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Existence of Weak Solutions for a Class of $(p(b(u)), q(b(u)))$-Laplacian Problems

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Abstract: In this paper, we consider the existence of weak solutions for some parabolic $(p(b(u)), q(b(u)))$-Laplacian problem when (p, q) is a nonlocal quantity. The novelty of this work is the study of some problems involving the (p, q)-Laplacian operator in the nonlocal case. The motivation to study these nonlocal problems relies in the fact that in reality, the measurements of some physical quantities are not made pointwise but through some local averages.

Keywords: *$(p(b(u)), q(b(u)))$-Laplacian; weak solutions; parabolic problem, generalised Sobolev spaces.*

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1 Introduction

The study of partial differential equations involving the (p, q)-Laplacian generalized several types of problems not only in physics, but also in biophysics, plasma physics, and in the study of chemical reactions. These problems appear, for example, in a general reaction–diffusion system

$$u_t = -\operatorname{div}\left[\left(a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}\right)\nabla u\right] + f(x, u),$$

where $a_p, b_q \in \mathbb{R}^+$ are some positive constants, the function u generally describes the concentration, the term $\operatorname{div}\left[\left(a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}\right)\nabla u\right]$ corresponds to the diffusion with coefficient $D(u) = a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}$, and $f(x, u)$ is the reaction term related to the source and loss processes. In general, the reaction term $f(x, u)$ has a polynomial form with respect to the concentration u.

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Because of the importance of this kind of problems, many authors have investigated the existence and uniqueness of different types of their solutions [5, 7, 14].

Our main interest in this work is to extend these results to the case when p, q may depend on the space variable x and the unknown solution u. We consider the case where the dependency of p, q on u is a nonlocal quantity. Namely, we study the following parabolic problem:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(b(u))-2}\nabla u) - \operatorname{div}(|\nabla u|^{q(b(u))-2}\nabla u) = f \ \text{ in } \Omega_T = \Omega \times (0,T), \\ u = 0 \quad \text{on } \Gamma = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N, N \geq 2, T > 0, f,\ u_0$ are given data, $p : \mathbb{R} \to [1,\ +\infty)$ and $q : \mathbb{R} \to [1,\ +\infty)$ are real functions such that

$$p, q \text{ are continuous} \quad \text{and} \quad 1 < \alpha \leq q < p \leq \beta < \infty \tag{2}$$

for some constants α, β. We denote by b a mapping from $W_0^{1,\alpha}(\Omega)$ into $\mathbb{R}$ such that

$$b \text{ is continuous and bounded,} \tag{3}$$

i.e., b sends the bounded sets of $W_0^{1,\alpha}(\Omega)$ into the bounded sets of $\mathbb{R}$. In this case, suitable examples for the mapping b in (3) can be chosen as

$$b(u) = \|\nabla u\|_{L^\alpha(\Omega)}.$$

This kind of problems was first introduced by Chipot and de Oliveira in [9]. The elliptic version of the problem (1) with local quantities p, q was studied by L. Yanru in [15], he obtained the existence of weak solutions by means of a singular perturbation technique and the Schauder fixed point theorem. We were inspired by the work of C. Zhang and X. Zhang (see [16]), where the authors proved the existence of weak solutions to the following parabolic $p(u)$-Laplacian problem:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) = f & \text{in } \Omega_T = \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

The fact that in reality, the physical measurements of certain quantities are not made in a pointwise way but through local averages, is always the motivation to study non-local problems. The main difficulty in the analysis of these $p(u)$-problems relies in the fact that their weak formulations cannot be written as equalities in terms of duality in fixed Banach spaces. For more interesting features and results, we refer to [6, 9, 13, 15] and references therein.

This paper is organized as follows. In Section 2, we introduce the basic assumptions and we recall some definitions, basic properties of generalised Sobolev spaces that we will use later. Section 3 is devoted to showing the existence of weak solutions to the problem (1).

