electro-viscoelastic materials were obtained by many authors using different functional methods. The novelty in this paper is to make the coupling of an electro-elasto-viscoplastic problem with damage and thermal effect. We employ the thermo-elastic-viscoplastic with damage constitutive law for the material. The damage of the material is caused by elastic deformations. The evolution of the damage is described by an inclusion of parabolic type. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement, a parabolic variational inequality for the damage and the heat equation for the temperature. We establish a variational formulation for the model and we prove the existence of a unique weak solution to the problem. A new law of behaviour for the so-called thermo-electro-elastic-viscoplastic material is given by

$$\sigma(t)=\mathcal{A}(\varepsilon(\dot{u}(t)))+\mathcal{B}(\varepsilon u(t),\alpha(t))+\int_0^t \mathcal{G}\left(\sigma(s)-\mathcal{A}(\varepsilon(\dot{u}(t))),\varepsilon\left(u\left(s\right)\right)\right)ds+\mathcal{E}^*\nabla\varphi\left(t\right)-\mathcal{M}\theta\left(t\right), \tag{1}$$

$$D\left(t\right)=\mathcal{E}\varepsilon\left(u\left(t\right)\right)-\mathcal{B}\nabla\left(\varphi\left(t\right)\right)-\mathcal{P}\theta\left(t\right), \tag{2}$$

where $\mathcal{A}$ and $\mathcal{B}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively, $\mathcal{G}$, $E(\varphi)=-\nabla\varphi$, $\mathcal{E}=(e_{ijk})$, $\mathcal{M}$, $\mathcal{B}$, and $\mathcal{P}$ are the relaxation operator, electric field, piezoelectric, thermal expansion, electric permittivity and pyroelectric tensors. $\mathcal{E}^*$ is the transpose of $\mathcal{E}$.

Many types of evolution of the temperature field are given by several authors, see for example [4, 5, 8]. In this paper, we use the evolution of the temperature field obtained from the conservation of energy and define it with the following differential equation:

$$\dot{\theta}(t) - \operatorname{div}(\mathcal{K}\nabla\theta(t)) = -\mathcal{M}\nabla\dot{\mathbf{u}}(t) + q,$$

where θ is the temperature, $\mathcal{K}$ denotes the thermal conductivity tensor, $\mathcal{M}$ is the thermal expansion tensor, q is the density of volume heat sources and ψ is a nonlinear function assumed here to depend on the thermal expansion tensor and the velocity.

The differential inclusion used for the evolution of the damage field is

$$\dot{\alpha} - k_1\triangle\alpha + \partial_\varphi(\alpha) \ni \Phi(\varepsilon(u), \alpha), \text{ in } \Omega \times (0, T), \tag{3}$$

where $\varphi_F(\alpha)$ denotes the sub-differential of the indicator function of the set F of an admissible damage function given as follows:

$$F = \{\alpha \in H^1(\Omega) : 0 \geq \alpha \geq 1, \text{ a.e. in } \Omega\}$$

and Φ are given constitutive functions which describe the sources of the damage in the system. When $\alpha = 0$, the material is completely damaged, when $\alpha = 1$, the material is undamaged, and for $0 < \alpha < 1$, there is partial damage. The Coulomb friction is one of the useful friction laws known from the literature. This law has two basic ingredients, namely, the concept of friction threshold and its dependence on the normal stress. Various versions of the normal compliance law were recently presented in the literature [1, 2, 12]. The paper is organized as follows. In Section 2, we present the model. In Section 3, we introduce the notations, some preliminary results, a list of the assumptions on the data and we give the variational formulation of the problem. In Section 4, we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on evolutionary elliptic variational inequalities, ordinary differential equations and fixed point arguments.

2 The Model

The physical setting is the following. A thermo-electro- elastic-viscoplastic body occupies a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with the outer Lipschitz surface Γ. This boundary is divided into three open disjoints Γ_1, Γ_2 and Γ_3, on one hand, and a partition of $\Gamma_1 \cup \Gamma_2$ into two open parts Γ_a and Γ_b, on the other hand. We assume that $\operatorname{meas}(\Gamma_1) > 0$ and $\operatorname{meas}(\Gamma_a) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. The body is subjected to the action of body forces of density f_0, a volume electric charges of density q_0 and a heat source of constant strength q.

The body is clamped on $\Gamma_1 \times (0, T)$, so the displacement field vanishes there. A surface traction of density f_2 acts on $\Gamma_2 \times (0, T)$. We also assume that the electrical potential vanishes on $\Gamma_a \times (0, T)$ and a surface electric charge of density q_b is prescribed on $\Gamma_b \times (0, T)$. Moreover, we suppose that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0, T)$. In the reference configuration, the body is in contact with an obstacle, or foundation, over the contact surface Γ_3. The contact is frictional and thermo-mechanical. The model of the contact is specified by the normal compliance and it is associated with Coulomb's law of dry friction for the mechanical contact and by an associated temperature boundary condition for the thermal contact.

The classical formulation of the mechanical problem is as follows.

Problem $\mathcal{P}$. Find the displacement field $\mathbf{u} : \Omega \times [0, T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : \Omega \times [0, T] \to \mathbb{S}^d$, the electric potential $\varphi : \Omega \times [0, T] \to \mathbb{R}$, the electric displacement field

$\mathbf{D} : \Omega \times [0,T] \to \mathbb{R}^d$, the temperature field $\theta : \Omega \times [0,T] \to \mathbb{R}$ and the damage field $\alpha : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t)))+\mathcal{B}(\varepsilon u(s), \alpha(t))+\int_0^t \mathcal{G}\left(\sigma(s)-\mathcal{A}(\varepsilon(\dot{u}(t))), \varepsilon\left(u\left(s\right)\right)\right) ds+\mathcal{E}^* \nabla\varphi\left(t\right)-\mathcal{M}\theta\left(t\right), \tag{4}$$

$$\mathbf{D}\left(t\right) = \mathcal{E}\epsilon\left(u\left(t\right)\right) - \mathcal{B}\nabla\left(\varphi\left(t\right)\right) - \mathcal{P}\theta\left(t\right), \tag{5}$$

$$\dot{\theta} - \operatorname{div}\left(K\nabla\theta\right) = -M\nabla\dot{u} + q \quad \text{in } \Omega \times (0,T), \tag{6}$$

$$\operatorname{div}\boldsymbol{\sigma} + \mathbf{f}_0 = \rho\ddot{u} \quad \text{in } \Omega \times (0,T), \tag{7}$$

$$\dot{\alpha} - K\triangle\alpha + \partial\varphi_K(\alpha) \ni \Phi(\varepsilon(u) - \alpha) \text{ in } \Omega \times (0,T), \tag{8}$$

$$\operatorname{div}\mathbf{D} - q_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{9}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_1 \times (0,T), \tag{10}$$

$$\boldsymbol{\sigma\nu} = \mathbf{f}_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{11}$$

$$\sigma_\tau = p_r\left(u_\nu - h\right) on\ \Gamma_3 \times (0,T), \tag{12}$$

$$\begin{cases} \|\sigma_\tau\| \leq \mu p\|R\sigma_\nu\|, \\ \|\sigma_\tau\| < \mu p\|R\sigma_\nu\| \Longrightarrow \dot{\mathbf{u}}_\tau = 0, \\ \|\sigma_\tau\| = \mu p\|R\sigma_\nu\| \Longrightarrow \exists\ \lambda > 0 : \quad \sigma_\tau = -\lambda\ \dot{\mathbf{u}}_\tau \text{ on } \ \Gamma_3 \times (0,T), \end{cases} \tag{13}$$

$$-K_{ij}\frac{\partial\theta}{\partial v}\nu_j = K_e(\theta - \theta_R) - h_\tau(|\dot{u}_\tau|) \quad \text{on } \Gamma_3 \times [0,T], \tag{14}$$

$$\frac{\partial\alpha}{\partial\nu} = 0 \text{ on } \Gamma \times (0,T), \tag{15}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = \mathbf{0} \quad \text{on } \ \Gamma_3 \times (0,T), \tag{16}$$

$$\theta = 0 \quad \text{on} \quad (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{17}$$

$$\varphi = 0 \quad \text{on} \quad \Gamma_a \times (0,T), \tag{18}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = q_b \quad \text{on } \ \Gamma_b \times (0,T), \tag{19}$$

$$\mathbf{D}\cdot\boldsymbol{\nu} = \psi\left(u_\nu - h\right)\phi_L\left(\varphi - \varphi_0\right) \quad \text{on } \Gamma_3 \times (0,T), \tag{20}$$

$$u(0) = u_0,\ \dot{u}(0) = v_0,\ \alpha(0) = \alpha_0 \text{ and } \theta(0) = \theta_0 \text{ in } \Omega. \tag{21}$$

We now describe problem (4)-(21) and provide the explanation of the equations and the boundary conditions. Equations (4) and (5) represent the thermo-electro-elastic-viscoplastic constitutive law, the evolution of the temperature field is governed by a differential equation given by the relation (6), assumed to be a rather general function of the strains. Next equations (20) and (9) are the steady equations for the stress and electric-displacement field, conditions (10) and (11) are the displacement and traction boundary conditions. Equation (17) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0,T)$ which implies that there is only an electro-mechanical effect on $(\Gamma_1 \cup \Gamma_2)$.

Next, (18) and (19) represent the electric boundary conditions for the electrical potential on Γ_a, and the electric charges on Γ_b, respectively. We use (19) as the electrical contact condition on Γ_3 which represents a regularized condition. Equation (20) represents the initial displacement field and the initial damage field, where u_0 is the initial displacement, and θ_0 is the initial temperature.

We turn to the contact conditions (12)-(14) and describe the frictional thermo-mechanical contact on the potential contact surface Γ_3. The relation (12) describes a normal compliance conditions with Coulomb's law. The equation (14) represents an associated temperature boundary condition on the contact surface. The equation (16) shows that there are no electric charges on the contact surface. R_ν is the truncation operator defined by

$$R_\nu(s) = \begin{cases} L & \text{if} \quad s < -L, \\ -s & \text{if} \quad -L \leq s \leq 0, \\ 0 & \text{if} \quad s > 0. \end{cases}$$

Here $L > 0$ is the characteristic length of the bond, beyond which it does not offer any additional traction. The introduction of the operator R_ν, together with the operator R_τ defined below, is motivated by mathematical arguments but it is not restrictive from a physical point of view since no restriction on the size of the parameter L is made in what follows, where $u^1_\tau - u^2_\tau$ stands for the jump of the displacements in the tangential direction. R_ν is the truncation operator given by

$$R_\nu(s) = \begin{cases} v & \text{if} \quad |v| \leq L, \\ L\frac{v}{|v|} & \text{if} \quad |v| > L. \end{cases}$$

3 Variational Formulation

In order to obtain the variational formulation of the Problem $\mathcal{P}$, we use the following notations and preliminaries

3.1 Notations and preliminaries.

In this short section, we recall some preliminary material and notations. For more details, we refer the reader to [7, 10]. The indices i, j, k and l run from 1 to d and summation over repeated indices is implied. An index that follows the comma represents the partial derivative with respect to the corresponding component of the spatial variable. We also use the following notations:

$$H = L^2(\Omega)^d = \{ \mathbf{u} = (u_i) : u_i \in L^2(\Omega)\}, \quad \mathcal{H} = \{ \sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$H^1(\Omega)^d = \{ \mathbf{u} = (v_i) \in H : \ \varepsilon(\mathbf{u}) \in \mathcal{H} \ \}, \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H}\text{:} \ \ \operatorname{Div}\sigma \in H\}.$$

The operators of deformation ε and Div are defined by

$$\varepsilon(\mathbf{u}) = (\varepsilon_{ij}(\mathbf{u})), \quad \varepsilon_{ij}(\mathbf{u}) = (u_{i,j} + u_{j,i})/2, \quad \operatorname{Div}\sigma = (\sigma_{ij,j}) \cdot$$

The associated norms on spaces H, $H^1(\Omega)^d$, $\mathcal{H}$, and $\mathcal{H}_1$ are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H^1(\Omega)^d}$, $\|\cdot\|_{\mathcal{H}}$, and $\|\cdot\|_{\mathcal{H}_1}$ respectively. Let $H_\Gamma = H^{1/2}(\Gamma)^d$ and $\gamma : H^1(\Omega)^d \to H_\Gamma$ be the trace map. For every element $v \in H^1(\Omega)^d$, we also use the notation v to denote the trace γv of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on Γ . Moreover, we use the dot above to indicate the derivative with respect to the time variable and, for a real number r, we use r_+ to represent its positive part, that is, $r_+ = \max(0, r)$. To obtain the variational formulation of the problem (4)-(21), we introduce, for the bonding field, the sets

$$W = \left\{\phi \in H^1(\Omega)^d : \phi = 0 \text{ on } \ \Gamma_a\right\}, \quad \mathcal{W} = \left\{D = (D_i) : \ D_i \in L^2(\Omega), \operatorname{div} D \ \in L^2(\Omega)\right\}.$$

On the spaces V, W, $\mathcal{W}$, we define the following inner products:

$$(\mathrm{u} \cdot \mathrm{v})_V = (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\mathbf{v}))_{\mathcal{H}}, \forall u, v \in V, \tag{22}$$

$$(\varphi, \phi)_W = (\nabla_\varphi, \nabla_\phi)_{\mathcal{W}}, \forall \varphi, \phi \in W, \tag{23}$$

$$(w, z)_E = (\nabla_w, \nabla_z)_{\mathcal{H}}, \forall w, z \in E, \tag{24}$$

where $E = \left\{ \gamma \in H^1(\Omega) : \gamma = 0 \text{ a.e. on } \Gamma_1 \cup \Gamma_2 \right\}$.

Therefore, the spaces $(V, (\cdot,\cdot)_V)$, $(W, (\cdot,\cdot)_W)$ and $(E, (\cdot,\cdot)_E)$ are real Hilbert spaces.

3.2 Assumptions on the data

We now list the assumptions on the problem's data.

The viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(a) There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \|\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)\| \leqslant L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\| \quad \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{a.e. } x \in \Omega \,, \\ \text{(b) There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ \quad (\mathcal{A}(x, \varepsilon_1) - \mathcal{A}(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geqslant m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2, \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega \,, \\ \text{(c) The mapping } x \longrightarrow \mathcal{A}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \,, \forall\ \varepsilon \in \mathbb{S}^d, \\ \text{(d) The mapping } x \longrightarrow \mathcal{A}(x, \varepsilon) \text{ belongs to } \mathcal{H}. \end{array} \right. \tag{25}$$

The elasticity operator $\mathcal{B} : \Omega \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(a) There exists } L_{\mathcal{B}} > 0 \text{ such that} \\ \|\mathcal{B}(x, \varepsilon_1) - \mathcal{B}(x, \varepsilon_2)\| \leqslant L_{\mathcal{B}} \|\varepsilon_1 - \varepsilon_2\| \quad \forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \text{ a.e. } x \in \Omega, \\ \text{(b) The mapping } x \longrightarrow \mathcal{B}(x, \varepsilon) \text{ is Lebesgue measurable on } \Omega \,, \forall\ \varepsilon \in \mathbb{S}^d, \\ \text{(c) The mapping } x \longrightarrow \mathcal{B}(x, 0) \text{ belongs to } \mathcal{H}. \end{array} \right. \tag{26}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \longrightarrow \mathbb{S}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(a) There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \|\mathcal{G}(x, \sigma_1, \varepsilon_1) - \mathcal{G}(x, \sigma_2, \varepsilon_2)\| \leqslant L_{\mathcal{G}} (\|\sigma_1 - \sigma_2\| + \|\varepsilon_1 - \varepsilon_2\|) \,, \\ \text{for all } \sigma_1, \sigma_2, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d \text{ , a.e. } x \in \Omega, \\ \text{(b) The mapping } x \longrightarrow \mathcal{G}(x, \sigma, \varepsilon) \text{ is Lebesgue measurable on } \Omega \,, \forall\ \varepsilon \in \mathbb{S}^d, \\ \text{ for any } \varepsilon, \sigma \in \mathbb{S}^d, \\ \text{(c) The mapping } x \longrightarrow \mathcal{G}(x, 0, 0) \in \mathcal{H}. \end{array} \right. \tag{27}$$

The piezoelectric operator $\mathcal{E}$: $\Omega \times \mathbb{S}^d \longrightarrow \mathbb{R}^d$ satisfies

$$\left\{ \begin{array}{l} \text{(a) } \mathcal{E}(x, \tau) = (e_{ijk}\,, \tau_{jk}) \,, \ \forall \tau = (\tau_{jk}) \in \mathbb{S}^d, \text{ a.e. } x \text{ in } \Omega \,, \\ \text{(b) } e_{ijk} = e_{ikj} \in L^\infty(\Omega) \,, \ 1 \leqslant i, j, k \leqslant d. \end{array} \right. \tag{28}$$

The thermal expansion operator $\mathcal{M}$:$\Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\left\{ \begin{array}{l} \text{(a) There exists a constant } L_{\mathcal{M}} > 0 \text{ such that} \\ \|\mathcal{M}(x, \theta_1) - \mathcal{M}(x, \theta_2)\| \leqslant L_{\mathcal{M}} \|\theta_1 - \theta_2\| \quad \forall\, \theta_1, \theta_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega \,, \\ \text{(b) The mapping } x \longrightarrow \mathcal{M}(x, \theta) \text{ is Lebesgue measurable on } \Omega, \forall\ \theta \in \mathbb{R}, \\ \text{(c) The mapping } x \longrightarrow \mathcal{M}(x, 0) \in \mathcal{H}. \end{array} \right. \tag{29}$$

The tangential function satisfies:

$$\begin{cases} h_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+ \text{ verifies:} \\ (a) : \exists \ \ L_\tau > 0 \text{ s.t. } \ |\, h_\tau(x, r_1 - h_\tau(x, r_2)\,| \leq L^\tau \,|\, r_1 - r_2 \,|\,, \\ \quad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3. \\ (b) : \text{The mapping } \mathbf{x} \mapsto h_\tau(x, r) \text{ belongs to } L^2(\Gamma_3). \end{cases} \tag{30}$$

The electric permittivity operator $B = (Bij) : \Omega \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ satisfies

$$\begin{cases} \text{(a)} \ \ B\,(x, E) = (Bij\,(x)\,E_j) \ \forall \ E = (E_i) \in \mathbb{R}^d \ \ , \text{ a.e. } x \in \Omega, \\ \text{(b)} \ Bij = Bji \in \ \ L^\infty\,(\Omega)\,, \ \ \ 1 \ \leqslant i, j \ \leqslant d, \\ \text{(c) There exists a constant } \ M_{\mathcal{B}} > 0 \ \text{ such that } BE.E \geqslant M_{\mathcal{B}}\,|E|^2\,, \\ \ \forall \ E = (E_i) \in \mathbb{R}^d \ \ , \text{ a.e. in } \Omega. \end{cases} \tag{31}$$

The thermal conductivity operator $\mathcal{K} : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } \ L_{\mathcal{K}} > 0 \text{ such that} \\ \ \|\mathcal{K}\,(x, r_1) - \mathcal{K}\,(x, r_2)\| \leqslant L_{\mathcal{K}}\,\|r_1 - r_2\| \quad \text{ for all } r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Omega, \\ \text{(b) } m_{ij} = m_{ji} \in \ \ L^\infty\,(\Omega)\,, \ \ \ 1 \ \leqslant i, j \ \leqslant d, \\ \text{(c) The mapping } x \longrightarrow S\,(x, 0, 0) \ \text{ belongs to } L^2\,(\Omega)\,. \end{cases} \tag{32}$$

The damage source function $\Phi : \Omega \times \mathbb{S}^d \times \mathbb{S}^d \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_\Phi > 0 \text{ such that} \\ |\Phi(x, \eta_1, \omega_1, \beta_1) - \Phi(x, \eta_2, \omega_2, \beta_2)| \leq L_\Phi(|\eta_1 - \eta_2| + |\omega_1 - \omega_2| + |\beta_1 - \beta_2|) \\ \text{for all } \eta_1, \ \eta_2, \ \omega_1, \ \omega_2 \in \mathbb{S}^d, \ \beta_1, \beta_2 \in \mathbb{R}, \ x \in \Omega, \\ \text{(b) The mapping } x \longrightarrow \Phi(x, \eta, \omega, \beta) \text{ is Lebesgue measurable on } \Omega, \\ \text{for any } \eta, \ \omega \in \mathbb{S}^d \text{ and for all } \beta \in \mathbb{R}, \\ \text{(c) The mapping } x \longrightarrow \Phi(x, 0, 0, 0) \text{ belongs to } \mathbb{L}^2(\Omega). \end{cases} \tag{33}$$

The function $\Psi : \varepsilon \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{S}_n \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \text{(a) There exists a constant } L_\Psi > 0 \text{ such that} \\ |\Psi(x, \sigma_1, \varepsilon_1, \theta_1, \xi_1) - \Psi(x, \sigma_2, \varepsilon_2, \theta_2, \xi_2)| \leq L_\Psi(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2| \\ \ +|\theta_1 - \theta_2| + |\xi_1 - \xi_2|), \text{ for all } \sigma_1, \ \sigma_2, \ \varepsilon_1, \ \varepsilon_2 \in \mathbb{S}_n, \ \theta_1, \theta_2, \xi_1, \xi_2 \in \mathbb{R}, \ x \in \Omega, \\ \text{(b) The mapping } x \longrightarrow \Psi(x, \sigma, \varepsilon, \theta, \xi) \text{ is Lebesgue measurable on } \Omega, \\ \text{for all } \sigma, \ \varepsilon \in \mathbb{S}_n \text{ and for all } \theta, \xi \in \mathbb{R}, \\ \text{(c) The mapping } x \longrightarrow \Psi(x, 0, 0, 0) \text{ belongs to } \mathbb{L}^2(\varepsilon). \end{cases} \tag{34}$$

We also suppose that the body forces and surface tractions have the regularity

$$f_0 \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)), f_2 \in \mathbb{L}^2(0, T; \mathbb{L}^2(\Omega)), \rho \in \mathbb{L}^\infty(\Omega), \tag{35}$$

$$q_0 \in C\left(0, T, L^2\,(\Omega)\right), \qquad q_2 \in C\left(0, T, L^2\,(\Gamma_b)\right), \tag{36}$$

$$q_2\,(t) = 0 \text{ on } \Gamma_3, \forall t \in [0, T]\,. \tag{37}$$

The functions g and μ have the following properties:

$$g \in L^2\,(\Gamma_3)\,, \quad g\,(x) \geqslant 0, \ \ a.e. \text{ on } \ \Gamma_3, \tag{38}$$

$$\mu \in L^\infty\,(\Gamma_3)\,, \quad \mu\,(x) > 0, \quad a.e. \text{ on } \Gamma_3, \tag{39}$$

here μ is the coefficient of friction. The initial displacement field satisfies

$$u_0 \in V, \tag{40}$$

and the initial temperature field satisfies

$$\theta_0 \in E, \quad \theta_F \in L^2\left(0, T, L^2\left(\Gamma_3\right)\right), k_e \in L^\infty\left(\Omega, \mathbb{R}_+\right), q_{th} \in L^2\left(0, T, E^{'}\right). \tag{41}$$

Using the above notation and Green's formulas, we obtain the variational formulation of the mechanical problem (4)-(21) for all functions $v \in V,\ w \in W_{th},\ \phi \in W_e$ and a.e. $t \in (0, T)$, given as follows.

Problem $\mathcal{PV}$. Find the displacement $\mathbf{u} : [0, T] \to V$, the stress $\sigma : [0, T] \to \mathcal{H}_1$, and an electric potential $\varphi : [0, T] \to W$, the electric displacement $D : [0, T] \longrightarrow H$ and the temperature $\theta : [0, T] \longrightarrow V$, and the damage $\alpha : [0, T] \longrightarrow E_1$ such that

$$\begin{aligned} \sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{B}(u(t), \alpha(t)) + \int_0^t \mathcal{G}\left(\sigma(s) - \mathcal{A}(\varepsilon(\dot{u}(t))), \varepsilon\left(u\left(s\right)\right)\right) ds \\ + \mathcal{E}^* \nabla\varphi\left(t\right) - \mathcal{M}\theta\left(t\right), \end{aligned} \tag{42}$$

$$(\ddot{u}(t), v - \dot{u}(t))_{V' \times V} + (\boldsymbol{\sigma}(t), \boldsymbol{\varepsilon}\ (\mathbf{v}(\mathbf{t}) - \dot{\mathbf{u}}(t))_{\mathcal{H}} + j(\mathbf{v}(t)) - j(\dot{\mathbf{u}}(t)) \geqslant (\mathbf{f}(t), \mathbf{v} - \dot{\mathbf{u}}(t))_V, \tag{43}$$

$$(\dot{\alpha}(t), \zeta - \alpha(t))_{L^2(\Omega)} + a(\alpha(t), \zeta - \alpha(t)) \geq (\Phi(\varepsilon(u(t))), \alpha(t), \zeta - \alpha(t))_{L^2(\Omega)}, \tag{44}$$

for all $\alpha(t) \in F$, $\zeta \in F$ and $t \in [0, T]$.

$$D\left(t\right) = \mathcal{E}\varepsilon\left(u\left(t\right)\right) - \mathcal{B}\nabla\left(\varphi\left(t\right)\right) - \mathcal{P}\theta\left(t\right), \tag{45}$$

$$\left(D\left(t\right), \nabla_\phi\right)_H = -\left(q_e\left(t\right), \phi\right)_W + \left(h\left(u\left(t\right), \varphi\right), \phi\right)_W, \ \forall\varphi \in W, \quad t \in [0, T], \tag{46}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t) \quad \text{ on } \ E^{'}, \tag{47}$$

$$u(0) = u_0,\ \dot{u}(0) = v_0,\ \alpha(0) = \alpha_0 \text{ and } \theta(0) = \theta_0 \text{ on } \Omega. \tag{48}$$

Here, the function $Q : [0, T] \to E'$ and the operators $K : E \to E',\ R : V \to\ E';\ M : E \to V'$ are defined by $\forall v \in V$, $\forall \tau \in E$, $\forall \eta \in E$:

$$\begin{aligned} \langle Q(t), \eta\rangle_{E' \times E} &= \int_{\Gamma_3} k_e \theta_R \eta\, ds + \int_\Omega q\eta\, dx, \\ \langle K\tau, \eta\rangle_{E' \times E} &= \sum_{i,j=1}^d \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \eta}{\partial x_i}\, dx + \int_{\Gamma_3} k_e \tau\eta\, ds, \\ \langle Rv, \eta\rangle_{E' \times E} &= \int_{\Gamma_3} h_\tau(|v_\tau|)\eta\, ds - \int_\Omega (M_e\, \nabla v)\eta\, dx, \\ \langle M\tau, v\rangle_{V' \times V} &= (-\tau M_e, \varepsilon(v))_{\mathcal{H}}, \end{aligned}$$

where $J_\varepsilon : V \times V \to \mathbb{R}$, $f : [0; T] \to V$, $q_e : [0; T] \to W$ and $\gamma : V \times W \to W$ are respectively defined by

$$J\varepsilon\ \left(N, v\right) = \int_{\Gamma_3} \mu p \left|R \times N_\nu\right| \sqrt{|v_\tau|^2 + \varepsilon^2} da, \quad \forall v \in V, \quad \forall\varepsilon > 0, \tag{49}$$

$$(\mathbf{f}(t), \mathbf{v})_V = \int_\Omega \mathbf{f}_0(t) \cdot \mathbf{v}\, dx + \int_{\Gamma_2} \mathbf{f}_2(t) \cdot \mathbf{v}\, da. \tag{50}$$

We define the bilinear form $a : H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$

$$a(\alpha, \zeta) = \kappa \int_\Omega \bigtriangledown\alpha \cdot \bigtriangledown\zeta \, dx, \tag{51}$$

$$(q_e(t), \phi)_W = \int_\Omega q_0(t)\phi dx - \int_{\Gamma_b} q_2(t)\phi \, da, \tag{52}$$

$$(\gamma(u, \varphi), \phi)_W = \int_{\Gamma_3} \psi(u_\nu - h)\, \phi_L(\varphi - \varphi_0)\phi \, da \tag{53}$$

for all u, $v \in V$, θ, $w \in W$, $\phi \in W$ and $t \in [0; T]$. We note that the definitions of f and q_e are based on the Riesz representation theorem. Moreover, the conditions (35) and (36) imply that

$$f \in C(0, T, V), \quad q_e \in C(0, T, We). \tag{54}$$

We denote by $\|.\|_V$, $\|.\|_H$ and $. \|.\|_{V'}$ the norms on the spaces V, H and V', respectively, and we use $(.,.)_{V' \times V}$ for the duality pairing between V' and V. Note that if $f \in H$, then

$$(f, v)_{V' \times V} = (f, v)_H, \forall v \in H. \tag{55}$$

The existence of the unique solution of problem $\mathcal{P}_V$ is stated and proved in the next section.

4 Existence and Uniqueness of the Solution

Our main existence and uniqueness result is the following.

Theorem 4.1 *Assume that (25)-(41) hold. Then, if $N_\psi < \frac{m_\beta}{a_0^2}$, there exists a unique solution $\{u, \sigma, \theta, \varphi, D\}$ to problem $\mathcal{P}_V$ satisfying*

$$u \in W^{1,2}(0; T; V) \cap C^1(0; T; V) \cap \; W^{2,2}(0; T; V^{'}), \sigma \in C(0; T; \mathcal{H}), \tag{56}$$

$$\varphi \in C(0; T; W), D \in C(0; T; \mathcal{W}), \tag{57}$$

$$\theta \in W^{1,2}(0; T; E^{'}) \cap L^2(0; T; E) \cap C(0; T; L^2(\Omega)). \tag{58}$$

$$\alpha \in W^{1,2}\left(0, T; L^2(\Omega)\right) \cap L^2\left(0, T; H^1(\Omega)\right). \tag{59}$$

Functions u, σ, θ, φ, D, θ and α, which satisfy (42)-(48), are called the weak solution to the contact problem $\mathcal{P}$. We conclude that, under the assumptions (25)-(40) and if $N_\psi < \frac{m_\beta}{a_0^2}$ is satisfied, the mechanical problem (4)-(21) has a unique weak solution satisfying (56)-(58).

The proof of Theorem 4.1 is carried out in several steps. It is based on the results of evolutionary variational inequalities, ordinary differential equations and fixed point arguments.

In the first step, we let $\eta \in \mathbb{L}^2(0, T; V)$ be given and consider the following variational inequality.

Problem $\mathcal{PV}u_\eta$. Find a displacement field $u_\eta : [0; T] \to V$ such that $\forall\ t \in [0, T]$,

$$\begin{aligned} (\ddot{u}(t), v - \dot{u}(t))_{V' \times V} + (\mathcal{A}\varepsilon\left(\dot{u_\eta}(t)\right), \varepsilon(v - \dot{u_\eta}(t)))_{\mathcal{H}} + j(u_\eta(t), v) - j(u_\eta(t), \dot{u}_\eta(t)) \\ + \left(\eta(t), v - \dot{u_\eta}(t)\right)_V \geq (f(t), v - \dot{u_\eta}(t))_V, \end{aligned} \tag{60}$$

$u_\eta(0) = u_0, \dot{u}_\eta(0) = v_0$ for all u_η, $v \in V$. In the study of the problem $\mathcal{PV}u_\eta$, we have the following result.

Lemma 4.1 *$\mathcal{PV}u_\eta$ has a unique solution satisfying the regularity expressed in* (56)*:*

$$u_\eta(t) = u_0 + \int_0^t v_{\eta g_\eta}(s)ds \quad \forall t \in [0,T].$$

We define the operator $A : V \to V'$ by

$$(Av, w)_{V'\times V} = (\mathcal{A}\varepsilon(v), \varepsilon(w))_{\mathcal{H}}, \quad \forall v, w \in V. \tag{61}$$

We consider the following variational inequality.
Problem $\mathcal{PV}v_\eta$. Find a displacement field $v_\eta : [0;T] \times \Omega \to V$ such that $\forall\, t \in [0,T]$.

$$\begin{aligned} &(\dot{v}_{N\eta}(t), w - v_{N\eta}(t))_{V'\times V} + (Av_{N\eta}(t)), w - v_{N\eta}(t))_{V'\times V} + j(N, w) \\ &-j(N, v_{N\eta}(t)) \geq (f_\eta(t), w - v_{N\eta}(t))_{V'\times V}, \ \forall w \in V, \end{aligned} \tag{62}$$

$$v_{N\eta}(0) = v_o. \tag{63}$$

In the study of **Problem** $\mathcal{PV}v_\eta$, we have the following result.

