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Exponential and Strong Stabilization for Inhomogeneous Semilinear Control Systems by Decomposition Method

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Abstract: In this work, we study, in a Hilbert state space, the stabilization problem of inhomogeneous semilinear control systems; the existence and uniqueness of solutions of the system are proved by the semigroup theory. The paper also gives a feedback control and sufficient conditions for exponential and strong stabilization using the decomposition method. Finally, an application to the heat equations is provided.

Keywords: *stability of control systems; stabilization of systems by feedback; heat equation.*

Mathematics Subject Classification (2010): 93-XX; 34-XX.

1 Introduction

Semilinear systems are special types of nonlinear systems. They are a transition class between linear and nonlinear systems and thus represent a wide range for modeling the dynamic behavior of various real-world phenomena. Stability is one of the most important concepts in dynamical systems theory, particularly semi-linear systems. This problem remains a major concern in the work of mathematicians and engineers. In this work, we study the stabilization of the inhomogeneous semi-linear system described by the equation

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + v(t)(Ny(t) + c), \\ y(0) = y_0, \in H, \end{cases} \tag{1}$$

where

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1. The state space is an infinite-dimensional Hilbert space H with the inner product $\langle ., . \rangle$ and the corresponding norm $\|.\|$;
2. $v(t)$ is a scalar-valued control;
3. A is an unbounded operator with the domain $D(A) \subset H$, it generates a semigroup of contractions $(S(t))_{t\geq 0}$ on H;
4. N is a nonlinear operator from H into H, which is locally Lipschitz and sequentially continuous operator such that $N(0) = 0$; since $N(.)$ is locally Lipschitz, there is $L > 0$ such that for all $z, y \in H$ satisfying $0 < \|y\| \leqslant \|z\| \leqslant R$, we have $\|Nz - Ny\| \leqslant L\|z - y\|$;
5. $c \neq 0$ is a fixed vector in H.

Remark 1.1 If N is linear, the system(1) is bilinear, and if N is not linear, the system (1) is semilinear; if $c = 0$, the system is homogeneous, and if $c \neq 0$, the system is inhomogeneous.

One of the most important concepts in systems theory is stability; we study the possibility of finding feedback $u(y(t))$ as "regular" as possible such that the system is stable; this stability can be strong, weak, or exponential. The study of the stability of homogeneous bilinear and semilinear systems has been considered in many works, and different results have been developed in finite and infinite dimensional cases, see J. Ball, M. Slemrod [1], M. Ouzahra, A. Tsouli and A. Boutoulout [2], M. Ouzahra [3], A. Benzaza and M. Ouzahra [4], E. Zerrik and M. Ouzahra [5], H. Bounit, and Hammouri [6], A. El Alami and M. Chqondi [7].

However, only a few works study the case of inhomogeneous systems; the stability of such systems has been studied in the bilinear case by Z. Hamidi and M. Ouzahra [8], who proved the necessary and sufficient conditions for weak and strong partial stabilization of inhomogeneous bilinear system by the control

$$v(t) = -\rho \frac{\langle y(t), Ny(t) + c\rangle}{|\langle y(t), Ny(t) + c\rangle| + 1}, \quad \forall t > 0, \tag{2}$$

where $\rho > 0$ is the gain control.

In this work, an exponential and strong stabilization result has been established using the same feedback control (2), provided that the following observation assumption is verified:

$$\exists \delta, T > 0 \quad \text{such that} \quad \int_0^T |\langle NS(s)y(t) + c, S(s)y(t)\rangle| \mathrm{d}s \geq \delta \|y(t)\|, \quad \forall y \in H.$$

This paper is organized as follows. In Section 2, we choose a control that ensures the stabilization of our system; in Section 3, we show the existence and uniqueness of the solution in the semilinear inhomogeneous case; in Section 4, we present an appropriate decomposition of the state space H and the system (1) via the spectral properties of the operator A. We apply this approach to study the exponential stabilization problem of the type (1); in Section 5, we look at the strong stabilization problem using the chosen control. In the last section, we give illustrations through examples governed by a heat equation.

2 Choice of Control

Let (1) be as given in the Introduction. Here, the state space is a Hilbert space H with the inner product $\langle .,. \rangle$ and corresponding norm $\|.\|$, $y(t)$ is the state, and $u(t)$ is a scalar valued control. The problem of stabilization consists of choosing a feedback control $u(t)$ such that the solution of the resulting feedback system satisfies in some sense $y(t) \to 0$ as $\to +\infty$. If we formally compute the time rate of change of the "energy", we get the following:

$$\begin{aligned}\frac{d}{dt}\|y(t)\|^2 &= 2 < y(t); \frac{d}{dt}y(t) > \\ &= 2 < y(t); Ay(t) > +2u(t) < y(t), Ny(t) + c >\end{aligned}$$

which implies, since $S(t)$ is a semigroup of contractions,

$$\frac{d}{dt}\|y(t)\|^2 \leq 2u(t) < y(t); Ny(t) + c >, \ \forall t \in [0,T].$$

Then, to make the energy nonincreasing, an obvious choice of the feedback control (though not the only one) is $v(t) = -\rho \frac{\langle y(t), Ny(t)+c \rangle}{|\langle y(t), Ny(t)+c \rangle|+1}; \big(\forall t > 0, \rho > 0\big)$ since this control yields the "dissipating energy inequality"

$$\frac{d}{dt}\|y(t)\|^2 \leq -2\frac{(\langle Ny(t)+c, y(t) \rangle)^2}{|\langle y(t), Ny(t)+c \rangle|+1}; \forall t \in [0,T]. \tag{3}$$

3 Well-Possedness

Let us consider the closed loop-system

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + f(y(t)), \quad \forall t > 0, \\ y(0) = y_0 \in H, \end{cases} \tag{4}$$

where $f(y(t)) = v(t)\big(Ny(t) + c\big)$, $\forall y \in H$, and $v(t)$ is the control given in (2).

