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A Note on the Controllability of Stochastic Partial Differential Equations Driven by Lévy Noise

D. D. Huan *

Than Nhan Trung High School, Bacgiang Agriculture and Forestry University, Bacgiang 26000, Vietnam

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Abstract: This paper discusses the exact controllability for impulsive neutral stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces. Under the Lipschitz conditions, the linear growth conditions are weakened and under the condition that the corresponding linear system is exactly controllable, a new set of sufficient conditions is derived by using a fixed point approach without imposing a severe compactness condition on the semigroup.

Keywords: *exact controllability; neutral stochastic partial differential equations; impulse; delay; Lévy noise.*

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1 Introduction

Exact controllability is one of the fundamental concepts in mathematical control theory, it plays an important role in both deterministic and stochastic control systems. It is well known that the controllability of deterministic systems is widely used in many fields of science and technology (for instance, see [4, 7, 21, 26, 28]). Stochastic control theory is a stochastic generalization of classic control theory. The theory of controllability of differential equations in infinite dimensional spaces has been extensively studied in the literature, and the details can be found in various papers and monographs, see [3, 16, 29] and the references therein. Besides white noise or stochastic perturbation, many systems, for example, predator-prey systems, arising from realistic models depend heavily on the histories or impulsive effect [10, 12, 13, 17, 20, 24, 28]. Therefore, there is a real need to discuss impulsive neutral partial differential systems with delays. Tai and Lun [25]

* Corresponding author: mailto:diemdanghuan@gmail.com

studied the exact controllability of fractional impulsive neutral infinite delay evolution integrodifferential systems in Banach spaces. However, in order to establish the results, the conditions presented in [25] state that the resolvent operator associated with the linear part is compact and the controllability operator is also compact, thus the induced inverse does not exist in the infinite dimensional state space. To relax that restriction, in this paper, we will study the exact controllability of stochastic nonlinear systems together with the condition that the compactness of the semigroup $S(t)$ is not assumed. Besides the environmental noise, sometimes, we have to consider the impulsive effects, which exist in many evolution processes, because the impulsive effects may bring an abrupt change at certain moments of time (see, e.g. [28]). Moreover, in most research on nonlinear stochastic systems, the control function $u_\alpha(t,x)$ is always constructed by its corresponding linear systems and the stochastic maximum principle [1], however, the stochastic maximum principle is not available in impulsive stochastic systems as a result of its linear form. Therefore, there is a real need to discuss impulsive differential control systems with memory (delay).

On the other hand, in recent years, stochastic partial differential equations with Poisson jumps have gained much attention since Poisson jumps not only exist widely but also can be used to study many phenomena in the real life. Therefore, it is necessary to consider the Poisson jumps into the stochastic systems. To be more precise, in [6], Cui et al. investigated the exponential stability of mild solutions to neutral stochastic partial differential equations with delays and Poisson jumps by using the Banach fixed point principle. Bao et al. [5] studied the existence, uniqueness and some sufficient conditions for stability in the distribution of mild solutions to stochastic partial differential delay equations with Poisson jumps. More recently, by using the successive approximations method, Yin and Xiao [27] considered the controllability of a stochastic partial equation driven by a Poisson random measure. For more details about the stochastic partial differential equations with Poisson jumps, we refer the reader to the monographs [2, 14, 15, 23] and the references therein.

However, to the best of our knowledge, the exact controllability problem for impulsive neutral stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces has not been investigated yet. Motivated by the above works, in this paper, we will study the exact controllability problem for impulsive neutral stochastic delay partial differential equations driven by Lévy noise, which are natural generalizations of controllability concepts well known in the theory of infinite dimensional deterministic control systems. More precisely, we consider the following form:

$$\begin{cases} d[x(t)-G(t,x(t-\tau))] = A[x(t)-G(t,x(t-\tau))]dt \\ \quad +\big[F(t,x(t),x(t-\tau))+Bu(t)\big]dt+\sigma(t,x(t),x(t-\tau))dW(t) \\ \quad +\int_{\mathbf{Z}} L(t,x(t),x(t-\tau),z)\widetilde{N}(dt,dz), t_k\neq t\in J:=[0,T], \\ \Delta x(t_k)=I_k(x(t_k^-)), \quad k\in\{1,2,...,m\}, \\ x(t)=\varphi(t)\in\mathcal{C}_\tau=\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H}), \quad -\tau\le t\le 0, \quad \tau>0, \end{cases} \tag{1}$$