Lemma 4.2 *For all $N \in \mathbb{L}^2(0,T,\mathcal{H}_1)$ and $\eta \in \mathbb{L}^2(0,T,V')$, the* **Problem** *$\mathcal{PV}v_\eta$ has a unique solution with the regularity $v_{N\eta} \in C(0,T,H) \cap \mathbb{L}^2(0,T,V) \cap W^{1,2}(0,T,V')$.*

Proof. We begin by the step of regularization we defined, for all $\varepsilon > 0$,

$$\dot{J}\varepsilon\ (N, v) = \int_{\Gamma_3} \mu p \left| R \times N_\nu \right| \sqrt{|v_\tau|^2 + \varepsilon^2} da, \quad \forall v \in V, \quad \forall \varepsilon > 0.$$

After some algebra, for all $\varepsilon > 0$, $\dot{J}_\varepsilon$ is C^1 convex on V, and its Frechet derivative satisfies

$$\forall c > 0,\ \forall w \in V \quad \left| \dot{J}'_\varepsilon(N, w) \right|_{V'} \leq C|N|_{\mathbb{L}^2(\Gamma_3)}. \tag{64}$$

From (25) and the monotonicity of $\dot{J}'_\varepsilon$, it follows from the classical first order evolution equation that

$$\forall \varepsilon > 0, \quad v^\varepsilon_{N\eta} \in \mathbb{L}^2(0,T,V) \cap C(0,T,H) \text{ and } \dot{v}^\varepsilon_{N\eta} \in \mathbb{L}^2(0,T,V')$$

such that

$$\left\{ \begin{array}{l} \dot{v}^\varepsilon_{N\eta}(t) + Av^\varepsilon_{N\eta} + j'_\varepsilon(N, v^\varepsilon_{N\eta}) = f_\eta(t) \text{ in } V', \text{ a.e. } t \in [0,T], \\ v^\varepsilon_{N\eta}(0) = v_0. \end{array} \right. \tag{65}$$

Therefore, $v^\varepsilon_{N\eta} \in \mathbb{L}^2(0,T;V) \cap W^{1,2}(0,T;V')$, we obtain

$$\left\{ \begin{array}{l} \left(\dot{v}^\varepsilon_{N\eta}(t), w - v^\varepsilon_{N\eta}\right)_{V'\times V} + \left(Av^\varepsilon_{N\eta}(t), w - v^\varepsilon_{N\eta}\right)_{V'\times V} + j_\varepsilon(N, w) \\ -j_\varepsilon(N, v^\varepsilon_{N\eta}(t)) \geq \left(f_\eta(t), w - v^\varepsilon_{N\eta}(t)\right)_{V'\times V} \forall w \in V, \text{ a.e. } t \in [0,T]. \end{array} \right. \tag{66}$$

Using (25) and the monotony of j'_ε, we deduce that

$$\exists C > 0,\ \forall t \in [0,T] : \left| v^\varepsilon_{N\eta}(t) \right| \leq C \int_0^T \left| v^\varepsilon_{N\eta}(t) \right|^2_V dt \leq C \int_0^T \left| \dot{v}^\varepsilon_{N\eta}(t) \right|^2_{V'} dt \leq C,$$

using a subsequence to find that

$$\begin{cases} v^{\varepsilon}_{N\eta} \rightharpoonup v_{N\eta} \text{ weakly in } \mathbb{L}^2(0,T;V) \text{ and weakly in } \mathbb{L}^{\infty}(0,T;H), \\ \dot{v}^{\varepsilon}_{N\eta} \rightharpoonup \dot{v}_{N\eta} \text{ star weakly in } \mathbb{L}^2(0,T;V'). \end{cases} \tag{67}$$

It follows that

$$v_{N\eta} \in C(0,T;H) \text{ and } v^{\varepsilon}_{N\eta}(t) \rightharpoonup v_{N\eta}(t) \text{ weakly in } H, \ \forall t \in [0,T]. \tag{68}$$

Integrating (66), we have $\forall w \in \mathbb{L}^2(0,T;V)$,

$$\int_0^T (\dot{v}^{\varepsilon}_{N\eta}(t), w)_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{N\eta}(t), w)_{V'\times V} dt + \int_0^T j_{\varepsilon}(N,w) dt \geq \int_0^T (f_\eta(t), w(t))_{V'\times V} dt,$$

then we have

$$\begin{array}{l} \int_0^T (\dot{v}^{\varepsilon}_{N\eta}(t), w)_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{N\eta}(t), w)_{V'\times V} dt + \int_0^T j_{\varepsilon}(N,w) dt \\ \geq \int_0^T (\dot{v}^{\varepsilon}_{N\eta}(t), v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{N\eta}(t), v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \\ \int_0^T j_{\varepsilon}(N, v^{\varepsilon}_{N\eta}(t)) dt + \int_0^T (f_\eta(t), w(t) - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt \\ \geq \frac{1}{2} \left| v^{\varepsilon}_{N\eta}(t) \right|^2_H - \frac{1}{2} \left| v^{\varepsilon}_{N\eta}(0) \right|^2_H + \int_0^T (Av^{\varepsilon}_{N\eta}(t), v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \\ \int_0^T j_{\varepsilon}(v^{\varepsilon}_{N\eta}(t)) dt + \int_0^T (f_\eta(t), w(t) - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt. \end{array}$$

From (67) and (68) we obtain that for all $w \in \mathbb{L}^2(0,T;V)$,

$$\begin{array}{l} \int_0^T (\dot{v}^{\varepsilon}_{N\eta}(t), w - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \int_0^T (Av^{\varepsilon}_{N\eta}(t), w - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt + \\ \int_0^T (j(N,w) - j(N, v_{N\eta}))\, dt \geq \int_0^T (f_\eta(t), w(t) - v^{\varepsilon}_{N\eta}(t))_{V'\times V} dt. \end{array}$$

The previous inequality implies (see [10]) that

$$\begin{array}{l} \left(\dot{v}^{\varepsilon}_{N\eta}(t), w - v^{\varepsilon}_{N\eta}\right)_{V'\times V} + \left(Av^{\varepsilon}_{N\eta}(t), w - v^{\varepsilon}_{N\eta}\right)_{V'\times V} + j_{\varepsilon}(N,w) \\ -j_{\varepsilon}(N, v^{\varepsilon}_{N\eta}(t)) \geq \left(f_\eta(t), w - v^{\varepsilon}_{N\eta}(t)\right)_{V'\times V} \forall w \in V, \ t \in [0,T]. \end{array}$$

We conclude that **Problem** $\mathcal{PV}v_\eta$ has at least a solution $v_{N\eta} \in C(0,T;H) \cap \mathbb{L}^2(0,T;V) \cap W^{1,2}(0,T;V')$ and $\dot{v}_{N\eta} \in \mathbb{L}^2(0,T;V')$. For the uniqueness, let $v^1_{N\eta}$, $v^2_{N\eta}$ be two solutions of **Problem** $\mathcal{PV}v_\eta$, we use (62) to obtain for a.e. $t \in [0,T]$,

$$(\dot{v}^2_{N\eta}(t) - \dot{v}^1_{N\eta}(t), v^2_{N\eta}(t) - v^1_{N\eta}(t)) - (Av^2_{N\eta}(t) - Av^1_{N\eta}(t), v^2_{N\eta}(t) - v^1_{N\eta}(t)) \leq 0.$$

Integrating the previous inequality, using (25) and (61), we find

$$\frac{1}{2}\left|v^2_{N\eta}(t) - v^1_{N\eta}(0)\right|^2_V + m_{\mathcal{A}} \int_0^T \left|v^2_{N\eta}(s) - v^1_{N\eta}(s)\right|^2_V ds \leq 0, \quad \forall t \in [0,T],$$

which implies $v^1_{N\eta} = v^2_{N\eta}$. Let us consider now $u_{N\eta} : [0,T] \to V$ is the function defined by

$$u_{N\eta}(t) = \int_0^T v_{N\eta}(s) ds + u_0, \quad \forall t \in [0,T]. \tag{69}$$

In the study of **Problem** $\mathcal{PV}u_\eta$, we have the following result.

Lemma 4.3 **Problem** $\mathcal{PV}u_\eta$ *has a unique solution satisfying the regularity expressed in* (56).

Proof. The proof of Lemma 4.3 is a consequence of Lemma 4.2 together with (69). In the second step, let $\eta \in C(0,T;V)$, we use the displacement field u_η obtained in Lemma 4.1 and consider the following variational problem.
Problem $\mathcal{PV}\varphi_\eta$. Find an electrical potential $\varphi_\eta : [0;T] \to W$ such that $\forall t \in [0,T]$,

$$(B\ \nabla\varphi_\eta\,(t)\,, \nabla_\phi) - (\mathcal{E}\ \varepsilon\,(u_\eta\,(t))\,, \nabla\phi)_H + (\gamma\,(u_\eta\,(t)\,, \varphi_\eta\,(t))\,, \phi)_W = (q_e\,(t)\,, \phi)\,w.\ \forall\ \phi \in W. \tag{70}$$

We have the following result.

Lemma 4.4 *$\mathcal{PV}\varphi_\eta$ has a unique solution φ_η which satisfies the regularity expressed in (57). Moreover, if φ_{η_1} and φ_{η_2} are solutions of (70) corresponding to $\eta_1, \eta_2 \in \mathbb{L}^2(0,T;V)$, then there exists $C > 0$ such that*

$$|\varphi_{\eta_1}(t) - \varphi_{\eta_2}(t)|_W \leq C\,|u_{\eta_1}\,(t) - u_{\eta_2}\,(t)|_V\,, \forall t \in [0,T]\,. \tag{71}$$

Proof. The same result for this Lemma 4.4 is given in [12]. In the third step, we let $\lambda \in \mathbb{L}^2\left(0,T;\mathbb{L}^2(\Omega)\right)$ be given and consider the following variational problem for the temperature field.
Problem $\mathcal{PV}\theta_\lambda$. Find a temperature field $\theta_\lambda : [0,T] \longrightarrow E$ such that

$$\begin{cases} \dot{\theta}_\lambda(t) + K\theta_\lambda(t) = R\dot{u}_\eta(t) + Q(t) \text{ in } E' \text{ a.e. } t \in [0,T], \\ \theta_\lambda(0) = \theta_0, \end{cases} \tag{72}$$

for all $\theta_\lambda, w \in E,$ a.e. $t \in (0,T)$. For the Problem $\mathcal{PV}\theta_\lambda$ we have the following result.

Lemma 4.5 *$\mathcal{PV}\theta_\lambda$ has a unique solution such that*

$$\theta_\lambda \in L^2(0,T;E) \cap \mathcal{C}(0,T;L^2(\Omega)) \cap W^{1,2}(0,T;E'). \tag{73}$$

Moreover, $\exists C > 0$ such that $\forall \lambda_1, \lambda_2 \in L^2(0,T;V')$,

$$\|\theta_1(t) - \theta_2(t)\|^2_{L^2(\Omega)} \leq C \int_0^T \|\lambda_1(s) - \lambda_2(s)\|^2_{E'}\,ds, \quad \forall t \in [0,T]. \tag{74}$$

Proof. The result follows from the classical first order evolution equation given in [3]. Here the Gelfand triple is given by

$$E \subset L^2\,(\Omega) = (L^2\,(\Omega))' \subset E'.$$

The operator K is linear continuous and coercive. By Korn's inequality, we have

$$|K(u)|_{\mathcal{H}} \geq C|u|_{H_1},\ \text{for all } u \in V,$$

with C being a strictly positive constant defined only on Ω and Γ_1. Therefore

$$(K\tau, \tau)_{E' \times E} \geq C|\tau|^2_E. \tag{75}$$

In the fourth step, we let $\mu \in \mathbb{L}^2(0,T;L^2(\Omega))$ be given and consider the following variational problem for the damage field.
Problem $\mathcal{PV}\alpha_\mu$. Find the damage field $\alpha_\mu : [0,T] \longrightarrow H^1(\Omega)$ such that $\alpha_\mu \in F$ and

$$(\dot{\alpha}_\mu(t), \zeta - \alpha_\mu(t))_{\mathbb{L}^2(\Omega)} + a\,(\alpha_\mu(t), \zeta - \alpha_\mu(t))\ \geq (\mathcal{S}(\varepsilon(u_\mu(t)), \alpha_\mu(t)), \zeta - \alpha_\mu(t))_{\mathbb{L}^2(\Omega)}\,, \tag{76}$$

$$\alpha_\mu(0) = \alpha_0, \tag{77}$$

for all $\alpha(t) \in F$, $\zeta \in F$ and $t \in [0,T]$. Note that if $f \in H$, then

$$(f,v)_{V' \times V} = (f,v)_H\,, \quad \forall v \in H.$$

Theorem 4.2 *Let $V \subset H \subset V'$ be a Gelfand triple. Let K be a nonempty, closed and convex set of V. Assume that $a(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$ is a continuous and symmetric form such that for some constants $\zeta > 0$ and c_0,*

$$a(v,v) = c_0\|v\|_H^2 \geq \zeta\|v\|_V^2, \quad \forall v \in H.$$

Then, for every $u_0 \in K$ and $f \in \mathbb{L}^2(0,T;H)$, there exists a unique function $u \in H^1(0,T;H) \cap \mathbb{L}^2(0,T,V)$ such that $u(0) = u_0$, $u(t) \in K$ for all $t \in [0,T]$ and for almost all $t \in [0,T]$,

$$(\dot{u}(t), v - u(t))_{V' \times V} + a(u(t), v - u(t)) \geq (f(t), v - u(t))_H, \quad \forall v \in K.$$

We apply this theorem to **Problem** $\mathcal{PV}\alpha_\mu$.

Lemma 4.6 *There exists a unique solution α_μ to the auxiliary problem $\mathcal{PV}\alpha_\mu$ such that*

$$\alpha_\mu \in W^{1,2}(0,T;\mathbb{L}^2(\Omega)) \cap \mathbb{L}^2(0,T;H^1(\Omega)). \tag{78}$$

The above lemma follows from the standard result for parabolic variational inequalities.

Proof. The inclusion mapping of $\left(H^1(\Omega), \|\cdot\|_{H^1(\Omega)}\right)$ into $(\mathbb{L}^2(\Omega), \|\cdot\|_{\mathbb{L}^2(\Omega)})$ is continuous and its range is dense. We denote by $\left(H^1(\Omega)\right)'$ the dual space of $H^1(\Omega)$ and, identifying the dual of $\mathbb{L}^2\Omega$ with itself, we can write the Gelfand triple

$$H^1(\Omega) \subset \mathbb{L}^2(\Omega) \subset \left(H^1(\Omega)\right)'.$$

We use the notation $(\cdot,\cdot)_{(H^1(\Omega))' \times H^1(\Omega)}$ to represent the duality pairing between $\left(H^1(\Omega)\right)'$ and $H^1(\Omega)$, we have

$$(\alpha,\beta)_{(H^1(\Omega))' \times H^1(\Omega)} = (\alpha,\beta)_{\mathbb{L}^2(\Omega)}, \quad \forall \alpha \in \mathbb{L}^2(\Omega),\ \beta \in H^1(\Omega),$$

and we note that F is a closed convex set in $H^1(\Omega)$. Then we use the definition of the bilinear form a given by (51), and the fact that $\alpha_\mu \in F$.

Problem $\mathcal{PV}\sigma_{\eta,\lambda,\mu}$. Find a stress field $\sigma_{\eta\lambda\mu} : [0,T] \longrightarrow \mathcal{H}$,

$$\sigma_{\eta\lambda\mu}(t) = \mathcal{B}(\varepsilon\left(u_\eta(t)\right), \alpha_\mu(t)\,(v))_{\mathcal{H}} + \int_0^t \mathcal{G}\left(\sigma(s), \varepsilon\left(u_\eta\left(s\right)\right)\right) ds - \mathcal{M}\theta_\lambda(t),\ \forall t \in [0,T]. \tag{79}$$

In the study of problem $\mathcal{PV}\sigma_{\eta\lambda\mu}$, we have the following result.

Lemma 4.7 *There exists a unique solution of problem $\mathcal{PV}\sigma_{\eta\lambda\mu}$, which satisfies* (56). *Moreover, if u_{η_i}, θ_{λ_i}, α_{μ_i} and $\sigma_{\eta_i,\lambda_i,\mu_i}$ represent the solution of problems $\mathcal{PV}u_{\eta_i}$, $\mathcal{PV}\theta_{\lambda_i}$, $\mathcal{PV}\alpha_{\mu_i}$ and $\mathcal{PV}\sigma_{\eta_i,\lambda_i,\mu_i}$, respectively, for $i = 1,2$, then there exists $c > 0$ such that*

$$\begin{aligned} &\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}^2} \leq C(\|u_{\eta_1(t) - u_{\eta_2}(t)}\|_V^2 \\ &\quad + \int_0^t (\|u_{\eta_1(s) - u_{\eta_2}(s)}\|_V^2 + \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|_V^2 + \|\alpha_{\mu_1}(s) - \alpha_{\mu_2}(s)\|_V^2)\, ds). \end{aligned} \tag{80}$$

Proof. Let $\Pi_{\eta,\lambda;\mu} : \mathbb{L}^2(0,T,\mathcal{H}) \longrightarrow \mathbb{L}^2(0,T;\mathcal{H})$ be the mapping given by

$$\Pi_{\eta,\lambda,\mu}\sigma(t) = \mathcal{B}(\varepsilon(u_\eta(t)),\alpha_\mu(t)) + \int_0^t \mathcal{G}(\sigma(s),\varepsilon(u_\eta(s)))\,ds - \mathcal{M}\theta_\lambda(t). \tag{81}$$

Let $\sigma_i \in \mathbb{L}^2(0,T;\mathcal{H}) : i = 1,2$ and $t_1 \in [0,T]$. Using hypothesis (27) and Hölder's inequality, we find

$$\|\Pi_{\eta,\lambda,\mu}\sigma_1(t) - \Pi_{\eta,\lambda,\mu}\sigma_2(t)\|^2_{\mathcal{H}} \leq L^2_{\mathcal{G}} T \int_0^t \|\sigma_1(s) - \sigma_2(s)\|^2_{\mathcal{H}}\,ds.$$

It follows from this inequality that for m large enough, a power $\Pi^m_{\eta,\lambda,\mu}$ of the mapping $\Pi_{\eta,\lambda,\mu}$ is a contraction of the Banach space $\mathbb{L}^2(0,T;\mathcal{H})$, and therefore there exists a unique element $\sigma_{\eta,\lambda,\mu} \in \mathbb{L}^2(0,T;\mathcal{H})$ such that $\Pi_{\eta,\lambda,\mu}\sigma_{\eta,\lambda,\mu} = \sigma_{\eta,\lambda,\mu}$. Moreover, $\sigma_{\eta,\lambda,\mu}$ is the unique solution of the problem $\mathcal{PV}\sigma_{\eta\lambda\mu}$. If u_{η_i}, θ_{λ_i} α_{μ_i} and $\sigma_{\eta_i,\lambda_i,\mu_i}$ represent the solution of the problems $\mathcal{PV}u_{\eta_i}$, $\mathcal{PV}\theta_{\lambda_i}$, $\mathcal{PV}\alpha_{\mu_i}$ and $\mathcal{PV}\sigma_{\eta_i\lambda_i\mu_i}$, respectively, for $i = 1,2$, then we use (4), (25), (26) and Young's inequality to obtain

$$\begin{aligned}
&\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}^2} \leq C(\|u_{\eta_1(t)-u_{\eta_2}(t)}\|^2_V \\
&+ \int_0^t (\|\sigma_{\eta_1,\lambda_1,\mu_1}(t) - \sigma_{\eta_2,\lambda_2,\mu_2}(t)\|_{\mathcal{H}^2} + \|u_{\eta_1(s)-u_{\eta_2}(s)}\|^2_V + \|\theta_{\lambda_1}(s) - \theta_{\lambda_2}(s)\|^2_V \\
&+ \|\alpha_{\mu_1}(s) - \alpha_{\mu_2}(s)\|^2_V)\,ds).
\end{aligned}$$

This permits us to obtain, using Gronwall's lemma, the inequality (80). Finally, we consider the operator Λ such that

$$\Lambda(\eta,\lambda,\mu)(t) = (\Lambda^1(\eta,\lambda,\mu)(t), \Lambda^2(\eta,\lambda,\mu)(t), \Lambda^3(\eta,\lambda,\mu)(t)), \tag{82}$$

where Λ^1, Λ^2 and Λ^3 are defined by

$$\begin{aligned}
&(\Lambda^1(\eta(t),\lambda(t),\mu(t),v(t))_{V'\times V} = \mathcal{B}(\varepsilon\left(u_\eta(t)\right),\varepsilon\left(v(t)\right))_{\mathcal{H}} + (\mathcal{E}^*\nabla\varphi_\eta\left(t\right),\varepsilon\left(v(t)\right))_{\mathcal{H}} \\
&+ \dot{J}_\varepsilon(u_\eta(t),v(t)) + \left(\int_0^t \mathcal{G}\left(\sigma_{\eta,\lambda,\mu}(s),\varepsilon\left(u_\eta(s)\right)\right)\,ds - \mathcal{M}\theta_\lambda(t),\varepsilon(v(t))\right)_{\mathcal{H}}, \quad \forall v \in V,
\end{aligned} \tag{83}$$

$$\Lambda^2(\eta(t),\lambda(t),\mu(t),v(t) = \Psi(\sigma_{\eta,\lambda,\mu}(t),\varepsilon(u_\eta(t)),\theta_\lambda(t))) \tag{84}$$

and

$$\Lambda^3(\eta(t),\lambda(t),\mu(t),v(t)) = \Phi(\sigma_{\eta,\lambda,\mu}(t),\varepsilon(u_\eta(t)),\alpha_\mu(t))). \tag{85}$$

Here, for $\eta \in \mathbb{L}^2(0,T;V)$, $\lambda \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Omega))$ and $\mu \in \mathbb{L}^2(0,T;\mathbb{L}^2(\Omega))$, u_η, ϕ_η, θ_λ, α_μ and $\sigma_{\eta,\lambda,\mu}$ represent the displacement field, the potential electric field, the temperature, the damage field and the stress field obtained in Lemmas 4.1, 4.4, 4.5, 4.6 and 4.7. We have the following result.

Lemma 4.8 *The operator Λ has a unique fixed point $(\eta^*,\lambda^*,\mu^*) \in \mathbb{L}^2(0,T;V \times \mathbb{L}^2(\Omega)) \times \mathbb{L}^2(\Omega))$.*

Proof. We show for a positive integer m, the mapping Λ^m is a contraction on $\mathbb{L}^2(0,T;V \times \mathbb{L}^2(\Omega)) \times \mathbb{L}^2(\Omega))$. To this end, we suppose that (η_1,λ_1,μ_1) and (η_2,λ_2,μ_2)

are two functions in $\mathbb{L}^2(0,T;V\times\mathbb{L}^2(\Omega))\times\mathbb{L}^2(\Omega))$ and denote $u_{\eta_i}=u_i$, $\dot{u}_{\eta_i}=v_i$, $\varphi_{\eta_i}=\varphi_i$, $\theta_{\lambda_i}=\theta_i$, $\alpha_{\mu_i}=\alpha_i$ and $\sigma_{\eta_i,\lambda_i,\mu_i}=\sigma_i$ for $i=1,2$. We have

$$\begin{aligned}\|\Lambda^1(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^1(\eta_2,\lambda_2,\mu_2)(t)\|_{V'}^2 &\le C\|R_\nu(u_\nu(t)-u_{2\nu}(t))\|_{\mathbb{L}^2(\Gamma_3)}^2\\ &+C\|R_\tau(u_\tau(t)-u_{2\tau}(t))\|_{\mathbb{L}^2(\Gamma_3)}^2+\|\mathcal{B}\varepsilon(u_1(t))-\mathcal{B}\varepsilon(u_2(t))\|_{\mathcal{H}}^2\\ &+\|\varepsilon^*\nabla\varphi_1(t)-\varepsilon^*\nabla\varphi_2(t)\|_{\mathcal{H}}^2+C\|\alpha_1(t)-\alpha_2(t)\|_{\mathbb{L}^2(\Omega)}^2\\ &+\int_0^t\|\mathcal{G}(\sigma_1(s),\varepsilon(u_2(s)))-\mathcal{G}(\sigma_2(s),\varepsilon(u_2(s)))\|_{\mathcal{H}}^2\,ds\\ &+C\|\theta_1(t)-\theta_2(t)\|_{\mathbb{L}^2\Omega}^2.\end{aligned}\tag{86}$$

Therefore, from (26), (27), (28) and the definition of R_ν, R_τ, we obtain

$$\begin{aligned}\|\Lambda^1(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^1(\eta_2,\lambda_2,\mu_2)(t)\|_{V'}^2 &\le C(\|u_1(t)-u_2(t)\|_V^2\\ &+\int_0^t\|u_1(s)-u_2(s)\|_V^2\,ds+\int_0^t\|\sigma_1(s)-\sigma_2(s)\|_{\mathcal{H}}^2\,ds\\ &+\int_0^t\|\theta_1(s)-\theta_2(s)\|_E^2\,ds+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|_F^2\,ds\\ &+\|\varphi_1(t)-\varphi_2(t)\|_W^2).\end{aligned}\tag{87}$$

We use estimate (81) to obtain

$$\begin{aligned}\|\Lambda^1(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^1(\eta_2,\lambda_2,\mu_2)(t)\|_{V'}^2 &\le C(\|u_1(t)-u_2(t)\|_V^2\\ &+\int_0^t\|u_1(s)-u_2(s)\|_V^2\,ds+\int_0^t\|\theta_1(s)-\theta_2(s)\|_E^2\,ds\\ &+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|_F^2\,ds+\|\varphi_1(t)-\varphi_2(t)\|_W^2).\end{aligned}\tag{88}$$

Recall that above u_{η_ν} and u_{η_τ} denote the normal and the tangential component of the function u_η, respectively. By similar arguments, from the function Φ and the definition of Λ^2, it follows that

$$\begin{aligned}\|\Lambda^2(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^2(\eta_2,\lambda_2,\mu_2)(t)\|_E^2 &\le C(\|u_1(t)-u_2(t)\|_V^2\\ &+\int_0^t\|\sigma_1(s)-\sigma_2(s)\|_V^2\,ds+\|\theta_1(s)-\theta_2(s)\|_E^2)\\ &\le C(\|u_1(t)-u_2(t)\|_V^2+\|\theta_1(s)-\theta_2(s)\|_E^2\\ &+\int_0^t\|u_1(s)-u_2(s)\|_V^2\,ds+\int_0^t\|\theta_1(s)-\theta_2(s)\|_E^2\,ds).\end{aligned}\tag{89}$$

On the other hand, by (33), (80) and the definition of Λ^3, we get

$$\begin{aligned}\|\Lambda^3(\eta_1,\lambda_1,\mu_1)(t)-\Lambda^3(\eta_2,\lambda_2,\mu_2)(t)\|_F^2 &\le C(\|u_1(t)-u_2(t)\|_V^2\\ &+\|\varphi_1(t)-\varphi_2(t)\|_W^2+\int_0^t\|u_1(s)-u_2(s)\|_V^2\,ds\\ &+\|\alpha_1(s)-\alpha_2(s)\|_F^2+\int_0^t\|\alpha_1(s)-\alpha_2(s)\|_F^2\,ds).\end{aligned}\tag{90}$$

Also, since

$$u_i(t) = \int_0^t v_i(s)\,ds + u_0, \quad t \in [0,T], \tag{91}$$

we have

$$u_i(t) = \|u_1(t) - u_2(t)\|_V^2 \leq \int_0^t \|v_i(s)\|_V^2\,ds + u_0, \tag{92}$$

which implies

$$\|u_1(t) - u_2(t)\|_V^2 + \int_0^t \|u_1(s) - u_2(s)\|_V^2\,ds \leq C\int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds. \tag{93}$$

Therefore

$$\begin{aligned} \|\Lambda(\eta_1,\lambda_1,\mu_1)(t) - \Lambda(\eta_2,\lambda_2,\mu_2)(t)\|_{V'\times E\times F}^2 &\leq C(\|u_1(t) - u_2(t)\|_V^2 \\ &+ \int_0^t \|u_1(s) - u_2(s)\|_V^2\,ds + \|\alpha_1(s) - \alpha_2(s)\|_F^2 \\ &+ \int_0^t \|\alpha_1(s) - \alpha_2(s)\|_F^2\,ds + \|\theta_1(s) - \theta_2(s)\|_E^2 \\ &+ \int_0^t \|\theta_1(s) - \theta_2(s)\|_E^2\,ds + \|\varphi_1(t) - \varphi_2(t)\|_W^2). \end{aligned} \tag{94}$$

Moreover, from (60), we obtain

$$\begin{aligned} (\dot{v}_1 - \dot{v}_2, v_1 - v_2)_{V'\times V} &= (\mathcal{A}\varepsilon(v_1) - \mathcal{A}\varepsilon(v_2), \varepsilon(v_2 - v_1))_{V'\times V} \\ &\quad + (\eta_1 - \eta_2, v_1 - v_2)_{V'\times V} \leq 0. \end{aligned} \tag{95}$$

We integrate this equality with respect to time, use the initial conditions, $v_1(0) = v_2(0) = v_0$, (27) and (61) to find

$$m_{\mathcal{A}}\int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds \leq C\int_0^t \|\eta_1(s) - \eta_2(s)\|_V \|v_1(s) - v_2(s)\|_V\,ds \tag{96}$$

for all $t \in [0,T]$. Then, using the inequality $2ab \leq \frac{a^2}{m_{\mathcal{A}}} + m_{\mathcal{A}}b^2$, we obtain

$$\int_0^t \|v_1(s) - v_2(s)\|_V^2\,ds \leq C\int_0^t \|\eta_1(s) - \eta_2(s)\|_V\,ds, \forall t \in [0,T]. \tag{97}$$

Since $u_1(0) = u_2(0) = u_0$, we have

$$\|u_1(s) - u_2(s)\|_V^2 \leq C\int_0^T \|v_1(s) - v_2(s)\|_V\,ds, \tag{98}$$

and from (74), we have

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leq C\int_0^T \|\lambda_1(s) - \lambda_2(s)\|_{E'}^2 ds, \quad \forall t \in [0,T], \tag{99}$$

and from (71), we have

$$\|\varphi_1(t) - \varphi_2(t)\|_W^2 \leq C\|u_1(t) - u_2(t)\|_V^2, \quad \forall t \in [0,T]. \tag{100}$$

We substitute (93) and (100) in (94) to obtain

$$\begin{aligned}\|\Lambda(\eta_1,\lambda_1,\mu_1)(t)-\Lambda(\eta_2,\lambda_2,\mu_2)(t)\|^2_{V'\times E\times F} &\le C(\int_0^t \|v_1(s)-v_2(s)\|^2_V\,ds\\ &+\|\theta_1(s)-\theta_2(s)\|^2_E+\int_0^t \|\theta_1(s)-\theta_2(s)\|^2_E\,ds\\ &+\|\alpha_1(s)-\alpha_2(s)\|^2_F+\int_0^t \|\alpha_1(s)-\alpha_2(s)\|^2_F\,ds),\end{aligned} \tag{101}$$

on the other hand, from (76), we deduce that

$$(\dot\alpha_1-\dot\alpha_2,\alpha_1-\alpha_2)_{F'\times F}+a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\le(\mu_1-\mu_2,\alpha_1-\alpha_2)_F,\ \text{a.e. } t\in[0,T]. \tag{102}$$

Integrating the previous inequality with respect to time, using the initial conditions $\alpha_1(0)=\alpha_2(0)=\alpha_0$ and inequality $a(\alpha_1-\alpha_2,\alpha_1-\alpha_2)\ge 0$, we find

$$\frac{1}{2}\|\alpha_1(s)-\alpha_2(s)\|^2_F\le\int_0^t(\mu_1(s)-\mu_2(s),\alpha_1(s)-\alpha_2(s))_F\,ds. \tag{103}$$

This inequality, combined with Gronwall's inequality, leads to

$$\|\alpha_1(s)-\alpha_2(s)\|^2_F\le C\int_0^t\|\mu_1(s)-\mu_2(s)\|^2_F\,ds,\quad \forall t\in[0,T]. \tag{104}$$

We substitute (97), (99) and (104) in (101) to obtain

$$\begin{aligned}\|\Lambda(\eta_1,\lambda_1,\mu_1)(t)-\Lambda(\eta_2,\lambda_2,\mu_2)(t)\|^2_{V'\times E\times F}\le C\int_0^t &\|((\eta_1,\lambda_1,\mu_1)(s)\\ &-(\eta_2,\lambda_2,\mu_2)(s))\|^2_{V'\times E\times F}\,ds.\end{aligned} \tag{105}$$

Reintegrating this inequality n times, we obtain

$$\begin{aligned}\|\Lambda^n(\eta_1,\lambda_1,\mu_1)-\Lambda^n(\eta_2,\lambda_2,\mu_2)\|^2_{\mathbb{L}^2(0,T;V'\times E\times F)}\le\frac{C^nT^n}{n!}&\|((\eta_1,\lambda_1,\mu_1)\\ &-(\eta_2,\lambda_2,\mu_2))\|^2_{\mathbb{L}^2(V'\times E\times F)},\end{aligned} \tag{106}$$

thus, for n sufficiently large, Λ^n is a contraction on the Banach space $\mathbb{L}^2(0,T;V'\times E\times F)$ and so Λ has a unique fixed point. Now, we have all ingredients to prove Theorem 4.1.

Proof. (of Theorem 4.1). Let $(\eta^*,\lambda^*,\mu^*)\in\mathbb{L}^2(0,T;V'\times\mathbb{L}^2(\Omega)\times\mathbb{L}^2(\Omega))$ be the fixed point of Λ defined by (82), (83), (84) and (85) and

$$u_*=u_{\eta^*},\quad \varphi_*=\varphi_{\eta^*},\quad \theta_*=\theta_{\eta^*}\quad\text{and}\quad \alpha_*=\alpha_{\eta^*}. \tag{107}$$

Let $\sigma_*:[0,T]\longrightarrow\mathcal{H}$ be the function defined by

$$\sigma_*=\mathcal{A}\varepsilon(\dot u_*)+\varepsilon^*\nabla\varphi_*+\sigma_{\eta^*,\lambda^*,\mu^*}. \tag{108}$$

We prove that $\{u_*,\sigma_*,\varphi_*,\theta_*,\alpha_*\}$ satisfies (42), (48) and the regularities (56)-(58). Indeed, we write (60) and use (107) to find

$$\begin{aligned}&(\ddot u_*(t),v)_{V'\times V}+(\mathcal{A}\varepsilon(\dot u_*(t)),\varepsilon(v))_{\mathcal{H}}+J_\varepsilon(\dot u_*(t),v)\\ &\qquad+(\eta^*(t),v)_{V'\times V}\ge(f(t),v)_{V'\times V},\quad\forall v\in V\ \text{a.e.},\ t\in 0,T,\end{aligned} \tag{109}$$

we use equalities $\Lambda^1(\eta^*,\lambda^*,\mu^*)=\mu^*$, $\Lambda^2(\eta^*,\lambda^*,\mu^*)=\lambda^*$ and $\Lambda^3(\eta^*,\lambda^*,\mu^*)=\eta^*$, it follows that

$$\begin{aligned}(\eta_*(t),v)_{V'\times V} &= (\mathcal{B}\varepsilon(u_*(t)),\varepsilon(v))_{\mathcal{H}}+(\varepsilon^*\nabla\varphi_*(t),\varepsilon(v))_{\mathcal{H}}\\ &+\left(\int_0^t \mathcal{G}(\sigma_{\eta^*,\lambda^*,\mu^*}(s),\varepsilon(u_*(s),\alpha_*(s)))\,ds-\mathcal{M}\theta_*(t),\varepsilon(v)\right)_{\mathcal{H}}\\ &+J_\varepsilon(u_*(t),v(t)),\end{aligned}\tag{110}$$

$$\lambda_*(t)=\Phi(\sigma_{\eta^*,\lambda^*,\mu^*}(t),\varepsilon(u_*(t)),\theta_*(t)),\tag{111}$$

$$\mu_*(t)=\Psi(\sigma_{\eta^*,\lambda^*,\mu^*}(t),\varepsilon(u_*(t)),\alpha_*(t)).\tag{112}$$

We now substitute (110) in (109) to obtain

$$\begin{aligned}&(\ddot{u}_*(t),v)_{V'\times V}+(\mathcal{A}\varepsilon(\dot{u}_*(t),\varepsilon(v))_{\mathcal{H}}+(\mathcal{B}\varepsilon(u_*)(t),\varepsilon(v),\alpha_*(t))_{\mathcal{H}}+(\varepsilon^*\nabla\varphi_*,\varepsilon(v))_{\mathcal{H}}\\ &+\left(\int_0^t \mathcal{G}(\sigma_{\eta^*,\lambda^*,\mu^*}(s),\varepsilon(u_*(s)))\,ds-\mathcal{M}\theta_*(t),\varepsilon(v)\right)_{\mathcal{H}}\\ &+J_\varepsilon(u_*(t),v)\geq(f(t),\dot{v})_{V'\times V},\quad\forall v\in V.\end{aligned}\tag{113}$$

It follows from Lemma 4.7 and (108) that $\sigma_*\in\mathbb{L}^2(0,T;\mathcal{H})$ and (43) implies that

$$\operatorname{div}\sigma_*+f_0(t)=\rho\ddot{u}_*(t),\ \text{a.e.},\ t\in[0,T].$$

We write (72) for $\lambda=\lambda^*$ to find that (74) is satisfied, also write (76) for $\mu=\mu^*$ to find that (76) is satisfied , we consider now (60) for $\eta=\eta^*$ to find that (60) is satisfied. Next, the regularities (56)-(59) follow from Lemmas 4.1, 4.2, 4.4, 4.5, 4.6 and the regularity (56) follows from Lemma 4.7, the uniqueness part of Theorem 4.1 is a consequence of the uniqueness of the fixed point of the operator Λ defined by (82)-(85) and thus follows the unique solvability of the problems $\mathcal{PV}u_\eta$, $\mathcal{PV}\varphi_\eta$, $\mathcal{PV}\theta_\lambda$, $\mathcal{PV}\alpha_\mu$ and $\mathcal{PV}\sigma_{\eta,\lambda,\mu}$, which completes the proof.