We set $\phi(s) = \frac{s}{|s|+1}$ for all $s \in \mathbb{R}$ so that $f(y(t)) = -\rho\phi(< y; Ny + c >)(Ny + c)$. In this section, we aim to study the existence, uniqueness, and regularity of the solution to the system (1).

We start our study with the well-posedness result.

Proposition 3.1 *Assume that A is the infinitesimal generator of a linear C^0 semi-group of contractions on a Hilbert space H and let N be locally Lipschitz and sequentially continuous operator; then the system (1) admits a unique global mild solution $y(t)$ defined on the infinite interval $[0, +\infty[$, which is given by the following variation of constants formula:*

$$y(t) = S(t)y_0 + \int_0^t v(\tau)(Ny(\tau) + c)S(t-\tau)d\tau.$$

Moreover, we have the following estimate:

$$\|z(t)\|^2 - \|z(s)\|^2 \leqslant -2\rho\int_s^t \frac{\langle y(\tau); Ny(\tau) + c \rangle^2}{|\langle y(\tau), Ny(\tau) + c \rangle| + 1}d\tau.$$

In particular, we have $\|y(t)\| \leq \|y_0\|$. $\forall t \geq 0$.

Proof. To establish the existence and uniqueness of the solution of (4), let us show that the function f is locally Lipschitz for all $y, z \in H$. Let $x, y, z \in H$ with $t \in [0, T]$ and $R > 0$ such that $\|x-y\|, \|x-z\| \leq R$. Without loss of generality, we can take $x = 0$. Since N is locally Lipschitz, there is $L_R(N) > 0$ such that for all $z, y \in H$, satisfying $0 < \|z\| \leqslant \|y\| \leqslant R$, we have $\|Nz - Ny\| \leqslant L_R(N)\|z - y\|$.

$$\begin{aligned}\text{We have } f(t,y) - f(t,z) &= \big(\rho\phi(< z; Nz + c >)(Nz + c) - \rho\phi(< y; Ny + c >)(Ny + c)\big)\\ &= \rho\big(\phi(< z; Nz + c >) - \phi(< y; Ny + c > \big)\big(Nz + c\big)\\ &+ \phi(< y; Ny + c > \big(Nz - Ny\big).\end{aligned}$$

By making use of the function $\phi(s) = \frac{s}{|s|+1}$, we have $|\phi(s) - \phi(r)| \leq |s - r|$ and $|\phi(s)| \leq |s|$ for all $(s, r) \in \mathbb{R}^2$ since $N()$ is locally Lipschitz, and when using Schwartz's inequality, it follows that

$$\begin{aligned}\|f(t,y) - f(t,z)\| &\leq \rho\Big(| < z; Nz + c > - < y; Ny + c > |\|Nz + c\|\Big)\\ &+ \rho|\phi(< y; Ny + c >)|\|Nz - Ny\|.\end{aligned}$$

$$\begin{aligned}\text{We have } \quad | < z; Nz + c >) - < y; Ny + c > | &= | < z - y; Nz + c >) + < y; Nz - Ny > |\\ &\leq \Bigg(\|c\| + 2L_R(N)\|y_0\|\Bigg)\|z - y\|\end{aligned}$$

$$\begin{aligned}\text{and } \quad |\phi(< y; Ny + c >)|\|Nz - Ny\| &\leq | < y; Ny + c > |L_R(N)\|z - y\|\\ &\leq \|y_0\|\Bigg(\|Ny\| + \|c\|\Bigg)|L_R(N)\|z - y\|\\ &\leq \|y_0\|L_R(N)\Bigg(L_R(N)\|y_0\| + \|c\|\Bigg)\|z - y\|.\end{aligned}$$

$$\begin{aligned}\text{So, } \|f(t,y) - f(t,z)\| &\leq \rho\Bigg(L_R(N)\|y_0\| + \|c\|\Bigg)\Bigg(\|y_0\||L_R(N) + \|c\| + 2L_R(N)\|y_0\|\Bigg)\|z - y\|\\ &\leq \rho\Bigg(L_R(N)\|y_0\| + \|c\|\Bigg)\Bigg(\|c\| + 3L_R(N)\|y_0\|\Bigg)\|z - y\|\\ &\leq \mathcal{M}_{(\|y_0\|;c)}\|z - y\|.\end{aligned}$$

$$\text{where } \quad \mathcal{M}_{(\|y_0\|;c)} = \rho\Bigg(L_R(N)\|y_0\| + \|c\|\Bigg)\Bigg(\|c\| + 3L_R(N)\|y_0\|\Bigg).$$

Remark 3.1 For c=0, we obtain the constant $\mathcal{M}_{(\|y_0\|;0)} = 3\rho\Bigg(L_R(N)\|y_0\|\Bigg)^2$ which is strictly smaller than that found in the homogeneous case in [9].