where $x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space $\mathbb{H}$; $A: D(A)\subset\mathbb{H}\to\mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup of the bounded linear operators $S(t)$, $t\ge 0$ in $\mathbb{H}$. Assume that the mappings $G: J\times\mathbb{H}\to\mathbb{H}$, $F: J\times\mathbb{H}\times\mathbb{H}\to\mathbb{H}$, $\sigma: J\times\mathbb{H}\times\mathbb{H}\to\mathcal{L}_2^0$, $L: J\times\mathbb{H}\times\mathbb{H}\times\mathbf{Z}\to\mathbb{H}$ are the Borel measurable functions and $I_k:\mathbb{H}\to\mathbb{H}$, $k=1,2,...,m$ are continuous functions. The control function $u(\cdot)$ takes values in $L_2^{\mathcal{F}}(J,U)$ of admissible control functions for a separable Hilbert space

U and B is a bounded linear operator from U into $\mathbb{H}$. Furthermore, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ be prefixed points, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represent the jump of the function x at time t_k with I_k determining the size of the jump, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left-hand limits of $x(t)$ at $t = t_k$, respectively. Let $\varphi(t) : [-\tau, 0] \to \mathbb{H}$ be càdlàg independent of the Wiener process W and the Poisson point process $p(\cdot)$ with $\mathbf{E}[\sup_{-\tau\le s\le 0} \|\varphi\|^2_{\mathbb{H}}] < \infty$.

The structure of this paper is as follows. In Section 2, we briefly present some basic notations, preliminaries and assumptions. The main results in Section 3 are devoted to the study of the exact controllability for the system (1) and supplied with their proofs.

2 Preliminaries

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, \langle\cdot,\cdot\rangle_{\mathbb{H}})$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle\cdot,\cdot\rangle_{\mathbb{K}})$ denote two real separable Hilbert space, with their vector norms and their inner products, respectively. We denote by $\mathcal{L}(\mathbb{K};\mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, which is equipped with the usual operator norm $\|\cdot\|$. In this paper, we use the symbol $\|\cdot\|$ to denote the norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t\ge0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual condition (i.e., it is right-continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). Let $W = (W(t))_{t\ge0}$ be a Q-Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ with the covariance operator Q such that $Tr(Q) < \infty$. We assume that there exist a complete orthonormal system $\{e_k\}_{k\ge1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, ...$, and a sequence of independent Brownian motions $\{\beta_k\}_{k\ge1}$ such that

$$\langle W(t), e\rangle_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle e_k, e\rangle_{\mathbb{K}}\beta_k(t), \quad e \in \mathbb{K}, t \ge 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K};\mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ into $\mathbb{H}$ with the inner product

$$\langle\Psi, \phi\rangle_{\mathcal{L}_2^0} = Tr[\Psi Q\phi^*],$$

where ϕ^* is the adjoint of the operator ϕ.

Let $p = p(t)$, $t \in D_p$ (the domain of $p(t)$), be a stationary $\mathcal{F}_t$-Poisson point process taking its value in a measurable space $(\mathbf{Z}, \mathcal{B}(\mathbf{Z}))$ with a σ-finite intensity measure $\lambda(dz)$. We will denote by $N(dt, dz)$ the Poisson counting measure of p such that

$$N(t, \mathbf{Z}) = \sum_{s\in D_p, s\le t} \mathbb{I}_{\mathbf{Z}}(p(s))$$

for any measurable set $\mathbf{Z} \in \mathcal{B}(\mathbb{K} - \{0\})$, which denotes the Borel σ-field of $(\mathbb{K} - \{0\})$. Let

$$\widetilde{N}(dt, dz) := N(dt, dz) - \lambda(dz)dt$$

be the compensated Poisson measure that is independent of $W(t)$.