5 Conclusion

As a conclusion, we can say that our model, which describes the contact problem with damage and thermal effect for an electro-elasto-viscoplastic problem, based on thermodynamics is developed to describe the self-heating and stress-strain behavior of thermoplastic polymers under tensile loading. The constitutive model considers temperature-dependent elasticity, nonlinear viscoplastic flow and damage evolution. The literature devoted to various aspects of the subject is considerable, it concerns the modelling and the mathematical analysis of the related problems. For example, many food materials used in process engineering are elastic-viscoplastic, mathematical models can be very helpful in understanding various problems related to the product development, packing, transport, shelf life testing, thermal effects, and heat transfer. It is thus important to study mathematical models that can be used to describe the dynamical behavior of a given elastic-viscoplastic material subjected to various highly nonlinear and even non-smooth phenomena like contact, friction, lubrication, adhesion, wear, damage, electrical and thermal effects. Thermal effects in contact processes affect the composition and stiffness of the contacting surfaces, and cause thermal stresses in the contacting bodies. Moreover, the contacting surfaces exchange heat and energy is lost to the surroundings.

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Nonlinear Dynamics and Systems Theory, **22** (5) (2022) 522–537

A Dynamic Contact Problem between Viscoelastic Piezoelectric Bodies with Friction and Damage

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Received: November 3, 2021;   Revised: November 7, 2022

Abstract: We consider a dynamic contact problem between two thermo-electro-viscoelastic bodies with damage and an internal state variable. The contact is bilateral and is modeled by Tresca's friction law. The damage of the materials is caused by elastic deformations. We derive a variational formulation for the model which is in the form of a system involving the displacement field, the electric potential, the internal state variable field, the temperature and the damage. Then we proved the existence of a unique weak solution to the model.

Keywords: *viscoelastic piezoelectric materials; internal state variable; damage; temperature; friction contact.*

Mathematics Subject Classification (2010): 74M10, 70K30, 70K75, 93-02.

1 Introduction

Our research paper tackles a frictional bilateral contact problem including the topic of piezoelectric, which can be explained as follows: when we apply mechanical pressure to some types of crystalline materials such as ceramics $BaTiO_3, BiFeO3$, a voltage proportional to the pressure is produced. Meanwhile, changes in shape and dimension occur if an electric field is applied to some types of crystalline materials. At present, there is a great interest in the study of piezoelectric materials for their importance in radio-electronics, electroacoustics and instrumentation. Thus, a big interest in the contact problems occurs because of the fact that parts of the equipment are in contact. So, many models have been developed to explain the interaction between the electrical and mechanical fields, see for example [2, 8] and the references therein. Frictional contact problem is a static problem of electro-elastic materials mentioned in [3] and [10], considering that the basis is

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isolated. Contact problems involving elasto-piezoelectric materials [3], viscoelastic piezoelectric materials [1] and the contact problem for electro-elastio-viscoplastic materials were studied in [7].

A mathematical investigation has been conducted for some models taking into consideration the influence of the internal damage of the material in the contact process. From the virtual power principle, general models for damage were derived in [6]. In [4], we can find the modes of mechanical damage which are derived from thermo-dynamical consideration. The ratio between the elastic moduli of the damage and damage-free materials is expressed by the function called the damage function $\zeta^\kappa = \zeta^\kappa(x,t)$ mentioned in [5,6]. In an isotropic and homogeneous elastic material, let E_Y^κ be the Young modulus of the original material and E_{eff}^κ be the current modulus, then the damage function is defined by $\zeta^\kappa = E_{eff}^\kappa / E_Y^\kappa$. This definition shows that the damage function ζ^κ is restricted to have values between zero and one; when $\zeta^\kappa = 1$, there is no damage in the material, when $\zeta^\kappa = 0$, the material is completely damaged, when $0 < \zeta^\kappa < 1$, there is partial damage and the system has a reduced load carrying capacity. The contact problem with damage has been mentioned in [9]. The differential inclusion used for the evolution of the damage field is

$$\dot{\zeta^\kappa} - \Delta\zeta^\kappa + \boldsymbol{k}^\kappa \partial\chi \boldsymbol{k}^\kappa(\zeta^\kappa) \ni \boldsymbol{S}^\kappa(\epsilon(\boldsymbol{u}^\kappa), \zeta^\kappa) \ in\ \Omega^\kappa \times [0,T], \tag{1.1}$$

where $\boldsymbol{k}^\kappa$ is a positive coefficient and $\boldsymbol{K}^\kappa$ is the set of admissible damages defined by

$$\boldsymbol{K}^\kappa = \{\zeta \in H^1(\Omega^\kappa); 0 \leq \zeta \leq 1.\ a.e \in \Omega^\kappa\}. \tag{1.2}$$

The paper is structured as follows. In Section 2, we present the physical setting and describe the mechanical problem. We derive a variational formulation, list the assumptions on the data, and give the variational formulation of the problem. In Section 3, we state our main existence and uniqueness result which is based on the classical result of non-linear first order evolution inequalities and equations with monotone operators and the fixed point arguments.

2 Problem Statement and Variational Formulation

The physical setting is the following. Let us consider two electro-thermovisco-elastic bodies, occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d$ ($d = 2,3$ in applications). We put a superscript κ to indicate that the quantity is related to the domain Ω^κ. In the following, the superscript κ ranges between 1 and 2. For each domain Ω^κ, the boundary Γ^κ is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^κ, Γ_2^κ and Γ_3^κ, on one hand, and in two measurable parts Γ_a^κ and Γ_b^κ, on the other hand, such that $meas\Gamma_1^\kappa > 0$, $meas\Gamma_a^\kappa > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The Ω^κ body is subject to f_0^κ forces and volume electric charges of density q_0^κ. The bodies are assumed to be clamped on $\Gamma_1^\kappa \times [0,T]$. The surface tractions f_2^κ act on $\Gamma_2^\kappa \times [0,T]$. We also assume that the electrical potential vanishes on $\Gamma_a^\kappa \times [0,T]$ and a surface electric charge of density $q_2^\kappa \times [0,T]$ is prescribed on $\Gamma_b^\kappa \times [0,T]$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$. The classical form of the bilateral contact with Tresca's friction and damage between two electro-thermovisco-elastic bodies with damage and an internal state variable is the following.

Problem P. For $\kappa = 1,2$, find a displacement field $u^\kappa : \Omega^\kappa \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma^\kappa : \Omega^\kappa \times [0,T] \to \mathbb{S}^d$, an electric potential $\psi^\kappa : \Omega^\kappa \times [0,T] \to \mathbb{R}$, an electric displacement field $D^\kappa : \Omega^\kappa \times [0,T] \longrightarrow \mathbb{R}^d$, a temperature $\tau^\kappa : \Omega^\kappa \times [0,T] \longrightarrow \mathbb{R}$, a damage $\alpha^\kappa : \Omega^\kappa \times [0,T] \to \mathbb{R}$ and an internal state variable field $\beta^\kappa : \Omega^\kappa \times [0,T] \longrightarrow \mathbb{R}^m$ such that for all $t \in (0,T)$, we have

$$\begin{aligned}
&\sigma^\kappa(t) = \mathcal{A}^\kappa \varepsilon(\dot{u}^\kappa(t)) + \mathcal{B}^\kappa(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)) - (\mathcal{E}^\kappa)^* E(\psi^\kappa(t)) \\
&+\mathcal{F}^\kappa(\beta^\kappa(t), \tau^\kappa(t)) \quad \text{in } \Omega^\kappa,
\end{aligned} \tag{2.1}$$

$$D^\kappa(t) = \mathcal{E}^\kappa \varepsilon(u^\kappa(t)) + \mathcal{R}^\kappa E(\psi^\kappa(t)) + \mathcal{G}^\kappa(\beta^\kappa(t), \tau^\kappa(t)) \quad \text{in } \Omega^\kappa, \tag{2.2}$$

$$\dot{\beta}^\kappa(t) = \Theta^\kappa\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t), \beta^\kappa(t), \tau^\kappa(t)\big) \quad \text{in } \Omega^\kappa, \tag{2.3}$$

$$\dot{\tau}^\kappa(t) - \mathcal{K}_0^\kappa \Delta \tau^\kappa(t) = \Psi^\kappa\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t), \beta^\kappa(t), \tau^\kappa(t)\big) + \chi^\kappa(t) \quad \text{in } \Omega^\kappa, \tag{2.4}$$

$$\dot{\alpha}^\kappa(t) - \mathcal{K}_1^\kappa \Delta \alpha^\kappa(t) + \partial \mathbb{I}_{\mathcal{Z}^\kappa}(\alpha^\kappa(t)) \ni S^\kappa(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)) \quad \text{in } \Omega^\kappa, \tag{2.5}$$

$$\operatorname{Div} \sigma^\kappa(t) + f_0^\kappa(t) = \rho^\kappa \ddot{u}^\kappa(t) \quad \text{in } \Omega^\kappa, \tag{2.6}$$

$$\operatorname{div} D^\kappa(t) = q_0^\kappa(t) \quad \text{in } \Omega^\kappa, \tag{2.7}$$

$$u^\kappa(t) = 0 \quad \text{on } \Gamma_1^\kappa, \tag{2.8}$$

$$\sigma^\kappa(t)\nu^\kappa = f_2^\kappa(t) \quad \text{on } \Gamma_2^\kappa, \tag{2.9}$$

$$\left\{\begin{array}{l}
u_\nu^1(t) + u_\nu^2(t) = 0, \quad \sigma_\tau^1(t) = -\sigma_\tau^2(t) \equiv \sigma_\tau(t), \quad |\sigma_\tau(t)| \le g, \\
|\sigma_\tau(t)| < g \Rightarrow \dot{u}_\tau^1(t) - \dot{u}_\tau^2(t) = 0 \\
|\sigma_\tau(t)| = g \Rightarrow \exists \lambda \ge 0 \text{ such that } \sigma_\tau(t) = -\lambda\big(\dot{u}_\tau^1(t) - \dot{u}_\tau^2(t)\big),
\end{array}\right. \quad \text{on } \Gamma_3, \tag{2.10}$$

$$\frac{\partial \alpha^\kappa(t)}{\partial \nu^\kappa} = 0 \quad \text{on } \Gamma^\kappa, \tag{2.11}$$

$$\mathcal{K}_0^\kappa \frac{\partial^\kappa \tau^\kappa(t)}{\partial \nu^\kappa} + \lambda_0^\kappa \tau^\kappa(t) = 0 \quad \text{on } \Gamma^\kappa, \tag{2.12}$$

$$\psi^\kappa(t) = 0 \quad \text{on } \Gamma_a^\kappa, \tag{2.13}$$

$$D^\kappa(t) \cdot \nu^\kappa = q_2^\kappa(t) \quad \text{on } \Gamma_b^\kappa, \tag{2.14}$$

$$u^\kappa(0) = u_0^\kappa,\ \dot{u}^\kappa(0) = v_0^\kappa,\ \alpha^\kappa(0) = \alpha_0^\kappa,\ \beta^\kappa(0) = \beta_0^\kappa,\ \tau^\kappa(0) = \tau_0^\kappa \quad \text{in } \Omega^\kappa. \tag{2.15}$$

First, equations (2.1)–(2.3) represent the electro-thermovisco-elastic constitutive law with damage and an internal state variable. The evolution of the damage field is governed by the inclusion given by the relation (2.5). Equation (2.4) represents the conservation of energy, where Ψ^κ is a nonlinear constitutive function which represents the heat generated by the work of internal forces and χ^κ is a given volume heat source. Next, equations (2.6) and (2.7) are the equations of motion written for the stress field and of balance written for the electric displacement field, respectively, in which Div and div denote the divergence operators for tensor and vector valued functions. Conditions (2.8) and (2.9) are the displacement and traction boundary conditions, respectively. Boundary conditions (2.11), (2.12) represent, respectively, on Γ^α, a homogeneous Neumann boundary condition for the damage field and a Fourier boundary condition for the temperature, (2.13) and (2.14) represent the electric boundary conditions, and (2.15) are the initial conditions. Conditions (2.10) represent the bilateral contact condition with Tresca's friction, where $[u_\nu] = u_\nu^1 + u_\nu^2$ and $[u_\tau] = u_\tau^1 - u_\tau^2$.

Now, to proceed with the variational formulation, we need the following function spaces:

$\mathbb{H}^\kappa = L^2(\Omega^\kappa)^d = \left\{u = (u_i)_{1\le i\le d};\ u_i \in L^2(\Omega^\kappa)\right\}$,

$\mathbb{H}_1^\kappa = W^{1,2}(\Omega^\kappa)^d = \left\{u = (u_i)_{1\le i\le d};\ u_i \in W^{1,2}(\Omega^\kappa)\right\}$,

$\mathcal{H}^\kappa = L^2(\Omega^\kappa)_s^{d\times d} = \left\{\sigma = (\sigma_{ij})_{1\le i,j\le d};\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega^\kappa)\right\}$,

$\mathcal{H}_1^\kappa = \left\{\sigma \in \mathcal{H}^\kappa;\ \operatorname{Div}\sigma \in \mathbb{H}^\kappa\right\}$,

$\mathbb{Y}^\kappa = L^2(\Omega^\kappa)^m = \left\{\beta = (\beta_i)_{1\le i\le m};\ \beta_i \in L^2(\Omega^\kappa)\right\}$,

$\mathcal{V}^\kappa = \left\{u \in W^{1,2}(\Omega^\kappa)^d;\ u = 0 \text{ on } \Gamma_1^\kappa\right\}$. These are real Hilbert spaces endowed with the

inner products $\langle u,v\rangle_{\mathbb{H}^\kappa} = \int_{\Omega^\kappa} u.vdx, \quad \forall u,v \in \mathbb{H}^\kappa, \langle \sigma,\theta\rangle_{\mathcal{H}^\kappa} = \int_{\Omega^\kappa} \sigma.\theta dx, \quad \forall \sigma,\theta \in \mathcal{H}^\kappa,$
$\langle u,v\rangle_{\mathbb{H}_1^\kappa} = \int_{\Omega^\kappa} u.vdx + \int_{\Omega^\kappa} \nabla u.\nabla vdx, \quad \forall u,v \in \mathbb{H}_1^\kappa,$
$\langle \sigma,\theta\rangle_{\mathcal{H}_1^\kappa} = \int_{\Omega^\kappa} \sigma.\theta dx + \int_{\Omega^\kappa} \operatorname{Div}\sigma.\operatorname{Div}\theta dx, \quad \forall \sigma,\theta \in \mathcal{H}^\kappa,$
$\langle \beta,k\rangle_{\mathbb{Y}^\kappa} = \int_{\Omega^\kappa} \beta.kdx, \ \forall \beta,k \in \mathbb{Y}, \langle u,v\rangle_{\mathcal{V}^\kappa} = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}^\kappa} \ \forall u,v \in \mathcal{V}^\kappa$ and the associated norms $\|.\|_{\mathbb{H}^\kappa}$, $\|.\|_{\mathcal{H}^\kappa}$, $\|.\|_{\mathbb{H}_1^\kappa}$, $\|.\|_{\mathcal{H}_1^\kappa}$, $\|.\|_{\mathbb{Y}^\kappa}$ and $\|.\|_{\mathcal{V}^\kappa}$, respectively. Here and below we use the notation

$$\nabla u = (u_{i,j}),\ \varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j}+u_{j,i}), \quad \forall u \in \mathbb{H}_1^\kappa,$$
$$\operatorname{Div}\sigma = (\sigma_{ij,j}), \quad \forall \sigma \in \mathcal{H}_1^\kappa.$$

Completeness of the space $(\mathcal{V}^\kappa, \|.\|_{\mathcal{V}^\kappa})$ follows from the assumption $meas(\Gamma_1^\kappa > 0)$, which allows the use of Korn's inequality. We denote u^κ as the trace of an element $u^\kappa \in \mathbb{H}_1^\kappa$ on Γ^κ. For every element $u^\kappa \in \mathcal{V}^\kappa$, we denote by u_ν^κ and u_τ^κ the normal and the tangential components of u on the boundary Γ^κ given by $u_\nu^\kappa = u^\kappa.\nu^\kappa$, $u_\tau^\kappa = u^\kappa - u_\nu^\kappa \nu^\kappa$. Also, for an element $\sigma^\kappa \in \mathcal{H}_1^\kappa$, we denote by $\sigma^\kappa\nu$, σ_ν^κ and σ_τ^κ the trace, the normal trace and the tangential trace of σ^κ to Γ^κ, respectively. In addition to the Sobolev trace theorem, there exists a constant $c_{tr} > 0$, depending only on Ω^κ, Γ_1^κ and Γ_3 such that

$$\|u^\kappa\|_{L^2(\Gamma_3)^d} \le c_{tr}\|u^\kappa\|_{\mathcal{V}^\kappa}, \quad \forall u^\kappa \in \mathcal{V}^\kappa. \tag{2.16}$$

Denote $E_0^\kappa = L^2(\Omega^\kappa)$, $E_1^\kappa = H^1(\Omega^\kappa)$, $\langle .,.\rangle_{E_0^\kappa} = \langle .,.\rangle_{L^2(\Omega^\kappa)}$, $\langle .,.\rangle_{E_1^\kappa} = \langle .,.\rangle_{H^1(\Omega^\kappa)}$, $\|.\|_{E_0^\kappa} = \|.\|_{L^2(\Omega^\kappa)}$ and $\|.\|_{E_1^\kappa} = \|.\|_{H^1(\Omega^\kappa)}$. For the electric unknowns ψ^κ and D^κ, we use the spaces

$$\mathbb{W}^\kappa = \{\psi^\kappa \in E_1^\kappa;\ \psi^\kappa = 0 \text{ on } \Gamma_a^\kappa\},$$
$$\mathcal{W}^\kappa = \left\{D^\kappa = (D_i^\kappa)_{1\le i\le d};\ D_i^\kappa \in L^2(\Omega^\kappa),\ \operatorname{div} D^\kappa \in L^2(\Omega^\kappa)\right\}.$$

These are real Hilbert spaces with the inner products

$$\langle \psi^\kappa,\varphi^\kappa\rangle_{\mathbb{W}^\kappa} = \int_{\Omega^\kappa} \nabla\psi^\kappa.\nabla\varphi^\kappa dx, \ \ \langle D^\kappa,E^\kappa\rangle_{\mathcal{W}^\kappa} = \int_{\Omega^\kappa} D^\kappa.E^\kappa dx + \int_{\Omega^\kappa} \operatorname{div} D^\kappa.\operatorname{div} E^\kappa dx, \tag{2.17}$$

where $\operatorname{div} D^\kappa = (D_{i,i}^\kappa)$, and the associated norms are denoted by $\|.\|_{\mathbb{W}^\kappa}$ and $\|.\|_{\mathcal{W}^\kappa}$, respectively. Completeness of the space $(\mathbb{W}^\kappa, \|.\|_{\mathbb{W}^\kappa})$ is a consequence of the assumption $meas(\Gamma_a^\kappa) > 0$ which allows the use of the Friedrichs-Poincaré inequality. When $\sigma^\kappa \in \mathcal{H}_1^\kappa$, $\tau^\kappa \in H^1(\Omega^\kappa)$ and $D^\kappa \in \mathcal{W}^\kappa$ are sufficiently regular functions, the following three Green's formulas hold

$$\langle \sigma^\kappa, \varepsilon(v^\kappa)\rangle_{\mathcal{H}^\kappa} + \langle \operatorname{Div}\sigma^\kappa, v^\kappa\rangle_{\mathbb{H}^\kappa} = \int_{\Gamma^\kappa} \sigma^\kappa\nu^\kappa.v^\kappa da, \quad \forall v^\kappa \in \mathbb{H}_1^\kappa, \tag{2.18}$$

$$\langle \Delta\tau^\kappa, \delta^\kappa\rangle_{\mathbb{H}^\kappa} + \langle \nabla\tau^\kappa, \nabla\delta^\kappa\rangle_{L^2(\Omega^\kappa)} = \int_{\Gamma^\kappa} \frac{\partial\tau^\kappa}{\partial\nu^\kappa}\delta^\kappa da, \quad \forall \delta^\kappa \in H^1(\Omega^\kappa), \tag{2.19}$$

$$(D^\kappa, \nabla\phi^\kappa)_{\mathbb{H}^\kappa} + (\operatorname{div} D^\kappa, \phi^\kappa)_{L^2(\Omega^\kappa)} = \int_{\Gamma^\kappa} D^\kappa\nu^\kappa\phi^\kappa da, \quad \forall \phi^\kappa \in H^1(\Omega^\kappa). \tag{2.20}$$

In order to simplify the notations, we define the spaces

$$\mathcal{V} = \{u = (u^1,u^2) \in \mathcal{V}^1 \times \mathcal{V}^2;\ \ u_\nu^1 + u_\nu^2 = 0 \text{ on } \Gamma_3\},$$
$$\mathbb{H} = \mathbb{H}^1 \times \mathbb{H}^2, \ \ \mathbb{H}_1 = \mathbb{H}_1^1 \times \mathbb{H}_1^2, \ \ \mathcal{H} = \mathcal{H}^1 \times \mathcal{H}^2, \ \ \mathcal{H}_1 = \mathcal{H}_1^1 \times \mathcal{H}_1^2, \ \ \mathbb{Y} = \mathbb{Y}^1 \times \mathbb{Y}^2,$$
$$E_0 = E_0^1 \times E_0^2, \ \ E_1 = E_1^1 \times E_1^2, \ \ \mathbb{W} = \mathbb{W}^1 \times \mathbb{W}^2, \ \ \mathcal{W} = \mathcal{W}^1 \times \mathcal{W}^2.$$

The spaces $\mathcal{V}$, $\mathbb{H}$, $\mathcal{H}$, $\mathbb{Y}$, E_0, E_1, $\mathbb{W}$ and $\mathcal{W}$ are real Hilbert spaces endowed with the canonical inner products denoted by $\langle .,.\rangle_{\mathcal{V}}$, $\langle .,.\rangle_{\mathbb{H}}$, $\langle .,.\rangle_{\mathcal{H}}$, $\langle .,.\rangle_{\mathbb{Y}}$, $\langle .,.\rangle_{E_0}$, $\langle .,.\rangle_{E_1}$, $\langle .,.\rangle_{\mathbb{W}}$, and $\langle .,.\rangle_{\mathcal{W}}$. The associate norms will be denoted by $\|.\|_{\mathcal{V}}$, $\|.\|_{\mathbb{H}}$, $\|.\|_{\mathcal{H}}$, $\|.\|_{\mathbb{Y}}$, $\|.\|_{E_0}$, $\|.\|_{E_1}$, $\|.\|_{\mathbb{W}}$, and $\|.\|_{\mathcal{W}}$, respectively.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $p \in [1,+\infty]$, $k \in [1,+\infty[$. We denote by $\mathcal{C}(0,T;X)$ and $\mathcal{C}^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|\pi\|_{\mathcal{C}(0,T;X)} = \max_{t\in[0,T]} \|\pi(t)\|_X, \quad \|\pi\|_{\mathcal{C}^1(0,T;X)} = \max_{t\in[0,T]} \|\pi(t)\|_X + \max_{\pi\in[0,T]} \|\dot{\pi}(t)\|_X,$$

respectively. Moreover, we use the dot above to indicate the derivative with respect to the time variable. Moreover, if X_1 and X_2 are real Hilbert spaces, then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $\langle .,.\rangle_{X_1\times X_2}$.

We now list assumptions on the data. Assume the operators $\mathcal{A}^\kappa$, $\mathcal{B}^\kappa$, $\mathcal{F}^\kappa$, $\mathcal{G}^\kappa$, $\mathcal{R}^\kappa$, Θ^κ, Ψ^κ, S^κ, and $\mathcal{E}^\kappa$ satisfy the following conditions $\big(L_{\mathcal{A}^\kappa}$, $m_{\mathcal{A}^\kappa}$, $L_{\mathcal{B}^\kappa}$, $L_{\mathcal{F}^\kappa}$, $L_{\mathcal{G}^\kappa}$, $m_{\mathcal{R}^\kappa}$, L_{Θ^κ}, L_{Ψ^κ} and L_{S^κ} being positive constants$\big)$ for $\kappa = 1,2$:

H(1): (a) $\mathcal{A}^\kappa : \Omega^\kappa \times \mathbb{S}^d \to \mathbb{S}^d$;
(b) $|\mathcal{A}^\kappa(x,\varepsilon_1) - \mathcal{A}^\kappa(x,\varepsilon_2)| \le L_{\mathcal{A}^\kappa}|\varepsilon_1-\varepsilon_2|$, $\forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d$, a.e. $x \in \Omega^\kappa$;
(c) $(\mathcal{A}^\kappa(x,\varepsilon_1) - \mathcal{A}^\kappa(x,\varepsilon_2)).(\varepsilon_1-\varepsilon_2) \ge m_{\mathcal{A}^\kappa}|\varepsilon_1-\varepsilon_2|^2$, $\forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d$ a.e $x \in \Omega^\kappa$;
(d) $\mathcal{A}^\kappa(.,\varepsilon)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d$;
(e) $\mathcal{A}^\kappa(.,0)$ belongs to $\mathcal{H}^\kappa$.

H(2): (a) $\mathcal{B}^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{S}^d$;
(b) $|\mathcal{B}^\kappa(x,\varepsilon_1,r_1) - \mathcal{B}^\kappa(x,\varepsilon_2,r_2)| \le L_{\mathcal{B}^\kappa}\big(|\varepsilon_1-\varepsilon_2| + |r_1-r_2|\big)$;
$\forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, r_1,r_1 \in \mathbb{R}$, a.e. $x \in \Omega^\kappa$;
(c) $\mathcal{B}^\kappa(.,\varepsilon,r)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d$, $r \in \mathbb{R}$;
(d) $\mathcal{B}^\kappa(.,0,0)$ belongs to $\mathcal{H}^\kappa$.

H(3): (a) $\mathcal{F}^\kappa : \Omega^\kappa \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{S}^d$;
(b) $|\mathcal{F}^\kappa(x,k_1,r_1) - \mathcal{F}^\kappa(x,k_2,r_2)| \le L_{\mathcal{F}^\kappa}\big(|k_1-k_2| + |r_1-r_2|\big)$;
$\forall k_1,k_2 \in \mathbb{R}^m, r_1,r_1 \in \mathbb{R}$, a.e. $x \in \Omega^\kappa$;
(c) $\mathcal{F}^\kappa(.,k,r)$ is measurable on Ω^κ, for all $k \in \mathbb{R}^m$, $r \in \mathbb{R}$;
(d) $\mathcal{F}^\kappa(.,0,0)$ belongs to $\mathcal{H}^\kappa$. H(4): (a) $\mathcal{G}^\kappa : \Omega^\kappa \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^d$;
(b) $|\mathcal{G}^\kappa(x,k_1,r_1) - \mathcal{G}^\kappa(x,k_2,r_2)| \le L_{\mathcal{G}^\kappa}\big(|k_1-k_2| + |r_1-r_2|\big)$;
$\forall k_1,k_2 \in \mathbb{R}^m, r_1,r_1 \in \mathbb{R}$, a.e. $x \in \Omega^\kappa$.
(c) $\mathcal{G}^\kappa(.,k,r)$ is measurable on Ω^κ, for all $k \in \mathbb{R}^m$, $r \in \mathbb{R}$;
(d) $\mathcal{G}^\kappa(.,0,0)$ belongs to $\mathcal{H}^\kappa$.

H(5): (a) $\mathcal{R}^\kappa : \Omega^\kappa \times \mathbb{R}^d \to \mathbb{R}^d$;
(b) $\mathcal{R}^\kappa = (r^\kappa_{ij})$, $r^\kappa_{ij} = r^\kappa_{ji} \in L^\infty(\Omega^\kappa)$, $1 \le i,j \le d$;
(c) $\mathcal{R}^\ell v.v \ge m_{\mathcal{R}^\kappa}|v|^2$, $\forall v \in \mathbb{R}^d$, a.e. $x \in \Omega^\kappa$.

H(6): (a) $\Theta^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}^m$;
(b) $|\Theta^\kappa(x,\varepsilon_1,r_1,k_1,d_1) - \Theta^\kappa(x,\varepsilon_2,r_2,k_2,d_2)| \le$;
$L_{\Theta^\kappa}\big(|\varepsilon_1-\varepsilon_2| + |r_1-r_2| + |k_1-k_2| + |d_1-d_2|\big)$;
$\forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, k_1,k_2 \in \mathbb{R}^m, r_1,r_2,d_1,d_2 \in \mathbb{R}$, *a.e.* $x \in \Omega^\kappa$;
(c) $\Theta^\kappa(.,\varepsilon,r,k,d)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d, k \in \mathbb{R}^m, r,d \in \mathbb{R}$;
(d) $\Theta^\kappa(.,0,0,0,0)$ belongs to $L^2(\Omega^\kappa)$.

H(7): (a) $\Psi^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R} \to \mathbb{R}$;
(b) $|\Psi^\kappa(x, \varepsilon_1, r_1, k_1, d_1) - \Psi^\kappa(x, \varepsilon_2, r_2, k_2, d_2)| \leq$;
$L_{\Psi^\kappa}\big(|\varepsilon_1 - \varepsilon_2| + |r_1 - r_2| + |k_1 - k_2| + |d_1 - d_2|\big)$;
$\forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, k_1, k_2 \in \mathbb{R}^m, r_1, r_2, d_1, d_2 \in \mathbb{R}$, *a.e.* $x \in \Omega^\kappa$;
(c) $\Psi^\kappa(., \varepsilon, r, k, d)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d, k \in \mathbb{R}^m, r, d \in \mathbb{R}$;
(d) $\Psi^\kappa(., 0, 0, 0, 0)$ belongs to $L^2(\Omega^\kappa)$.

H(8): (a) $S^\kappa : \Omega^\kappa \times \mathbb{S}^d \times \mathbb{R} \to \mathbb{R}$;
(b) $|S^\kappa(x, \varepsilon_1, r_1) - S^\kappa(x, \varepsilon_2, r_2)| \leq L_{S^\kappa}\big(|\varepsilon_1 - \varepsilon_2| + |r_1 - r_2|\big)$;
$\forall\, \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, \forall r_1, r_2 \in \mathbb{R}$ a.e. $x \in \Omega^\kappa$;
(c) $S^\kappa(., \varepsilon, r)$ is measurable on Ω^κ, for all $\varepsilon \in \mathbb{S}^d,\ r \in \mathbb{R}$;
(d) $S^\kappa(., 0, 0)$ belongs to $L^2(\Omega^\kappa)$.

H(9): (a) $\mathcal{E}^\kappa : \Omega^\kappa \times \mathbb{S}^d \to \mathbb{R}^d$;
(b) $\mathcal{E}^\kappa = (e^\kappa_{ijk}),\ e^\kappa_{ijk} = e^\kappa_{ikj} \in L^\infty(\Omega^\kappa),\ 1 \leq i, j, k \leq d$;
(c) $\mathcal{E}^\kappa \varepsilon.\upsilon = \varepsilon.(\mathcal{E}^\kappa)^*\upsilon,\ \ \forall \varepsilon \in \mathbb{S}^d, \upsilon \in \mathbb{R}^d$.

We suppose that the mass density, the forces, the traction densities and the foundation's temperatures satisfy

H(10): (a) $\rho^\kappa \in L^\infty(\Omega^\kappa)$, $\exists \rho_0 > 0$; $\rho^\kappa(x) \geq \rho_0$ a.e. $x \in \Omega^\kappa$;
(b) $f^\kappa_0 \in L^2(0, T; L^2(\Omega^\kappa)^d)$, $f^\kappa_2 \in L^2(0, T; L^2(\Gamma^\kappa_2)^d)$;
(c) $q^\kappa_0 \in \mathcal{C}(0, T; L^2(\Omega^\kappa))$, $q^\kappa_2 \in \mathcal{C}(0, T; L^2(\Gamma^\kappa_b))$;
(d) $\chi^\kappa \in L^2(0, T; L^2(\Omega^\kappa))$.

The energy coefficient, microcrack diffusion coefficient and the friction yield limit g satisfy

H(11): $\mathcal{K}^\kappa_0,\ \mathcal{K}^\kappa_1 > 0$, $g \in L^\infty(\Gamma_3),\ g \geq 0$, a.e. on Γ_3.

Finally, we assume that the initial values satisfy the regularity

H(12): $\beta^\kappa_0 \in \mathbb{Y}^\kappa$, $u^\kappa_0 \in \mathcal{V}^\kappa$, $v^\kappa_0 \in \mathbb{H}^\kappa$, $\alpha^\kappa_0 \in \mathcal{Z}^\kappa$, $\tau^\kappa_0 \in E^\kappa_1$.