2 Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N, N \geq 2$, with the Lipschitz-continuous boundary $\partial\Omega$. Given a measurable function $h : \Omega \to [1, \infty)$, we introduce the variable exponent Lebesgue space by

$$L^{h(\cdot)}(\Omega) = \{ u : \Omega \to \mathbb{R} \ / \ \rho_{h(\cdot)}(u) := \int_\Omega |u(x)|^{h(x)} dx < \infty\}. \tag{4}$$

Equipped with the Luxembourg norm

$$\|u\|_{h(\cdot)} := \inf \left\{ \lambda > 0 \ : \ \rho_{h(\cdot)}\left(\frac{u}{\lambda}\right) \leq 1 \right\}, \tag{5}$$

$L^{h(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 < h_- \leq h_+ < \infty, \tag{6}$$

then $L^{h(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{h(\cdot)}(\Omega)$ is $L^{h'(\cdot)}(\Omega)$, where $h'(x)$ is the generalised Hölder conjugate of $h(x)$,

$$\frac{1}{h(x)} + \frac{1}{h'(x)} = 1.$$

From the definitions of the modular $\rho_{h(\cdot)}(u)$ and the norm (5), it can be proved that if (6) holds, then

$$\min\left\{\|u\|_{h(\cdot)}^{h_-}, \ \|u\|_{h(\cdot)}^{h_+}\right\} \leq \rho_{h(\cdot)}(u) \leq \max\left\{\|u\|_{h(\cdot)}^{h_-}, \ \|u\|_{h(\cdot)}^{h_+}\right\}. \tag{7}$$

One very useful consequence of (7) is

$$\|u\|_{h(\cdot)}^{h_-} - 1 \leq \rho_{h(\cdot)}(u) \leq \|u\|_{h(\cdot)}^{h_+} + 1. \tag{8}$$

For any functions $u \in L^{h(\cdot)}(\Omega)$ and $v \in L^{h'(\cdot)}(\Omega)$, the generalized Hölder inequality holds:

$$\int_\Omega uv dx \leq (\frac{1}{h_-} + \frac{1}{h'_-})\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)} \leq 2\|u\|_{h(\cdot)}\|v\|_{h'(\cdot)}. \tag{9}$$

We define also the generalised Sobolev space by

$$W^{1,h(\cdot)}(\Omega) := \{u \in L^{h(\cdot)}(\Omega) : \nabla u \in L^{h(\cdot)}(\Omega)\},$$

which is a Banach space for the norm

$$\|u\|_{1,h(\cdot)} := \|u\|_{h(\cdot)} + \|\nabla u\|_{h(\cdot)}. \tag{10}$$

Now, we introduce the following function space:

$$W_0^{1,h(\cdot)}(\Omega) := \{u \in \mathrm{W}_0^{1,1}(\Omega) : \nabla u \in L^{h(\cdot)}(\Omega)\},$$

which we endow with the norm

$$\|u\|_{W_0^{1,h(\cdot)}(\Omega)} := \|u\|_1 + \|\nabla u\|_{h(\cdot)}. \tag{11}$$

If $h \in C(\overline{\Omega})$, then the norm in $W_0^{1,h(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{h(\cdot)}$. If h is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,h(\cdot)}(\Omega)$. Let h be a measurable function in Ω satisfying $1 \leq h_- \leq h_+ < d$ and being log-Hölder continuous, then

$$\|u\|_{h^*(\cdot)} \leq C\|\nabla u\|_{h(\cdot)} \quad \forall u \in W_0^{1,h(\cdot)}(\Omega),$$

for some positive constant C, where

$$h^*(x) := \begin{cases} \frac{Nh(x)}{N-h(x)} & \text{if } h(x) < N, \\ \infty & \text{if } h(x) \geq N. \end{cases}$$

On the other hand, if $h_- > N$, then

$$\|u\|_\infty \leq C\|\nabla u\|_{h(\cdot)} \quad \forall u \in W_0^{1,h(\cdot)}(\Omega)\ ,$$

where C is another positive constant.

Lemma 2.1 *[9] Assume that*

$$1 < \alpha \leq q_n(x) \leq \beta < \infty \quad \forall n \in \mathbb{N},$$

$$\textit{for a.e. } x \in \Omega, \textit{ for some constants } \alpha \textit{ and } \beta, \tag{12}$$

$$q_n \to q \ \textit{ a.e. in } \Omega, \textit{ as } n \to \infty, \tag{13}$$

$$\nabla u_n \to \nabla u \textit{ in } L^1(\Omega)^d, \textit{ as } n \to \infty, \tag{14}$$

$$\||\nabla u_n|^{q_n(x)}\|_1 \leq C, \textit{ for some positive constant } C \textit{ not depending on } n. \tag{15}$$