So $f(t; y(t))$ satisfies a local Lipschitz condition in y, uniformly in t on bounded intervals. Thus we may apply Theorem 1.4 [10] (p.185), to obtain that there is a $t_{\max} \leq \infty$ such that (8) has a unique mild solution y on $[0, t_{\max}[$, which is given by the following variation of constants formula:

$$\begin{aligned}y(t) &= S(t)y_0 + \int_0^t f(y(\tau))S(t - \tau)d\tau\\ &= S(t)y_0 - \rho\int_0^t \frac{\langle y(\tau), Ny(\tau) + c\rangle}{|\langle y(\tau), Ny(\tau) + c\rangle| + 1}(Ny(\tau) + c)S(t - \tau)d\tau.\end{aligned}$$

To show that $t_{\max} = +\infty$, it is sufficient to prove that for each $T > 0$, the mild solution $y(t)$ is bounded by a constant independent of T. To do this, we discuss two cases:
(i) If the initial value $y_0 \in D(A)$, then the function $w(t) := \frac{1}{2}\|y(t)\|^2$ is continuously differentiable and we can write for all $t \geq 0$, the following: since this control yields the "dissipating energy inequality",

$$\frac{d}{dt}\|y(t)\|^2 \leq -2\rho \frac{(\langle y(t); Ny(t) + c\rangle)^2}{|\langle y(t), Ny(t) + c\rangle| + 1}; \forall t \in [0, T]. \tag{5}$$

When integrating the last inequality over the interval [s,t], it follows that

$$\|y(t)\|^2 - \|y(s)\|^2 \leq -2\rho \int_s^t \frac{\langle y(\tau); Ny(\tau) + c\rangle^2}{|\langle y(\tau), Ny(\tau) + c\rangle| + 1} d\tau, \forall t \geq s \geq 0. \tag{6}$$

It follows that $\|y(t)\| \leq \|y_0\|, \forall t \geq 0$.
ii) Let $y_0 \in H$ and consider a sequence $(y_0^n)_n$ of elements in H converging to y_0. For each $T > 0$, let $y(t)$ and $y^n(t)$ be the mild solutions of (S) associated, respectively, to the initial values y_0 and y_0^n. Then one can prove that for each $t \in [0, T]$, the sequence $(y^n(t))_n$ converges in H to $y(t)$, see [11].

So, if $y_0 \notin D(A)$, then we can find a sequence $(y_0^n)_n$ of elements in $D(A)$ converging to y_0 in H (because $\overline{D(A)} = H$).

$\forall t \in [0, T]$ and $\forall n \in \mathbb{N}$, we know from i) that $\|y_0^n(t)\| \leq \|y_0^n\|$.
Now, we conclude that $\|y(t)\| \leq \|y_0\|$ for all $t \in [0, T]$.

$\|z(t)\| \leqslant \|z_0\|, \ \forall t \in [0, t_{\max}[$.

Hence $t_{\max} = +\infty$ and from (6), we have $\|y(\tau)\|^2 - \|y(t)\|^2 \geq 2\rho \int_\tau^t \frac{\langle y(s); Ny(s)+c\rangle^2}{|\langle y(t), Ny(t)+c\rangle|+1} ds$ for all $0 \leq \tau \leq t$. This completes the proof.

4 Exponential Stabilisation

4.1 Decomposition of the state space and the system

Let $\delta > 0$ be fixed in advance. We suppose that the spectrum $\sigma(A)$ of A can be decomposed into $\sigma_u(A)$ and $\sigma_s(A)$

$$\text{such that } \sigma_u(A) = \sigma(A) \cap \{\lambda : \operatorname{Re}\lambda \geqslant -\delta\}, \quad \sigma_s(A) = \sigma(A) \cap \{\lambda : \operatorname{Re}\lambda < -\delta\}.$$

Then $\sigma(A) = \sigma_u(A) \cup \sigma_s(A)$ such that $\sigma_u(A)$ can be separated from $\sigma_s(A)$ by a simple and closed curve C.

It has been shown in [12](p. 178) and [13] that the operator A may be decomposed according to the decomposition:

$$H = H_u \oplus H_s \tag{7}$$

meaning $P\mathcal{D}(A) \subset \mathcal{D}(A); AH_s \subset H_s$, $AH_u \subset H_u$ (invariance of H_s and H_u under A), where $H_u = P_uH$ and $H_s = P_sH$ with P_u being the projection operator

$$P_u = \frac{1}{2\pi i}\int_C (\lambda I - A)^{-1} d\lambda \text{ and } P_s = I - P_u.$$

Then the operator A can be decomposed as

$$A = A_u + A_s \tag{8}$$

with $A_u = P_u A$ and $A_s = P_s A$. Here, A_s and A_u are the restrictions of A on H_s and H_u, respectively.

We consider N_s and N_u, the restrictions of the operator N on H_s and H_u, respectively, such that

$(\mathbf{H}_1): \ NH_u \subseteq H_u$.

$(\mathbf{H}_2): \ NH_s \subseteq H_s$.

In the sequel, we suppose that the operator A may be decomposed according to the decomposition (8). Under the hypotheses $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, the system (1) can be decomposed into the two following systems:

$$\begin{cases} \frac{dy_u(t)}{dt} = A_u y_u(t) + v_u(t)(N_u y_u(t) + c_u), \quad \forall t > 0, \\ y_u(0) = (y_{u_0}) \in H_u, \end{cases} \tag{9}$$

and

$$\begin{cases} \frac{dy_s(t)}{dt} = A_s y_s(t) + v_s(t)(N_s y_s(t) + c_s), \quad \forall t > 0, \\ y_s(0) = (y_{s_0}) \in H_s, \end{cases} \tag{10}$$

where y_u and y_s are the components of the solution $y \in H$ on H_u and H_s, respectively. By [13], the semigroup $S(t)$ generated by A also commutes with P_u and P_s, and induces a C_0-semigroup $S_u(t)$ (resp. $S_s(t)$) on H_u (resp. H_s).