Let $\tau > 0$ and $\mathcal{C} := \mathcal{C}([-\tau, 0];\mathbb{H})$ denote the family of all right-continuous functions with left-hand limits (càdlàg) from $[-\tau, 0]$ to $\mathbb{H}$. The space $\mathcal{C}$ is assumed to be equipped with the norm

$$\|\varsigma\|_{\mathcal{C}} := \sup_{-\tau\le t\le 0} \|\varsigma(t)\|_{\mathbb{H}}, \quad \varsigma(t) \in \mathcal{C}.$$

We also assume that $\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H})$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $\mathcal{C}([-\tau,0];\mathbb{H})$-valued random variables. For all $t \geq 0$,

$$x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\}$$

is regarded as the $\mathcal{C}([-\tau,0];\mathbb{H})$-valued stochastic process. Further, we consider the Banach space $\boldsymbol{B}_T$ of all $\mathbb{H}$-valued $\mathcal{F}_t$-adapted càdlàg process $x(t)$ defined on $[0,T]$ with

$$x(t) = \varphi(t), \quad t \in [-\tau, 0]$$

such that

$$\|x\|^2_{\boldsymbol{B}_T} := \mathbf{E}[\sup_{0\leq t\leq T} \|x(t)\|^2_{\mathbb{H}}] < \infty.$$

Next, let us recall the definition of a mild solution for (1).

Definition 2.1 An $\mathcal{F}_t$-adapted càdlàg stochastic process $x : J \to \mathbb{H}$ is called a mild solution of (1) if for each $u \in L_2^{\mathcal{F}}(J,U)$ and for arbitrary $t \in J$, $\mathbf{P}\{\omega : \int_J \|x(s)\|^2_{\mathbb{H}} ds < +\infty\} = 1$, it satisfies the integral equation

$$\begin{aligned} x(t) =& S(t)[\varphi(0) - G(0,\varphi)] + G(t, x(t-\tau)) \\ &+ \int_0^t S(t-s)[F(s,x(s),x(s-\tau)) + Bu(s)]ds \\ &+ \int_0^t S(t-s)\sigma(s,x(s),x(s-\tau))dW(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) \\ &+ \int_0^t \int_{\mathbf{Z}} S(t-s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz) \end{aligned} \tag{2}$$

for any $x_0(\cdot) = \varphi(\cdot) \in \mathcal{C}_\tau$.

Consider the following linear stochastic system of the form:

$$\begin{cases} dx(t) = [Ax(t) + Bu(t)]dt + \sigma(t)dW(t), \quad t \in J, \\ x(0) = x_0. \end{cases} \tag{3}$$

It is convenient to introduce the relevant operators and the basic controllability condition.

(i) The operator $L_0^T \in \mathcal{L}(L_2^{\mathcal{F}}(J,\mathbb{H}), L_2(\Omega,\mathcal{F}_T,\mathbb{H}))$ is defined by

$$L_0^T u = \int_0^T S(T-s)Bu(s)ds,$$

where $L_2^{\mathcal{F}}(J,\mathbb{H})$ is the space of all $\mathcal{F}_t$-adapted, H-valued measurable square integrable processes on $J \times \Omega$. Clearly, the adjoint $(L_0^T)^* : L_2(\Omega,\mathcal{F}_T,\mathbb{H}) \to L_2^{\mathcal{F}}(J,\mathbb{H})$ is defined by

$$[(L_0^T)^* z](t) = B^* S^*(T-t)\mathbf{E}\{z \mid \mathcal{F}_t\}.$$

(ii) The controllability operator Π_0^T associated with (3) is defined by

$$\Pi_0^T\{\cdot\} = L_0^T(L_0^T)^*\{\cdot\} = \int_0^T S(T-t)BB^*S^*(T-t)\mathbf{E}\{\cdot \mid \mathcal{F}_t\}dt$$

and belongs to $\mathcal{L}(L_2(\Omega, \mathcal{F}_T, \mathbb{H}), L_2(\Omega, \mathcal{F}_T, \mathbb{H}))$, and the controllability operator $\Gamma_s^T \in \mathcal{L}(\mathbb{H}, \mathbb{H})$ is

$$\Gamma_s^T = \int_s^T S(T-t)BB^*S^*(T-t)dt, \quad 0 \le s < t.$$

Lemma 2.1 *([19]) The linear stochastic system (3) is exactly controllable on J iff there exists a $\gamma > 0$ such that*

$$\mathbf{E}\langle \Pi_0^T x, x\rangle \ge \gamma \mathbf{E}\|x\|^2, \quad \forall x \in L_2(\Omega, \mathcal{F}_T, \mathbb{H}).$$