Then $\nabla u \in L^{q(\cdot)}(\Omega)^d$ *and*

$$\liminf_{n\to\infty} \int_\Omega |\nabla u_n|^{q_n(x)}dx \geq \int_\Omega |\nabla u|^{q(x)}dx. \tag{16}$$

3 Main Results

In this section, we will give a reasonable definition for weak solutions and prove the existence of weak solutions to problem (1). We introduce the functional space

$$X(\Omega_T) := \left\{u \in L^\infty\left(0,T;L^2(\Omega)\right) : |\nabla u| \in L^{p(b(u))}(\Omega_T), u(\cdot,t) \in V_t(\Omega) \text{ a.e. } t \in (0,T)\right\},$$

where

$$V_t(\Omega) := \left\{u \in L^2(\Omega) \cap W_0^{1,\alpha}(\Omega) : |\nabla u| \in L^{p(b(u(\cdot,t)))}(\Omega)\right\}.$$

In the same way, we define $Y(\Omega_T)$ associated with the nonlinear exponent function $q(b(u))$. We denote their dual spaces by $X(\Omega_T)^*$ and $Y(\Omega_T)^*$, respectively.
Now, we give a definition of weak solutions for the parabolic problem (1).

Definition 3.1 A function $u \in X(\Omega_T) \cap Y(\Omega_T) \cap C\left([0,T];L^2(\Omega)\right)$ is said to be a weak solution to problem (1) if for any $\varphi \in C^1\left(\overline{\Omega}_T\right)$ with $\varphi(\cdot,T) = 0$, we have

$$\begin{aligned} &-\int_\Omega u_0(x)\varphi(x,0)dx + \int_0^T\int_\Omega -u\varphi_t dxdt \\ &+\int_0^T\int_\Omega \left(|\nabla u|^{p(b(u))-2}\nabla u + |\nabla u|^{q(b(u))-2}\nabla u\right)\cdot\nabla\varphi dxdt = \int_0^T\int_\Omega f\varphi dxdt. \end{aligned} \tag{17}$$

Theorem 3.1 *Assume that (2) and (3) hold together with $\alpha > 2N/(N+2)$, $u_0 \in L^2(\Omega)$ and $f \in L^{\alpha'}(\Omega_T)$. Then there exists at least one weak solution to problem (1) in the sense of Definition 3.1.*

Proof. We denote $h = T/N_0$, where N_0 is a positive integer. We consider the following time-discrete problem:

$$\begin{cases} \frac{u_k - u_{k-1}}{h} - \operatorname{div}\left(|\nabla u_k|^{p(b(u_k))-2}\nabla u_k\right) \\ \qquad\qquad - \operatorname{div}\left(|\nabla u_k|^{q(b(u_k))-2}\nabla u_k\right) = [f]_h\left((k-1)h\right), \quad x \in \Omega, \\ u_k|_{\partial\Omega} = 0, \quad k = 1, 2, \ldots, N_0, \end{cases} \tag{18}$$

where the Steklov average $[f]_h$ of f is defined as

$$[f]_h(x,t) = \frac{1}{h}\int_t^{t+h} f(x,\tau)d\tau \in L^{\alpha'}(\Omega).$$

For $k = 1$, we consider the problem

$$\begin{cases} \frac{u-u_0}{h} - \operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) - \operatorname{div}\left(|\nabla u|^{q(b(u))-2}\nabla u\right) = [f]_h(0), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{19}$$

Set

$$W = W_0^{1,p(b(u))}(\Omega) \cap W_0^{1,q(b(u))}(\Omega) \cap L^2(\Omega).$$

First, we show that the problem (19) has a weak solution $u_1 \in W$.