We further suppose that A_u generates a C_0-semigroup of contractions $S_u(t)$, and A_u generates a C_0-semigroup of contractions $S_u(t)$. If A_s satisfies the following spectrum-determined growth assumption:
$\lim_{t\to+\infty} \frac{\ln\|S_s(t)\|}{t} = \sup \operatorname{Re}(\sigma(A_s))$, then $\exists \eta, K_\eta > 0$ such that $\|S_s(t)\| \leqslant K_\eta e^{-\eta t}, t \geqslant 0$.

The aim of what follows is to study the problem of weak and strong stabilization of (1) via the properties of the systems (10) and (9). We begin with the component $y_s(t)$ of the solution $y(t)$ of the system (10).

4.2 Exponential stabilization of the component $y_s(t)$

Theorem 4.1 *Let A generate a C_0-semigroup $S(t)$, and suppose that the following conditions hold:*

1. *The operator A may be decomposed according to the decomposition (8).*
2. *A_s satisfies the following spectrum-determined growth assumption:*

$$\exists \eta, K_\eta > 0 \text{ such that: } \quad \|S_s(t)\| \leqslant K_\eta e^{-\eta t}, \quad t \geqslant 0.$$

Then if $\rho < \dfrac{\eta}{K_\eta \Big(L\|y_s(0)\| + \|c_s\| \Big)^2}$, the feedback

$$v_u(t) = -\rho \frac{\langle y_u(t), N_u y_u(t) + c_u \rangle}{|\langle y_u(t), N_s y(t) + c_u \rangle| + 1}, \ \forall t > 0, \tag{11}$$

exponentially stabilizes the system (10).
More precisely, $\exists \beta_s > 0$ such that $\quad \|y_s(t)\| \leq K_\eta \|y_s(0)\| e^{-\beta_s t}, \ \forall t > 0.$

Proof. Using Proposition 3.1, we deduce that the system (10) admits a unique global mild solution given by

$$\begin{cases} y_s(t) = S_s(t)y_s(0) + \int_0^t S_s(t-\tau)v_s(\tau)\left(N_s y_s(\tau) + c_s\right)d\tau, t \geq 0, \\ y_s(0) \in H_s. \end{cases}$$

So, $\|y_s(t)\| \leq \|S_s(t)\|\|y_s(0)\| + \displaystyle\int_0^t |v_s(\tau)|\,\|S_s(t-\tau)\|\big(\|N_s y_s(\tau)\| + \|c_s\|\,\big)d\tau.$

We have $\|S_s(t)\| \leqslant K_\eta e^{-\eta t}$. So,

$$\begin{aligned} \|y_s(t)\| &\leq K_\eta e^{-\eta t}\|y_s(0)\| + \int_0^t |v_s(\tau)|\, K_\eta e^{-\eta(t-\tau)}\big(\|N_s y_s(\tau)\| + \|c_s\|\big)d\tau \\ &\leq K_\eta e^{-\eta t}\|y_s(0)\| + \Big(L_s\|y_s(0)\| + \|c_s\|\Big)\int_0^t |v_s(\tau)|\, K_\eta e^{-\eta(t-\tau)}d\tau \end{aligned}$$

$\Big(L_s$ is a Lipschitz constant of N_u in the ball $\mathcal{B}_{(0,\|z_0\|)}\Big)$.

The feedback (11) is a bounded function in time and is uniformly bounded with respect to the initial states, and we have

$$\begin{aligned} |v_u(t)| &\leq \rho|\langle y_u(t), Ny_u(t) + c_u\rangle| \quad \forall t > 0, \\ &\leq \rho\|y_u(t)\|\Big(L_u\|y_u(0)\| + \|c_u\|\Big); \\ &\Big(L_u \text{ is a Lipschitz constant of } N_u \text{ in the ball } \mathcal{B}_{(0,\|z_0\|)}\Big) \end{aligned}$$

So,

$$\begin{aligned} \|y_s(t)\| &\leq K_\eta e^{-\eta t}\|y_s(0)\| \\ &\quad + \rho K_\eta\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)\Big(L_s\|y_s(0)\| + \|c_s\|\,\Big)\int_0^t \|y_s(\tau)\|e^{-\eta(t-\tau)}d\tau \\ &\leq K_\eta e^{-\eta t}\|y_s(0)\| + \mathcal{A}e^{-\eta t}\int_0^t \|y_s(\tau)\|e^{\tau}d\tau, \end{aligned}$$

where $\mathcal{A} = \rho K_\eta\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)$.

So, $\|y_s(t)\|\,e^{\eta t} \leq K_\eta\|y_s(0)\| + \mathcal{A}\int_0^t \|y_s(\tau)\|e^{\tau}d\tau$.

By using Gronwall's inequality, we have

$$\begin{aligned} \|y_s(t)\|\,e^{\eta t} &\leq K\|y_s(0)\|\exp\left(\int_0^t \mathcal{A}\mathrm{d}s\right) \\ &\leq K_\eta\|y_s(0)\|e^{\mathcal{A}t}. \end{aligned}$$

So, $\|y_s(t)\| \leq K_\eta\|y_s(0)\|e^{(\mathcal{A}-\eta)t}$.

We set: $\beta_s = \eta - \mathcal{A} = \eta - \rho K\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)$ if $\rho < \dfrac{\eta}{K_\eta\Big(L_u\|y_s(0)\|+\|c_s\|\Big)\Big(L_u\|y_s(0)\|+\|c_s\|\Big)}$.

Then $\rho K_\eta\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big) < \eta$, so $\beta_s > 0$.

Finally, $\|y_s(t)\| \leq K_\eta\|y_s(0)\|e^{-\beta_s t}$, $K_\eta, \beta_s > 0$. This completes the proof of the theorem.

Now let us study the component $y_u(t)$ of the solution $y(t)$ of the system (10).