Then

$$\mathbf{E}\|(\Pi_0^T)^{-1}\|^2 \le \frac{1}{\gamma}.$$

Let $x(t;\varphi,u)$ denote the state value of the system (1) at time t corresponding to the control $u \in L_2^{\mathcal{F}}(J,U)$ and the initial value φ. In particular, the state of system (1) at $t = T$, $x(T;\varphi,u)$, is called the terminal state with the control u.

$$\mathcal{R}_T := \mathcal{R}(T,\varphi) = \{x(T;\varphi,u) : u(\cdot) \in L_2^{\mathcal{F}}(J,U)\}$$

is called the reachable set of the system (1).

Definition 2.2 The stochastic system (1) is said to be exactly controllable on the interval J if

$$\mathcal{R}_T = L_2(\Omega, \mathcal{F}_T, \mathbb{H}).$$

To prove our main results, we list the following basic assumptions of this paper.
(**H**1) A is the infinitesimal generator of a contraction C_0-semigroup $S(t)$, $t \ge 0$, in $\mathbb{H}$.
(**H**2) There exists a positive constant C_0 such that for all $t \in J$, $x, y \in \mathbb{H}$,

$$\|G(t,x) - G(t,y)\|_{\mathbb{H}}^2 \le C_0\big(\|x-y\|_{\mathbb{H}}^2\big).$$

(**H**3) There exist a positive constant C_1 such that for all $t \in J$, $x_1, y_1, x_2, y_2 \in \mathbb{H}$,

$$\begin{aligned}\|F(t,x_1,y_1) - F(t,x_2,y_2)\|_{\mathbb{H}}^2 &\vee \|\sigma(t,x_1,y_1) - \sigma(t,x_2,y_2)\|_{\mathcal{L}_2^0}^2 \\ &\vee \int_{\mathbf{Z}} \|L(t,x_1,y_1,z) - L(t,x_2,y_2,z)\|_{\mathbb{H}}^2 \lambda(dz) \\ &\le C_1\big(\|x_1-x_2\|_{\mathbb{H}}^2 + \|y_1-y_2\|_{\mathbb{H}}^2\big).\end{aligned}$$

(**H**4) There exists a positive constant C_2 such that for all $t \in J$, $x_1, y_1, x_2, y_2 \in \mathbb{H}$,

$$\int_{\mathbf{Z}} \|L(t,x_1,y_1,z) - L(t,x_2,y_2,z)\|_{\mathbb{H}}^4 \lambda(dz) \le C_2\big(\|x_1-x_2\|_{\mathbb{H}}^4 + \|y_1-y_2\|_{\mathbb{H}}^4\big).$$

(**H**5) There exists some positive constants Q_k, $k = 1, 2, ..., m$ such that for all $t \in J$, $x, y \in \mathbb{H}$,

$$\|I_k(x) - I_k(y)\|_{\mathbb{H}}^2 \le Q_k\|x-y\|_{\mathbb{H}}^2.$$

(**H**6) For all $t \in J$, there exists a positive constant M such that

$$\|G(t,0)\|_{\mathbb{H}}^2 \vee \|F(t,0,0)\|_{\mathbb{H}}^2 \vee \|\sigma(t,0,0)\|_{\mathbb{H}}^2 \vee \|I_k(0)\|_{\mathbb{H}}^2$$
$$\vee \int_{\mathbf{Z}} \|L(t,0,0,z)\|_{\mathbb{H}}^2 \lambda(dz) \vee \int_{\mathbf{Z}} \|L(t,0,0,z)\|_{\mathbb{H}}^4 \lambda(dz) \leq M.$$

(**H**7) The linear stochastic system (3) is exactly controllable.

We now note that for the proof of our main results, we need the following lemmas.