Step 1: Approximation

For each $\varepsilon > 0$, we consider the auxiliary problem

$$\begin{cases} \frac{u-u_0}{h} - \operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) - \operatorname{div}\left(|\nabla u|^{q(b(u))-2}\nabla u\right) \\ \qquad\qquad -\varepsilon \operatorname{div}\left(|\nabla u|^{\beta-2}\nabla u\right) = [f]_h(0), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{20}$$

where

$$\frac{2N}{N+2} < \alpha < q(b(u)) \leqslant p(b(u)) \leqslant \beta < \infty \quad \forall u \in \mathbb{R}.$$

Lemma 3.1 *For each $\varepsilon > 0$, there exists a weak solution u_ε to the problem (20).*

Proof. Let $\omega \in L^2(\Omega)$ be given. We have

$$\frac{2N}{N+2} < \alpha < q(b(w)) \leqslant p(b(w)) \leqslant \beta < \infty \quad \text{for a.e. } x \in \Omega.$$

Observing that $[f]_h(0) \in L^{\alpha'}(\Omega) \subset W^{-1,\alpha'}(\Omega) \subset W^{-1,\beta'}(\Omega)$, by the usual theory of monotone operators, there exists a unique solution $u_w \in W_0^{1,\beta}(\Omega)$ such that

$$\int_\Omega \frac{u_w - u_0}{h} v dx + \int_\Omega |\nabla u_w|^{p(b(w))-2} \nabla u_w \cdot \nabla v dx \\ + \int_\Omega |\nabla u_w|^{q(b(w))-2} \nabla u_w \cdot \nabla v dx + \varepsilon \int_\Omega |\nabla u_w|^{\beta-2} \nabla u_w \cdot \nabla v dx = \int_\Omega [f]_h(0) v dx \quad (21)$$

for all $v \in W_0^{1,\beta}(\Omega)$.

By taking $v = u_w$ in (21), we get

$$\frac{1}{2h} \int_\Omega u_w^2 dx + \int_\Omega |\nabla u_w|^{p(b(w))} dx + \int_\Omega |\nabla u_w|^{q(b(w))} dx \\ + \varepsilon \int_\Omega |\nabla u_w|^\beta dx \leqslant \frac{1}{2h} \int_\Omega u_0^2 dx + C \|\nabla u_w\|_{L^\beta(\Omega)}$$

for some positive constant $C = C(\alpha, \beta, \Omega, [f]_h(0))$. Then, using Young's inequality, we obtain

$$\|u_w\|_{L^2(\Omega)} + \|\nabla u_w\|_{L^\beta(\Omega)} \leqslant C \quad (22)$$

for some positive constant $C = C(\alpha, \beta, \Omega, [f]_h(0), \varepsilon, h, N)$. Hence

$$\|u_w\|_{L^2(\Omega)} \leqslant C.$$

Let us now consider the mapping

$$T \ni w \to u_w \in T,$$

where $T := \{v \in L^2(\Omega) \ : \ \|v\|_2 \leqslant C\}$. Firstly, we prove that this mapping is continuous, then by Schauder's fixed point theorem, it will have a fixed point. We suppose that w_n is a sequence in $L^2(\Omega)$ such that

$$w_n \to w \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad n \to \infty. \quad (23)$$

For every $n \in \mathbb{N}$, let u_n be the solution to the problem (20) associated to $w = w_n$. From (22), we have

$$\|\nabla u_n\|_\beta \leq C$$

for some positive constant C which does not depend on n. It follows that

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\beta}(\Omega), \quad \text{as} \quad n \to \infty, \quad (24)$$

$$u_n \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty. \quad (25)$$

From (21), one has

$$\int_\Omega \frac{u_n - u_0}{h} v dx + \int_\Omega |\nabla u_n|^{p(b(w_n))-2} \nabla u_n \cdot \nabla v dx + \int_\Omega |\nabla u_n|^{q(b(w_n))-2} \nabla u_n \cdot \nabla v dx \\ + \varepsilon \int_\Omega |\nabla u_n|^{\beta-2} \nabla u_n \cdot \nabla v dx = \int_\Omega [f]_h(0) v dx, \ \forall v \in W_0^{1,\beta}(\Omega). \quad (26)$$