We will use a modified inner product on $\mathbb{H}$, given by

$$\langle\!\langle u, v \rangle\!\rangle_{\mathbb{H}} = \sum_{\kappa=1}^{2} \langle \rho^\kappa u^\kappa, v^\kappa \rangle_{\mathbb{H}^\kappa}, \quad \forall u, v \in \mathbb{H} \tag{2.21}$$

and let $|\!|\!|.|\!|\!|_{\mathbb{H}}$ be the associated norm. It follows from assumption H(8)(a) that $|\!|\!|.|\!|\!|_{\mathbb{H}}$ and $\|.\|_{\mathbb{H}}$ are equivalent norms on $\mathbb{H}$, and the inclusion mapping of $(\mathcal{V}, \|.\|_{\mathcal{V}})$ into $(\mathbb{H}, |\!|\!|.|\!|\!|_{\mathbb{H}})$ is continuous and dense. We denote by $\mathcal{V}'$ the dual of $\mathcal{V}$. Identify $\mathbb{H}$ with its own dual. Then

$$\langle u, v \rangle_{\mathcal{V}' \times \mathcal{V}} = \langle\!\langle u, v \rangle\!\rangle_{\mathbb{H}}, \quad \forall u \in \mathbb{H}, v \in \mathcal{V}. \tag{2.22}$$

We define five mappings $F : [0, T] \to \mathcal{V}'$, $Q : [0, T] \to \mathbb{W}$, $a_0 : E_1 \times E_1 \to \mathbb{R}$, $a_1 :$

$E_1 \times E_1 \to \mathbb{R}$ and $J : \mathcal{V} \to \mathbb{R}$, respectively, by

$$\langle F(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} = \sum_{\kappa=1}^{2} \int_{\Omega^\kappa} f_0^\kappa(t)\cdot v^\kappa dx + \sum_{\kappa=1}^{2} \int_{\Gamma_2^\kappa} f_2^\kappa(t)\cdot v^\kappa da \quad \forall v \in \mathcal{V}, \tag{2.23}$$

$$\langle Q(t), \zeta\rangle_{\mathbb{W}} = \sum_{\kappa=1}^{2} \int_{\Omega^\kappa} q_0^\kappa(t)\zeta^\kappa dx - \sum_{\kappa=1}^{2} \int_{\Gamma_b^\kappa} q_2^\kappa(t)\zeta^\kappa da \quad \forall \zeta \in \mathbb{W}, \tag{2.24}$$

$$a_0(\xi,\zeta) = \sum_{\kappa=1}^{2} \mathcal{K}_0^\kappa \int_{\Omega^\kappa} \nabla\xi^\kappa.\nabla\zeta^\kappa dx + \sum_{\kappa=1}^{2} \lambda_0^\kappa \int_{\Gamma^\kappa} \xi^\kappa\zeta^\kappa da, \tag{2.25}$$

$$a_1(\xi,\zeta) = \sum_{\kappa=1}^{2} \mathcal{K}_1^\kappa \int_{\Omega^\kappa} \nabla\xi^\kappa.\nabla\zeta^\kappa dx, \tag{2.26}$$

$$J(u) = \int_{\Gamma_3} g\left|u_\tau^1 - u_\tau^2\right| da. \tag{2.27}$$

We note that conditions H(10)(b) and H(10)(c) imply

$$F \in L^2(0,T;\mathcal{V}'), \quad Q \in \mathcal{C}(0,T;\mathbb{W}). \tag{2.28}$$

We now turn to deriving a variational formulation of the mechanical problem P. To that end we assume that $\{u^\kappa, \sigma^\kappa, \psi^\kappa, D^\kappa, \tau^\kappa, \alpha^\kappa, \beta^\kappa\}$ with $\kappa = 1, 2$ are sufficiently smooth functions satisfying (2.1)–(2.15) and let $w = (w^1, w^2) \in \mathcal{V}$ and $t \in [0,T]$. First, we use Green's formula (2.18) and by (2.6) (2.8), (2.9) and (2.21)–(2.23), we find

$$\begin{aligned}\langle \ddot{u}(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum_{\kappa=1}^{2} \langle \sigma^\kappa, \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} &= \langle F(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} \\ &+ \sum_{\kappa=1}^{2} \int_{\Gamma_3} \sigma^\kappa(t)\nu^\kappa.(w^\kappa - \dot{u}^\kappa(t))da.\end{aligned} \tag{2.29}$$

Using now (2.9) and definition of $\mathcal{V}$, we achieve

$$\sum_{\kappa=1}^{2} \sigma^\kappa(t)\nu^\kappa.(w^\kappa - \dot{u}^\kappa(t)) = \sigma_\tau(t).\big((w_\tau^1 - w_\tau^2) - (\dot{u}_\tau^1(t) - \dot{u}_\tau^2(t))\big)$$

and use the frictional contact conditions (2.9) and the definition (2.27) to obtain

$$\sum_{\kappa=1}^{2} \int_{\Gamma_3} \sigma^\kappa(t)\nu^\kappa.(w^\ell - \dot{u}^\kappa(t))da \geq -J(w) + J(\dot{u}(t)). \tag{2.30}$$

Finally, we combine (2.1), (2.29) and (2.30) to deduce that

$$\begin{aligned}&\langle \ddot{u}(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum_{\kappa=1}^{2} \langle \mathcal{A}^\kappa\varepsilon(\dot{u}^\kappa) + \mathcal{B}^\kappa(\varepsilon(u^\kappa), \alpha^\kappa), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} \\ &+ \sum_{\kappa=1}^{2} \langle (\mathcal{E}^\kappa)^* \nabla\psi^\kappa, \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} + \sum_{\kappa=1}^{2} \langle \mathcal{F}^\kappa(\beta^\kappa, \tau^\kappa), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} \\ &+ J(w) - J(\dot{u}(t)) \geq \langle F(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}}.\end{aligned} \tag{2.31}$$

Similarly, let $\phi = (\phi^1, \phi^2) \in \mathbb{W}$ and $t \in [0, T]$, from (2.2), (2.7), (2.14), (2.19) and (2.24), we deduce that

$$\sum_{\kappa=1}^{2} \big\langle \mathcal{R}^\kappa \nabla \psi^\kappa(t) - \mathcal{E}^\kappa \varepsilon(u^\kappa(t)) - \mathcal{G}^\kappa(\beta^\kappa(t), \tau^\kappa(t)), \nabla \phi^\kappa \big\rangle_{\mathbb{H}^\kappa} = \langle Q(t), \phi \rangle_{\mathbb{W}}. \tag{2.32}$$

On the other hand, let $\xi = (\xi^1, \xi^2) \in \mathcal{Z}$ and $t \in [0, T]$. Then, using (2.5), we have

$$\begin{aligned} &\sum_{\kappa=1}^{2} \langle \dot{\alpha}^\kappa(t), \xi^\kappa - \alpha^\kappa(t) \rangle_{L^2(\Omega^\kappa)} - \sum_{\kappa=1}^{2} \langle \mathcal{K}_1^\kappa \triangle \alpha^\kappa(t), \xi^\kappa - \alpha^\kappa(t) \rangle_{L^2(\Omega^\kappa)} \\ &\geq \sum_{\kappa=1}^{2} \langle S^\kappa(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)), \xi^\kappa - \alpha^\kappa(t) \rangle_{L^2(\Omega^\kappa)}. \end{aligned}$$

Combining this inequality with (2.11), (2.19) and (2.26), we obtain

$$\begin{aligned} &\sum_{\kappa=1}^{2} \langle \dot{\alpha}^\kappa(t), \xi^\kappa - \alpha^\kappa(t) \rangle_{L^2(\Omega^\kappa)} + a_1(\alpha(t), \xi - \alpha(t)) \\ &\geq \sum_{\kappa=1}^{2} \langle S^\kappa(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)), \xi^\kappa - \alpha^\kappa(t) \rangle_{L^2(\Omega^\kappa)}. \end{aligned} \tag{2.33}$$

For the temperature, let $\delta = (\delta^1, \delta^2) \in E_1$ and $t \in [0, T]$. Using (2.4), (2.12) and (2.19), we have

$$\begin{aligned} &\sum_{\kappa=1}^{2} \Big\langle \Psi^\ell\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t), \beta^\kappa(t), \tau^\kappa(t)\big) + \chi^\kappa(t), \delta^\kappa \Big\rangle_{L^2(\Omega^\kappa)} \\ &= \sum_{\kappa=1}^{2} \langle \dot{\tau}^\kappa(t), \delta^\kappa \rangle_{L^2(\Omega^\kappa)} - \sum_{\kappa=1}^{2} \int_{\Omega^\kappa} \mathcal{K}_0^\kappa \Delta \tau^\kappa(t) \delta^\kappa dx \\ &= \sum_{\kappa=1}^{2} \langle \dot{\tau}^\kappa(t), \delta^\kappa \rangle_{L^2(\Omega^\kappa)} + \sum_{\kappa=1}^{2} \int_{\Omega^\kappa} \mathcal{K}_0^\kappa \nabla \tau^\kappa(t) \nabla \delta^\kappa dx + \sum_{\kappa=1}^{2} \int_{\Gamma^\kappa} \lambda_0^\kappa \tau^\kappa(t) \delta^\kappa da. \end{aligned}$$

We use now (2.25) in the previous equality to obtain

$$a_0(\tau(t), \delta) = \sum_{\kappa=1}^{2} \Big\langle \Psi^\ell\big(\varepsilon(u^\kappa), \alpha^\kappa, \beta^\kappa, \tau^\kappa\big)(t), \delta^\kappa \Big\rangle_{E_0^\kappa} - \sum_{\kappa=1}^{2} \langle \dot{\tau}^\kappa(t) - \chi^\kappa(t), \delta^\kappa \rangle_{E_0^\kappa}. \tag{2.34}$$

We now gather the constitutive law (2.3), the initial condition (2.15), inequalities (2.31), (2.33), and equalities (2.32), (2.34) to obtain the following weak formulation of the piezoelectric contact problem P.

Problem PV. Find $u = (u^1, u^2) : [0, T] \to \mathcal{V}$, $\psi = (\psi^1, \psi^2) : [0, T] \to \mathbb{W}$, $\tau = (\tau^1, \tau^2) : [0, T] \to E_1$, $\alpha = (\alpha^1, \alpha^2) : [0, T] \to E_1$ and $\beta = (\beta^1, \beta^2) : [0, T] \to \mathbb{Y}$

such that for a.e.$t \in (0,T)$,

$$\dot{\beta}^\kappa(t) = \Theta^\kappa\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t), \beta^\kappa(t), \tau^\kappa(t)\big) \quad \text{in } \Omega^\kappa, \kappa = 1,2, \tag{2.35}$$

$$\left.\begin{array}{l} \langle \ddot{u}(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum\limits_{\kappa=1}^{2} \langle \mathcal{A}^\kappa \varepsilon(\dot{u}^\kappa(t)) + \mathcal{B}^\kappa(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} \\ + \sum\limits_{\kappa=1}^{2} \langle (\mathcal{E}^\kappa)^* \nabla\psi^\kappa(t), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} + \sum\limits_{\kappa=1}^{2} \langle \mathcal{F}^\kappa(\beta^\kappa(t), \tau^\kappa(t)), \varepsilon(w^\kappa - \dot{u}^\kappa(t))\rangle_{\mathcal{H}^\kappa} \\ + J(w) - J(\dot{u}(t)) \geq \langle F(t), w - \dot{u}(t)\rangle_{\mathcal{V}'\times\mathcal{V}} \quad \forall w \in \mathcal{V}, \end{array}\right\} \tag{2.36}$$

$$\begin{array}{l} \sum\limits_{\kappa=1}^{2} \big\langle \mathcal{R}^\kappa \nabla\psi^\kappa(t) - \mathcal{E}^\kappa \varepsilon(u^\kappa(t)) - \mathcal{G}^\kappa(\beta^\kappa(t), \tau^\kappa(t)), \nabla\phi^\kappa \big\rangle_{\mathbb{H}^\kappa} = \langle Q(t), \phi\rangle_{\mathbb{W}}, \\ \forall \phi \in \mathbb{W}, \end{array} \tag{2.37}$$

$$\left.\begin{array}{l} \alpha(t) \in \mathcal{Z}, \quad \sum\limits_{\kappa=1}^{2} \langle \dot{\alpha}^\kappa(t), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\kappa)} + a(\alpha(t), \xi - \alpha(t)) \\ \geq \sum\limits_{\kappa=1}^{2} \langle S^\kappa\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t)\big), \xi^\kappa - \alpha^\kappa(t)\rangle_{L^2(\Omega^\ell)} \quad \forall \xi \in \mathcal{Z}, \end{array}\right\} \tag{2.38}$$

$$\left.\begin{array}{l} a_0(\tau(t), \delta) = \sum_{\kappa=1}^{2} \big\langle \Psi^\kappa\big(\varepsilon(u^\kappa(t)), \alpha^\kappa(t), \beta^\kappa(t), \tau^\kappa(t)\big), \delta^\kappa \big\rangle_{E_0^\kappa} \\ - \sum_{\kappa=1}^{2} \langle \dot{\tau}^\kappa(t) - \chi^\kappa(t), \delta^\kappa\rangle_{E_0^\kappa} \quad \forall \delta \in E_1, \end{array}\right\} \tag{2.39}$$

$$\begin{array}{l} u(0) = (u_0^1, u_0^2), \ \dot{u}(0) = (v_0^1, v_0^2), \ \alpha(0) = (\alpha_0^1, \alpha_0^2), \ \beta(0) = (\beta_0^1, \beta_0^2), \\ \tau(0) = (\tau_0^1, \tau_0^2). \end{array} \tag{2.40}$$

The existence of a unique solution to Problem **PV** will be presented in the next section.

3 Main Existence and Uniqueness Result

Now, we propose our existence and uniqueness result.

Theorem 3.1 *Under the assumptions H(1)–H(12), there exists a unique solution* $\{u, \psi, \tau, \alpha, \beta\}$ *to problem PV. Moreover, the solution satisfies*

$$u \in W^{1,2}(0,T;\mathcal{V}) \cap \mathcal{C}^1(0,T;\mathbb{H}) \cap W^{2,2}(0,T;\mathcal{V}'), \tag{3.1}$$

$$\psi \in \mathcal{C}(0,T;\mathbb{W}), \tag{3.2}$$

$$\tau \in W^{1,2}(0,T;E_0) \cap L^2(0,T;E_1), \tag{3.3}$$

$$\alpha \in W^{1,2}(0,T;\mathbb{Y}), \tag{3.4}$$

$$\beta \in W^{1,2}(0,T;E_0) \cap L^2(0,T;E_1). \tag{3.5}$$

The functions $\{\sigma, D, u, \psi, \tau, \alpha, \beta\}$, which satisfy (2.1), (2.2) and (2.35)–(2.40) , are called the weak solution of the thermo-piezoelectric contact Problem P. We conclude by Theorem 3.1 that, under the assumptions H(1)–H(12), the mechanical problem (2.1)–(2.15) has a unique weak solution $\{\sigma, D, u, \psi, \tau, \alpha, \beta\}$. To precuse the regularity of the weak solution, we note that the constitutive relation (2.1)–(2.2), the assumptions H(1)–H(5), H(9) and the regularities (3.1)–(3.3) show that $\sigma \in \mathcal{C}(0,T;\mathcal{H})$ and $D \in \mathcal{C}(0,T;\mathbb{H})$. We test (2.36) with $v^\kappa \in \mathcal{C}_0^\infty(\Omega^\kappa;\mathbb{R}^d)$ and $v^{3-\kappa} = 0$. Then we take $\phi^\kappa \in \mathcal{C}_0^\infty(\Omega^\kappa)$ and $\phi^{3-\kappa} = 0$ in (2.37) to obtain that

$$\operatorname{Div} \sigma^\kappa(t) + f_0^\kappa(t) = \rho^\kappa \ddot{u}^\kappa(t), \quad \operatorname{div} D^\kappa(t) = q_0^\kappa(t),$$

almost everywhere in Ω^κ for a.e. $t \in (0,T)$ and $\kappa = 1,2$. Next, we use assumptions H(10) to deduce that $\operatorname{Div}\sigma^\kappa \in L^2(0,T;\mathbb{H}^\kappa)$, $\operatorname{div} D^\kappa \in \mathcal{C}(0,T;E_0^\kappa)$, $\kappa = 1,2$, which shows that

$$\sigma \in L^2(0,T;\mathcal{H}_1), \quad D \in \mathcal{C}(0,T;\mathcal{W}). \tag{3.6}$$

We conclude that the weak solution $\{\sigma, D, u, \psi, \tau, \alpha, \beta\}$ of the thermo-piezoelectric contact Problem P has the regularity (3.1)–(3.6).

The proof of Theorem 3.1 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that assumptions of Theorem 3.1 hold, and let a $\eta = (\eta^1, \eta^2) \in L^2(0,T;\mathcal{V}')$ be given. In the first step, we consider the following variational problem.

Problem P_{u_η}. Find $u_\eta = (u_\eta^1, u_\eta^2) : [0,T] \to \mathcal{V}$ such that for a.e. $t \in (0,T)$,

$$\left.\begin{array}{l}
\langle \ddot{u}_\eta(t), w - \dot{u}_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \sum\limits_{\kappa=1}^{2} \left\langle \mathcal{A}^\kappa \varepsilon(\dot{u}_\eta^\kappa(t)), \varepsilon(w^\kappa - \dot{u}_\eta^\kappa(t))\right\rangle_{\mathcal{H}^\kappa} \\
+J(w) - J(\dot{u}_\eta(t)) \geq \langle F(t) - \eta(t), w - \dot{u}_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}}, \ \forall w \in \mathcal{V}, \\
u_\eta(0) = (u_0^1, u_0^2), \quad \dot{u}_\eta(0) = (v_0^1, v_0^2).
\end{array}\right\} \tag{3.7}$$

We define the mappings $\mathcal{A} : \mathcal{V} \to \mathcal{V}'$ and $F_\eta : [0,T] \longrightarrow \mathcal{V}'$, respectively, by

$$\begin{array}{l}
\langle \mathcal{A}u, v\rangle_{\mathcal{V}'\times\mathcal{V}} = \sum\limits_{\kappa=1}^{2} \langle \mathcal{A}^\kappa \varepsilon(u^\kappa), \varepsilon(v^\kappa)\rangle_{\mathcal{H}^\kappa}, \quad \forall u, v \in \mathcal{V}, \\
\langle F_\eta(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} = \langle F(t) - \eta(t), v\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad \forall t \in [0,T], \ v \in \mathcal{V}.
\end{array} \tag{3.8}$$

Use velocities $v_\eta^\kappa = \dot{u}_\eta^\kappa$ with $\kappa = 1,2$. So, Problem P_{u_η} has been rewritten.

Problem P_{v_η}. Find $v_\eta = (v_\eta^1, v_\eta^2) : [0,T] \to \mathcal{V}$ such that for a.e. $t \in (0,T)$,

$$\left.\begin{array}{l}
\langle \dot{v}_\eta(t), w - v_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + \langle \mathcal{A}v_\eta(t), w - v_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}} + J(w) - J(v_\eta(t)) \\
\geq \langle F_\eta(t), w - v_\eta(t)\rangle_{\mathcal{V}'\times\mathcal{V}}, \quad \forall w \in \mathcal{V}. \\
v_\eta(0) = (v_0^1, v_0^2).
\end{array}\right\} \tag{3.9}$$

Lemma 3.1 *Assume that H(1) and H(11) hold, then the mappings $\mathcal{A}$ and J defined, respectively, by (3.8) and (2.27) satisfy*

$$\left\{\begin{array}{ll}
(a) & \mathcal{A} : \mathcal{V} \to \mathcal{V}' \textit{ is semi-continuous and strongly monotonous,} \\
(b) & \exists\, C_{\mathcal{A}}^1 \geq 0, \ \exists C_{\mathcal{A}}^2 \geq 0 \textit{ such that } \ \|\mathcal{A}u\|_{\mathcal{V}'} \leq C_{\mathcal{A}}^1 \|u\|_{\mathcal{V}} + C_{\mathcal{A}}^2, \quad \forall u \in \mathcal{V}, \\
(c) & \textit{for all sequence } (u_k) \textit{ and } u \textit{ in } L^2(0,T;\mathcal{V}) \textit{ such that } \ u_k \rightharpoonup u \textit{ weakly in } L^2(0,T;\mathcal{V}), \\
& \quad \mathcal{A}u_k \rightharpoonup \mathcal{A}u \textit{ weakly in } L^2(0,T;\mathcal{V}') \\
& \quad \textit{and} \quad \lim_{k\to+\infty} \ \inf \int_0^T \langle \mathcal{A}u_k(s), u_k(s)\rangle_{\mathcal{V}'\times\mathcal{V}} ds \geq \int_0^T \langle \mathcal{A}u(s), u(s)\rangle_{\mathcal{V}'\times\mathcal{V}} ds
\end{array}\right.$$

$$\left\{\begin{array}{l}
(a') \ J : \mathcal{V} \to \mathbb{R} \textit{ is a convex and lower semi-continuous functional.} \\
\qquad \textit{There exists a sequence of } \mathcal{C}^1 \textit{ convex functions } (J_k) : \mathcal{V} \to \mathbb{R} \\
(b') \ \ \exists\, C_g \geq 0 \quad \textit{such that} \quad \|J_k'(u)\|_{\mathcal{V}'} \leq C_g, \quad \forall k \in \mathbb{N}, \quad \forall u \in \mathcal{V}, \\
(c') \ \lim_{k\to+\infty} \int_0^T J_k(u(s)) ds = \int_0^T J(u(s)) ds, \quad \forall u \in L^2(0,T;\mathcal{V}), \\
(d') \ \ \textit{There exists a sequence } (u_k) \textit{ and } u \textit{ in } L^2(0,T;\mathcal{V}) \textit{ such that} \\
\qquad u_k \rightharpoonup u \textit{ weakly in } L^2(0,T;\mathcal{V}), \ \textit{then} \lim_{k\to+\infty} \ \inf \int_0^T J_k(u_k(s)) ds \geq \int_0^T J(u(s)) ds,
\end{array}\right.$$

where $J_k'(u)$ is the Fréchet derivative of J_k at u.

Proof. From the definition (3.8) and assumption H(1), we can verify that $\mathcal{A}$ satisfies the conditions (a)-(b), and applying the Lebesgue theorem, we deduce the condition (c). On the other hand, by using the continuous embedding $\mathcal{V} \hookrightarrow L^2(\Gamma_3)^d$, we find that J is convex and continuous. To approximate the function J, we use the following functional $J_k : \mathcal{V} \to \mathbb{R}$ defined by

$$J_k(u) = \int_{\Gamma_3} g\sqrt{\left|u_\tau^1 - u_\tau^2\right|^2 + k^{-1}}\, da, \quad \forall u = (u^1, u^2) \in \mathcal{V},\ \forall k \in \mathbb{N}^*.$$

We verify that the Fréchet derivative of J_k at $u = (u^1, u^2)$ is given by

$$\langle J_k'(u), h\rangle_{\mathcal{V}'\times\mathcal{V}} = \int_{\Gamma_3} g\frac{(u_\tau^1 - u_\tau^2, h_\tau^1 - h_\tau^2)_{\mathbb{R}^d}}{\sqrt{\left|u_\tau^1 - u_\tau^2\right|^2 + k^{-1}}} da, \quad \forall h = (h^1, h^2) \in \mathcal{V}. \tag{3.10}$$

Then J_k is of class $\mathcal{C}^1$. Direct algebraic computations show that for all $a \geq 0$, $b \geq 0$ such that $a + b = 1$, and for all reals x and y, $k \geq 1$,

$$\sqrt{(ax + by)^2 + k^{-1}} \leq a\sqrt{x^2 + k^{-1}} + b\sqrt{y^2 + k^{-1}}.$$

Then J_k is convex for all $k \in \mathbb{N}^*$. From (3.10), it follows that

$$\exists c \geq 0,\ \forall u \in \mathcal{V},\ \|J_k'(u)\|_{\mathcal{V}'} \leq c\|g\|_{L^\infty(\Gamma_3)},$$

therefore (b') is satisfied. From the definition of J_k, we have $\lim_{k\to+\infty} J_k(u) = J(u)$ and as J_k is continuous on $\mathcal{V}$, applying the Lebesgue theorem, we deduce the property (c'). Finally, (d') is a consequence of the fact that

$$\forall u \in \mathcal{V},\ \forall k \in \mathbb{N}^*, \quad J_k(u) \geq J(u),$$

which finishes the proof.

Lemma 3.2 *Problem P_{v_η} has a unique solution v_η which satisfies*

$$v_\eta \in \mathcal{C}(0, T; \mathbb{H}) \cap L^2(0, T; \mathcal{V}) \cap W^{1,2}(0, T; \mathcal{V}').$$

The proof of Lemma 3.2 is found in [9, p.48].

Let now $u_\eta = (u_\eta^1, u_\eta^2) : [0, T] \to \mathcal{V}$ be the function defined by

$$u_\eta^\kappa(t) = \int_0^t v_\eta^\kappa(s)ds + u_0^\kappa, \quad \forall t \in [0, T], \quad \kappa = 1, 2. \tag{3.11}$$

In the study of Problem P_{u_η}, we have the following result.

Lemma 3.3 *P_{u_η} has a unique solution satisfying the regularity expressed in (3.1).*

Proof. The proof of Lemma 3.3 is a consequence of Lemma 3.2 and the relation (3.11). In the second step, let $\pi = (\pi^1, \pi^2) \in L^2(0, T; E_0)$ and consider the auxiliary problem.

Problem P_{τ_π}. Find $\tau_\pi = (\tau_\pi^1, \tau_\pi^2) : [0, T] \to E_0$ such that for a.e. $t \in (0; T)$,

$$\textstyle\sum_{\kappa=1}^2 \langle \dot\tau_\pi^\kappa(t) - \pi^\kappa(t) - \chi^\kappa(t), \delta^\kappa\rangle_{E_0^\kappa} + a_0(\tau_\pi(t), \delta) = 0,\ \forall\delta \in E_0, \tag{3.12}$$

$$\tau_\pi(0) = (\tau_0^1, \tau_0^2). \tag{3.13}$$

Lemma 3.4 *There exists a unique solution τ_π to the auxiliary problem P_{τ_π} satisfying* (3.3).

Proof. The proof of Lemma 3.4 is a consequence of the Poincaré-Friedrichs inequality and the definitions (2.25) of the operator $a_0(.,.)$.

In the third step, let $\mu = (\mu^1, \mu^2) \in L^2(0, T, \mathbb{Y})$ be given, and define $\beta_\mu = (\beta^1_\mu, \beta^2_\mu) \in W^{1,2}(0, T, \mathbb{Y})$ by

$$\beta^\kappa_\mu(t) = \beta^\kappa_0 + \int_0^t \mu^\kappa(s)ds, \quad \kappa = 1, 2. \tag{3.14}$$

We use $u_\eta = (u^1_\eta, u^2_\eta)$ obtained in Lemma 3.3 and $\tau_\pi = (\tau^1_\pi, \tau^2_\pi)$ obtained in Lemma 3.4 to construct the following variational problem.

Problem $P_{\psi_{\eta\pi\mu}}$. Find $\psi_{\eta\pi\mu} = (\psi^1_{\eta\pi\mu}, \psi^2_{\eta\pi\mu}) : [0, T] \to \mathbb{W}$ such that for a.e. $t \in (0, T)$,

$$\begin{aligned}&\textstyle\sum_{\kappa=1}^2 \left\langle \mathcal{R}^\kappa \nabla \psi^\kappa_{\eta\pi\mu}(t), \nabla\phi^\kappa \right\rangle_{\mathbb{H}^\kappa} - \sum_{\kappa=1}^2 \left\langle \mathcal{E}^\kappa \varepsilon(u^\kappa_\eta(t)) + \mathcal{G}^\kappa(\beta^\kappa_\mu(t), \tau^\kappa_\pi(t)), \nabla\phi^\kappa \right\rangle_{\mathbb{H}^\kappa}\\ &= (Q(t), \phi)_{\mathbb{W}} \quad \forall \phi \in \mathbb{W}.\end{aligned} \tag{3.15}$$

We have the following result.

Lemma 3.5 *Problem $P_{\psi_{\eta\pi\mu}}$ has a unique solution $\psi_{\eta\pi\mu} = (\psi^1_{\eta\pi\mu}, \psi^2_{\eta\pi\mu})$ which satisfies the regularity* (3.2).

Proof. We define a bilinear form $b(.,.) : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ by

$$b(\psi, \phi) = \sum_{\kappa=1}^2 \langle \mathcal{R}^\kappa \nabla\psi^\kappa, \nabla\phi^\kappa \rangle_{\mathbb{H}^\kappa}, \quad \forall \psi, \phi \in \mathbb{W}. \tag{3.16}$$

We use H(5) and (3.16) to show that the bilinear form $b(.,.)$ is continuous, symmetric and coercive on $\mathbb{W}$, moreover, using (2.24) and the Riesz representation theorem, we may define an element $Q_{\eta\pi\mu} : [0, T] \to \mathbb{W}$ such that

$$\begin{aligned}&\langle Q_{\eta\pi\mu}(t), \phi \rangle_{\mathbb{W}} = \langle Q(t), \phi \rangle_{\mathbb{W}} + \textstyle\sum_{\kappa=1}^2 \left\langle \mathcal{E}^\kappa \varepsilon(u^\kappa_\eta(t)) + \mathcal{G}^\kappa(\beta^\kappa_\mu(t), \tau^\kappa_\pi(t)), \nabla\phi^\kappa \right\rangle_{\mathbb{H}^\kappa}\\ &\quad \forall \phi \in \mathbb{W},\ t \in (0, T).\end{aligned}$$

We apply the Lax-Milgram theorem to deduce that there exists a unique element $\psi_{\eta\pi\mu}(t) = (\psi^1_{\eta\pi\mu}(t), \psi^2_{\eta\pi\mu}(t)) \in \mathbb{W}$ such that

$$b(\psi_{\eta\pi\mu}(t), \phi) = \langle Q_{\eta\pi\mu}(t), \phi \rangle_{\mathbb{W}} \quad \forall \phi \in \mathbb{W}. \tag{3.17}$$

We conclude that $\psi_{\eta\pi\mu}$ is a solution of Problem $P_{\psi_{\eta\pi\mu}}$. Let $t_1, t_2 \in [0, T]$, it follows from (3.15) that

$$\begin{aligned}\|\psi_{\eta\pi\mu}(t_1) - {}& \psi_{\eta\pi\mu}(t_2)\|_{\mathbb{W}} \le C\big(\|u_\eta(t_1) - u_\eta(t_2)\|_{\mathcal{V}} + \|\beta_\mu(t_1) - \beta_\mu(t_2)\|_{\mathbb{Y}}\\ &+ \|\tau_\pi(t_1) - \tau_\pi(t_2)\|_{E_0} + \|Q(t_1) - Q(t_2)\|_{\mathbb{W}}\big).\end{aligned} \tag{3.18}$$

Due to (2.28), (3.2), (3.3) and $\beta_\mu \in W^{1,2}(0, T; \mathbb{Y})$, inequality (3.18) implies that $\psi_{\eta\pi\mu} \in \mathcal{C}(0, T; \mathbb{W})$. In the fourth step, let $\theta = (\theta^1, \theta^1) \in L^2(0.T; E_0)$ be given and consider the following initial-value problem.

Problem P_{α_θ}. *Find* $\alpha_\theta = (\alpha_\theta^1, \alpha_\theta^2) : [0,T] \to E_1$ such that for a.e. $t \in (0,T)$,

$$\alpha_\theta(t) \in \mathcal{Z}, \quad \sum_{\kappa=1}^{2} \langle \dot{\alpha}_\theta^\kappa(t) - \theta^\kappa(t), \mu^\kappa - \alpha_\theta^\kappa(t) \rangle_{L^2(\Omega^\kappa)} + a_1(\alpha_\theta(t), \mu - \alpha_\theta(t)) \geq 0, \quad \forall \mu \in \mathcal{Z}. \tag{3.19}$$

In the study of Problem P_{α_θ}, we have the following result.

Lemma 3.6 *The problem P_{α_θ} has a unique solution $\alpha_\theta = (\alpha_\theta^1, \alpha_\theta^2)$ which satisfies the regularity (3.5).*

Proof. We use a standard result for parabolic variational inequalities [9, p.47]. Finally, we now pass to the final step of the proof of Theorem 3.1 in which we use a fixed point argument. To this end, we consider the mapping

$$\Sigma : L^2(0,T; \mathcal{V}' \times \mathbb{Y} \times E_0 \times E_0) \to L^2(0,T; \mathcal{V}' \times \mathbb{Y} \times E_0 \times E_0)$$

defined by

$$\Sigma(\eta,\mu,\pi,\theta) = \big(\Sigma_1(\eta,\mu,\pi,\theta), \Sigma_2(\eta,\mu,\pi,\theta), \Sigma_3(\eta,\mu,\pi,\theta), \Sigma_4(\eta,\mu,\pi,\theta)\big) \tag{3.20}$$

with

$$\langle \Sigma_1(\eta,\mu,\pi,\theta)(t), v\rangle_{\mathcal{V}'\times\mathcal{V}} = \sum_{\kappa=1}^{2} \Big\langle \mathcal{B}^\kappa(\varepsilon(u_\eta^\kappa(t)), \alpha_\theta^\kappa(t)) + \big(\mathcal{E}^\kappa\big)^* \nabla \psi_{\eta\pi\mu}^\kappa(t),\ \varepsilon(v^\kappa) \Big\rangle_{\mathcal{H}^\kappa}$$
$$+ \sum_{\kappa=1}^{2} \big\langle \mathcal{F}^\kappa(\beta_\mu^\kappa(t), \tau_\pi^\kappa(t)),\ \varepsilon(v^\kappa) \big\rangle_{\mathcal{H}^\kappa}, \quad \forall v \in \mathcal{V}, \tag{3.21}$$

$$\Sigma_2(\eta,\mu,\pi,\theta)(t) = \Big(\Theta^1\big(\varepsilon(u_\eta^1(t)), \alpha_\theta^1(t), \beta_\mu^1(t), \tau_\pi^1(t)\big)\ ,\ \Theta^2\big(\varepsilon(u_\eta^2(t)), \alpha_\theta^2(t), \beta_\mu^2(t), \tau_\pi^2(t)\big)\Big), \tag{3.22}$$

$$\Sigma_3(\eta,\mu,\pi,\theta)(t) = \Big(\Psi^1\big(\varepsilon(u_\eta^1(t)), \alpha_\theta^1(t), \beta_\mu^1(t), \tau_\pi^1(t)\big)\ ,\ \Psi^2\big(\varepsilon(u_\eta^2(t)), \alpha_\theta^2(t), \beta_\mu^2(t), \tau_\pi^2(t)\big)\Big), \tag{3.23}$$

$$\Sigma_4(\eta,\mu,\pi,\theta)(t) = \Big(S^1(\varepsilon(u_\eta^1(t)), \alpha_\theta^1(t))\ ,\ S^2(\varepsilon(u_\eta^2(t)), \alpha_\theta^2(t))\Big). \tag{3.24}$$

We have the following result.