Lemma 2.2 *(see [8], Proposition 7.3). Suppose that $\Phi(t)$, $t \geq 0$, is a $\mathcal{L}_2^0$-valued predictable process and let $W_A^\Phi = \int_0^t S(t-s)\Phi(s)dW(s)$, $t \in [0,T]$. Then for any arbitrary $p > 2$, there exists a constant $C(p,T) > 0$ such that*

$$\mathbf{E} \sup_{t\in[0,T]} \|W_A^\Phi\|_{\mathbb{H}}^p \leq \overline{C} \sup_{t\in[0,T]} \|S(t)\|^p \mathbf{E} \int_0^T \|\Phi(s)\|_{\mathcal{L}_2^0}^p ds,$$

where $\overline{C} = C(p,T)$. Moreover, if $\mathbf{E}\int_0^T \|\Phi(s)\|^p ds < +\infty$, then there exists a continuous version of the process $\{W_A^\Phi\}_{t\geq 0}$. If $(S(t))_{t\geq 0}$ is a contraction semigroup, then the above result is true for $p \geq 2$.

Lemma 2.3 *(see [18], Lemma 2.2). Let the space $M_\nu^p([0,T]\times\Omega\times(\mathbb{K}-\{0\}),\mathbb{H})$, $p \geq 2$, denote the set of all random process $J(x,y)$ with values in $\mathbb{H}$, predictable with respect to $\{\mathcal{F}_t\}$ such that*

$$\mathbf{E}\Big(\int_0^T \int_{\mathbf{Z}} \|J(t,y)\|_{\mathbb{H}}^p \nu(dy)dt\Big) < \infty.$$

Suppose $J \in M_\nu^2([0,T]\times\Omega\times(\mathbb{K}-\{0\}),\mathbb{H}) \cap M_\nu^4([0,T]\times\Omega\times(\mathbb{K}-\{0\}),\mathbb{H})$. Then for any $t \in [0,T]$,

$$\mathbf{E}\Big[\sup_{\theta\in[0,t]} \Big\| \int_0^\theta \int_{\mathbf{Z}} S(\theta-s)J(s,y)\widetilde{N}(ds,dy)\Big\|_{\mathbb{H}}^2\Big] \leq C\Bigg\{\mathbf{E}\Big(\int_0^t \int_{\mathbf{Z}} \|J(t,y)\|_{\mathbb{H}}^2 \nu(dy)dt\Big) + \mathbf{E}\Big(\int_0^t \int_{\mathbf{Z}} \|J(t,y)\|_{\mathbb{H}}^4 \nu(dy)dt\Big)^{\frac{1}{2}}\Bigg\}$$

for some constant $C = C(T) > 0$, dependent on $T > 0$.

3 Main Results

In this section, we shall investigate the exact controllability for impulsive neutral stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces.

The main result of this section is the following theorem.

Theorem 3.1 *Let the assumptions* (**H1**)$-$(**H7**) *hold. Then the system (1) is exactly controllable on J provided that*

$$\begin{cases} \Big(1 + \frac{30T^2C_B^2}{\gamma}\Big)\Big[6(C_0 + 2T^2C_1 + 2T\overline{C}C_1 + m\sum_{k=1}^m Q_k) \\ +\frac{1}{2} + (2CC_1 + C^2C_2)T\Big] < 1, \ \textit{where } \|B\|^2 = C_B. \end{cases} \quad (4)$$

Proof. Using the assumptions, for an arbitrary function $x(\cdot)$, choose the feedback control function

$$
\begin{aligned}
u_x^T(t) =& B^*S^*(T-t)\mathbf{E}\Big\{(\Pi_0^T)^{-1}\big[x_T - S(T)[\varphi(0)-G(0,\varphi)] - G(t,x(t-\tau)) \\
&- \int_0^T S(T-s)F(s,x(s),x(s-\tau))ds \\
&- \int_0^T S(T-s)\sigma(s,x(s),x(s-\tau))dW(s) \\
&- \int_0^T \int_{\mathbf{Z}} S(T-s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz) \\
&- \sum_{0<t_k<T} S(T-t_k)I_k(x(t_k^-))\big] \mid \mathcal{F}_t\Big\}.
\end{aligned} \tag{5}
$$