4.3 Decay estimate and exponential stabilization of the component $y_u(t)$

Lemma 4.1 *Let A_u generate a semigroup $S_u(t)$ of contractions on H_u and let N_u be locally Lipschitz. Then the system (9) controlled by (2) possesses a unique mild solution $y_u(t) \in H_u$ for each $y_u(0) \in H_u$ which satisfies, when $t \to +\infty$,*

$$\int_0^T |\langle S_u(\tau)y_u(t), N_u S_u(\tau)y_u(t) + c_u\rangle| d\tau = \mathcal{O}\left(\sqrt{\int_t^{t+T} \frac{|\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle|^2}{1 + |\langle N_u y_u(\tau) + c_u, y_u(\tau)\rangle|} d\tau}\right). \quad (12)$$

Proof. Using Proposition 3.1, we deduce that the system (9) admits a unique global mild solution given by the following formula of variation of the constants:

$$y_u(t) = S_u(t)y_u(0) - \rho \int_0^t \frac{\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle}{|\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle| + 1} (N_u y_u(\tau) + c_u) S_u(t-\tau) d\tau,$$

and using the fact that $S_u(t)$ is a semigroup of contractions, and Schwartz's inequality, for all $t \in [0, T]$, we have

$$\|y_u(t) - S_u(t)y_u(0)\| \le \rho\sqrt{T}\Big(L\,\|y_u(0)\| + \|c_u\|\Big)\left(\int_0^T \frac{|\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle|^2}{1 + |\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle|} d\tau\right)^{\frac{1}{2}}. \quad (13)$$

From the relation

$$\begin{aligned}\langle N_u S_u(t)y_0 + c_u, S_u(t)y_0\rangle &= \langle N_u S_u(t)y_0, S_u(t)y_u(0) - y_u(t)\rangle + \langle c_u, S_u(t)y_u(0) - y_u(t)\rangle \\ &+ \langle N_u S_u(t)y_0 - N_u y_u(t), y_u(t)\rangle + \langle N_u y_u(t) + c, y_u(t)\rangle,\end{aligned}$$

when using $\|y_u(t)\| \le \|y_u(0)\|, \forall t \in [0, t_{\max}[$, the fact that $S_u(t)$ is a semigroup of contraction, N_u is locally Lipschitz, and Schwartz's inequality, it comes

$$|\langle N_u S_u(s)y_u(0)+c_u, S_u(s)y_u(0)\rangle| \le \Big(2L_u\|y_u(0)\| + \|c_u\|\Big)\|y_u t) - S_u(t)y_u(0)\| + |\langle N_u y_u(s)+c_u, y_u(s)\rangle|.$$

Using (13),

$$\begin{aligned}|\langle N_u S_u(s)y_u(0) + c_u, S_u(s)y_u(0)\rangle| \le \mathcal{C}_{(\|y_u(0)\|;c_u)}&\left(\int_0^t \frac{|\langle y_u(s), N_u y_u(s) + c_u\rangle|^2}{1 + |\langle y_u(s), N_u y_u(s) + c_u\rangle|} ds\right)^{\frac{1}{2}} \\ &+ |\langle N_u y_u(s) + c_u, y_u(s)\rangle|,\end{aligned} \quad (14)$$

where $\mathcal{C}_{(\|y_u(0)\|;c_u)} = \rho\sqrt{T}\left(2L_u\|y_0\| + \|c_u\|\right)\left(L_u\|y_0\| + \|c_u\|\right)$. Replacing y_0 by $y_u(t)$ in (14) and using the fact that $\|y_u(t)\| \le \|y_u(0)\|\ \forall t \ge 0$, we get

$$\begin{aligned}|\langle N_u S_u(s)y_u(t) + c_u, S_u(s)y_u(t)\rangle| \le \mathcal{C}_{(\|y_0\|;c_u)}&\left(\int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|^2}{1 + |\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|} ds\right)^{\frac{1}{2}} \\ &+ |\langle N_u y_u(s+t) + c_u, y_u(s+t)\rangle|.\end{aligned}$$

Integrating the last inequality over the interval $[0, T]$ and using the semigroup property of the solution $y(t)$ and Schwartz's inequality, remarking that the mappings $x \mapsto C_x = (2L_u x + \|c\|)(L_u x + \|c_u\|)$ are increasing $\left(C_{\|y_u(t)\|} \le C_{\|y_0\|}\right)$, we get

$$\begin{aligned}\int_0^T |\langle N_u S_u(s)y_u(t)+c_u, S_u(s)y_u(t)\rangle| ds \le T^3\mathcal{C}_{(\|y_0\|;c_u)}&\left(\int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|^2}{1 + |\langle y_u(s+t), N_u y(s+t) + c_u\rangle|} ds\right)^{\frac{1}{2}} \\ &+ \int_0^T |\langle N_u y_u(s+t) + c_u, y_u(s+t)\rangle| ds.\end{aligned}$$

Since
$\int_0^T |\langle N_u y_u(s+t) + c_u, y_u(s+t)\rangle| ds \le \left(1 + L_u\|y_0\|^2 + \|c_u\|\|y_0\|\right) \int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|^2}{1 + |\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|} ds,$

by Schwartz's inequality, we get

$\int_0^T |\langle N_u y_u(s+t)+c_u, y_u(s+t)\rangle| ds \leq T\left(1+L_u\|y_0\|^2+\|c_u\|\|y_0\|\right)\left(\int_0^T \frac{|\langle y_u(s+t),N_u y_u(s+t)+c_u\rangle|^2}{1+|\langle y_u(s+t),N_u y_u(s+t)+c_u\rangle|} ds\right)^{\frac{1}{2}}.$

We deduce that

$$\int_0^T |\langle N_u S_u(s) y_u(t)+c_u, S_u(s) y_u(t)\rangle| ds \leq \mathcal{M}\left(\int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t)+c_u\rangle|^2}{1+|\langle y_u(s+t), N_u y_u(s+t)+c_u\rangle|} ds\right)^{\frac{1}{2}},$$

where $\mathcal{M}=\left(\rho T^{\frac{3}{2}} \mathcal{C}_{(\|y_0\|;c_u)}\right)+T(1+L_u\|y_0\|^2+\|c_u\|\|y_0\|)$. This gives the estimate (12).