Using the monotonicity, one also has

$$\int_\Omega \frac{(u_n-v)^2}{h}dx$$
$$+\int_\Omega \Big(|\nabla u_n|^{p(b(w_n))-2}\nabla u_n+|\nabla u_n|^{q(b(w_n))-2}\nabla u_n+\varepsilon|\nabla u_n|^{\beta-2}\nabla u_n\Big)\nabla(u_n-v)\,dx$$
$$-\int_\Omega \Big(|\nabla v|^{p(b(w_n))-2}\nabla v+|\nabla v|^{q(b(w_n))-2}\nabla v+\varepsilon|\nabla v|^{\beta-2}\nabla v\Big)\nabla(u_n-v)\,dx\geqslant 0 \quad (27)$$

for all $v\in W_0^{1,\beta}(\Omega)$. Taking u_n-v as a test function in (26) and using (27), we get

$$\int_\Omega \frac{u_0-v}{h}(u_n-v)\,dx+\int_\Omega [f]_h(0)(u_n-v)\,dx$$
$$-\int_\Omega \Big(|\nabla v|^{p(b(w_n))-2}\nabla v+|\nabla v|^{q(b(w_n))-2}\nabla v+\varepsilon|\nabla v|^{\beta-2}\nabla v\Big)\cdot\nabla(u_n-v)\,dx\geqslant 0 \quad (28)$$

for all $v\in W_0^{1,\beta}(\Omega)$. From (23), we may assume that for some subsequence

$$w_n\to w \quad a.e. \text{ in } \Omega, \quad \text{as} \quad n\to\infty.$$

According to the assumptions of p, q and Lebesgue's theorem, we know that for any $v\in W_0^{1,\beta}(\Omega)$,

$$|\nabla v|^{p(b(w_n))-2}\nabla v\to|\nabla v|^{p(b(w))-2}\nabla v \quad \text{strongly in} \quad L^{\beta'}(\Omega)^d, \quad \text{as} \quad n\to\infty,$$
$$|\nabla v|^{q(b(w_n))-2}\nabla v\to|\nabla v|^{q(b(w))-2}\nabla v \quad \text{strongly in} \quad L^{\beta'}(\Omega)^d, \quad \text{as} \quad n\to\infty. \quad (29)$$

Using (25), (28) and (29), we obtain

$$\int_\Omega \frac{u_0-v}{h}(u-v)\,dx+\int_\Omega [f]_h(0)(u-v)\,dx$$
$$-\int_\Omega \Big(|\nabla v|^{p(b(w))-2}\nabla v+|\nabla v|^{q(b(w))-2}\nabla v+\varepsilon|\nabla v|^{\beta-2}\nabla v\Big)\cdot\nabla(u-v)\,dx\geqslant 0$$

for all $v\in W_0^{1,\beta}(\Omega)$. We take $v=u\pm\theta z$ in (3), with $z\in W_0^{1,\beta}(\Omega)$ and $\theta>0$, we get

$$\pm\Big[\int_\Omega \frac{u_0-(u\mp\delta z)}{h}zdx+\int_\Omega [f]_h(0)zdx-\int_\Omega \Big(|\nabla(u\mp\delta z)|^{p(b(w))-2}\nabla(u\mp\delta z)$$
$$+|\nabla(u\mp\delta z)|^{q(b(w))-2}\nabla(u\mp\delta z)+\varepsilon|\nabla(u\mp\delta z)|^{\beta-2}\nabla(u\mp\delta z)\Big)\cdot\nabla zdx\Big]\geqslant 0.$$

By letting $\theta\to 0$, we obtain that

$$\int_\Omega \frac{u-u_0}{h}zdx+\int_\Omega |\nabla u|^{p(b(w))-2}\nabla u\cdot\nabla zdx+\int_\Omega |\nabla u|^{q(b(w))-2}\nabla u\cdot\nabla zdx$$
$$+\varepsilon\int_\Omega |\nabla u|^{\beta-2}\nabla u\cdot\nabla zdx=\int_\Omega [f]_h(0)zdx, \quad \forall z\in W_0^{1,\beta}(\Omega).$$

Hence $u=u_w$. Since the limit is uniquely determined and by (25), we get

$$u_n\to u_w \quad \text{strongly in} \quad L^2(\Omega), \quad \text{as} \quad n\to\infty,$$

which proves the continuity of the mapping. By Schauder's fixed point theorem, this mapping has a fixed point, and thus concludes the proof of Lemma 3.1.

Step 2: Passage to the limit as $\varepsilon \to 0$
From Lemma 3.1, it can be obtained that for each $\varepsilon > 0$, there exists $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ such that

$$\int_\Omega \frac{u_\varepsilon - u_0}{h} v dx + \int_\Omega |\nabla u_\varepsilon|^{p(b(u_\varepsilon))-2} \nabla u_\varepsilon \cdot \nabla v dx + \int_\Omega |\nabla u_\varepsilon|^{q(b(u_\varepsilon))-2} \nabla u_\varepsilon \cdot \nabla v dx$$
$$+\varepsilon \int_\Omega |\nabla u_\varepsilon|^{\beta-2} \nabla u_\varepsilon \cdot \nabla v dx = \int_\Omega [f]_h(0) v dx \tag{30}$$

for all $v \in W_0^{1,\beta}(\Omega)$.