We transform (1) into a fixed point problem. Consider the operator $\Pi : \mathbf{B}_T \to \mathbf{B}_T$ defined by

$$
\begin{aligned}
\Pi(x)(t) =& S(t)[\varphi(0)-G(0,\varphi)] + G(t,x(t-\tau)) \\
&+ \int_0^t S(t-s)[F(s,x(s),x(s-\tau)) + Bu_x^T(s)]ds \\
&+ \int_0^t S(t-s)\sigma(s,x(s),x(s-\tau))dW(s) \\
&+ \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) \\
&+ \int_0^t \int_{\mathbf{Z}} S(t-s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz).
\end{aligned} \tag{6}
$$

In what follows, we shall show that when using the control $u_x^T(\cdot)$, the operator Π has a fixed point, which is then a mild solution for system (1).

By our assumptions, Hölder's inequality, Lemma 2.1, Lemma 2.2, Lemma 2.3 and the basic inequality $\Big(\sum_{i=1}^n x_i\Big)^p \le n^{(p-1)\vee 0}\sum_{i=1}^n x_i^p$, $p>0$, we obtain that for $x \in \mathbf{B}_T$,

$$
\begin{aligned}
\|\Pi(x)(t)\|_{\mathbf{B}_T}^2 \le 7\Big\{& \mathbf{E}\big(\sup_{t\in J}\|S(t)[\varphi(0)-G(0,\varphi)]\|^2\big) \\
&+ \mathbf{E}\big(\sup_{t\in J}\|G(t,x(t-\tau))\|^2\big) \\
&+ \mathbf{E}\Big(\sup_{t\in J}\Big\|\int_0^t S(t-s)F(s,x(s),x(s-\tau))ds\Big\|^2\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbf{E}\big(\sup_{t\in J}\Big\|\int_0^t S(t-s)Bu_x^T(s)ds\Big\|^2\big)\\
&+\mathbf{E}\big(\sup_{t\in J}\Big\|\int_0^t S(t-s)\sigma(s,x(s),x(s-\tau))dW(s)\Big\|^2\big)\\
&+\mathbf{E}\big(\sup_{t\in J}\Big\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)\Big\|^2\big)\\
&+\mathbf{E}\big(\sup_{t\in J}\Big\|\int_0^t\int_{\mathbf{Z}} S(t-s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz)\Big\|^2\big)\Big\}\\
\le 7\Big\{&\Big[2(1+C_0)\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+2\big[M+C_0(\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+\|x\|_{\mathbf{B}_T}^2)\big]\\
&+2T\big[MT+C_1(\tau\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+2T\|x\|_{\mathbf{B}_T}^2)\big]\\
&+2\overline{C}\big[MT+C_1(\tau\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+2T\|x\|_{\mathbf{B}_T}^2)\big]+2m\sum_{k=1}^m Q_k(M+\|x\|_{\mathbf{B}_T}^2)\\
&+8C\big[MT+C_1(\tau\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+2T\|x\|_{\mathbf{B}_T}^2)\big]\\
&+8C\big[MT+\sqrt{C_2}(\sqrt{\tau}\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+\sqrt{2T}\|x\|_{\mathbf{B}_T}^2)\big]\Big]\\
&\times\Big(1+\frac{7T^2C_B^2}{\gamma}\Big)+\frac{7T^2C_B^2}{\gamma}\mathbf{E}\|x_T\|_{\mathbb{H}}^2\Big\}<\infty.
\end{aligned}
$$

Thus, Π maps $\mathbf{B}_T$ into itself.