Lemma 3.7 *The operator Σ has a unique fixed point $(\eta_*, \mu_*, \pi_*, \theta_*) \in L^2(0,T; \mathcal{V}' \times \mathbb{Y} \times E_0 \times E_0)$.*

Proof. Let $(\eta_1, \mu_1, \pi_1, \theta_1)$, $(\eta_2, \mu_2, \pi_2, \theta_2)$ in $L^2(0,T; \mathcal{V}' \times \mathbb{Y} \times E_0 \times E_0)$ and let $t \in [0,T]$. For simplicity, we use the notation $u_i = u_{\eta_i}$, $v_i = \dot{u}_{\eta_i}$, $\psi_i = \psi_{\eta_i\pi_i\mu_i}$, $\beta_i = \beta_{\mu_i}$, $\tau_i = \tau_{\pi_i}$ and $\alpha_i = \alpha_{\theta_i}$ for $i = 1, 2$. From the definition (3.20)–(3.24) combined with the assumptions H(2), H(3) and H(6)–H(9), we conclude that there is $C > 0$ such that

$$\big\|\Sigma\big(\eta_1,\mu_1,\pi_1,\theta_1\big)(t) - \Sigma\big(\eta_2,\mu_2,\pi_2,\theta_2\big)(t)\big\|^2_{\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0} \leq C\big(\big\|u_1(t) - u_2(t)\big\|^2_{\mathcal{V}}$$
$$+\big\|\psi_1(t) - \psi_2(t)\big\|^2_{\mathbb{W}} + \big\|\beta_1(t) - \beta_2(t)\big\|^2_{\mathbb{Y}} + \big\|\tau_1(t) - \tau_2(t)\big\|^2_{E_0} + \big\|\alpha_1(t) - \alpha_2(t)\big\|^2_{E_0}\big). \tag{3.25}$$

Moreover, from (3.11), we have

$$\|u_1(t) - u_2(t)\|_{\mathcal{V}} \leq \int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{V}} ds, \quad \forall t \in [0, T]. \tag{3.26}$$

Substituting $\eta = \eta_1$, $w = v_2$ and $\eta = \eta_2$, $w = v_1$ in (3.7), we find

$$\langle \dot{v}_1 - \dot{v}_2, v_1 - v_2 \rangle_{\mathcal{V}' \times \mathcal{V}} + \sum_{\kappa=1}^{2} \langle \mathcal{A}^\kappa \varepsilon(v_1^\kappa) - \mathcal{A}^\kappa \varepsilon(v_2^\kappa), \varepsilon(v_1^\kappa - v_2^\kappa) \rangle_{\mathcal{H}^\kappa} + \langle \eta_1 - \eta_2, v_1 - v_2 \rangle_{\mathcal{V}' \times \mathcal{V}} \leq 0.$$

We integrate this inequality with respect to time, use the initial conditions $v_1(0) = v_2(0) = (v_0^1, v_0^2)$, the assumption H(1)(c) and the inequality $\langle \dot{v}_1 - \dot{v}_2, v_1 - v_2 \rangle_{\mathcal{V}' \times \mathcal{V}} \geq 0$ to find that

$$\min(m_{\mathcal{A}^1}, m_{\mathcal{A}^2}) \int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{V}}^2 ds \leq - \int_0^t \langle \eta_1(s) - \eta_2(s), v_1(s) - v_2(s) \rangle_{\mathcal{V}' \times \mathcal{V}} ds.$$

Then, using the inequality $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$, we obtain

$$\int_0^t \|v_1(s) - v_2(s)\|_{\mathcal{V}}^2 ds \leq C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{V}'}^2 ds, \tag{3.27}$$

where C is a positive constant that may change from line to line.
From (3.26) and (3.27), we deduce

$$\|u_1(t) - u_2(t)\|_{\mathcal{V}}^2 \leq C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{V}'}^2 ds. \tag{3.28}$$

The definition (3.14) yields

$$\|\beta_1(t) - \beta_2(t)\|_{\mathbb{Y}}^2 \leq \int_0^t \|\mu_1(s) - \mu_2(s)\|_{\mathbb{Y}}^2 ds. \tag{3.29}$$

On the other hand, from (3.12), we can write

$$\begin{aligned} &\big\langle \dot{\tau}_1(t) - \dot{\tau}_2(t), \tau_1(t) - \tau_2(t) \big\rangle_{E_0} + a_0(\tau_1(t) - \tau_2(t), \tau_1(t) - \tau_2(t)) \\ &= \big\langle \pi_1(t) - \pi_2(t), \tau_1(t) - \tau_2(t) \big\rangle_{E_0} \quad a.e. t \in (0, T). \end{aligned}$$

We integrate this equality with respect to time, and use the initial conditions $\tau_1(0) = \tau_2(0) = (\tau_0^1, \tau_0^2)$ and inequality $a_0(\tau_1 - \tau_2, \tau_1 - \tau_2) \geq 0$ to find

$$\frac{1}{2} \|\tau_1(t) - \tau_2(t)\|_{E_0}^2 \leq \int_0^t \|\pi_1(s) - \pi_2(s)\|_{E_0} . \|\tau_1(s) - \tau_2(s)\|_{E_0} ds.$$

Then, using the inequality $2ab \leq a^2 + b^2$, we obtain

$$\|\tau_1(t) - \tau_2(t)\|_{E_0}^2 \leq \int_0^t \|\pi_1(s) - \pi_2(s)\|_{E_0}^2 ds + \int_0^t \|\tau_1(s) - \tau_2(s)\|_{E_0}^2 ds$$

and, by using Gronwall's inequality, we obtain

$$\|\tau_1(t) - \tau_2(t)\|_{E_0}^2 \leq C \int_0^t \|\pi_1(s) - \pi_2(s)\|_{E_0}^2 ds \;\; \text{a.e. } t \in (0, T). \tag{3.30}$$

Also, (3.15) and the arguments similar to those used in the proof of (3.18) yield

$$\|\psi_1(t)-\psi_2(t)\|_{\mathbb{W}} \le C\big(\|u_1(t)-u_2(t)\|_{\mathcal{V}}+\|\beta_1(t)-\beta_2(t)\|_{\mathbb{Y}}+\|\tau_1(t)-\tau_2(t)\|_{E_0}\big) \ a.e.\ t\in(0,T). \tag{3.31}$$

Furthermore, by substituting $\theta=\theta_1,\ \mu=\alpha_1$ and $\theta=\theta_2,\ \mu=\alpha_2$ in (3.19) and subtracting the two inequalities obtained, we find

$$\begin{aligned}(\dot\alpha_1(t)-\dot\alpha_2(t),\alpha_1(t)-\alpha_2(t))_{E_0}+a_1(\alpha_1(t)-\alpha_2(t),\alpha_1(t)-\alpha_2(t))\\ \le(\theta_1(t)-\theta_2(t),\alpha_1(t)-\alpha_2(t))_{E_0},\quad \text{a.e. } t\in(0,T).\end{aligned}$$

We integrate the previous inequality and applying the inequality of Hölder and Young with Gronwall's lemma, we deduce that

$$\|\alpha_1(t)-\alpha_2(t)\|^2_{E_0}\le C\int_0^t\|\theta_1(s)-\theta_2(s)\|^2_{E_0}ds \quad \text{a.e. } t\in(0,T). \tag{3.32}$$

We substitute (3.28)–(3.32) in (3.25), we obtain

$$\begin{aligned}&\big\|\Sigma\big(\eta_1,\mu_1,\pi_1,\theta_1\big)(t)-\Sigma\big(\eta_2,\mu_2,\pi_2,\theta_2\big)(t)\big\|^2_{\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0}\le\\ &C\textstyle\int_0^t\big\|(\eta_1,\mu_1,\pi_1,\theta_1\big)(s)-\big(\eta_2,\mu_2,\pi_2,\theta_2\big)(s)\big\|^2_{\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0}ds \quad \text{a.e. } t\in(0,T).\end{aligned}$$

Reiterating this inequality n times leads to

$$\begin{aligned}&\big\|\Sigma^n\big(\eta_1,\mu_1,\pi_1,\theta_1\big)-\Sigma^n\big(\eta_2,\mu_2,\pi_2,\theta_2\big)\big\|^2_{L^2(0,T;\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0)}\le\\ &\tfrac{C^nT^n}{n!}\big\|(\eta_1,\mu_1,\pi_1,\theta_1\big)-\big(\eta_2,\mu_2,\pi_2,\theta_2\big)\big\|^2_{L^2(0,T;\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0)}.\end{aligned}$$

Thus, for n sufficiently large, Σ^n is a contraction on the Banach space $L^2(0,T;\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0)$, and so Σ has a unique fixed point.

Now, we have all the ingredients to prove Theorem 3.1.

Proof. Let $(\eta_*,\mu_*,\pi_*,\theta_*\big)\in L^2(0,T;\mathcal{V}'\times\mathbb{Y}\times E_0\times E_0)$ be the fixed point Σ defined by (3.20)–(3.24) and denote

$$u_*=u_{\eta_*},\quad \tau_*=\tau_{\pi_*},\quad (\dot u^\kappa_*(t)),\varepsilon(w^\kappa-\psi_*=\psi_{\eta_*\pi_*\mu_*},\quad \alpha_*=\alpha_{\theta_*},\quad \beta_*=\beta_{\mu_*}. \tag{3.33}$$

We prove $\{u_*,\psi_*,\tau_*,\alpha*,\beta_*\}$ satisfies (2.35)–(2.40) and the regularities (3.1)–(3.5). Indeed, we write (3.7) for $\eta=\eta_*$ and use (3.33) to find

$$\begin{aligned}&\langle\ddot u_*(t),w-\dot u_*(t)\rangle_{\mathcal{V}'\times\mathcal{V}}+\sum_{\kappa=1}^2\langle\mathcal{A}^\kappa\varepsilon\dot u^\kappa_*(t))\rangle_{\mathcal{H}^\kappa}+J(w)-J(\dot u_*(t))\\ &+\langle\eta_*(t),w-\dot u_*(t)\rangle_{\mathcal{V}'\times\mathcal{V}}\ge\langle F(t),w-\dot u_*(t)\rangle_{\mathcal{V}'\times\mathcal{V}},\quad \forall w\in\mathcal{V},\ a.e.\ t\in(0,T).\end{aligned} \tag{3.34}$$

Equation $\Sigma_1\big(\eta_*,\mu_*,\pi_*,\theta_*\big)=\eta_*$ combined with (3.21) shows that $\ for\ \ a.e.t\in(0,T)$,

$$\langle\eta_*(t),v\rangle_{\mathcal{V}'\times\mathcal{V}}=\sum_{\kappa=1}^2\Big\langle\mathcal{B}^\kappa(\varepsilon(u^\kappa_*(t)),\alpha^\kappa_*(t))+\mathcal{F}^\kappa(\beta^\kappa_*(t),\tau^\kappa_*(t))+\big(\mathcal{E}^\kappa\big)^*\nabla\psi^\kappa_*(t),\ \varepsilon(v^\kappa)\Big\rangle_{\mathcal{H}^\kappa}, \tag{3.35}$$

We substitute (3.35) in (3.34) and use (3.33) to see that (2.36) is satisfied. From $\Sigma_2\big(\eta_*,\mu_*,\pi_*,\theta_*\big)=\mu_*$ and (3.14), we see that (2.35) is satisfied. We write now (3.15) for

$(\eta,\pi,\mu)=(\eta_*,\pi_*,\mu_*)$ and use (3.33) to find (2.37). The equalities $\Sigma_3\big(\eta_*,\mu_*,\pi_*,\theta_*\big)=\pi_*$ and $\Sigma_4\big(\eta_*,\mu_*,\pi_*,\theta_*\big)=\theta_*$, combined with (3.12), (3.19) show that (2.38)–(2.39) are satisfied. Next, (2.40) and the regularity (3.1)–(3.5) follow from Lemmas 3.1, 3.4, 3.5 and 3.6 and the relation (3.14), which concludes the existence part of Theorem 3.1. The uniqueness of the solution follows from the uniqueness of the fixed point of the operator Σ defined by (3.20)–(3.24) combined with the unique solvability of Problems P_{u_η}, P_{τ_π}, $P_{\varphi_{\eta\pi\mu}}$ and P_{α_θ}.

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Exact Controllability of the Reaction-Diffusion Equation under Bilinear Control

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Received: May 12, 2019; Revised: November 10, 2022

Abstract: The goal of this paper is to study the global exact controllability of a reaction-diffusion equation in a bounded domain with Dirichlet boundary conditions. We will first consider the case of bilinear equation, then we identify a set of target states that can be exactly reached at any a priori given time. This result is then applied to prove the exact controllability of semilinear reaction-diffusion equation under distributed controls. The approach is constructive and based on linear semigroup theory and null controllability properties of linear problems.

Keywords: *exact controllability; reaction-diffusion equation; bilinear control.*

Mathematics Subject Classification (2010): 35K57, 35K58, 93C20.

1 Introduction

This paper deals with the controllability of the following semilinear reaction-diffusion equation:

$$\begin{cases} y_t = \Delta y + q(x,t)y + f(y), & \text{in } Q_T \ \ (T > 0), \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n, n \geq 1$ with a boundary $\partial\Omega$, $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \partial\Omega \times (0,T)$. Here, $q \in L^\infty(Q_T)$ is a control function with the corresponding solution $y = y(x,t)$. The nonlinearity $f : \mathbb{R} \longrightarrow \mathbb{R}$ is assumed to be a Lipschitz function such that $f(0) = 0$.

In terms of applications, the equation like (1) provides the practical description of various real problems such as chemical reactions, nuclear chain reactions, biomedical

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models etc. (see [2, 9, 10, 14, 19, 20] and the references therein). Equation (1) can be also used to describe a diffusion process with $y(x,t)$ being the concentration of a substance at the point x at time t, or a heat-transfer process, with $y(x,t)$ describing the temperature at the point x and time t (see [5] and [14], p. 17). It is shown in [2] that the equation like (1) can be also used to study the insect dispersal with constant random motion and an $(x,t)-$dependent emigration parameter. It may be also used as a model for the growth of avascular tumor [9].

The question of controllability of the bilinear reaction-diffusion equation has attracted many researchers (see e.g., [4, 5, 8] and [14]- [18]). In [4], the approximate controllability properties have been derived for the truncated bilinear version of (1) (i.e., $f = 0$) for the initial and target states y_0, y_d with finitely many changes of sign. The same question has been discussed by Fernàndez and Khapalov in [8] when the support of the bilinear control is allowed to depend on time. The exact controllability of the bilinear part of equation (1) with inhomogeneous Dirichlet conditions has been considered in [15, 17]. However, the assumptions of [15, 17] are not compatible when dealing with homogeneous Dirichlet conditions. Recently, the approximate and exact controllability have been studied for the truncated bilinear version of equation (1) under the sign condition $y_0(x)y_d(x) \geq 0$, for almost every $x \in \Omega$ in [18]. Moreover, the partial controllability of bilinear reaction-diffusion equation has been studied in [12]. According to the maximum principle, it is not possible to steer the bilinear part of (1) from an initial state which has a constant sign to a target state that change its sign. In [13], Khapalov studied the global approximate controllability of the semilinear convection-diffusion-reaction equation by multiplicative controls while dealing with nonnegative initial and target states. In [5], Cannarsa, Floridia and Khapalov have studied the global approximate controllability properties of system (1) in the one-dimensional case for suitable classes of initial and target states that change their sign at a finite number of points. However, in the works above, the time of steering depends on the given initial and target states. In this paper, we are interested in the multiplicative controllability of the semilinear reaction diffusion system (1) at a priori given time, when the initial and target states have the same sign at almost every $x \in \Omega$ and satisfy $\ln(\frac{y^d}{y_0}) \in L^\infty(\Omega)$. We will first deal with a bilinear case ($f = 0$), then we proceed to the full equation (1). Moreover, we will see that the exact steering of the semilinear system (1) can be reduced to the controllability of its bilinear part since the nonlinear term f can be absorbed by the control in an appropriate way.

The paper is organized as follows. In the next section, we first consider the problem of exact controllability of the bilinear part of the system (1), and we will show that the steering time can be arbitrary small and uniform for all initial and reached states. Then, we apply this result to solve the problem of exact controllability of the semilinear system (1) at a priori fixed time. In the third section, we present a numerical example with simulations.

2 The Main Results

Our goal in this section is to study the exact controllability properties of the system (1) at a given time $T > 0$.

2.1 Exact controllability of the bilinear equation

Here we consider the following system:

$$\begin{cases} y_t = \Delta y + v(x,t)y, & \text{in } Q_T, \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{2}$$

From [18], one may deduce the following approximate controllability result regarding the bilinear part (2) of the system (1).

Lemma 2.1 *[18] For any initial state $y_0 \in L^2(\Omega)$, for any function $g \in W^{2,\infty}(\Omega)$ and for all $\varepsilon > 0$, there exists a time $T = T(y_0, y_d, \epsilon)$ such that the respective solution to (2) controlled with $v := \frac{g}{T}$ satisfies*

$$\|y(T) - e^g y_0\| < \varepsilon.$$

We also recall the following null-controllability of the linear heat equation.

Lemma 2.2 *[7, 11] Consider the system*

$$\begin{cases} \psi_t = \Delta\psi + b(x)\psi + \mathbf{1}_\omega u_2(x,t), & in\, \Omega \times (t_0, T), \\ \psi = 0, & on\ \partial\Omega \times (t_0, T), \\ \psi(\cdot, t_0) = \xi \in L^2(\Omega), & in\, \Omega, \end{cases} \tag{3}$$

where $0 \le t_0 < T$, $b \in L^\infty(\Omega)$ and ω is a nonempty open subset of Ω. Then there is a control $u_2 \in L^\infty(\Omega \times (t_0, T))$ such that the corresponding solution to (3) vanishes at T. Furthermore, we have

$$\|u_2\|_{L^\infty(\Omega\times(t_0,T))} \le C\, \|\xi\|_{L^2(\Omega)}, \tag{4}$$

where $C = C_{T-t_0}$ is a positive constant depending on $T - t_0$ and such that C_{T-t_0} is bounded near $t_0 \to 0^+$.

We now state the exact controllability result of the bilinear system (2).

Theorem 2.1 *Let $y_0 \in L^p(\Omega)$, ($p \ge 2$ and $p > \frac{n}{2}$) and let $y_d \in H^2(\Omega)$ such that*
i) for a.e. $x \in \Omega$, $y_0(x)y_d(x) \ge 0$ and $y_d(x) = 0 \Leftrightarrow y_0(x) = 0$,
ii) $ln(\frac{y_d}{y_0})\mathbf{1}_{E_{y_0}} \in L^\infty(\Omega)$, where $\mathbf{1}_{E_{y_0}}$ denotes the characteristic function of the set $E_{y_0} = \{x \in \Omega /\ y_0(x) \ne 0\}$,
iii) $\frac{\Delta y_d}{y_d}\mathbf{1}_{E_{y_d}} \in L^\infty(\Omega)$ and $|y_d| \ge \alpha > 0$, a.e. on some open subset $\omega \subset \Omega$.
Then for any $T > 0$, there exists a control $v \in L^\infty(Q_T)$ such that the respective solution to (2) satisfies $y(T) = y_d$, a.e. in Ω.

Proof. Let $T > 0$.
1. Approximate steering.
Let $g := ln(\frac{y_d}{y_0})\mathbf{1}_E$. It follows from the assumption (i) that $e^g y_0 = y_d$. Then, in the case where $g \in W^{2,\infty}(\Omega)$, we deduce from Lemma 2.1 that for any $\varepsilon > 0$, there exists $0 < T_1 < T$ small enough such that the corresponding solution to (2) controlled with $v_1 = \frac{g}{T}$ verifies

$$\|y(T_1) - y_d\| < \varepsilon. \tag{5}$$

Moreover, in the general case $g \in L^\infty(\Omega)$, one can construct a sequence $(g_k) \subset W^{2,\infty}(\Omega)$ which is uniformly bounded in Ω such that $g_k \to g$ in $L^2(\Omega)$, as $k \to +\infty$. We will

consider the control $v_1(x) = \dfrac{g_k}{T_1}$ for a suitably selected $k \in I\!N$ (large enough integer). Let $y(t)$ be the solution of (2) corresponding to $v_1(x)$ and to the initial state $y(0) = y_0$. Finally, let $(y_{0l}) \in L^\infty(\Omega)$ such that $y_{0l} \to y_0$ in $L^2(\Omega)$, as $l \to +\infty$.
We have the following triangular inequality:

$$\|y(T_1) - e^g y_0\| \le \|y(T_1) - e^{g_k} y_0\| + \|e^{g_k} y_0 - e^{g_k} y_{0l}\| + \|e^{g_k} y_{0l} - e^g y_{0l}\| + \|e^g y_{0l} - e^g y_0\|$$
$$\le \|y(T_1) - e^{g_k} y_0\| + \|e^{g_k} y_{0l} - e^g y_{0l}\| + \big(\sup_{k\in I\!N} \|e^{g_k}\|_{L^\infty(\Omega)} + e^{\|g\|_{L^\infty(\Omega)}}\big)\|y_{0l} - y_0\|.$$

Let $L \in I\!N$ be such that

$$\big(\sup_{k\in I\!N} \|e^{g_k}\|_{L^\infty(\Omega)} + e^{\|g\|_{L^\infty(\Omega)}}\big)\|y_{0L} - y_0\| < \frac{\epsilon}{3},$$

and for such value of L, we consider K such that

$$\|e^{g_K} - e^g\|\|y_{0L}\|_{L^\infty(\Omega)} < \frac{\epsilon}{3}.$$

Finally, for this value of K, it comes from Lemma 2.1 that there exists $T > 0$ such that

$$\|y(T_1) - e^{g_K} y_0\| < \frac{\epsilon}{3}.$$

We conclude that

$$\|y(T_1) - e^g y_0\| < \epsilon.$$

Hence, since $e^g y_0 = y_d$, it comes that (5) holds for some $0 < T_1 < T$.
2. Exact steering.
Let us consider the following system:

$$\begin{cases} y_t = \Delta y + v(x,t)y, & \text{in } \Omega \times (T_1, T), \\ y(0,t) = 0, & \text{on } \partial\Omega \times (T_1, T), \\ y(T_1) = y(T_1^-), & \text{in } \Omega\cdot \end{cases} \tag{6}$$

Let $z = y - y_d$, where y satisfies (6). Thus z satisfies

$$\begin{cases} z_t = \Delta z + v(x,t)(z + y_d) + \Delta y_d, & \text{in } \Omega \times (T_1, T), \\ z(0,t) = 0, & \text{on } \partial\Omega \times (T_1, T), \\ z(T_1) = y(T_1^-) - y_d, & \text{in } \Omega. \end{cases} \tag{7}$$

In order to prove Theorem 2.1, it is sufficient to prove that (7) is exact null controllable. Let $T_2 \in (T_1, T)$ be close to T_1, so we can assume in the sequel that $0 < T_2 - T_1 < 1$. Then consider the following time-independent control in (T_1, T_2) :

$$v_2(x) = -\frac{\Delta y_d}{y_d}\mathbf{1}_{E_{y_d}},\ a.e., \text{ in } \Omega.$$

From the definition of v_2, we have $v_2 y_d + \Delta y_d = 0$, a.e. in Ω. Thus the system (7) can be reduced to the following one:

$$\begin{cases} z_t = \Delta z + v_2(x)z, & \text{in } \Omega \times (T_1, T_2), \\ z = 0, & \text{on } \partial\Omega \times (T_1, T_2), \\ z(T_1) = y(T_1^-) - y_d, & \text{in } \Omega, \end{cases} \tag{8}$$

whose solution is given by

$$z(t) = S(t - T_1)z(T_1) + \int_{T_1}^{t} S(t - s)v_2(x)z(s)ds, \ \forall t \in [T_1, T_2]. \tag{9}$$

Then, since $S(t)$ is a contraction semigroup,

$$\|z(t)\| \leq \|z(T_1)\| + \|v_2\|_{L^\infty(\Omega)} \int_{T_1}^{t} \|z(s)\|ds$$

for all $t \in [T_1, T_2]$. Gronwall's inequality gives

$$\|z(t)\| \leq C\|z(T_1)\|, \ \forall t \in [T_1, T_2], \ C > 0. \tag{10}$$

Moreover, we know that $S(t)(L^p(\Omega)) \subset L^\infty(\Omega)$, and for all $\xi \in L^p(\Omega)$, we have

$$\|S(t)\xi\|_{L^\infty(\Omega)} \leq C\, t^{-\frac{n}{2p}} \|\xi\|_{L^p(\Omega)}, \forall t > 0, \tag{11}$$

where the constant C is independent of ξ. We also have $S(t)(L^\infty(\Omega)) \subset L^\infty(\Omega)$, and for all $\xi \in L^\infty(\Omega)$, we have (see [6], p. 44)

$$\|S(t)\xi\|_{L^\infty(\Omega)} \leq \|\xi\|_{L^\infty(\Omega)}, \forall t \geq 0.$$

Using the smooth effect of the heat semigroup $S(t)$, we can take the mild solution $z(t)$ in the space of continuous function equipped with the supremum norm. Then, by taking the $L^\infty-$norm in (9) and using (11), we get

$$\|z(t)\|_{L^\infty(\Omega)} \leq C(t - T_1)^{-\frac{n}{2p}} \|z(T_1)\|_{L^1(\Omega)} + C\|v_2\|_{L^\infty(\Omega)} \int_{T_1}^{t} (t - s)^{-\frac{n}{2p}} \|z(s)\|_{L^1(\Omega)} ds$$

for all $t \in [T_1, T_2]$, and for some constant $C > 0$ which is independent of $\eta := T_2 - T_1$. Then, when using (10), it comes

$$\|z(t)\|_{L^\infty(\Omega)} \leq C(t - T_1)^{-\frac{n}{2p}} \|z(T_1)\|_{L^1(\Omega)} + C\|v_2\|_{L^\infty(\Omega)} \|z(T_1)\| \int_{T_1}^{t} (t - s)^{-\frac{n}{2p}} ds$$

for all $t \in [T_1, T_2]$, and in particular,

$$\|z(T_2)\|_{L^\infty(\Omega)} \leq C\eta^{-\frac{n}{2p}} \|z(T_1)\|,$$

where C is a positive constant which is independent of $\eta \in (0, 1)$. Thus (5) implies

$$\|z(T_2)\|_{L^\infty(\Omega)} \leq C\eta^{-\frac{n}{2p}} \epsilon \tag{12}$$

for some constant $C > 0$ which is independent of η.

Let us now consider the control $v(x, t) = v_2(x) + v_3(x, t)$ on $[T_2, T]$, $v_3 \in L^\infty(\Omega \times (T_2, T))$ (with $v_3(t) = 0, \ t \in (T_1, T_2)$). When using this control, the system (7) becomes

$$\begin{cases} z_t = \Delta z + v_2(x)z + v_3(x, t)(z + y_d), & \text{in } \Omega \times (T_2, T), \\ z = 0, & \text{on } \partial\Omega \times (T_2, T), \\ z(T_2) = y(T_2^-) - y_d, & \text{in } \Omega. \end{cases} \tag{13}$$

Let us consider the following linear system:

$$\begin{cases} \psi_t = \Delta\psi + v_2(x)\psi + \mathbf{1}_\omega u_1(x,t), & \text{in } \Omega \times (T_2, T), \\ \psi(0,t) = 0, & \text{on } \partial\Omega \times (T_2, T), \\ \psi(T_2) = z(T_2), & \text{in } \Omega. \end{cases} \tag{14}$$

From Lemma 2.2, there exists a control $u_1 \in L^\infty(\Omega\times(T_2,T))$ such that the corresponding solution to (14) satisfies $\psi(.,T) = 0$. Furthermore, the steering control u_1 satisfies the estimate

$$\|u_1\|_{L^\infty(\Omega\times(T_2,T))} \leq C\|\psi(T_2)\|_{L^2(\Omega)} \tag{15}$$

for some positive constant C which is independent of T_2. In other words, for some positive constant C (depending though on $T - T_2$). Moreover, since T_2 is small enough, we can (according to the Lemma 2.2) choose C only dependent on T.
The solution of (14) is given, for all $t \in [T_2, T]$, by

$$\psi(t) = S(t-T_2)\psi(T_2) + \int_{T_2}^t S(t-s)(v_2(x)\psi(s) + u_1(\cdot,s))ds. \tag{16}$$

Since $\psi(T_2) \in L^\infty(\Omega)$, we have (see [6], p. 44)

$$\|S(t)\psi(T_2)\|_{L^\infty(\Omega)} \leq \|\psi(T_2)\|_{L^\infty(\Omega)}.$$

Since $\psi(T_2) \in L^p(\Omega)$ and $u_1 \in L^\infty(\Omega \times (T_2,T))$, we can see that $\psi(t) \in L^p(\Omega),\ T_2 \leq t \leq T$.

Moreover, from (11), we have

$$\|S(t-s)\psi(s)\|_{L^\infty(\Omega)} \leq C(t-s)^{-\frac{n}{2p}}\|\psi(s)\|_{L^p(\Omega)},\ 0 \leq s < t,$$

and

$$\|S(t-s)\mathbf{1}_\omega u_1(\cdot,s)\|_{L^\infty(\Omega)} \leq C(t-s)^{-\frac{n}{2p}}\|u_1(\cdot,s)\|_{L^p(\Omega)},\ 0 \leq s < t\cdot$$

Thus from (16), we have $\psi(t) \in L^\infty(\Omega)$ for all $t \in (T_2, T]$, and

$$\|\psi(t)\|_{L^\infty(\Omega)} \leq \|\psi(T_2)\|_{L^\infty(\Omega)} + C\|u_1\|_{L^\infty(\Omega\times(T_2,T))} + C\int_{T_2}^t \|\psi(s)\|_{L^\infty(\Omega)}$$

for some C which is independent of η.

Gronwall's inequality yields, via (12) and (15),

$$\|\psi(t)\|_{L^\infty(\Omega)} \leq C_*\eta^{-\frac{n}{2p}}\varepsilon,\ t \in [T_2, T],$$

for η small enough and for some constant $C_* > 0$ which is independent of η.
Since $|y_d| \geq \alpha > 0$ a.e. in ω, we can choose ε and η small enough such that $\eta > (\frac{C_*\epsilon}{\alpha})^{\frac{2p}{n}}$. Hence

$$|\psi(x,t) + y_d| > 0, \text{ a.e } \omega \times (T_2, T).$$

This enables us to define a control v_3 in $\Omega \times (T_2, T)$ through the following relation:

$$v_3(x,t)(\psi(x,t) + y_d) = u_1(x,t).$$

Since $u_1 \in L^\infty(\Omega \times (T_2, T))$, it follows that $v_3 \in L^\infty(\Omega \times (T_2, T))$.
Using the control $v(x,t) = v_2(x) + v_3(x,t),\ t \in (T_2, T)$ in the system (13), leads to the following one:

$$\begin{cases} z_t = \Delta z + v_2(x) z + \dfrac{u_1(x,t)}{\psi(x,t) + y_d}(z + y_d), & \text{in } \Omega \times (T_2, T), \\ z(0,t) = 0, & \text{on } \partial\Omega \times (T_2, T), \\ z(T_2) = y(T_2^-) - y_d, & \text{in } \Omega, \end{cases} \tag{17}$$

which admits ψ as a solution. Hence, by uniqueness, we have $z = \psi$ a.e in $\Omega \times (T_2, T)$.
Finally, returning to initial system (2), the control is then defined by

$$v(x,t) = \begin{cases} v_1(x), & \text{in } (0, T_1), \\ v_2(x), & \text{in } (T_1, T_2), \\ v_2(x) + \dfrac{u_1(x,t)}{\psi(x,t) + y_d}, & \text{in } \ (T_2, T), \end{cases}$$

so that $v \in L^\infty(Q_T)$ and $y(T) = y_d$.

Remark 2.1 The result of Theorem 2.1 improves the results from the literature in terms of the steering time which is here independent of the initial and target states (see for instance [4, 18]).

2.2 Exact controllability of the semilinear system

Presently, the system (1) is considered. The next theorem introduces significant differences with respect to the literature in terms of the proof techniques. Indeed, the method used in [5] consists of shifting the points of sign change by making use of a finite sequence of initial-value pure diffusion problems. In [18], a static control was used to study the approximate controllability of the truncated part of (1), and the equation at hand becomes linear so that one can apply the linear semigroup theory. In the context of equation (1), the central idea of our method is to try to select the bilinear control in such a way that the corresponding trajectory of (1) can be approximated by the bilinear term $v(x,t)y(t)$ on a small interval of steering $[0, T]$. In other words, the effect of the pure diffusion (i.e. $v = 0$ and $f = 0$) as well as the one of the nonlinearity becomes negligible as $T \to 0^+$.

Our exact controllability result for semilinear case is as follows.

Theorem 2.2 *Let $T > 0$. If y_0 and y_d satisfy the assumptions of Theorem 2.1, then there exists a control $q(\cdot,\cdot) \in L^\infty(Q_T)$ for which the respective solution to (1) is such that $y(T) = y_d$.*

Proof. The idea consists in looking for a control that makes the system (1) equivalent to its bilinear part (2) so that one may apply the results of the previous section.
Let us observe that (at least formally) the system (1) can be written as follows:

$$\begin{cases} y_t = \Delta y + (q(x,t) + \frac{f(y)}{y}\mathbf{1}_{E_y})y, & \text{in } Q_T, \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{18}$$

where $E_y = \{(x,t) \in Q_T : y(x,t) \neq 0\}$. This leads us to consider the following bilinear system:

$$\begin{cases} \varphi_t = \Delta\varphi + v(x,t)\varphi, & \text{in } Q_T, \\ \varphi(0,t) = 0, & \text{on } \Sigma_T, \\ \varphi(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{19}$$

According to Theorem 2.1, there exists $v \in L^\infty(Q_T)$ for which the solution of the system (19) is such that $\varphi(T) = y_d$.

From the assumptions on f, we deduce that $|f(y(x))| \leq L|y(x)|$ for a.e $x \in \Omega$, where L is a Lipschitz constant of f. Thus we have $\dfrac{f(\varphi)}{\varphi}\mathbf{1}_{E_\varphi} \in L^\infty(\Omega)$, where $E_\varphi = \{(x,t): \ \varphi(x,t) \neq 0\}$.

Consider the following bilinear system:

$$\begin{cases} y_t = \Delta y + \big(v(x,t) - \frac{f(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}\big)y + f(y), & \text{in } Q_T, \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega, \end{cases} \tag{20}$$

and let us set $q(x,t) = v(x,t) - \dfrac{f(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}$ in (20), where φ is the solution of (19) corresponding to the steering control v. It is apparent that φ is a solution of (20). Hence, by uniqueness, we have that $y = \varphi$ is the unique solution of (20) corresponding to the control $q(x,t) = v(x,t) - \dfrac{f(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}$. Then the controllability result of the theorem follows from Theorem 2.1.

Remark 2.2 1. In the case where $f(0) \neq 0$, we can use the control $q(x,t) = v(x,t) - \dfrac{f(\varphi) - f(0)}{\varphi}\mathbf{1}_{E_\varphi}$.