By taking $v = u_\varepsilon$ in (30), we get

$$\frac{1}{2h} \int_\Omega u_\varepsilon^2 dx + \int_\Omega |\nabla u_\varepsilon|^{p(b(u_\varepsilon))} dx + \int_\Omega |\nabla u_\varepsilon|^{q(b(u_\varepsilon))} dx + \varepsilon \int_\Omega |\nabla u_\varepsilon|^{\beta} dx$$
$$\leqslant \frac{1}{2h} \int_\Omega u_0^2 dx + \int_\Omega [f]_h(0) u_\varepsilon dx.$$

Then we conclude that

$$\int_\Omega |\nabla u_\varepsilon|^{p(b(u_\varepsilon))} dx + \int_\Omega |\nabla u_\varepsilon|^{q(b(u_\varepsilon))} dx + \frac{\varepsilon}{2} \|\nabla u_\varepsilon\|^\beta_{L^\beta(\Omega)} \leqslant C,$$

and

$$\|\nabla u_\varepsilon\|_{L^\alpha(\Omega)} \leqslant C,$$

where C is a positive constant which does not depend on ε.

Using the compact embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ due to the fact that $\alpha > 2N/(N+2)$, we have

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,\alpha}(\Omega), \\ &\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } \left(L^\alpha(\Omega)\right)^N, \\ &u_\varepsilon \to u \quad \text{strongly in } L^2(\Omega), \\ &u_\varepsilon \to u \quad \text{a.e. in } \Omega, \end{aligned}$$
$$p\left(b\left(u_\varepsilon\right)\right) \to p(b(u)) \quad \text{a.e. in } \Omega, \qquad q\left(b\left(u_\varepsilon\right)\right) \to q(b(u)) \quad \text{a.e. in } \Omega.$$

By the application of Lemma 2.1, we get

$$u \in W_0^{1,p(b(u))}(\Omega) \text{ and } u \in W_0^{1,q(b(u))}(\Omega).$$

Therefore,

$$u \in W_0^{1,p(b(u))}(\Omega) \cap W_0^{1,q(b(u))}(\Omega).$$

By using the monotonicity trick in [9], we can establish that (19) has a weak solution $u_1(x)$ in W.

In the same way, we show that (18) has weak solutions u_k for $k = 2, \ldots, N_0$. It means that, for every $\varphi \in W$, we have

$$\int_\Omega \frac{u_k - u_{k-1}}{h} \varphi dx + \int_\Omega |\nabla u_k|^{p(b(u_k))-2} \nabla u_k \cdot \nabla \varphi dx + \int_\Omega |\nabla u_k|^{q(b(u_k))-2} \nabla u_k \cdot \nabla \varphi dx$$
$$= \int_\Omega [f]_h\left((k-1)h\right) \varphi dx. \tag{31}$$

For any $h = T/N_0$, we define $u_h(x,t)$ by

$$u_h(x,t) = \begin{cases} u_0(x), & t = 0, \\ u_1(x), & 0 < t \leqslant h, \\ \vdots & \vdots \\ u_j(x), & (j-1)h < t \leqslant jh, \\ \vdots & \vdots \\ u_{N_0}(x), & (N_0 - 1)\,h < t \leqslant N_0 h = T. \end{cases}$$

By taking $\varphi = u_k$ in (31), we obtain

$$\begin{aligned} \frac{1}{2}\int_\Omega u_k^2 dx + h\int_\Omega |\nabla u_k|^{p(b(u_k))}\,dx + h\int_\Omega |\nabla u_k|^{q(b(u_k))}\,dx \\ \leqslant \frac{1}{2}\int_\Omega u_{k-1}^2 dx + h\,\|[f]_h((k-1)h)\|_{L^{\alpha'}(\Omega)}\cdot\|u_k\|_{L^\alpha(\Omega)} \\ \leqslant \frac{1}{2}\int_\Omega u_{k-1}^2 dx + Ch\,\|[f]_h((k-1)h)\|_{L^{\alpha'}(\Omega)}\cdot\|\nabla u_k\|_{L^\alpha(\Omega)}. \end{aligned} \tag{32}$$