Remark 4.1 For c=0, we obtain the same content found in the homogeneous case, see [2], $\mathcal{M}=\rho T^{\frac{3}{2}}\left(2L_u\|y_u(0)\|\right)\left(L(N_u)\|y_u(0)\|\right)+T(1+L_u\|y_u(0)\|^2)$.

Theorem 4.2 *Let A generate a C_0-semigroup $S_u(t)$, and suppose that the following conditions hold:*
1. $S_u(t)$ is a contraction semigroup;
2. there exist $\delta, T>0$ such that

$$\int_0^T |\langle N_u S_u(s) y_u(t)+c_u, S(s) y_u(t)\rangle| ds \geq \delta\|y_u(t)\|, \quad \forall y_u \in H_u. \tag{15}$$

Then the feedback (5) exponentially stabilizes the system (6).
More precisely, there exists $\beta_u>0$ such that $\|y_u(t)\| \leq e^{-\beta_u}\|y_0\| e^{-\frac{\beta_u}{T}(t)}\ \forall t>0$.

Proof. Integrate now the following inequality over the interval $[kT,(k+1)T]$, for $k \in \mathbb{N}$ and $T>0$,

$$\frac{d}{dt}\|y_u(t)\|^2 \leq -2\rho \frac{(\langle y_u(t); N_u y(t)+c_u\rangle)^2}{|\langle y_u(t), N_u y(t)+c_u\rangle|+1}; \forall t \in [0,T].$$

We get $\|y_u((k+1)T)\|^2-\|y_u(kT)\|^2 \leq -2\rho \int_{kT}^{(k+1)T} \frac{|\langle y(\tau),N_u y(\tau)+c_u\rangle|^2}{|\langle y_u(\tau),N_u y_u(\tau)+c_u\rangle|+1}\, \mathrm{d}\tau$.
Using now the estimate (12), we deduce that

$$\|y_u((k+1)T)\|^2-\|y_u(kT)\|^2 \leq \frac{-2\rho}{\mathcal{M}}\left(\int_0^T |\langle S_u(\tau) y_u(t), N_u S_u(\tau) y_u(t)+c_u\rangle| \,\mathrm{d}\tau\right)^2.$$

According to the inequality (15), we have

$$\|y_u((k+1)T)\|^2-\|y_u(kT)\|^2 \leq \frac{-2\rho\delta^2}{\mathcal{M}}\|y_u(kT)\|^2. \tag{16}$$

Letting $s_k=\|y(kT)\|^2,\ k \in \mathbb{N}$, the inequality (16) can be written as

$$s_{k+1}-s_k \leq \frac{-2\rho\delta^2}{\mathcal{M}} s_k, \quad \forall k \geq 0,$$

$$s_{k+1} \leq C s_k, \quad \forall k \geq 0,$$

where $C=\left(1-\frac{2\rho}{\mathcal{M}}\delta^2\right)<1$, which gives $s_k \leq e^{-k\ln\frac{1}{C}} s_0$.

So, $\|y_u(t)\| \leq e^{-\frac{\ln(\frac{1}{C})}{2}}\|y_0\| e^{-\frac{\ln(\frac{1}{C})}{2T}t}$ for all $t \geq 0$, $\|y_u(t)\| \leq e^{-\beta_u}\|y_0\| e^{-\frac{\beta_u}{T}t}$ for all $t \geq 0$, where $\beta_u=\frac{\ln(\frac{1}{C})}{2}>0$.

4.4 Exponential stabilization

Theorem 4.3 *Suppose that the assumptions of both Theorems 4.1 and 4.2 are verified. Then the feedback (2) exponentially stabilizes the system (1). More precisely, there exist* $\beta > 0$ *and* $\alpha > 0$ *such that* $\|y(t)\| \leq \alpha e^{-\beta t}, \forall t \geq 0$.

Proof. Using Proposition 3.1, we deduce that the system (1) admits a unique global mild solution $y(t)$; according to the decomposition (8), we have $y(t) = y_u(t) + y_s(t)$.
It follows from Theorems 4.1 and 4.2 that

$$\begin{aligned} y(t) &\leqslant e^{-\beta_u} \|y_0\| e^{-\frac{\beta_u}{T} t} + K_\eta \|y_s(0)\| e^{-\beta_s t} \\ &\leqslant 2(e^{-\beta_u} \ \|y_0\| + K_\eta \|y_s(0)\|) e^{-min(e^{-\frac{\beta_u}{T}}, \beta_s)t}. \end{aligned}$$

So, $\|y(t)\| \leq \alpha e^{-\beta t}, \ \forall t \geq 0$, where $\alpha = 2(e^{-\beta_u} \ \|y_0\| + K_\eta \|y_s(0)\|)$ and $\beta = min(\frac{\beta_u}{T}, \beta_s)$.