2. The result of Theorem 2.2 extends the results of approximate multiplicative controllability of semilinear systems established in [5] to the case of several dimensions. Moreover, the result of Theorem 2.2 also holds for a nonlinearity $f = f(t,x,y,\nabla y)$ which is globally Lipschitz in y uniformly w.r.t the other parameters (see [13]).

The next result provides a set of states that can be reached with additive controls through the following semilinear system:

$$\begin{cases} y_t = \Delta y + f(y) + u(x,t), & \text{in } Q_T, \\ y(0,t) = 0, & \text{on } \Sigma_T, \\ y(x,0) = y_0(x), & \text{in } \Omega. \end{cases} \tag{21}$$

Corollary 2.1 *Let assumptions of Theorem 2.1 hold. Then for any $T > 0$, there exists a control $u \in L^2(0,T;L^2(\Omega))$ for which the respective solution to (21) satisfies $y(T) = y_d$.*

Proof. It suffices to take $u(x,t) = q(x,t)y(x,t)$, where q is the steering control of (1) from y_0 to y_d at T and y is the corresponding solution of (1).

3 Simulation

In this section, we investigate the exact controllability for the one dimensional version of system (1). Note that the approximate controllability of such models has been considered in the bilinear and semilinear context in [4, 5] (see also [14] for different interpretations of these equations).

Let us consider the system (1) with $f(x) = \sin(x), \ x \in \mathbb{R}$. This function constitutes a prototype of (non trivial) smooth Lipschitz nonlinearities that vanish at the origin, which is widely used for illustrative and numerical examples (see for instance [1, 3, 16]). As initial and final data, let us take $y_0 = 10^{-2}(x + 10^{-2})(1.01 - x)$ and $y_d = y_0 e^x$ in $\Omega = (0, 1)$.

Thus, we have $a(x) = \frac{1}{T}\ln\frac{y_d}{y_0} = \frac{x}{T}$ and $g(x) = -\dfrac{\Delta y_d}{y_d} = -1 + \dfrac{4x}{(x + 10^{-2})(1.01 - x)}$.

According to Theorem 2.2, we deduce that for every $T > 0$, there are positive real numbers T_1 and T_2 which are small enough and for which the control

$$q(x,t) = \begin{cases} \dfrac{x}{T_1} - \dfrac{\sin(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}, & (0, T_1), \\ -1 + \dfrac{4x}{(x + 10^{-2})(1.01 - x)} - \dfrac{\sin(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}, & (T_1, T_2), \\ -1 + \dfrac{4x}{(x + 10^{-2})(1.01 - x)} + \dfrac{u(x,t)}{\psi(x,t) + y_d} - \dfrac{\sin(\varphi)}{\varphi}\mathbf{1}_{E_\varphi}, & (T_2, T), \end{cases}$$

($E_\varphi = \{(x,t) : \varphi(x,t) \neq 0\}$) achieves the exact steering of system (1) from y_0 to y_d at T, where φ solves (19) with

$$v(x,t) = \begin{cases} \frac{x}{T_1}, & (0, T_1), \\ -1 + \frac{4x}{(x+10^{-2})(1.01-x)}, & (T_1, T_2), \\ -1 + \frac{4x}{(x+10^{-2})(1.01-x)} + \frac{u(x,t)}{\psi(x,t)+y_d}, & (T_2, T), \end{cases}$$

and $u(x,t)$ is the control of null-controllability of the linear system

$$\begin{cases} \psi_t = \Delta\psi - \psi + u(x,t), & \text{in } \Omega \times (T_2, T), \\ \psi(0,t) = 0, & \text{on } \partial\Omega \times (T_2, T), \\ \psi(T_2) = y(T_2) - y_d, & \text{in } \Omega, \end{cases}$$

and ψ is the corresponding solution.

Here we consider a globally distributed control $u(x,t)$, which can be taken time-independent (see [14], p.57)

$$u(x,t) = -\sum_{k=1}^{\infty} \frac{(\pi^2 k^2 + 1)e^{-(T-T_2)(\pi^2 k^2+1)}}{e^{-(T-T_2)(k^2\pi^2+1)} - 1}\Big(\int_0^\pi (y(\xi, T_2) - y_d(\xi))\varphi_k(\xi)d\xi\Big)\varphi_k(x), \quad (22)$$

where $\varphi_k(x) = \sqrt{2}\sin(k\pi x)$, $k \geq 1$, are the eigenfunctions of A associated to the eigenvalues $\lambda_k = -k^2\pi^2$.

Now, note that system (1) with control $q(x,t)$ and system (19) with control $v(x,t)$ have the same state and it suffices to simulate the latter. We will give simulations for $T = 1$, $T_1 = 0.01$ and $T_2 = 0.02$, and we will follow the three steps given below.

Step 1. Approximate steering: Solve system (1), controlled on the time interval $(0, T_1)$, by $v(x,t) = v_1(x) = \frac{1}{T_1}\ln(\frac{y_d}{y_0}) = 100x$ to get $y(x, T_1)$.

Step 2. Computation of the additive control $u(x,t)$: Solve (19) on the time interval $(0, T_2)$, by taking the control

$$v(x,t) = v_1(x,t) = \begin{cases} 100x, & (0, T_1), \\ -1 + \dfrac{4x}{(x + 10^{-2})(1.01 - x)}, & (T_1, T_2). \end{cases}$$

This gives $y(\xi, T_2)$, which enables us to compute the control $u(x,t)$ using the formula (22).

Step 3. Exact steering: Consider the solution ψ:

$$\psi(x,t) = e^{T_2 - t} S(t - T_2) y(x, T_2) + \int_{T_2}^{t} e^{-(t-s)} S(t-s) u(x,s) ds$$

of the equation

$$\psi_t = \Delta\psi - \psi + u(x,t), \ t \in (T_2, T),$$

with $\psi(T_2^+) = y(T_2^-)$ as the initial state. Then, we use the relation $y(x,t) = \psi(x,t) + y^d, \ t \in (T_2, T)$ to get $y(x,T) = y_d$. Below are the figures corresponding to the exact steering with the error function.

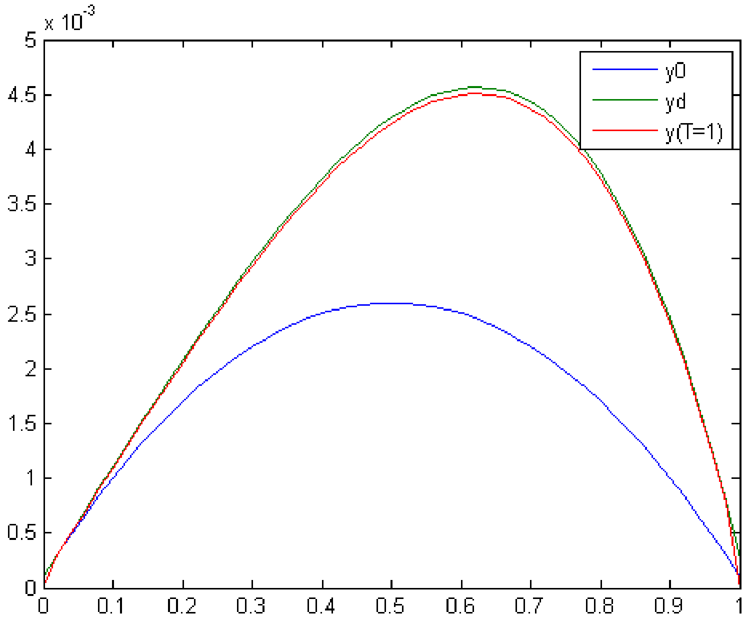

Figure 1: The evolution of the state at T.

• Figure 1 reflects the exact steering and shows that the trajectory $y(t)$ becomes very close to the desirable state at time T.

• Figure 2 describes the variation of the error function defined by $E(t) = \|y(t) - y_d\|, t \in [0,T]$, and shows that $E(t)$ tends to zero when t becomes close to the time of steering T.

4 Conclusion

In the present paper, the multiplicative controllability of a semilinear reaction-diffusion equation is considered in several space dimensions. The approach is constructive and consists in using a set of three controls applied subsequently in time. First, a static

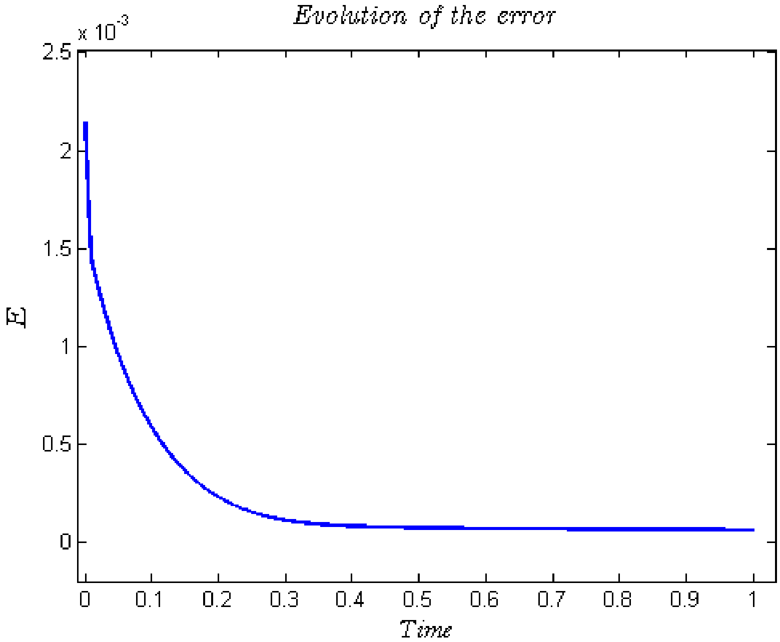

Figure 2: The variation of the error.

control is used to achieve the approximate steering in L^2 at a small time T_1. Then, a second static control is used in a small time interval $[T_1, T_2]$ so that the approximate steering becomes in L^∞ sense. Finally, in the remaining time interval $[T_2, T]$, we exploit a $(x, t)-$dependent control law that ensures the zero controllability of an appropriate linear system (with an additive control) to guarantee the exact steering. Furthermore, the considered methods allow us to achieve the approximate and exact steering (for a given couple of the initial and desirable states) at arbitrary small control time which can be fixed in advance.

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The Analysis of Demand and Supply of Blood in Hospital in Surabaya City Using Panel Data Regression

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Received: October 11, 2021; Revised: November 25, 2022

Abstract: Blood is a vital component in body health because it distributes oxygen, food, and hormones in the whole body. However, there are some cases such as the lack of blood, accidents, or other diseases when humans need blood transfusions, which depend on the demand and supply of blood in hospitals. In this research, panel data regression is used to analyse the demand and supply of blood in hospitals in Surabaya city. There are three models in panel data regression, namely, common effect (CE), fixed effect (FE), and random effect (RE). In this panel data regression, the number of demands of blood type O, A, B, and AB is the independent variable. In contrast, the blood supply is the dependent variable. First, we will determine the best model, common effect (CE), fixed effect (FE), or random effect (RE), through the Chow test, Hausman test, and Lagrange Multiplier test. From the result, the best model of the quantity of blood supply is fixed effect (FE). Then, the fixed effect (FE) model parameters are tested by using the F-test and T-test for testing the impact of independent variables on the dependent variable and R-squared for finding the proportion of effectiveness of independent variables. According to our simulation results, the R-squared is 0.998, which is very satisfactory.

Keywords: *panel data regression; demand and supply of blood; fixed effect model; statistics.*

Mathematics Subject Classification (2010): 62J02, 62J05, 62J07, 62M10.

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1 Introduction

Blood is a vital component in body health because it distributes oxygen, food, and hormones in the whole body. However, there are some cases such as the lack of blood, accidents, or other diseases when a human needs blood transfusion which depends on the demand and supply of blood in hospital [1, 2].

In this research, panel data regression is used to analyse the demand and supply of blood in hospitals in Surabaya city. Panel data is the data combining time-series data and cross-section data. Time-series data cover an object for an extended period. Cross-section data consist of many things such as a company, factory, restaurant, a place with some attributes. Thus, panel data regression is the regression using panel data. There are three models in panel data regression, namely, common effect (CE), fixed effect (FE), and random effect (RE).

From the previous research, the effects of independent variables and dependent variables have been applied by the correlation method in a Neural Network (NN) [3, 4] and Adaptive Neuro-Fuzzy Inference System (ANFIS) [5, 6]. There is a work on stability analysis of stochastic neural networks [7]. Let us also mention the results on control design using Sliding PID [8] and Linear Quadratic Regulator [9].

In this panel data regression, the number of demands of blood type O, A, B, AB is the independent variable, while blood supply is the dependent variable.

First, we determine the best model, common effect (CE), fixed effect (FE), or random effect (RE), through the Chow test for determining a better model between the common effect (CE) model and the fixed effect (FE) model in the panel data model, the Hausman test for determining a better model between the random effect (RE) model and the fixed effect (FE) model in the panel data model, and the Lagrange Multiplier test for determining a better model between the common effect (CE) model and the random effect (RE) model in the panel data model. Then, the simulation results are applied by EViews software.

From the result, the best model of the quantity of blood supply is fixed effect (FE). The F-test and T-test test the parameters of the fixed effect (FE) model for testing the impact of independent variables on the dependent variable, and R-squared is used for finding the proportion of effectiveness of independent variables.

2 Panel Data Modeling

Panel data is the data combining time-series data and cross-section data. Time-series data cover an object for a long time. Cross-section data consist of many entities (for example, a company, factory, restaurant, place) with some attributes (for example, cost, benefit, the volume of production, the number of workers) in one period. Thus, panel data regression is the regression using panel data.

The regression model of time-series data is as follows:

$$Y_t = \alpha + \sum_{j=1}^{P} \beta^j x_t^j + \varepsilon_t, \qquad t = 1, 2, ..., T,$$

where T is the number of time-series data and P is the number of independent variables.

The regression model of cross-section data is as follows:

$$Y_i = \alpha + \sum_{j=1}^{P} \beta^j x_i^j + \varepsilon_i, \qquad i = 1, 2, ..., N,$$

where N is the number of cross-section data and P is the number of independent variables. So the regression model of panel data is as follows:

$$Y_{it} = \alpha + \sum_{j=1}^{P} \beta^j x_{it}^j + \varepsilon_{it}, \qquad t = 1, 2, ..., T, \qquad i = 1, 2, ..., N,$$

where N is the number of cross-section data, T is the number of time-series data, and P is the number of independent variables.

3 Estimation of Panel Data Regression

For estimating the parameters of the panel data model, there are three techniques.

3.1 Common Effect (CE) model (Pooled model)

In this model, time-series data and cross-section data are merged. By joining both of them, one can use the Ordinal Least Square (OLS) method or the least square technique to estimate the data panel model. It is assumed that the properties of data of the objects are similar in the interval of time [10].

However, this assumption deviates from the actual conditions because the characteristics of the objects are very different. Therefore, this model can be constructed as follows:

$$Y_{it} = \alpha + \sum_{j=1}^{P} \beta^j x_{it}^j + \varepsilon_{it}, \qquad t = 1, 2, ..., T, \qquad i = 1, 2, ..., N,$$

where N is the number of cross-section data, T is the number of time-series data, P is the number of independent variables, Y_{it} is the dependent variable of the i-th object in the t-th time, x_{it}^j is the j-th independent variables of the i-th object in the t-th time, β^j is the coefficient (parameter) of the j-th independent variables, α is the intercept, ε_{it} is the error component of the i-th object in the t-th time.

3.2 Fixed Effect (FE) model

This model estimates panel data by adding dummy variables. There are different effects among objects through the difference of their intercepts. In the fixed effect (FE) model, an object is an unknown parameter, and it will be estimated by dummy variables. This model can be constructed as follows [10]:

$$Y_{it} = \alpha + \sum_{j=1}^{P} \beta^j x_{it}^j + \sum_{k=2}^{n} \alpha_k D_k + \varepsilon_{it}, \qquad t = 1, 2, ..., T, \qquad i = 1, 2, ..., N,$$

where D_k is the dummy variable.

3.3 Random Effect (RE) model

In this method, the differences in object and time characteristics are formed by the error from the model. Because two components contribute to the error results, such as object and time, this method needs to be expanded to become the error from the object component, the error from the time component, and the combined error. The random effect (RE) model is as follows [11]:

$$Y_{it} = \alpha + \sum_{j=1}^{P} \beta^j x_{it}^j + \varepsilon_{it}, \qquad t = 1, 2, ..., T, \qquad i = 1, 2, ..., N,$$

where $\varepsilon_{it} = u_i + v_t + w_{it}$, u_i is the error from the object component, v_t is the error from the time component, w_{it} is the combined error.

4 The Selection of Best Model

For selecting the best model, the common effect (CE) model, fixed effect (FE) model or random effect (RE) model, there are some tests such as the Chow test, Hausman test, and Lagrange Multiplier test.

4.1 Chow test

The Chow test is used for determining a better model between the common effect (CE) model and the fixed effect (FE) model in the panel data model [12].

The hypothesis used in the Chow test is as follows. The null hypothesis (H_0) represents the common effect (CE) model, whereas the alternative hypothesis (H_1) represents the fixed effect (FE) model. The Chow statistics is given by

$$\frac{(ESS1 - ESS2)/(N-1)}{(ESS2)/(NT - N - P)},$$

where

$ESS1$: Residual Sum Square of the fixed effect (FE) model,
$ESS2$: Residual Sum Square of the common effect (CE) model,
N : the number of cross-section data,
T : the number of time-series data,
P : the number of independent variables.

The Chow statistics follows the F-statistics distribution with the degree of freedom $(N - 1, NT - N - P)$. If the Chow statistics is larger than the critical value of the F-statistics distribution or the p-value is less than the significance level α, then H_1 is accepted and H_0 is rejected so that the selected model is the fixed effect (FE) model.

4.2 Hausman test

The Hausman test is used for determining a better model between the random effect (RE) model and the fixed effect (FE) model in the panel data model [12].

The hypothesis used in the Hausman test is as follows. The null hypothesis (H_0) represents the random effect (RE) model, whereas the alternative hypothesis (H_1) represents the fixed effect (FE) model. The m statistics is given by

$$m = (\beta - b)(M0 - M1)^{-1}(\beta - b),$$

where:

β : Statistics vector of fixed effect (FE) variables,
b : Statistics vector of random effect (RE) variables,
$M0$: covariance matrix for the fixed effect (FE) model,
$M1$: covariance matrix for the random effect (RE) model.

The m statistics follows the chi-square distribution with the degree of freedom equal to P. If m statistics is larger than the critical value of the chi-square distribution or the p-value is less than the significance level α, then H_1 is accepted and H_0 is rejected so that the selected model is the fixed effect (FE) model.

4.3 Lagrange Multiplier test

The Lagrange Multiplier test is used to determine a better model between the common effect (CE) model and the random effect (RE) model in the panel data model [13].

The hypothesis used in the Lagrange Multiplier test is as follows. The null hypothesis (H_0) represents the common effect (CE) model, whereas the alternative hypothesis (H_1) represents the random effect (RE) model. The LM statistics is given by

$$LM = \frac{NT}{2(T-1)} \left[\frac{\sum_{i=1}^{N} \left(\sum_{t=1}^{T} \bar{e}_{it} \right)^2}{\sum_{i=1}^{N} \sum_{t=1}^{T} e_{it}^2} - 1 \right],$$

where

N : the number of cross-section data,
T : the number of time-series data,
e_{it} : residual of the common effect (CE) model.

The LM statistics follows the chi-square distribution with the degree of freedom equal to P.

If the LM statistics is larger than the critical value of the chi-square distribution or the p-value is less than the significance level α, then H_1 is accepted and H_0 is rejected so that the selected model is the random effect (RE) model.

The Lagrange Multiplier test is not applied when the Chow test and the Hausman test show a better model is the fixed effect (FE) model [10].

5 Significance Test

After the best model is obtained, it is required to apply the significance test as follows.

5.1 F-test

The F-test is applied for testing the estimation results on whether the independent variables have effects on the dependent variable globally [14].

The hypothesis used in the F-test is as follows. The null hypothesis (H_0) represents "independent variables do not affect the dependent variable", whereas the alternative hypothesis (H_1) represents "independent variables affect the dependent variable".

The F-statistics is given by

$$F_{test} = \frac{MS(y)}{MS(e)},$$

where

$MS(e)$: mean square of regression,
$MS(y)$: mean square of residual.

The F_{test} statistics follows the F-statistics distribution with the degree of freedom equal to $(N + P - 1, NT - N - P)$. If the F_{test} statistics is larger than the critical value of the F-statistics distribution or the p-value is less than the significance level α, then H_1 is accepted and H_0 is rejected so that the conclusion is that there is the effect of the independent variables on the dependent variable.

5.2 T-test

The T-test is applied for testing the estimation results on whether the independent variables have effects on the dependent variable partially [14].

The hypothesis used in the T-test is as follows. The null hypothesis (H_0) represents "independent variables do not affect the dependent variable", whereas the alternative hypothesis (H_1) represents "independent variables affect the dependent variable".

The T-statistics is given by

$$T_{test} = \frac{\hat{\beta}_k}{SE(\hat{\beta}_k)},$$

where

$\hat{\beta}_k$: the k-th parameter,
$SE(\hat{\beta}_k)$: standard deviation of the k-th parameter.

The T_{test} statistics follows the T-statistics distribution with the degree of freedom equal to $(NT - N - P)$. If the T_{test} statistics is larger than the critical value of the T-statistics distribution or the p-value is less than the significance level $\alpha/2$, then H_1 is accepted and H_0 is rejected so that the conclusion is that there is the effect of the independent variables on the dependent variable.

5.3 R-squared

The determination coefficient (R^2) is used for measuring the fitness rate of panel data regression. It is a proportion of the contribution of independent variables and dummy variables to that of the dependent variable [15].

The coefficient (R^2) is determined using

$$R^2 = \frac{ESS}{TSS},$$

where

ESS : sum of square of regression,
TSS : total of sum of square.

The value of R^2 is between 0 and 1. If R^2 approaches 1, then in this model, the effect of independent variables is stronger.

6 Results

This research shows the effects of the demand for blood types O, A, B, AB on the quantity of blood supply in five hospitals in Surabaya city from January 2015 until December 2017. In creating panel data regression, the data used are as follows. Cross-section or object data:

1. Angkatan Laut hospital,
2. Unair hospital,
3. Haji hospital,
4. Adi Husada hospital,
5. Darmo hospital.

The data used are monthly data from January 2015 until December 2017.

In this research, the analysis of demand and supply of blood in some hospitals in Surabaya city is done by panel data regression using EViews software. There are three models in panel data regression, such as common effect (CE), fixed effect (FE), and random effect (RE). In this panel data regression, the number of demands of blood type O, A, B, AB is an independent variable, while blood supply is the dependent variable.

6.1 The selection of best model

First, we use the Chow test to determine a better model between the common effect (CE) model and the fixed effect (FE) model in the panel data model.

The Chow test results can be seen in Figure 1. Figure 1 shows the p-value of cross-section F is $0.0039 < 0.05$. Therefore, H_1 is accepted and H_0 is rejected so that the selected model is the fixed effect (FE) model.

Second, we use the Hausman test to determine a better model between the random effect (RE) model and the fixed effect (FE) model in the panel data model.

The Hausman test results can be seen in Figure 2. Figure 2 shows the p-value of cross-section random is $0.003 < 0.05$. Therefore, H_1 is accepted and H_0 is rejected so that the selected model is the fixed effect (FE) model.

Because both the Chow and the Hausman test show that the fixed effect (FE) model is the best model, the Lagrange Multiplier test is not required.

Redundant Fixed Effects Tests
Equation: Untitled
Test cross-section fixed effects

Effects Test	Statistic	d.f.	Prob.
Cross-section F	4.003197	(4,171)	0.0039
Cross-section Chi-square	16.112418	4	0.0029

Figure 1: Chow test result.

Correlated Random Effects - Hausman Test
Equation: Untitled
Test cross-section random effects

Test Summary	Chi-Sq. Statistic	Chi-Sq. d.f.	Prob.
Cross-section random	16.012786	4	0.0030

Figure 2: Hausman test result.

The fixed effect (FE) panel data regression model for the quantity of blood supply used is shown in Figure 3. The coefficient of the quantity of blood type O is 0.982566, the coefficient of the quantity of blood type A is 1.003406, the coefficient of the quantity of blood type B is 0.976337, the coefficient of the quantity of blood type AB is 0.990383, the intercept is 1.376641.

Dependent Variable: S_BLOOD
Method: Panel Least Squares
Date: 06/19/20 Time: 11:36
Sample: 2015M01 2017M12
Periods included: 36
Cross-sections included: 5
Total panel (balanced) observations: 180

Variable	Coefficient	Std. Error	t-Statistic	Prob.
BLOOD_O	0.982566	0.009145	107.4399	0.0000
BLOOD_A	1.003406	0.013104	76.57403	0.0000
BLOOD_B	0.976337	0.008611	113.3822	0.0000
BLOOD_AB	0.990383	0.027919	35.47346	0.0000
C	1.376641	0.951421	1.446931	0.1497

Figure 3: Fixed effect (FE) model panel data regression.

6.2 Significance test

In fixed effect (FE) model panel data regression, we will test the impact of independent variables on the dependent variables using the F-test and T-test. Furthermore, we also measure the proportion of the independent and dummy variables' contribution and that of dependent variables using R-squared.

The F-test is applied for testing the estimation results on whether the independent variables have effects on the dependent variables globally. The F-test results can be seen in Figure 4. Figure 4 shows the p-value of the F-test is $0.000 < 0.05$. Therefore, H_1 is accepted and H_0 is rejected, i.e., there is an effect of the independent variables on the dependent variables.

Effects Specification

Cross-section fixed (dummy variables)			
Root MSE	3.490123	R-squared	0.998008
Mean dependent var	193.3889	Adjusted R-squared	0.997915
S.D. dependent var	78.41534	S.E. of regression	3.580790
Akaike info criterion	5.437751	Sum squared resid	2192.572
Schwarz criterion	5.597399	Log likelihood	-480.3976
Hannan-Quinn criter.	5.502481	F-statistic	10708.81
Durbin-Watson stat	1.469845	Prob(F-statistic)	0.000000

Figure 4: F-test results.

The T-test is applied for testing the estimation results on whether the independent variables have effects on the dependent variables partially. The T-test results can be seen in Figure 5. Figure 5 shows the p-value of the T-test on the number of demands of blood type O is $0.000 < 0.05$, the number of requests of blood type A is $0.000 < 0.05$, the number of requests of blood type B is $0.000 < 0.05$, the number of demands of blood type AB is $0.000 < 0.05$. Therefore, H_1 is accepted and H_0 is rejected, i.e., there is an effect of the independent variables on the dependent variables.

Dependent Variable: S_BLOOD
Method: Panel Least Squares
Date: 06/19/20 Time: 11:36
Sample: 2015M01 2017M12
Periods included: 36
Cross-sections included: 5
Total panel (balanced) observations: 180

Variable	Coefficient	Std. Error	t-Statistic	Prob.
BLOOD_O	0.982566	0.009145	107.4399	0.0000
BLOOD_A	1.003406	0.013104	76.57403	0.0000
BLOOD_B	0.976337	0.008611	113.3822	0.0000
BLOOD_AB	0.990383	0.027919	35.47346	0.0000
C	1.376641	0.951421	1.446931	0.1497

Figure 5: T-test results.

R-squared is used to measure the proportion of the contribution of independent and dummy variables and that of the dependent variables. The R-squared test can be seen in Figure 6. Figure 6 shows R-squared is 0.998. It means that the effects of the independent variables on the dependent variables are 99.8%.

7 Conclusions

There are three models in panel data regression, namely, common effect (CE), fixed effect (FE), and random effect (RE). In this panel data regression, the number of demands of

Effects Specification

Cross-section fixed (dummy variables)			
Root MSE	3.490123	R-squared	0.998008
Mean dependent var	193.3889	Adjusted R-squared	0.997915
S.D. dependent var	78.41534	S.E. of regression	3.580790
Akaike info criterion	5.437751	Sum squared resid	2192.572
Schwarz criterion	5.597399	Log likelihood	-480.3976
Hannan-Quinn criter.	5.502481	F-statistic	10708.81
Durbin-Watson stat	1.469845	Prob(F-statistic)	0.000000

Figure 6: T-test results.

blood type O, A, B, AB is an independent variable, while blood supply is the dependent variable. First, we determine the best model, common effect (CE), fixed effect (FE), or random effect (RE), through the Chow test, Hausman test, and Lagrange Multiplier test. From the result, the best model of the quantity of blood supply is fixed effect (FE). Then, the parameters of the fixed effect (FE) model are tested by the F-test and T-test for testing the impact of the independent variables on the dependent variable and R-squared is used for finding the proportion of effectiveness of the independent variables. In our simulation, the R-squared is 0.998, which is a very good result. As a future work, we are planning to employ some machine learning techniques to analyze the demand and supply of blood. Furthermore, by combining mathematical science and business management, we strive to provide a feedback for stakeholders before making any decision.

Acknowledgment

This research was supported by Kemendikbud with contract number 313/E4.1/AK.04.PT/2021,027/AMD-SP2H/LT-MULTI-TERAPAN/LL7/2021, 895/UNUSA/Adm-LPPM/VII/2021 year 2021 and thanks to LPPM Universitas Nahdlatul Ulama Surabaya.

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Nonlinear Dynamics and Systems Theory, **22** (5) (2022) 561–572

Effect of Water Scarcity in the Society: A Mathematical Model

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Received: July 21, 2021; Revised: November 14, 2022

Abstract: Water scarcity is one of the major problems faced by all those living around the world. So, there should be a multiple way approach to be adopted to conquer the water scarcity effects in future. Keeping this in mind, we developed a mathematical model and demonstrated the effect of water scarcity through a deterministic and stochastic formats. The equilibrium point of the model is found and its stability is analyzed analytically. Numerical simulation of both the deterministic and the stochastic model is exhibited to validate our analytical findings. The attainment level of the equilibrium point is demonstrated by using the Runge-Kutta method. The comparison is also made for this equilibrium. The effect of few parameters of the model was exhibited in different figures in the numerical simulation section. Particularly the effect of the water draining rate and the rate of human population affected by water scarcity on each compartment were shown visually through plotting time vs particular compartments. Our results show the better ways for water recovery through the compartments of the model.

Keywords: *water scarcity; local stability; global stability; stochastic model.*

Mathematics Subject Classification (2010): 70Kxx; 70K42; 70K70; 93-XX; 93E03; 93C10; 93C15; 34D20; 34D23; 65C30.

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1 Introduction

Water shortages are a severe shortage in which the rates of water availability do not meet certain basic requirements specified. Water is one of the most essential natural renewable resources, and no one, neither humans nor animals, can live without it. Water comes from numerous sources, including runoff, groundwater, and surface water. The main contributor to the world growth and development are water supplies [1].

The paper concludes that there is a fixed amount of water on our planet. But so little of it is at our disposal to use. 70 percent of the earth surface is filled with 1400 million cubic kilometers of water (m km3): 2.5% is freshwater and 97.5% is saltwater, 2.5 percent is groundwater, 0.3 percent are lakes and rivers, 68.9 percent is frozen in ice caps. One-third of the population of the world currently resides in countries where the quality of the water is not adequately compromised, but by 2025, it is projected to increase by two-thirds [2].

The primary objective of this paper is to determine the scarcity of water in selected Middle East countries. For Iran, Iraq, and Saudi Arabia, the Anomaly Standardized Precipitation (WASP) index was spatially computed from 1979 to 2017. The water scarcity situation has been investigated in cities with a population of more than one million. This was done by using the methodology of the composite index to make water-related statistics more intelligible. A forecast was created for the years 2020 to 2030 to show potential improvements in the supply and demand for water in selected Middle East countries. With rising urbanization, there is a moderate to high water shortage risk for all countries at present [3]. Water shortage is a common issue in many parts of the world. Many previous water shortage evaluation strategies only considered the volume of water, and overlooked the quality of water. Moreover, the Environmental Flow Criterion (EFR) was not usually considered directly in the evaluation. In this paper, we have developed an approach to assess water scarcity by considering both water quantity and quality [4].

The formulation of a corruption control model and its analysis using the theory of differential equations are presented in paper [5]. The equilibria of the model and the stability of these equilibria are discussed in detail. Yadav, A. et al. [6] propose and evaluate mathematical models to research the dynamics of smoking activity under the influence of educational programs and also the willingness of the person to quit smoking. A nonlinear mathematical model is formulated and analyzed in paper [7] to research the relationship between the criminal population and non-criminal population by taking into account the rate of non-monotone incidence. See also [8, 9].

[10] suggested and analyzed a mathematical model using oncolytic virotherapy for cancer care. The growth of tumor cells is presumed to obey logistic growth and the interaction between tumor cells and viruses is of saturation type. Several nonlinear mathematical models are proposed and analyzed in paper [11] to study the spread of asthma due to inhaled industrial pollutants [12, 13] are also referenced.

This paper aims to illustrate the requirements to and the availability of water. As a result of growing population, rising urbanization, and rapid industrialization, combined with the need to increase agricultural production, water demand has been found to increase significantly. Water per capital supply is also slowly declining. More than 2.2 million people are expected to die every year from diseases related to polluted drinking water and poor sanitation.

As mentioned above, we have analyzed and proved that water scarcity is one of the major problems that has been proved statistically and theoretically. We are here giving a new try to prove the same by using the mathematical model.

Using the principle of an ordinary differential equation, we analyze our model and record comprehensive results of numerical simulations to support the analytical results. First, our model is expanded to the model of stochastic differential equations. The outcomes of deterministic and stochastic models were also compared. The remainder of this paper is structured as follows, Section 2 explains the model and the presence of equilibria and illustrates local stability, global equilibrium stability. Section 3 addresses the remaining stochastic model. Section 4 displays the effects of simulation for deterministic and stochastic models. Our results are summarized in Section 5 as a conclusion.

2 The Model and Analysis

We proposed and analyzed a non linear model for water scarcity by dividing into four different compartments [14], namely, the total usage of water (W), the human (H), water scarcity (W_s), water recover (W_r). All variables are time t functions. The transfer diagram of the model is described in Figure 1. The mathematical model is suggested as follows, in view of the above considerations:

$$\begin{aligned} \frac{dW}{dt} &= \Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r, \\ \frac{dH}{dt} &= \alpha_2 WH - \beta H - \mu H - \mu_1 H, \\ \frac{dW_s}{dt} &= \alpha_1 W + \beta H - \delta_1 W_s, \\ \frac{dW_r}{dt} &= \delta_1 W_s - \delta_2 W_r. \end{aligned} \tag{1}$$

In Table 1, the parameters used in model (1) are defined.

Table 1: Description of parameters.