By the Hölder inequality, one has

$$\begin{aligned} \int_\Omega |\nabla u_k|^\alpha\,\mathrm{d}x &\leqslant C\,\||\nabla u_k|^\alpha\|_{L^{(p(b(u_k))/\alpha}(\Omega)} \\ &\leqslant C\left(\int_\Omega |\nabla u_k|^{p(b(u_k))}\,dx + 1\right). \end{aligned}$$

By using Young's inequality, one deduces that

$$\|[f]_h((k-1)h)\|_{L^{\alpha'}(\Omega)}\cdot\|\nabla u_k\|_{L^\alpha(\Omega)} \leqslant \varepsilon\int_\Omega |\nabla u_k|^{p(b(u_k))}\,dx + C.$$

From (32), we get

$$\int_\Omega u_k^2 dx + h\int_\Omega |\nabla u_k|^{p(b(u_k))}\,dx + h\int_\Omega |\nabla u_k|^{q(b(u_k))}\,dx \leqslant \int_\Omega u_{k-1}^2 dx + Ch. \tag{33}$$

By summing up inequalities in (33), we deduce that

$$\int_\Omega u_h^2(x,t)dx + \int_0^T\int_\Omega \left(|\nabla u_h(x,t)|^{p(b(u_h))} + |\nabla u_h(x,t)|^{q(b(u_h))}\right)dxdt \leqslant \int_\Omega u_0^2 dx + CT.$$

Hence

$$\begin{aligned} \|u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u_h\|_{L^{p(b(u_h))}(\Omega_T)} + \|u_h\|_{L^\alpha\left(0,T;W_0^{1,p(b(u_h))}(\Omega)\right)} \\ + \|\nabla u_h\|_{L^{q(b(u_h))}(\Omega_T)} + \|u_h\|_{L^\alpha\left(0,T;W_0^{1,q(b(u_h))}(\Omega)\right)} \leqslant C. \end{aligned}$$

Thus we have for some subsequence still labeled with h and some u,

$$\begin{aligned} &u_h \longrightarrow u \quad \text{weakly-* in } L^\infty\left(0,T;L^2(\Omega)\right),\\ &u_h \longrightarrow u \quad \text{in } L^\alpha\left(0,T;W_0^{1,\alpha}(\Omega)\right),\\ &|\nabla u_h|^{p(b(u_h))-2}\nabla u_h \longrightarrow \xi \quad \text{in } \left(L^{\alpha'}(\Omega_T)\right)^N,\\ &|\nabla u_h|^{q(b(u_h))-2}\nabla u_h \longrightarrow \chi \quad \text{in } \left(L^{\alpha'}(\Omega_T)\right)^N. \end{aligned}$$

Lemma 3.2 *u is a weak solution to problem* (1).

Proof. For every $k \in \{1,2,\ldots,N_0\}$, we take $\varphi(x,kh)$ as a test function in (31), where $\varphi \in C^1\left(\overline{\Omega}_T\right)$, $\varphi(\cdot,T)=0$ and $\varphi(x,t)|_\Gamma = 0$, we obtain

$$\begin{aligned} &\frac{1}{h}\int_\Omega u_k(x)\varphi(x,kh)dx - \frac{1}{h}\int_\Omega u_{k-1}(x)\varphi(x,kh)dx\\ &-\int_\Omega \left(|\nabla u_k|^{p(b(u_k))-2}\nabla u_k\right)(x)\cdot\nabla\varphi(x,kh)dx - \int_\Omega \left(|\nabla u_k|^{q(b(u_k))-2}\nabla u_k\right)(x)\cdot\nabla\varphi(x,kh)dx\\ &\qquad = \int_\Omega [f]_h((k-1)h)\varphi(x,kh)dx. \end{aligned}$$