5 Strong Stabilisation

Theorem 5.1 *Let* A *generate a semigroup* $S(t)$ *of contractions on* H. *Suppose that*
(i) N *is locally Lipschitz;*
(ii) $\exists \delta, T > 0$ *such that*

$$\int_0^T |\langle NS(s)y(t) + c, S(s)y(t)\rangle| \mathrm{d}s \geq \delta \|y(t)\|^2, \quad \forall y \in H. \tag{17}$$

Then the feedback (2) strongly stabilises the system (1) with the following decay estimate:

$$\|y(t)\| = O\left(t^{-\frac{1}{2}}\right) \ \text{as } t \to +\infty.$$

Proof. If $H = H_u$ is of finite dimension, then we retrieve the result of Theorem 4.2. In the case $\dim H_u = +\infty$, following the techniques used in the proof of Lemma 4.1, we can obtain the following estimate when $t \to +\infty$:

$$\int_0^T |\langle S(\tau)y(t), NS(\tau)y(t) + c\rangle| d\tau = \mathcal{O}\left(\sqrt{\int_t^{t+T} \frac{|\langle y(\tau), Ny(\tau) + c\rangle|^2}{1 + |\langle Ny(\tau) + c, y(\tau)\rangle|} d\tau}\right). \tag{18}$$

Integrating now the inequality $\frac{d}{dt}\|y(t)\|^2 \leq -2\rho \frac{(\langle y(t); Ny(t)+c\rangle)^2}{|\langle y(t), Ny(t)+c\rangle|+1}; \forall t \in [0, T]$, over the interval $[kT, (k+1)T]$, for $k \in \mathbb{N}$ and $T > 0$, we get

$$\|y((k+1)T)\|^2 - \|y(kT)\|^2 \leq -2\rho \int_{kT}^{(k+1)T} \frac{|\langle y(\tau), Ny(\tau) + c\rangle|^2}{|\langle y(\tau), Ny(\tau) + c\rangle| + 1} \, \mathrm{d}\tau.$$

Using now the estimate (18), we deduce that

$$\|y((k+1)T)\|^2 - \|y(kT)\|^2 \leq \frac{-2\rho}{\mathcal{M}} \left(\int_0^T |\langle S(\tau)y, NS(\tau)y + c\rangle| \mathrm{d}\tau\right)^2.$$

From (17), we have $\|y((k+1)T)\|^2 - \|y(kT)\|^2 \leq \frac{-2\rho\delta^2}{\mathcal{M}} \|y(kT)\|^4$.
Letting $s_k = \|y(kT)\|^2, k \in \mathbb{N}$, the last inequality can be written as

$$s_{k+1} \leq s_k - \frac{2\rho\delta^2}{\mathcal{M}} s_k^2, \quad \forall k \geq 0.$$

Using the fact that $t \mapsto \|y(t)\|$ is a decreasing function on $[0, +\infty[$, we get

$$s_{k+1} \leq s_k - \frac{2\rho\delta^2}{\mathcal{M}} s_{k+1}^2, \quad \forall k \geq 0.$$

The last inequality can be written as follows: $s_{k+1} \leq s_k - Cs_{k+1}^2, \quad \forall k \geq 0$, where $C = \frac{2\rho\delta^2}{\mathcal{M}} > 0$. Now, to obtain the decay rate for solutions of (1), we recall the following lemma, see [14] and [15].

Lemma 5.1 *Let the sequence of non-negative real numbers s_k, $k = 0, 1, 2, \ldots$, satisfy $s_{k+1} \leqslant s_k - C(k+1)^r s_{k+1}^2$, where C is a positive real number and r is a non-negative integer. Then there exists a positive number $M = M(M, r, u_0)$ such that $s_k \leqslant \frac{M}{(k+1)^{r+1}}$, $k = 0, 1, 2, 3, \ldots$.*

So, from the lemma and for $r = 0$, there exists a positive constant K (depending on C) such that $s_k \leq \frac{M}{k+1}$, so $\|y(kT)\|^2 \leq \frac{M}{k+1}$. For $k = E(\frac{t}{T})$, $\big(E\left(\frac{t}{T}\right)$ designed the integer part of $\frac{t}{T}\big)$, we obtain

$$\|y(E(\frac{t}{T})T)\|^2 \leq \frac{M}{E(\frac{t}{T}) + 1}.$$

Using the fact that $E(\frac{t}{T})T \leq t$ and $t \mapsto \|y(t)\|$ is a decreasing function on $[0, +\infty[$, we get $\|y(t)\|^2 \leq \frac{TM}{t}$. So, $\|y(t)\| = \mathcal{O}\left(t^{-\frac{1}{2}}\right)$ as $t \to +\infty$.

6 Applications

Example 6.1 One-dimensional heat equation.
Let us consider the following semilinear heat equation:

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial x^2} + v(y(t))(Ny(t) + c), & x \in (0,1), t > 0, \\ \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0, & \forall t > 0, \end{cases} \tag{19}$$

where $y(t)$ is the temperature profile at time t. Here we take the state space $H = L^2(0,1)$ and the operator A is defined by

$$Ay = \frac{\partial^2 y}{\partial x^2} \text{ with } \mathcal{D}(A) = \left\{ y \in H^2(0,1) \mid \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0 \right\}.$$

The domain of A gives the homogeneous Neumann boundary condition imposed at the ends of the bar, which requires specifying how the heat flows out of the bar and means that both ends are insulated. The control $v(y(t))$ is defined by

$$v(y(t)) = -\rho \frac{\langle y(t), Ny(t) + c \rangle}{|\langle y(t), Ny(t) + c \rangle| + 1}, \quad \forall t \geq 0. \tag{20}$$

The operator of control N is defined by $Ny = \frac{1}{1+\|y\|} \sum_{j=1}^{+\infty} \alpha_j \langle y, \varphi_j \rangle \varphi_j, \alpha_j \geq 0, \forall j \geq 1$, such that $\sum_{j=1}^{+\infty} \alpha_j^2 < \infty$; N is a nonlinear sequentially continuous and locally Lipschitz operator such that $N(0) = 0$.