Parameter	Description
Λ	Recruitment rate
α_1	Water draining rate
α_2	The rate of consumption of water by a human
δ_1	The recovery rate of water resource
δ_2	The rate at which water becomes normal level water
β	Rate of human population affected by water scarcity
μ	Natural death rate
μ_1	Death rate due to water scarcity

2.1 Existence of equilibria

Our model's equilibrium is calculated by setting the right-hand side of the model to zero [15]. The system has the following equilibria, namely, the endemic equilibrium (EE)

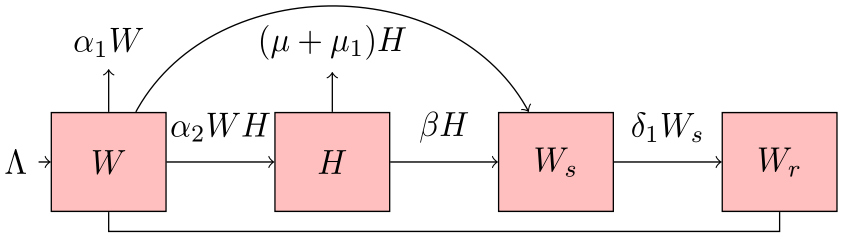

Figure 1: Transfer Diagram of the Model.

$E^*\left(W^*, H^*, W_s^*, W_r^*\right)$, where

$$W^* = \frac{k_1}{\alpha_2}, \tag{2}$$

$$H^* = \frac{\Lambda}{k_1-\beta}, \tag{3}$$

$$W_s^* = \frac{\Lambda\alpha_2\beta - \alpha_1 k_1\beta + \alpha_1 k_1^2}{\delta_1\alpha_2(k_1-\beta)}, \tag{4}$$

$$W_r^* = \frac{\Lambda\alpha_2\beta - \alpha_1 k_1\beta + \alpha_1 k_1^2}{\delta_2\alpha_2(k_1-\beta)}, \tag{5}$$

where $\quad k_1 = \beta+\mu+\mu_1$.

2.2 Stability analysis

The system's variational matrix is given by

$$M = \begin{pmatrix} -(\alpha_1+\alpha_2 H) & -\alpha_2 W & 0 & \delta_2 \\ \alpha_2 H & \alpha_2 W - k_1 & 0 & 0 \\ \alpha_1 & \beta & -\delta_1 & 0 \\ 0 & 0 & \delta_1 & -\delta_2 \end{pmatrix}.$$

2.2.1 Stability analysis of EE point

The variation matrix M* corresponding to the point E^* of the endemic equilibrium, is given by

$$M^* = \begin{pmatrix} n_{11} & n_{12} & 0 & n_{14} \\ n_{21} & n_{22} & 0 & 0 \\ n_{31} & n_{32} & n_{33} & 0 \\ 0 & 0 & n_{43} & n_{44} \end{pmatrix},$$

where

$n_{11} = -(\alpha_1+\alpha_2 H), \quad n_{12} = -\alpha_2 W, \quad n_{14} = \delta_2$
$n_{21} = \alpha_2 H, \quad n_{22} = \alpha_2 W - k_1$
$n_{31} = \alpha_1, \quad n_{32} = \beta, \quad n_{33} = -\delta, n_{43} = \delta_1, \quad n_{44} = -\delta_2.$

The bi-quadratic equation is

$$\lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where

$$\begin{aligned}
a_1 &= -(n_{11} + n_{22} + n_{33} + n_{44}),\\
a_2 &= n_{11}n_{22} + n_{22}n_{33} + n_{33}n_{44} + n_{11}n_{33} + n_{11}n_{44} + n_{22}n_{44} - n_{12}n_{21},\\
a_3 &= -n_{11}n_{22}n_{33} - n_{11}n_{22}n_{44} - n_{11}n_{33}n_{44} - n_{22}n_{33}n_{44} + n_{12}n_{21}n_{33}\\
&+ n_{12}n_{21}n_{44} - n_{14}n_{43}n_{31},\\
a_4 &= n_{11}n_{22}n_{33}n_{44} + n_{14}n_{22}n_{31}n_{43} - n_{14}n_{21}n_{32}n_{43} - n_{12}n_{21}n_{33}n_{44}.
\end{aligned}$$

E^* will be locally asymptotically stable by using the Routh-Hurwitz criteria if the following conditions are satisfied: $a_1 > 0$, $a_3 > 0$, $a_1a_2a_3 - a_3^2 - a_1^2a_4 > 0$, $a_3 > 0$.

If two other inequalities referred to above are satisfied, E^* is locally asymptotically stable [16].

2.2.2 Global stability of endemic equilibrium

In order to analyze the global stability of the endemic equilibrium E^*, we adopt the approach developed by Korobeinikov [8] and successfully applied in [9]. E^* exists for all $x, y, z, u > \epsilon$, for some $\epsilon > 0$.

Let $k_1y = [\beta+\mu+\mu_1]y = g(x, y, z, u)$ be positive and monotonic functions in $\mathbb{R}^4_+$ (for more details, see [8, 9]).

$$\begin{aligned}
V(x,y,z,u) &= x - \int_\epsilon^x \frac{g(x^*,y^*,z^*,u^*)}{g(\eta,y^*,z^*,u^*)}\,d\eta + y - \int_\epsilon^y \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,\eta,z^*,u^*)}\,d\eta\\
&+z - \int_\epsilon^z \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,y^*,\eta,u^*)}\,d\eta + u - \int_\epsilon^w \frac{g(x^*,y^*,z^*,u^*)}{h(x^*,y^*,z^*,\eta)}\,d\eta. \qquad (6)
\end{aligned}$$

If $g(x, y, z, u)$ is monotonic with respect to its variables, then the state E is the only extreme and the global minimum of this function. So, obviously,

$$\begin{aligned}
&\frac{\partial V}{\partial x} = 1 - \frac{g(x^*,y^*,z^*,u^*)}{g(x,y^*,z^*,u^*)}, \frac{\partial V}{\partial y} = 1 - \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,y,z^*,u^*)},\\
&\frac{\partial V}{\partial z} = 1 - \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,y^*,z,u^*)}, \frac{\partial V}{\partial u} = 1 - \frac{g(x^*,y^*,z^*,u^*)}{g(x^*,y^*,z^*,u)}. \qquad (7)
\end{aligned}$$

The functions $g(x, y, z, u)$ and $g(x, y, z, u)$ grow monotonically, then have only one stationary point. Further, since

$$\begin{aligned}
\frac{\partial^2 V}{\partial x^2} &= \frac{g(x^*,y^*,z^*,u^*)}{[g(x,y^*,z^*,u^*)]^2}.\frac{g(x,y^*,z^*,u^*)}{\partial x},\\
\frac{\partial^2 V}{\partial y^2} &= \frac{g(x^*,y^*,z^*,u^*)}{[g(x^*,y,z^*,u^*)]^2}.\frac{g(x^*,y,z^*,u^*)}{\partial y},\\
\frac{\partial^2 V}{\partial z^2} &= \frac{g(x^*,y^*,z^*,u^*)}{[g(x^*,y^*,z,u^*)]^2}.\frac{g(x^*,y^*,z,u^*)}{\partial z},
\end{aligned}$$

$$\frac{\partial^2 V}{\partial u^2} = \frac{g(x^*, y^*, z^*, u^*)}{[g(x^*, y^*, z^*, u)]^2} . \frac{g(x^*, y^*, z^*, u)}{\partial u}$$

are non negative, $g(x, y, z, u)$ and $h(x, y, z, u)$ have minimum. That is,

$$V(x, y, z, u) \geq V(x^*, y^*, z^*, u^*)$$

and hence, V is a Lyapunov function, and its derivative is given by

$$\begin{aligned}
\frac{dV}{dt} &= x' - x' \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)} + y' - y' \frac{h(x^*, y^*, z^*, u^*)}{g(x^*, y, z^*, u^*)} + z' - z' \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y^*, z, u^*)} + \\
&\quad u' - u' \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u)} \\
&= \alpha_1 x^* (1 - \frac{x}{x^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)}\right) - \delta_2 u^* (1 - \frac{u}{u^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)}\right) \\
&\quad + k_1 y^* (1 - \frac{y}{y^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y, z^*, u^*)}\right) - \alpha_1 x^* (1 - \frac{x}{x^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y^*, z, u^*)}\right) \\
&\quad - \beta y^* (1 - \frac{y}{y^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y^*, z, u^*)}\right) + \delta_1 z^* (1 - \frac{z}{z^*}) \left(1 - \frac{h(x^*, y^*, z^*, u^*)}{h(x^*, y^*, z, u^*)}\right) \\
&\quad - \delta_1 z^* (1 - \frac{z}{z^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y^*, z^*, u)}\right) + \delta_2 u^* (1 - \frac{u}{u^*}) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y^*, z^*, u)}\right) \\
&\quad + g(x^*, y^*, z, u) \left(1 - \frac{g(x, y, z, u)}{g(x^*, y^*, z, u)}\right) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x, y^*, z^*, u^*)}\right) \\
&\quad - g(x^*, y^*, z, u) \left(1 - \frac{g(x, y, z, u)}{g(x^*, y^*, z, u)}\right) \left(1 - \frac{g(x^*, y^*, z^*, u^*)}{g(x^*, y, z^*, u^*)}\right). \qquad (8)
\end{aligned}$$

It is noted here that $g(x^*, y^*, z^*, u^*) = h(x^*, y^*, z, u)$ is explicitly given as g and h in terms of x, y, z and u.

Since $E > 0$, the function $g(x, y, z, u)$ is concave with respect to y, z and u and

$$\frac{\partial^2 g(x, y, z, u)}{\partial y^2} \leq 0, \quad \frac{\partial^2 g(x, y, z, u)}{\partial z^2} \leq 0,$$

then $\frac{dV}{dt} \leq 0$ for all $x, y, z, u > 0$. Also, the monotonicity of $g(x, y, z, u)$ with respect to x, y, z and u ensures that

$$\begin{array}{c}
(1 - \frac{x}{x^*}) \left(1 - \frac{g(x^*,y^*,z^*,u^*)}{g(x,y^*,z^*,u^*)}\right) \leq 0, (1 - \frac{y}{y^*}) \left(1 - \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,y,z^*,u^*)}\right) \leq 0, \\
(1 - \frac{z}{z^*}) \left(1 - \frac{h(x^*,y^*,z^*,u^*)}{h(x^*,y^*,z,u^*)}\right) \leq 0, (1 - \frac{u}{u^*}) \left(1 - \frac{g(x^*,y^*,z^*,u^*)}{g(x^*,y^*,z^*,u)}\right) \leq 0
\end{array} \qquad (9)$$

holds for all $x, y, z, u > 0$. Thus, we establish the following result.

Theorem 2.1 *The endemic equilibrium E^* of model (1) is globally asymptotically stable whenever conditions outlined in Eq.* (9) *are satisfied [17].*

3 Stochastic Model

We are expanding our deterministic model to stochastic systems here, as stochastic models are more able to capture random variations of the biological dynamics of the problem. The derivation of an SDE model is based on the method developed by Yuan et al. [18]. Let $X(t) = (X_1(t), X_2(t), X_3(t), X_4(t))^T$ be a continuous random variable for $(W(t), H(t), W_s(t), W_r(t))^T$ and T denote the transpose of a matrix.

Let $\Delta X = X(t+\Delta t) - X(t) = (\Delta X_1, \Delta X_2, \Delta X_3, \Delta X_4)^T$ denote the random vector for the change in random variables during the time interval Δt. Here, we'll write transition maps that define all possible changes in the SDE model between states. Based on our ODE model system (1), here we see that within a small time interval Δt, there are 9 possible changes between states. Changes in the state and their probabilities are discussed in Table 2. In the case, the state change ΔX is denoted by $\Delta X = (-1, 1, 0, 0)$. The probability of this change is determined by

Prob $(\Delta X_1, \Delta X_2, \Delta X_3, \Delta X_4) = (-1, 1, 0, 0)|(X_1, X_2, X_3, X_4) = P_3 = \alpha_2 X_1 X_2 + o(\Delta t)$ by neglecting terms higher than o(Δ t), the following expectation change $E(\Delta X)$ and its covariance matrix $V(\Delta X)$ associated with ΔX, can be identified. The expectation of ΔX is

$$E(\Delta X) = \sum_{i=1}^{8} P_i (\Delta X)_i \Delta t = \begin{pmatrix} \Lambda - \alpha_1 X_1 - \alpha_2 X_1 X_2 + \delta_2 X_4 \\ \alpha_2 X_1 X_2 - \beta X_2 - \mu X_2 - \mu_1 X_2 \\ \alpha_1 X_1 + \beta X_2 - \delta_1 X_3 \\ \delta_1 X_3 - \delta_2 X_4 \end{pmatrix} \Delta t$$

$$= f(X_1, X_2, X_3, X_4)\Delta \mathrm{t}.$$

Table 2: Possible changes of states and their probabilities.

Possible stage change	Probability of state changes
$(\Delta x)_1 = (1, 0, 0, 0)^T$	$P_1 = \Lambda \Delta t + o(\Delta t)$
$(\Delta x)_2 = (-1, 0, 1, 0)^T$	$P_2 = \alpha_1 X_1 \Delta t + o(\Delta t)$
$(\Delta x)_3 = (-1, 1, 0, 0)^T$	$P_3 = \alpha_2 X_1 X_2 \Delta t + o(\Delta t)$
$(\Delta x)_4 = (1, 0, 0, -1)^T$	$P_4 = \delta_2 X_4 \Delta t + o(\Delta t)$
$(\Delta x)_5 = (0, -1, 1, 0)^T$	$P_5 = \beta X_2 \Delta t + o(\Delta t)$
$(\Delta x)_6 = (0, -1, 0, 0)^T$	$P_6 = \mu X_2 \Delta t + o(\Delta t)$
$(\Delta x)_7 = (0, -1, 0, 0)^T$	$P_7 = \mu_1 X_2 \Delta t + o(\Delta t)$
$(\Delta x)_8 = (0, 0, -1, 1)^T$	$P_8 = \delta_1 X_3 \Delta t + o(\Delta t)$
$(\Delta x)_9 = (0, 0, 0, 0)^T$	$P_9 = (1 - \sum_{i=1}^{8} P_i) + o(\Delta t)$

It can be noted here that the expectation vector and also the function f are in the same form as those of the ODE system (1).

Since the covariance matrix $V(\Delta X) = E((\Delta X)(\Delta X)^T) - E(\Delta X)(E(\Delta X)^T)$ and $E((\Delta X)(\Delta X)^T) = f(X)(f(X)^T)\Delta t$, it can be approximated with the diffusion matrix Ω times Δt by neglecting the term of $(\Delta t)^2$ so that $V(\Delta X) \approx E((\Delta X)(\Delta X)^T)$. That is,

$$E((\Delta X)(\Delta X)^T) = \sum_{i=1}^{8} P_i(\Delta X)_i(\Delta X)_i^T \Delta t = \begin{pmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & 0 \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & 0 & V_{43} & V_{44} \end{pmatrix} .\Delta t = \Omega.\Delta t,$$

where each component of the diffusion matrix of 4×4 is symmetric, positive-definite, and can be obtained by

$$V_{11} = P_1 + P_2 + P_3 + P_4 = \Lambda + \alpha_1 X_1 + \alpha_2 X_1 X_2 + \delta_2 X_4, \qquad V_{34} = V_{43} = -P_4 = -\delta_1 X_3,$$
$$V_{22} = P_3 + P_5 + P_6 + P_7 = \alpha_2 X_1 X_2 + \beta X_2 + \mu X_2 + \mu_1 X_2, \qquad V_{14} = V_{41} = -P_4 = -\delta_2 X_4,$$
$$V_{33} = P_2 + P_5 + P_8 = \alpha_1 X_1 + \beta X_2 + \delta_1 X_3, \qquad V_{13} = V_{31} = -P_2 = -\alpha_1 X_1,$$
$$V_{44} = P_4 + P_8 = \delta_2 X_4 + \delta_1 X_3, \qquad V_{23} = V_{32} = -P_5 = -\beta X_2.$$
$$V_{12} = V_{21} = -P_3 = -\alpha_2 X_1 X_2,$$

A matrix D square root of the symmetric, positive-definite diffusion matrix Ω is such that $K = \Omega^{1/2}$. Use an equivalent matrix K such that $\Omega = KK^T$, where K has the dimension of a 4×7 matrix.

$$K = \begin{pmatrix} \sqrt{\Lambda} & -\sqrt{\alpha_1 X_1} & -\sqrt{\alpha_2 X_1 X_2} & \sqrt{\delta_2 X_4} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\alpha_2 X_1 X_2} & 0 & -\sqrt{\beta X_2} & -\sqrt{(\mu+\mu_1)X_2} & 0 \\ 0 & \sqrt{\alpha_1 X_1} & 0 & 0 & \sqrt{\beta X_2} & 0 & -\sqrt{\delta_1 X_3} \\ 0 & 0 & 0 & -\sqrt{\delta_2 X_4} & 0 & 0 & \sqrt{\delta_1 X_3} \end{pmatrix}.$$

Then, the Ito stochastic differential model has the following form:

$$dX(t) = f(X_1, X_2, X_3, X_4)dt + K.dW(t)$$

with the initial condition $X(0) = (X_1(0), X_2(0), X_3(0), X_4(0))^T$ and a Wiener process, $W(t) = (W_1(t), W_2(t), W_3(t), W_4(t), W_5(t), W_6(t), W_7(t))^T$. We get the stochastic differential equation model as follows:

$$\begin{aligned} dW &= [\Lambda - \alpha_1 W - \alpha_2 WH + \delta_2 W_r]dt + \sqrt{\Lambda}dW_1 - \sqrt{\alpha_1 W}dW_2 - \sqrt{\alpha_2 WH}dW_3 + \sqrt{\delta_2 W_r}dW_4, \\ dH &= [\alpha_2 WH - \beta H - \mu H - \mu_1 H]dt + \sqrt{\alpha_2 WH}dW_3 - \sqrt{\beta H}dW_5 - \sqrt{(\mu+\mu_1)H}dW_6, \\ dW_s &= [\alpha_1 W + \beta H - \delta W_s]dt + \sqrt{\alpha_1 W}dW_2 + \sqrt{\beta H}dW_5 - \sqrt{\delta_1 W_s}dW_7, \\ dW_r &= [\delta_1 W_s - \delta_2 W_r]dt - \sqrt{\delta_2 W_r}dW_4 + \sqrt{\delta_1 W_s}dW_7. \end{aligned} \tag{10}$$

4 Numerical Simulation

Here, we simulate both deterministic and stochastic models for the following set of parameters: $\Lambda = 200$, $\alpha_1 = 0.02$, $\alpha_2 = 0.04$, $\mu = 0.0143$, $\mu_1 = 0.08$, $\beta = 0.093$, $\delta_1 = 0.02$, $\delta_2 = 0.0001$.

The system (1) is simulated for various sets of parameters satisfying the condition of local and globally asymptotic stability of equilibrium E^*. For both deterministic and stochastic models, the simulation results are shown in Fig. 2. The stochastic model (SDE model) is simulated by the method of Euler-Maruyama, and Fig. 2 plots the mean of the 100 runs. Here, the results of the stochastic model seem better than those of the deterministic model as the curve corresponding to scarcity lies below the one that corresponds to the deterministic model $\Lambda = 100$, $\alpha_1 = 0.00002$, $\alpha_2 = 0.004$, $\mu = 0.0143$,

$\mu_1 = 0.08$, $\beta = 0.093$, $\delta_1 = 0.02$, $\delta_2 = 0.9$. The system (1) is simulated for different sets of parameters satisfying the condition of local and globally asymptotic stability of equilibrium E^* (see Fig. 3).

Figs. 4 – 7 demonstrate the impact of various parameters on the equilibrium level of water scarcity and recovery.

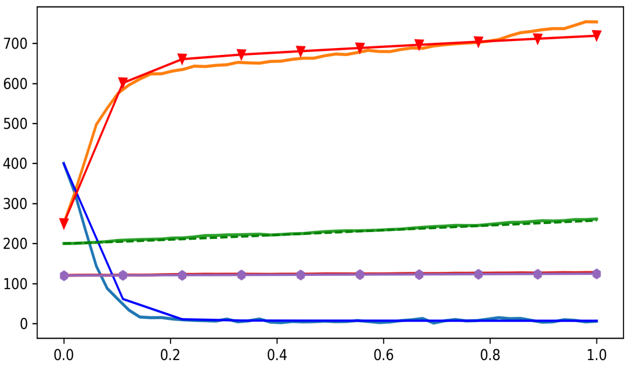

Figure 2: Variation of all compartments of the model showing the effect of stochastic and deterministic models.

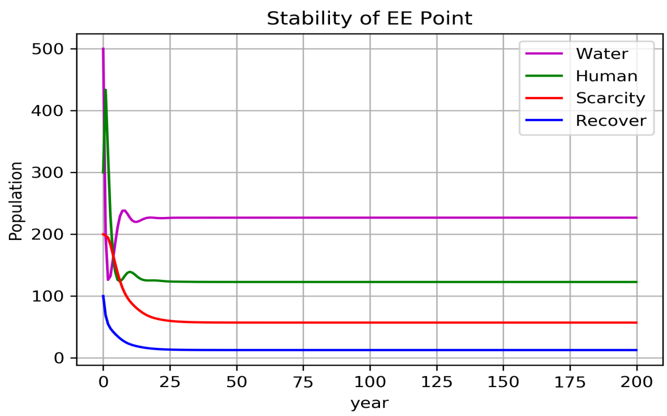

Figure 3: Variation of all compartments of the model showing the stability.

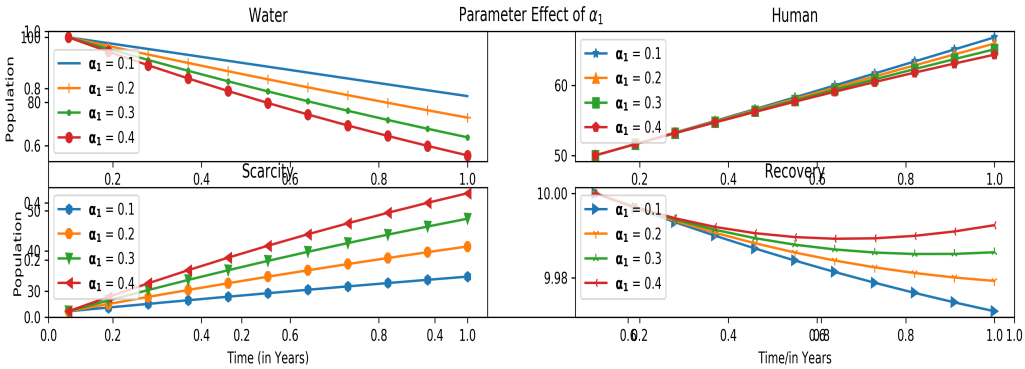

Figure 4: Effect of α_1 on the variation of all compartments of the model.

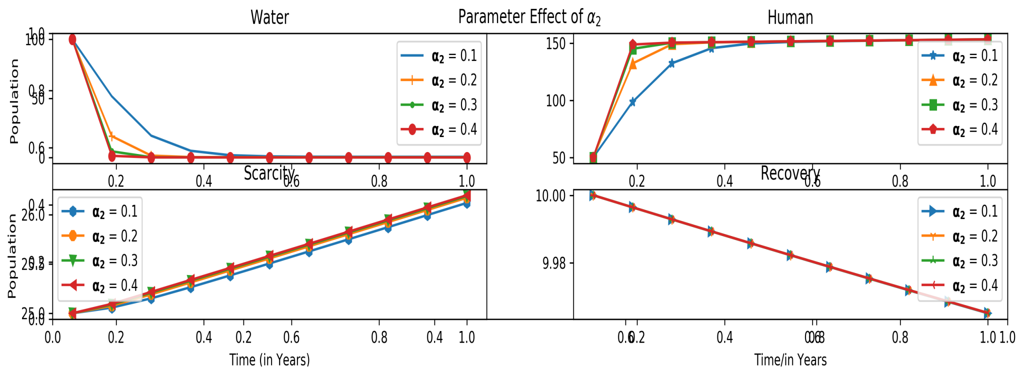

Figure 5: Effect of α_2 on the variation of all compartments of the model.

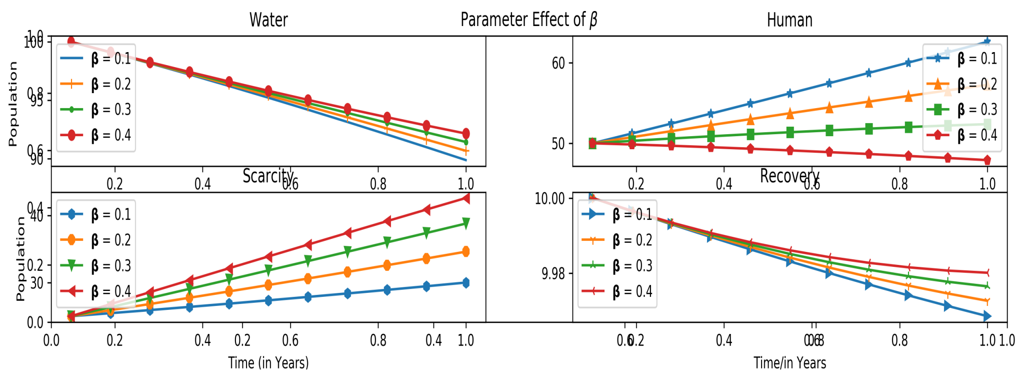

Figure 6: Effect of β on the variation of all compartments of the model.

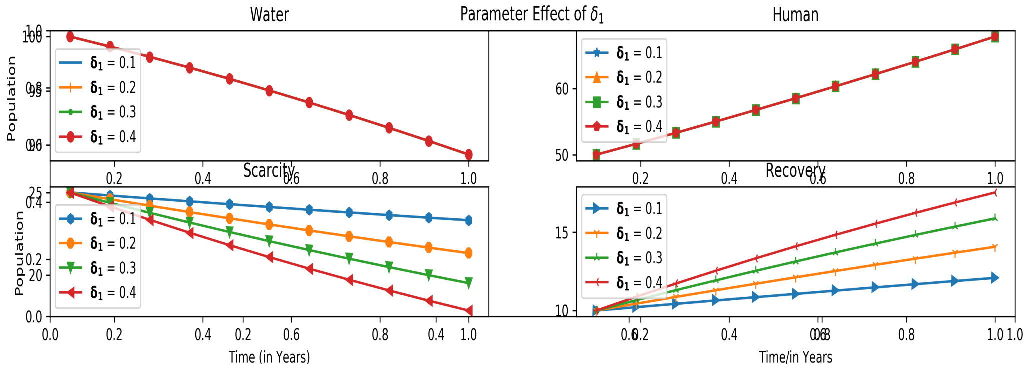

Figure 7: Effect of δ_1 on the variation of all compartments of the model.

5 Result of Discussion and Conclusion

In this paper, a deterministic mathematical model on water resource-related water scarcity problems was proposed and analyzed. We calculate the equilibrium of the proposed model and analyze in detail the local stability and global stability of endemic equilibria.

Further, we extended the deterministic model to a stochastic model and compared numerical simulation results of both models. The resuls of the stochastic model showed that the water scarcity decreased comparatively to the deterministic model. The impact of various parameters on the equilibrium point of water scarcity and recovery is demonstrated. As a society, we have a social responsibility to reduce the scarcity of water. Therefore, we have developed a model of possible strategies to predict better results. Simulations using this model showed the effectiveness of progressing from human to water scarcity.

When the value β (the rate of human population affected by water scarcity) increases in time, the stable point is differed in all compartment (see Fig. 6). Figs. 4 and 5 depict if the values α_1 and α_2 increase or decrease, there is no major difference in all compartments. Fig. 7 depicts if the parameter δ_1 (the rate of water recovery) is increasing in time, the water scarcity is decreased and the recovery is increased.

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Frictional Contact Problem for Thermoviscoelastic Materials with Internal State Variable and Wear

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Received: April 1, 2022; Revised: November 2, 2022

Abstract: In this paper, we study a contact problem between a deformable viscoelastic body and a rigid foundation. Thermal effects, wear and friction between surfaces are taken into account. We model the material's behavior by a nonlinear thermo-viscoelastic law with the internal state variable. The problem is formulated as a coupled system of an elliptic variational inequality for the displacement and the heat equation for the temperature. Our proof is based on nonlinear evolution equations with monotone operators, differential equations and fixed point arguments.

Keywords: *thermo-viscoelastic materials; internal state variable; variational inequality of evolution; fixed point; wear.*

Mathematics Subject Classification (2010): 74M15, 74D10, 70K70, 70K75, 93-05, 93-10.

1 Introduction

During the last decades, the analysis of mathematical models in Contact Mechanics is rapidly growing. These models are suggested for different materials using different boundary conditions modelling friction, lubrication, adhesion, wear, damage, etc.

The aim of this paper is to model and establish the variational analysis of a contact problem for viscoelastic materials within the infinitesimal strain theory. The process is supposed to be subject to thermal effects, friction and wear of contacting surfaces. Mathematical models in Contact Mechanics can be found in [3, 4, 9, 11, 13].

Wear of surfaces is the degradation phenomenon of the superficial layer caused by many factors such as pressure, lubrication, friction and corrosion. Moreover, wear is a

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loss of use as a result of plastic deformations, material removal or fractures. Analysis of contact problems with wear can be found in [6, 7, 12, 16].

The constitutive laws with k internal variables have been used in various publications in order to model the effect of internal variables in the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials. Here, we consider a general model for the dynamic process of a bilateral frictional contact between a deformable body and an obstacle which results in the wear of the contacting surface. Recent models of frictional contact problems can be found in [2, 11, 14, 15]. The material obeys a viscoelastic constitutive law with thermal effects. Models taking into account thermal effects can be found in [5, 12]. We derive a variational formulation of the problem which includes a variational second order evolution inequality. We establish the existence and the uniqueness of a weak solution of the problem. The idea is to reduce the second order evolution nonlinear inequality of the system to the first order evolution inequality. After this, we use classical results on first order evolution nonlinear inequalities, a parabolic variational inequality and equations and the fixed point arguments. The novelty of this paper consists in the coupling of k internal state variable, the thermal effect and wear.

The paper is structured as follows. In Section 2, we present the thermo-viscoelastic contact model with friction and provide comments on the contact boundary conditions. In Section 3, we list the assumptions on the data and derive the variational formulation. In Section 4, we present our main results on existence and uniqueness which state the unique weak solvability.

2 Problem Statement

The physical setting is the following. A viscoelastic body occupies a bounded domain $\Omega \subset \mathbb{R}_d$ $(d = 2, 3)$ with a smooth Γ. The body is acted upon by body forces of density f_0. It is also constrained mechanically on the boundary. We consider a partition of Γ into three disjoint measurable parts Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. Let $T > 0$ and let $[0, T]$ be the time interval of interest. We assume that the body is fixed on Γ_1, surface traction of density f_2 acts on Γ_2 and a body force of density f_0 acts in Ω. Moreover, the process is dynamic, and thus the inertial terms are included in the equation of motion. Then, the classical formulation of the mechanical contact problem of a thermo-visco-elastic material with an internal state variable is as follows.

Problem P. Find a displacement field $u : \Omega \times [0, T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0, T] \to S_d$, an internal state variable field $k : \Omega \times [0, T] \to \mathbb{R}^m$, a temperature field $\theta : \Omega \times [0, T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0, T] \to \mathbb{R}_+$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{F}(\varepsilon(u(t))) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \text{in } \Omega \times [0, T] \quad (1)$$

$$\dot{k}(t) = \phi\left(\sigma(t) - \mathcal{A}\varepsilon\left(\dot{u}(t)\right), \varepsilon\left(u(t)\right), k(t)\right), \quad (2)$$

$$\dot{\theta} - div\,(K_c \nabla\theta) = -M\nabla\dot{u} + q, \tag{3}$$

$$\mathrm{Div}\,\sigma + f_0 = \rho\ddot{u} \quad \text{in } \Omega\times(0,T), \tag{4}$$

$$u = 0 \quad \text{on } \Gamma_1\times(0,T), \tag{5}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2\times(0,T), \tag{6}$$

$$\left\{\begin{array}{l} \sigma_\nu = -\alpha\left|\dot{u}_\nu\right|,\ \ |\sigma_\tau| = -\mu\sigma_\nu, \\ \sigma_\tau = -\lambda\left(\dot{u}_\tau - v^*\right),\ \lambda\geq 0,\ \dot{\omega} = -kv^*\sigma_\nu,\ k>0, \end{array}\right. \quad \text{on } \Gamma_3\ \times[0,T], \tag{7}$$

$$-k_{ij}\frac{\partial\theta}{\partial x_i}\upsilon_j \ = k_e\left(\theta-\theta_R\right)\ - h_\Gamma\left(\left|\dot{u_\Gamma}\right|\right) \quad \text{on } \Gamma_3\times(0,T), \tag{8}$$

$$\theta = 0 \quad \text{on } \Gamma_1\cup\Gamma_2\times(0,T), \tag{9}$$

$$u(0) = u_0,\ \dot{u}(0) = v_0,\ k(0) = k_0,\ \theta\,(0) = \theta_0 \quad \text{in } \Omega, \tag{10}$$

$$\omega(0) = \omega_0 \text{ on } \Gamma_3. \tag{11}$$

First, (1) represents the thermal viscoelastic constitutive law with long-term memory, θ represents the temperature, $M := (m_{ij})$ represents the thermal expansion tensor. We denote by $\varepsilon(u)$ (respectively, by $\mathcal{A}$, $\mathcal{F}$, $\mathcal{B}$) the linearized strain tensor (respectively, the viscosity nonlinear tensor, the elasticity operator, the relaxation function), ϕ is also a nonlinear constitutive function which depends on k. There is a variety of choices for the internal state variables, for reference in the field, see [8,10]. Equation (3) describes the evolution of the temperature field, where $K_c := (k_{ij})$ represents the thermal conductivity tensor, q is the density of volume heat sources. (4) represents the equation of motion, where ρ represents the mass density; we mention that Div is the divergence operator. (5) - (6) are the displacement and the traction boundary condition, respectively. (7) describes the frictional bilateral contact with wear described above on the potential contact surface. (8) represents the associated temperature boundary condition on Γ_3, where θ_R is the temperature of the foundation, and k_e is the exchange coefficient between the body and the obstacle. The equation (9) means that the temperature vanishes on $\Gamma_1\cup\Gamma_2\times(0,T)$. In (10), u_0 is the initial displacement, v_0 is the initial velocity, k_0 is the initial internal state variable and θ_0 is the initial temperature. In (11), ω_0 is the initial wear.