Using the definition of $u_h(x,t)$ and the fact that $\varphi(\cdot,N_0h)=0$, we get

$$\begin{aligned} &h\sum_{k=1}^{N_0-1}\int_\Omega u_h(x,kh)\frac{\varphi(x,kh)-\varphi(x,(k+1)h)}{h}dx - \int_\Omega u_0(x)\varphi(x,h)dx\\ &-h\sum_{k=1}^{N_0}\int_\Omega \left(|\nabla u_h|^{p(b(u_h))-2}\nabla u_h + |\nabla u_h|^{q(b(u_h))-2}\nabla u_h\right)(x,kh)\cdot\nabla\varphi(x,kh)dx\\ &\qquad = h\sum_{k=1}^{N_0}\int_\Omega [f]_h((k-1)h)\varphi(x,kh)dx. \end{aligned} \tag{34}$$

Since $C^1\left(\overline{\Omega}_T\right)$, one has

$$\begin{aligned} &h\sum_{k=1}^{N_0}\int_\Omega \left(|\nabla u_h|^{p(b(u_h))-2}\nabla u_h + |\nabla u_h|^{q(b(u_h))-2}\nabla u_h\right)(x,kh)\cdot\nabla\varphi(x,kh)dx\\ &= \int_0^T\int_\Omega \left(|\nabla u_h|^{p(b(u_h))-2}\nabla u_h + |\nabla u_h|^{q(b(u_h))-2}\nabla u_h\right)(x,\tau)\cdot\nabla\varphi(x,\tau)dxd\tau\\ &+\sum_{k=1}^{N_0}\int_{(k-1)h}^{kh}\int_\Omega \left(|\nabla u_h|^{p(b(u_h))-2}\nabla u_h\right)(x,\tau)\cdot(\nabla\varphi(x,kh)-\nabla\varphi(x,\tau))dxd\tau\\ &+\sum_{k=1}^{N_0}\int_{(k-1)h}^{kh}\int_\Omega \left(|\nabla u_h|^{q(b(u_h))-2}\nabla u_h\right)(x,\tau)\cdot(\nabla\varphi(x,kh)-\nabla\varphi(x,\tau))dxd\tau\\ &\longrightarrow \int_0^T\int_\Omega \xi\cdot\nabla\varphi(x,\tau)dxd\tau + \int_0^T\int_\Omega \chi\cdot\nabla\varphi(x,\tau)dxd\tau, \quad \text{as } h\to 0. \end{aligned}$$

From (34), we deduce that

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}dx\ \mathrm{d}\tau-\int_\Omega u_0(x)\varphi(x,0)dx-\int_0^T\int_\Omega \xi\cdot\nabla\varphi dxd\tau$$
$$-\int_0^T\int_\Omega \chi\cdot\nabla\varphi dxd\tau=\int_0^T\int_\Omega f\varphi dxd\tau.$$

By using the monotonicity method as in [9,15], we show that $\xi=|\nabla u|^{p(b(u))-2}\nabla u$ a.e. in Ω_T and $\chi=|\nabla u|^{q(b(u))-2}\nabla u$ a.e. in Ω_T. By applying Lemma 2.1, we can show that $\nabla u\in\left(L^{p(b(u))}\left(\Omega_T\right)\right)^N$ and $\nabla u\in\left(L^{q(b(u))}\left(\Omega_T\right)\right)^N$.
Choosing $\varphi\in C_0^\infty\left(\Omega_T\right)$, we get

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}dxd\tau=\int_0^T\int_\Omega \xi\cdot\nabla\varphi dxd\tau+\int_0^T\int_\Omega \chi\cdot\nabla\varphi dxd\tau+\int_0^T\int_\Omega f\varphi dxd\tau,$$

therefore $u_t\in X\left(\Omega_T\right)^*$ and $u_t\in Y\left(\Omega_T\right)^*$. Since $u\in X\left(\Omega_T\right)\cap Y\left(\Omega_T\right)$, we can deduce that $u\in C\left([0,T];L^2(\Omega)\right)$ (see [10,12]). Then u is a weak solution to problem (1) in the sense of Definition 3.1.

4 Conclusion

In this paper, we proved the existence of weak solutions to some parabolic $(p(b(u)),q(b(u)))$-Laplacian problems stated in (1). By using a singular perturbation technique, we proved the existence of weak solutions for the discretized problem associated with problem (1). We finished this paper by proving the existence of a solution for problem (1) as a limit of the solutions of the approximated problem (18). This work provides a qualitative addition to the study of problems involving the (p,q)-Laplacian operators, especially the general reaction–diffusion system [5,7,14].

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