The spectrum of A is given by the simple eigenvalues $\lambda_j = -\pi^2(j-1)^2, j \in \mathbb{N}^*$, and eigenfunctions $\varphi_1(x) = 1$ and $\varphi_j(x) = \sqrt{2}\cos((j-1)\pi x)$ for all $j \geq 2$. Then the subspace H_u is the one-dimensional space spanned by the eigenfunction φ_1, and we have $S_u(t)y_u = \langle y_u, \varphi_1\rangle \varphi_1$, so $S_u(t) = I_{H_u}$ (the identity) and hence $(S_u(t))_{t\geq 0}$ is a semigroup of isometries.

N_u is the restriction of the operator N on H_u defined by

$$\begin{aligned} N_u y_u(t) &= \tfrac{\alpha_1 y_u(t)}{1+\|y_u(t)\|} \\ &= \tfrac{\alpha_1}{1+|\langle y_u(t),\varphi_1\rangle|}\langle y_u(t),\varphi_1\rangle \varphi_1, \end{aligned}$$

N_u is a nonlinear sequentially continuous and locally Lipschitz operator,

$$\begin{aligned} \|N_u y_u(t) - N_u z_u(t)\| &= \|\tfrac{\alpha_1 y_u(t)}{1+\|y_u(t)\|} - \tfrac{\alpha_1 z_u(t)}{1+\|z_u(t)\|}\| \\ &\leqslant \tfrac{\alpha_1 y_u(t)}{1+\|y_u(t)\|} - \tfrac{\alpha_1 z_u(t)}{1+\|y_u(t)\|} + \tfrac{\alpha_1 z_u(t)}{1+\|y_u(t)\|} - \tfrac{\alpha_1 z_u(t)}{1+\|z_u(t)\|} \\ &\leqslant \|\tfrac{\alpha_1}{1+\|y_u(t)\|}(y_u(t)-z_u(t))\| + \|\alpha_1 z_u(t)\big(\tfrac{1}{1+\|y_u(t)}-\tfrac{1}{1+\|z_u(t)\|}\big)\| \\ &\leqslant L_u\|y_u(t)-z_u(t)\|, \text{ where } L_u = |\alpha_1|. \end{aligned}$$

Here we can see that $NH^u \subset H^u$. We have

$$\begin{aligned} < NS_u(\tau)y_u(t)+c_u, S_u(\tau)y_u(t) >_H | &= | < N_u y_u(t), y_u(t) > + < c_u, y_u(t) > | \\ &= \left(\tfrac{\alpha_1\|y_u(t)\|}{1+\|y_u(t)\|} + \|c_u\|\right)\|y_u(t)\| \end{aligned}$$

if $\alpha_1 > 0$, we have $\left(\frac{\alpha_1\|y_u(t)\|}{1+\|y_u(t)\|} + \|c_u\|\right)\|y_u(t)\| > \|c_u\|\|y_u(t)\|$, then
$| < NS_u(\tau)y_u(t)+c_u, S_u(\tau)y_u(t) > | > \|c_u\|\|y_u(t)\|$, so
$\int_0^T | < NS_u(\tau)y_u(t)+c_u, S_u(\tau)y_u(t) > |d\tau > \delta\|y_u(t)\|$ with $\delta = T\|c_u\|$.

We can see that (15) holds, and the assumptions of Theorem 4.1 are verified.
For $j \geq 2$, the subspace H_s is spanned by the eigenfunctions $(\varphi_j)_{j\geq 2}$; N_s is the restriction of the operator N on H_s defined by $N_s y_s = \frac{1}{1+\|y_s\|}\sum_{j=2}^{+\infty}\alpha_j \langle y_s, \varphi_j\rangle \varphi_j$.

We have $S_s(t) = \sum_{j=2}^{\infty} e^{-\pi^2(j-1)^2 t}$. Since $j-2 \geq 0$, one has $e^{-2\pi^2 j(j-2)t} \leqslant 1$, then $\|S_s(t)\| \leq \mathrm{e}^{-\pi^2 t}$, $\forall t \geq 0$.

So, A_s satisfies the spectrum-determined growth assumption.

$$\text{If } \rho < \frac{\pi^2}{(L_s\|y_s(0)\| + \|c_s\|)(L_u\|y_s(0)\| + \|c_s\|)}, \text{ where } L_u = \alpha_1 \text{ and } L_s = \sum_{j=2}^{+\infty}\alpha_j^2,$$

then the assumptions of Theorem 4.2 are verified.

Finally, the assumptions of both Theorems 4.1 and 4.2 are verified. Then by applying Theorem 4.3, we deduce that (19) is exponentially stabilizable by the control

$$v(y(t)) = -\rho\frac{(\alpha_1+|c_u|)y_u(t)^2 + |c_u|y_u(t)}{(1+\alpha_1+|c_u|)y_u(t)^2 + (1+|c_u|)y_u(t)}, \quad \forall t \geq 0.$$

7 Conclusion

In this work, the sets of necessary and sufficient conditions for the exponential stabilization of inhomogeneous semilinear systems are given. The stabilizing controls may be chosen bounded with respect to time and initial states and can be applied to systems subject to constraints on the control input. Though the exponential stabilization of bounded operators enables us to discuss various stabilization problems, the present study does not cover other situations. This is the case of exponential stabilization of unbounded operators in a Banach space. Also, the issue of unbounded operator control is of great interest.

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