3 Variational Formulation and Preliminaries

For a weak formulation of the problem, first, we introduce some notations. The indices i, j, k, l range from 1 to d and summation over the repeated indices is implied. The index that follows the comma represents the partial derivative with respect to the corresponding component of the spatial variable, e.g., $u_{i.j} = \frac{\partial u_i}{\partial x_j}$. We also use the following notations:

$$\begin{array}{l} H = \mathbb{L}^2(\Omega)^d, \mathcal{H} = \{\sigma = (\sigma_{ij})/\sigma_{ij} = \sigma_{ji}\in\mathbb{L}^2(\Omega)\}, \\ H_1 = \{u = (u_i)/\varepsilon(u)\in\mathcal{H}\}, \mathcal{H}_1 = \{\sigma\in\mathcal{H}/Div\sigma\in H\}. \end{array}$$

The operators of deformation ε and divergence Div are defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j}+u_{j,i}), Div\sigma = (\sigma_{ij,j}).$$

The spaces $H, \mathcal{H}, H_1$, and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$(u,v)_H = \int_\Omega u_i v_i dx, \forall u,v \in H, (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx, \forall \sigma,\tau \in \mathcal{H},$
$(u,v)_{H_1} = (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \forall u,v \in H_1, (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H,$
$\sigma,\tau \in \mathcal{H}_1.$

We denote by $|.|_H$ (respectively, by $|.|_{\mathcal{H}}$, $|.|_{H_1}$, and $|.|_{\mathcal{H}_1}$) the associated norm on the space H (respectively, $\mathcal{H}$, H_1, and $\mathcal{H}_1$).

The following Green's formula holds:

$$(\sigma,\varepsilon(v))_{\mathcal{H}} + (\mathrm{Div}(\sigma),v)_H = \int_\Gamma \sigma\nu\cdot v da \qquad \forall v \in H^1(\Omega)^d,$$

and for the displacement field, we need the closed subspace of H_1 defined by

$$V = \{v \in H_1(\Omega) : v = 0 \text{ on } \Gamma_1\}.$$

The set of admissible internal state variables is given by

$$Y = \{\alpha = (\alpha_i)\,/\alpha_i \in L^2(\Omega), 1 \le i \le m\}.$$

Let us define

$$E = \{\eta \in H_1(\Omega) : \eta = 0 \text{ on } \Gamma_1 \cup \Gamma_2\}.$$

Since $meas(\Gamma_1) > 0$, Korn's inequality holds, i.e., there exists a positive constant C_k, which depends only on Ω, Γ_1, such that

$$|\varepsilon(v)|_{\mathcal{H}} \ge C_k\,|v|_{H_1(\Omega)^d}\,, \quad \forall v \in V.$$

On the space V, we consider the inner product and the associated norm given by

$$(u,v)_V = (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad |v|_V = |\varepsilon(v)|_{\mathcal{H}} \quad \forall u,v \in V. \tag{12}$$

It follows that $|.|_{H_1}$ and $|.|_V$ are equivalent norms on V . Therefore $(V,|.|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and Korn's inequality, there exists a positive constant C_0 which depends only on Ω, Γ_1 and Γ_3 such that

$$|v|_{L^2(\Gamma_3)^d} \le C_0\,|v|_V \quad \forall v \in V. \tag{13}$$

In the study of the mechanical problem $(1)-(11)$, we make the following assumptions that the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies:

$$\begin{cases} \text{a) } \exists\, L_{\mathcal{A}} > 0 : |\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)| \le L_{\mathcal{A}}\,|\varepsilon_1 - \varepsilon_2|\,, \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d,\ p.p.\ x \in \Omega, \\ \text{b) } \exists\, m_{\mathcal{G}} > 0 : (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\,, \varepsilon_1 - \varepsilon_2) \ge m_{\mathcal{A}}\,|\varepsilon_1 - \varepsilon_2|^2\,, \forall \varepsilon_1,\varepsilon_2 \in \mathbb{S}^d, \\ \text{c) The mapping } x \to \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega,\ \forall \varepsilon \in \mathbb{S}^d, \\ \text{d) The mapping } \mathbf{x} \mapsto \mathcal{A}(\mathbf{x},0) \in \mathcal{H}. \end{cases} \tag{14}$$

The elasticity operator $\mathcal{F} : \Omega \times S_d \to S_d$ satisfies

$$\begin{cases} \text{a) There exists a constant } L_F > 0 \text{ such that} \\ \qquad |\mathcal{F}(x,\varepsilon_1) - \mathcal{F}(x,\varepsilon_2)| \le L_F\,(|\varepsilon_1 - \varepsilon_2|) \\ \qquad \forall \varepsilon_1,\varepsilon_2 \in S_d, \quad \text{a.e. } x \in \Omega. \\ \text{b) The mapping } x \to \mathcal{F}(x,\varepsilon) \text{ is Lebesgue measurable} \\ \qquad \text{on } \Omega, \text{ for any } \varepsilon \in S_d. \\ \text{c) The mapping } x \mapsto \mathcal{F}(x,0) \text{ is in } \mathcal{H}. \end{cases} \tag{15}$$

The relaxation function $\mathcal{B} : [0,T] \times \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies

$$\left\{ \begin{array}{l} \text{a) } \mathcal{B}_{ijkh} \in W^{1.\infty}(0,T;\mathbb{L}^\infty(\Omega)), \\ \text{b) } \mathcal{B}(t)\,\sigma \cdot \tau = \sigma \cdot \mathcal{B}(t)\,\tau, \forall \sigma, \tau \in \mathbb{S}^d,\ p.p.t \in [0,T]\,,\ p.p.\text{on } \Omega. \end{array} \right. \tag{16}$$

The function $\phi : \Omega \times S_d \times S_d \times \mathbb{R}^m \to \mathbb{R}^m$ satisfies

$$\left\{ \begin{array}{l} \text{a) There exists a constant } L_\phi > 0 \text{ such that} \\ \quad |\phi(x,\sigma_1,\xi_1,k_1) - \phi(x,\sigma_2,\xi_2,k_2)\ | \le\ L_\phi\left(|\sigma_1-\sigma_2| + |\xi_1-\xi_2| + |k_1-k_2|\right), \\ \quad \forall \sigma_1, \sigma_{2,}\varepsilon_1, \varepsilon_2 \in S_d \ \text{ and } k_1, k_2 \in \mathbb{R}^m,\ \text{a.e. } x \in \Omega. \\ \text{b) For any } \sigma, \varepsilon \in S_d \text{ and } k \in \mathbb{R}^m,\ \ x \to \phi(x,\sigma,\varepsilon,k) \text{ is Lebesgue measurable on } \Omega. \\ \text{c) The mapping } \mathbf{x} \mapsto \phi(x,0,0,0) \text{ is in } L^2(\Omega)^m. \end{array} \right. \tag{17}$$

The function $h_\tau : \Gamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies

$$\left\{ \begin{array}{l} \text{ a) There exists a constant } L_\tau > 0 \text{ such that} \\ |h_\tau(x,r_1) - h_\tau(x,r_2)| \le L_h\,|r_1 - r_2| \quad \forall r_1, r_2 \in \mathbb{R}_+\ ,\ \text{a.e. } x \in \Gamma_3. \\ \text{b) } x \mapsto p_\tau(.,0) \text{ is Lebesgue measurable on } \Gamma_3,\ \forall r \in \mathbb{R}_+. \end{array} \right. \tag{18}$$

For the temperature, we use the following Green's formula:

$$\int\limits_\Omega \dot\theta \tau dx - \int\limits_\Omega div\,(K_c \nabla\theta) = \int_\Omega -\,(M_e \nabla\dot u)\,\tau dx + \int\limits_\Omega q\tau dx \quad \forall \tau \in E. \tag{19}$$

The mass density satisfies

$$\rho \in L^\infty(\Omega),\ \text{there exists } \rho^* > 0 \text{ such that} \quad \rho \ge \rho^* \text{ a.e. } x \in \Omega. \tag{20}$$

We also suppose that the forces, the tractions, the volume, the surface free charges densities and the functions α and μ have the regularity

$$\left\{ \begin{array}{l} f_0 \in L^2(0,T;H), \quad f_2 \in L^2(0,T;L^2(\Gamma_2)^d), \\ \alpha \in \mathbb{L}^\infty(\Gamma_3) \quad \alpha(x) \ge \alpha^* > 0, \quad p.p.\text{on } \Gamma_3, \\ \mu \in \mathbb{L}^\infty(\Gamma_3), \quad \mu(x) > 0, \quad p.p.\text{on } \Gamma_3, \end{array} \right. \tag{21}$$

$$q \in W^{1,2}(0,T;L^2(\Omega)),\ \theta_R \in W^{1,2}(0,T;L^2(\Gamma_3)),\ k_e \in L^\infty(\Omega,\mathbb{R}_+)\ , \tag{22}$$

$$\left\{ \begin{array}{l} K_c \ = (k_{ij})\,, (k_{ij} = k_{ji} \in L^\infty(\Omega)\,, \\ \forall c_k \ge 0, \zeta_i \in \mathbb{R}^d, k_{ij}\zeta_i\zeta_j \ge c_k\zeta_i\zeta_j, \end{array} \right. \tag{23}$$

$$M = (m_{ij})\,, m_{ij} = m_{ji} \in L^\infty(\Omega)\,. \tag{24}$$

The initial data satisfy

$$u_0 \in V,\ v_0 \in H,\ \theta_0 \in E,\ k_0 \in Y,\ \omega_0 \in L^\infty(\Gamma_3)\,. \tag{25}$$

We will use a modified inner product on the Hilbert space, given by

$$((u,v))_H = (\rho u, v)_H \quad \forall\ u, v \in H, \tag{26}$$

and we let $\|.\|_H$ be the associated norm given by

$$\|v\|_H = (\rho v, v)^{\frac12} \quad \forall v \in H. \tag{27}$$

It follows from assumption (20) that $\| \,.\, \|_H$ and $|\cdot|_H$ are equivalent norms on H, and also the inclusion mapping of $(V, |\cdot|_V)$ into $(H, \|.\|_H)$ is continuous and dense. We denote by V' the dual space of V. Identifying H with its own dual, we can write the Gelfand triple

$$V \subset H \subset V'.$$

We use the notation $(.,.)_{V'\times V}$ to represent the duality pairing between V' and V , recall that

$$(u, v)_{V'\times V} = ((u, v))_H \quad \forall u \in H, \forall v \in V. \tag{28}$$

Let $f :\ [0, T] \to V'$ be the function defined by

$$(f(t), v)_{V'\times V} = \int_\Omega f_0(t).v dx + \int_{\Gamma_2} f_2(t).v da \ \ \forall \ \mathbf{v} \in V. \tag{29}$$

Next, we denote by $j : L^2(\Gamma_3) \times V \times V \to \mathbb{R}$

$$j(u, v) = \int_{\Gamma_3} \alpha \, |u_\nu| \, (\mu \, |v_\tau - v^*|) \, da \ . \tag{30}$$

Let $\varphi : V \times V \to \mathbb{R}$ be the function defined by

$$\varphi(u, v) = \int_{\Gamma_3} \alpha \, |u_\nu| \, |v_\nu| \, da, \forall v \in V. \tag{31}$$

Let us introduce the operator $A : V \to V'$

$$(Au, v)_{V'\times V} = (\mathcal{A}(\varepsilon(u)), \varepsilon(v))_{\mathcal{H}}$$

for all $u, v \in V$ and $t \in [0, T]$. Note that

$$f \in L^2(0, T; V'). \tag{32}$$

Using standard arguments based on Green's formulas we can derive the following variational formulation of problem P.

Problem PV. Find a displacement field $u : [0, T] \to V$, a stress field $\sigma : [0, T] \to \mathcal{H}$, an internal state variable field $k : [0, T] \to Y$, a temperature field $\theta : \Omega \times [0, T] \to \mathbb{R}_+$ and the wear $\omega : \Gamma_3 \times [0, T] \to \mathbb{R}_+$ such that

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{F}(\varepsilon(u(t))) + \int\limits_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}, \text{in } \Omega \times [0, T] \,, \tag{33}$$

$$\dot{k}\,(t) = \phi\left(\sigma\,(t) - \mathcal{A}\varepsilon\left(\dot{u}\,(t)\right), \varepsilon\left(u\,(t)\right), k\,(t)\right), \tag{34}$$

$$\begin{aligned}&(\ddot{u}(t), w - \dot{u}(t))_{V'\times V} + \ (\sigma(t), \varepsilon(w - \dot{u}(t)))_{\mathcal{H}} + j(\dot{u}, w) - j(\dot{u}, \dot{u}(t)) + \varphi(\dot{u}, w) - \varphi(\dot{u}, \dot{u}(t))\\ &\geq \big(f(t), w - \dot{u}(t)\big), \ \forall u, w \in V,\end{aligned} \tag{35}$$

$$\dot{\theta}\,(t) + K\theta\,(t) = R\dot{u}\,(t) + Q\,(t) \quad t \in (0, T), \tag{36}$$

$$\dot{\omega} = -kv^*\sigma_\nu, \tag{37}$$

$$u(0) = u_0, \ \dot{u}(0) = v_0, \ k(0) = k_0, \ \theta\,(0) = \theta_0, \tag{38}$$

where $Q : [0,T] \to E', K : E \to E', R : V \to E'$ are given by

$$(Q(t), \mu)_{E' \times E} = \int_{\Gamma_3} k_e \theta_R(t) \, \mu da + \int_{\Omega} q(t) \, \mu dx, \tag{39}$$

$$(K\tau, \mu)_{E' \times E} = \sum_{i,j=1}^{d} \int_{\Omega} k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \mu}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \mu da, \tag{40}$$

$$(Rv, \mu)_{E' \times E} = \int_{\Gamma_3} h_\tau(|v_\tau|) \mu da - \int_{\Omega} (M \nabla v) \, dx \tag{41}$$

for all $v \in V, \mu, \tau \in E$.

The proof of the existence and uniqueness of solution to problem PV will be given in the next section.

4 Existence and Uniqueness Result

Now, we propose our existence and uniqueness result.

Theorem 4.1 *Let the assumptions (14)−(25) hold. Then the problem has a unique solution* $\{u, \sigma, k, \omega, \theta\}$ *satisfying*

$$u \in C^1(0,T;H) \cap W^{1.2}(0,T;V) \cap W^{2.2}(0,T;V') \tag{42}$$

$$\sigma \in L^2(0,T;\mathcal{H}), \quad Div\sigma \in L^2(0,T;V'), \tag{43}$$

$$k \in W^{1,2}(0,T;Y), \tag{44}$$

$$\omega \in C^1(0,T;\mathbb{L}^2(\Gamma_3)), \tag{45}$$

$$\theta \in W^{1,2}(0,T;E') \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)). \tag{46}$$

We conclude that under the assumptions (14)−(25), the mechanical problem (1)-(11) has a unique weak solution with the regularity (42)-(46).

The proof of this theorem will be carried out in several steps.

The first step: let $g \in L^2(0,T;V)$ and $\eta = (\eta^1, \eta^2) \in L^2(0,T;V' \times Y)$ be given, and prove that there exists a unique solution $u_{g\eta}$ of the following intermediate problem.

Problem PV$_{g\eta}$. Find the displacement field $u_{g\eta} : [0,T] \to V$ such that for a.e. $t \in (0,T)$,

$$\begin{cases} (\ddot{u}_{g\eta}(t), w - \dot{u}_{g\eta}(t))_{V' \times V} + (\mathcal{A}\varepsilon(\dot{u}_{g\eta}(t)), \varepsilon(w - \dot{u_{g\eta}}(t)))_{\mathcal{H}} + \\ (\eta^1, w - \dot{u_{g\eta}}(t))_{V' \times V} + j(g,w) - j(g, \dot{u_{g\eta}}(t)) \geq (f_{g\eta}(t), w - \dot{u}_{g\eta}(t)), \ \forall w \in V, \end{cases} \tag{47}$$

$$u_{g\eta}(0) = u_0, \quad \dot{u}_{g\eta}(0) = v_0. \tag{48}$$

We define $f_{g\eta}(t) \in V$ for $a.e.t \in [0.T]$ by

$$(f_{g\eta}(t), w)_{V' \times V} = (f(t) - \eta^1(t), w)_{V' \times V}, \quad \forall w \in V. \tag{49}$$

From (29), we deduce that

$$f_\eta \in L^2(0,T;V'). \tag{50}$$

Let now $u_\eta : [0.T] \to V$ be the function defined by

$$u_\eta(t) = \int_0^t v_\eta(s)\,ds + u_0, \forall t \in [0,T]. \tag{51}$$

Concerning Problem $\mathrm{PV}_{g\eta}$, we have the following result.

Lemma 4.1 *There exists a unique solution to problem $PV_{g\eta}$ with the regularity.*

$$v_\eta \in L^2(0,T;V) \text{ and } \dot{v}_\eta \in L^2(0,T;V'). \tag{52}$$

Proof. The proof by nonlinear first order evolution inequalities is given in [9].

The second step: we use the displacement $u_{g\eta}$ to consider the following variational problem.

Let us consider now the operator $\Lambda_\eta(g) : \mathbb{L}^2(0,T;V) \to \mathbb{L}^2(0,T;V)$ defined by

$$\Lambda_\eta(g) = v_{g\eta}\,. \tag{53}$$

We have the following lemma.

Lemma 4.2 *The operator Λ_η has a unique fixed point $g_\eta^* \in \mathbb{L}^2(0,T;V)$.*

Proof. Let $g_1, g_2 \in \mathbb{L}^2(0,T;V)$ and let $\eta = (\eta^1, \eta^2) \in L^2(0,T;V' \times Y)$. Using similar arguments as in (47), (51), we find

$$\begin{aligned}&(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)) + (\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))) + \\ &+ j(g_1, v_1(t)) - j(g_1, v_2(t)) - j(g_2, v_1(t)) + j(g_2, v_2(t)) \le 0.\end{aligned} \tag{54}$$

From the definition of the functional j given by (30), we have

$$\begin{aligned}&j(g_1, v_2(t)) - j(g_1, v_1(t)) - j(g_2, v_2(t)) + j(g_2, v_1(t)) = \textstyle\int_{\Gamma_3} (\alpha\,|g_{1\nu}| - \alpha\,|g_{2\nu}|) \\ &(\mu\,|v_{1\tau} - v^*| - \mu\,|v_{2\tau} - v^*|)\,da.\end{aligned} \tag{55}$$

From (13), (21) we find

$$j(g_1, v_2(t)) - j(g_1, v_1(t)) - j(g_2, v_2(t)) + j(g_2, v_1(t)) \le C\,|g_1 - g_2|_V\,|v_1 - v_2|_V\,. \tag{56}$$

Integrating the inequality (54) with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (14) , (56) and the inequality

$$2ab \le \frac{C}{m_{\mathcal{A}}} a^2 + \frac{m_{\mathcal{A}}}{C} b^2,$$

we find

$$|v_2(t) - v_1(t)|_V^2 \le C \int_0^t |g_2(s) - g_1(s)|_V^2\,ds. \tag{57}$$

From (53) and (57), we find that

$$|\Lambda_\eta g_2(t) - \Lambda_\eta g_1(t)|_V^2 \le C \int_0^t |g_2(s) - g_1(s)|_V^2\,ds.$$

Reiterating this inequality m times, we obtain

$$\left|\Lambda_\eta^m g_2\left(t\right)-\Lambda_\eta^m g_1\left(t\right)\right|_{\mathbb{L}^2(0,T;V)}\leq\frac{C^mT^m}{m!}\left|g_2\left(t\right)-g_1\right|_{\mathbb{L}^2(0,T;V)}. \tag{58}$$

Since $\lim\limits_{m\to+\infty}\frac{C^mT^m}{m!}=0$, it follows that exists a positive integer m such that $\frac{C^mT^m}{m!}<1$ and, therefore, (58) shows that Λ_η^m is a contraction on the Banach space $\mathbb{L}^2(0,T;V)$. Thus, from Banach's fixed point theorem, the operator Λ_η has a unique fixed point $g_\eta^*\in\mathbb{L}^2(0,T;V)$.

Lemma 4.3 *Now, define $k_\eta\in W^{1,2}(0,T;Y)$ by*

$$k_\eta\left(t\right)=k_0+\int_0^t\eta^2\left(s\right)ds. \tag{59}$$

Then there exists $C>0$ such that

$$\left|k_1\left(s\right)-k_2\left(s\right)\right|_Y^2\leq C\int_0^t\left|\eta_1^2\left(s\right)-\eta_2^2\left(s\right)\right|_{Y'}^2ds. \tag{60}$$

In the third step, we use the displacement field u_η obtained in Lemma 4.1 and k_η defined in (59) to consider the following variational problem for the temperature field.

Problem PV$_\theta$. Find $\theta_\eta:[0,T]\to E'$ satisfying for a.e. $t\in(0,T)$,

$$\begin{aligned}\dot\theta_\eta(t)+K\theta_\eta\left(t\right)&=R\dot u_\eta\left(t\right)+Q\left(t\right)\quad t\in(0,T),\ \text{in } E', &(61)\\ \theta_\eta\left(0\right)&=\theta_0. &(62)\end{aligned}$$

Lemma 4.4 *Problem $\boldsymbol{PV}_\theta$ has a unique solution*

$$\theta_\eta\in W^{1,2}(0;T;E^{'})\cap L^2(0;T;E)\cap C(0;T;L^2(\Omega)),\quad \forall\eta\in L^2(0,T;V'),$$

satisfying

$$\left|\theta_{\eta_1}\left(t\right)-\theta_{\eta_2}\left(t\right)\right|_{L^2(\Omega)}^2\leq C\int_0^t\left|\upsilon_1(s)-\upsilon_2(s)\right|_V^2ds\qquad\forall\ t\in(0,T). \tag{63}$$

Proof. The existence and uniqueness result verifying (61) follows from the classical result on the first order evolution equation, applied to the Gelfand evolution triple

$$E\subset F\equiv F^{'}\subset E^{'}.$$

We verify that the operator K is linear continuous and strongly monotone. Now from the expression of the operator R, $v_\eta\in W^{1,2}(0,T;V)\Rightarrow Rv_\eta\in W^{1,2}(0,T;F)$, as $Q\in W^{1,2}(0,T;E)$, then $Rv_\eta+Q\in W^{1,2}(0,T;E)$, we deduce (63), (See [1]).

Finally, as a consequence of these results, and using the properties of $\mathcal{F}$, $\mathcal{E}$, $\mathcal{G}$, ϕ, and j for $t\in[0,T]$, we consider the element

$$\Lambda\eta\left(t\right)=\left(\Lambda^1\eta\left(t\right),\Lambda^2\eta\left(t\right)\right)\in V'\times Y, \tag{64}$$

$$\begin{aligned}&\left(\Lambda^1(\eta),w\right)_{V'\times V}=\left(\mathcal{F}(\varepsilon(u_\eta(t)),w\right)_V+\\&+\left(\int_0^t\mathcal{B}(t-s)\varepsilon(u_\eta(s))ds,w\right)_V-(\theta_\eta(t)\mathcal{M},\varepsilon(w))_{\mathcal{H}}+\varphi(\dot u,w)\ \forall w\in V,\end{aligned} \tag{65}$$

$$\Lambda^2\eta\left(t\right)=\phi\left(\sigma_\eta\left(t\right),\varepsilon\left(u_\eta\left(t\right)\right),k_\eta\left(t\right)\right). \tag{66}$$

Here, for every $\eta \in L^2(0,T;V' \times Y)$, u_η, θ_η represent the displacement field and the temperature field obtained in Lemmas 4.1, 4.4, respectively, and k_η is the internal state variable given by (59). We have the following result.

Lemma 4.5 *The operator Λ has a unique fixed point $\eta^* \in L^2(0,T;V' \times Y)$.*

Proof. Let η_1, $\eta_2 \in L^2(0,T;V' \times Y)$. Write for $i = 1.2, u_{\eta i} = u_i$, $\dot{u}_{\eta i} = v_{\eta i} = v_i$, $\sigma_{\eta i} = \sigma_i$, $k_{\eta i} = k_i$, $\theta_{\eta i} = \theta_i$. Using (12), (15), (16), (24), (31), we have

$$\begin{aligned}&\left|\Lambda^1\eta_1(t) - \Lambda^1\eta_2(t)\right|_{V'}^2 \leq C(|u_1(t) - u_2(t)|_V^2 + \textstyle\int_0^t |u_1(s) - u_2(s)|_V^2 \; ds + \\ &|\theta_1(t) - \theta_2(t)|_{L^2(\Omega}^2 + |v_1(t) - v_2(t)|_V^2).\end{aligned} \tag{67}$$

By similar arguments, from (66), (33) and (17), it follows that

$$\left|\Lambda^2\eta_1(t) - \Lambda^2\eta_2(t)\right|_Y^2 \leq C(|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}}^2 + |u_1(t) - u_2(t)|_V^2 + |k_1(t) - k_2(t)|_Y^2). \tag{68}$$

Taking into account that

$$\sigma_i(t) = \mathcal{A}(\varepsilon(\dot{u}_i(t))) + \eta_i^1(t), \; \forall t \in [0,T], \tag{69}$$

by (14), and using (69), we find

$$|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}_1}^2 \leq C\left(|v_1(t) - v_2(t)|_V^2 + \left|\eta_1^1(t) - \eta_2^1(t)\right|_{V'}^2\right). \tag{70}$$

So

$$\begin{aligned}\left|\Lambda^2\eta_1(t) - \Lambda^2\eta_2(t)\right|_Y^2 \quad &\leq \quad C(|v_1(t) - v_2(t)|_V^2 + \left|\eta_1^1 - \eta_2^1\right|_{V'}^2 + |u_1(t) - u_2(t)|_V^2 \\ &\qquad + |k_1(t) - k_2(t)|_Y^2).\end{aligned} \tag{71}$$

Consequently,

$$\begin{aligned}|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{V'\times Y}^2 &\leq C(|u_1(t) - u_2(t)|_V^2 + |k_1(t) - k_2(t)|_Y^2 + \left|\eta_1^1(t) - \eta_2^1(t)\right|_{V'}^2 \\ &+ |\theta_1(t) - \theta_2(t)|_{L^2(\Omega}^2 + |v_1(t) - v_2(t)|_V^2 + \textstyle\int_0^t |u_1(s) - u_2(s)|_V^2 \, ds.\end{aligned} \tag{72}$$

Since u_1 and u_2 have the same initial value, we get

$$|u_1(t) - u_2(t)|_V^2 \leq C\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds.$$

From this inequality, (72) and (63), we obtain

$$\begin{aligned}&|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{V'\times Y}^2 \leq C(\textstyle\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds + |v_1(t) - v_2(t)|_V^2 + \\ &|k_1(t) - k_2(t)|_Y^2 + \left|\eta_1^1(t) - \eta_2^1(t)\right|_{V'}^2), \forall t \in [0,T].\end{aligned}$$

Moreover, from (54), we obtain

$$\begin{aligned}&\left(\dot{v}_1(t) - \dot{v}_2(t), v_1(t) - v_2(t)\right) + (\mathcal{A}\varepsilon(v_1(t)) - \mathcal{A}\varepsilon(v_2(t)), \varepsilon(v_1(t)) - \varepsilon(v_2(t))) + \\ &+ (\eta_1(t) - \eta_2(t), v_1(t) - v_2(t)) \leq j(v_1(t), v_2(t)) - j(v_1(t), v_1(t)) \\ &-j(v_2(t), v_2(t)) + j(v_2(t), v_1(t)).\end{aligned} \tag{73}$$

From the definition of the functional j given by (30), and using (13),(23), we get

$$j(v_1(t), v_2(t)) - j(v_1(t), v_1(t)) - j(v_2(t), v_2(t)) + j(v_2(t), v_1(t)) \le C\, |v_1(t) - v_2(t)|_V^2. \tag{74}$$

Integrating the inequality(73) with respect to time, using the initial conditions $v_2(0) = v_1(0) = v_0$, using (14),(74) and using the Cauchy-Schwartz inequality and the inequality

$$2ab \le m_{\mathcal{A}} a^2 + \frac{1}{m_{\mathcal{A}}} b^2,$$

we find

$$\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds \le C \int_0^t \left|\eta_1^1(s) - \eta_2^1(s)\right|_{V'}^2 ds. \tag{75}$$

It follows now from (59),(63) and (75) that

$$|\Lambda\eta_1(t) - \Lambda\eta_2(t)|_{V'\times Y}^2 \le C \int_0^t |\eta_1(s) - \eta_2(s)|_{V'\times Y'}^2 \, ds.$$

Reiterating the previous inequality n times, we find that

$$|\Lambda^n\eta_1 - \Lambda^n\eta_2|_{L^2(0,T;\; V'\times Y)}^2 \le \tfrac{C^n T^n}{n!} \int_0^t |\eta_1(s) - \eta_2(s)|_{V'\times Y}^2 \, ds.$$

This inequality shows that for n large enough, the operator Λ^n is a contraction on the Banach space $L^2(0,T;\; V' \times Y)$, and so Λ has a unique fixed point. Next, we consider the operator $\mathcal{L} : C(0,T;\mathbb{L}^2(\Gamma_3)) \to C(0,T;\mathbb{L}^2(\Gamma_3))$ defined by

$$\mathcal{L}\omega(t) = -kv^* \int_0^t \sigma_\nu(s)\, ds, \forall t \in [0,T]. \tag{76}$$

Lemma 4.6 *The operator $\mathcal{L} : C(0,T;\mathbb{L}^2(\Gamma_3)) \to C(0,T;\mathbb{L}^2(\Gamma_3))$ has a unique point element $\omega^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ such that $\mathcal{L}\omega^* = \omega^*$.*

Proof. Using ω_1, $\omega_2 \in C(0,T;\mathbb{L}^2(\Gamma_3))$, we have

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 \le kv^* \int_0^t |\sigma_1(s) - \sigma_2(s)|^2 \, ds.$$

From (12) and using (14)-(16), we find

$$\begin{aligned}&|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}_1}^2 \le C(|u_1(t) - u_2(t)|_V^2 + |v_1(t) - v_2(t)|_V^2 + \\ &\textstyle\int_0^t |u_1(s) - u_2(s)|_V^2 \, ds + |\theta_1(t) - \theta_2(t)|_{H^1(\Omega)}^2).\end{aligned} \tag{77}$$

Using (63), we obtain

$$\begin{aligned}&|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}_1}^2 \le C(\textstyle\int_0^t |u_1(s) - u_2(s)|_V^2 \, ds + |u_1(t) - u_2(t)|_V^2 + \\ &|v_1(t) - v_2(t)|_V^2 + \textstyle\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds.\end{aligned} \tag{78}$$

From (51), we have

$$\begin{gathered}\textstyle\int_0^t |u_1(s) - u_2(s)|_V^2 \, ds + |u_1(t) - u_2(t)|_V^2 + |v_1(t) - v_2(t)|_V^2 \le \\ C \textstyle\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds.\end{gathered}$$

So

$$\begin{aligned}&\int_0^t |u_1(s) - u_2(s)|_V^2 \, ds + |u_1(t) - u_2(t)|_V^2 + |v_1(t) - v_2(t)|_V^2 \leq \\ &C(\int_0^t |v_1(s) - v_2(s)|_V^2 \, ds + |\omega_1(t) - \omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2).\end{aligned} \tag{79}$$

By Gronwall's inequality, we find

$$\int_0^t |u_1(s) - u_2(s)|_V^2 \, ds + |u_1(t) - u_2(t)|_V^2 + |v_1(t) - v_2(t)|_V^2 \leq C \, |\omega_1(t) - \omega_2(t)|_{\mathbb{L}^2(\Gamma_3)}^2 .$$

So, we have

$$|\sigma_1(t) - \sigma_2(t)|_{\mathcal{H}_1}^2 \leq C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)}^2 \, ds. \tag{80}$$

Using (80), we find

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leq C \int_0^t |\omega_1(s) - \omega_2(s)|_{\mathbb{L}^2(\Gamma_3)} \, ds.$$

Reiterating the previous inequality p times, we find that

$$|\mathcal{L}\omega_1(t) - \mathcal{L}\omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} \leq \frac{(Ct)^p}{p!} \, |\omega_1(t) - \omega_2(t)|_{\mathbb{L}^2(\Gamma_3)} .$$

This inequality shows that for p large enough, the operator $\mathcal{L}^p$ is a contraction on the Banach space $C(0,T;\mathbb{L}^2(\Gamma_3))$, and so $\mathcal{L}$ has a unique fixed point $\omega^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$.

Now we have all the ingredients to prove Theorem 4.1.

Existence. Let $g^* = g^*_{\eta^*}$ be the fixed point of Λ_{η^*} defined by Lemma 4.2, let $\eta_* = (\eta^1_*, \eta^2_*) \in L^2\,(0,T;V' \times Y)$ be the fixed point of Λ defined by (65) and (66), $k_{\eta^*}(t) = k_0 + \int_0^t \eta^2_*(s)\,ds$, and let $\omega^* \in C(0,T;\mathbb{L}^2(\Gamma_3))$ be the fixed point $\mathcal{L}$ defined by (76) and let $(u_{\eta_*}, \theta_{\eta_*})$ be the solution to Problems $\mathbf{PV}_{g\eta}$, $\mathbf{PV}_\theta$ for $\eta = \eta_*$, that is, $u = u_{\eta_*}, k = k_{\eta_*}$, $\theta = \theta_{\eta_*}$, and

$$\sigma(t) = \mathcal{A}(\varepsilon(\dot{u}(t))) + \mathcal{F}(\varepsilon(u(t))) + \int_0^t \mathcal{B}(t-s)\varepsilon(u(s))ds - \theta(t)\mathcal{M}.$$

It results from (65) and (66), for $\Lambda^1(\eta_*) = \eta^1$ and $\Lambda^2(\eta_*) = \eta^2$, that $(u, \sigma, k, \theta, \omega)$ is a solution of Problem **PV**. The regularities (42)-(46) follow from Lemmas 4.1, 4.3, 4.4 and 4.6.

Uniqueness. The uniqueness of the solution follows from the uniqueness of the fixed point of the operators Λ_η, Λ and $\mathcal{L}$.

5 Concluding Remark

Scientific research and recent papers in mechanics are articulated around two main components, one devoted to the laws of behavior and the other devoted to the boundary conditions imposed on the body.

The constitutive laws with internal variables have been used in various publications in order to model the effect of internal variables on the behavior of real bodies like metals, rocks, polymers and so on, for which the rate of deformation depends on the internal variables. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials. Our model is obtained by combining the thermoviscoelastic constitutive law with a long memory term, wear, friction and the internal state variable k. The model is developed to describe the self-heating and stress-strain behavior of thermoviscoelastic polymers under tensile loading when the rate of deformation depends on the internal variable k.

Mathematically, the idea is to reduce the second order nonlinear evolution inequality of the system to the first order evolution inequality. After this, we use classical results on first order evolution nonlinear inequalities, parabolic inequalities, differential equations and fixed point arguments.

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