Now, we shall prove that Π is a contraction mapping in $\mathbf{B}_T$. For any $x,y\in\mathbf{B}_T$, in the same ways as above, we can get

$$
\begin{aligned}
&\|\Pi(x)(t)-\Pi(y)(t)\|_{\mathbf{B}_T}^2\\
&\le\Big[6(C_0+2T^2C_1+2T\overline{C}C_1+m\sum_{k=1}^m Q_k)+\frac{1}{2}+(2CC_1+C^2C_2)T\Big]\|x-y\|_{\mathbf{B}_T}^2\\
&\quad+6T^2C_B\mathbf{E}\big(\sup_{t\in J}\|u_x^T-u_y^T\|_{\mathbb{H}}^2\big)\\
&\le\Big(1+\frac{30T^2C_B^2}{\gamma}\Big)\Big[6(C_0+2T^2C_1+2T\overline{C}C_1+m\sum_{k=1}^m Q_k)\\
&\quad+\frac{1}{2}+(2CC_1+C^2C_2)T\Big]\|x-y\|_{\mathbf{B}_T}^2.
\end{aligned}
$$

By assumption (4), we conclude that Π is a contraction mapping on $\mathbf{B}_T$. On the other hand, by the Banach fixed point theorem, there exists a unique fixed point $x(\cdot)\in\mathbf{B}_T$ such that $(\Pi x)(t)=x(t)$. This fixed point is then the mild solution of the system (1). Thus, the system (1) is exactly controllable on J. The proof of Theorem 3.1 is complete.

Remark 3.1 From the assumptions $(\mathbf{H1})-(\mathbf{H6})$, for every $u(\cdot)\in U$, the system (1) has a unique solution in $\mathbf{B}_T$.

Now let us consider a special case for the system (1). If $G\equiv 0$, $m\equiv 0$, then the system (1) becomes the following stochastic delay partial differential equations driven by

Lévy noise:

$$\begin{cases} dx(t) = Ax(t)dt + \big[F(t,x(t),x(t-\tau)) + Bu(t)\big]dt + \sigma(t,x(t),x(t-\tau))dW(t) \\ \qquad\qquad + \int_{\mathbf{Z}} L(t,x(t),x(t-\tau),z)\widetilde{N}(dt,dz), \quad t\in[0,T], \\ x(t) = \varphi(t) \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H}), \quad -\tau\le t\le 0, \quad \tau>0. \end{cases} \tag{7}$$

Corollary 3.1 *Suppose that the assumptions* (**H1**), (**H3**), (**H4**), (**H6**), (**H7**) *hold. Then the system (10) is exactly controllable on J provided that*

$$\Big(1+\frac{12T^2C_B^2}{\gamma}\Big)\Big[\frac{1}{2} + 8(T^2C_1 + T\overline{C}C_1) + (2CC_1 + C^2C_2)T\Big] < 1. \tag{8}$$

Remark 3.2 As we all know, the mathematical formulation of many physical phenomena contains integro-differential equations arisen in various applications such as viscoelasticity, heat equations, fluid dynamics, chemical kinetics and so on. More recently, M.A. Diop et al. [9, 10] studied the asymptotic stability of neutral impulsive stochastic partial integro-differential equations with delays and Poisson jumps in Hilbert spaces. In this remark, we consider the exact controllability for impulsive neutral stochastic delay partial integro-differential equations driven by Lévy noise in the form

$$\begin{cases} d[x(t) - G(t,x(t-\tau))] = A[x(t) - G(t,x(t-\tau))]dt \\ \qquad\qquad\qquad\qquad + \int_0^t K(t-s)[x(s) - G(s,x(s-\tau))]dsdt \\ \qquad\qquad\qquad\qquad + \big[F(t,x(t),x(t-\tau)) + Bu(t)\big]dt \\ \qquad\qquad\qquad\qquad + \sigma(t,x(t),x(t-\tau))dW(t) \\ \qquad\qquad\qquad\qquad + \int_{\mathbf{Z}} L(t,x(t),x(t-\tau),z)\widetilde{N}(dt,dz), \quad t_k\ne t\in[0,T], \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k\in\{1,2,...,m\}, \\ x(t) = \varphi(t)\in\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H}), \quad -\tau\le t\le 0, \end{cases} \tag{9}$$

where $K(t)$, $t\ge 0$, is a closed linear operator defined on a common domain which is dense in a Banach space X.

Further, we assume that the integro-differential abstract Cauchy problem

$$\frac{dx(t)}{dt} = Ax(t) + \int_0^t K(t-s)x(s)ds, \quad x(0) = x_0\in X, \tag{10}$$

has an associated resolvent operator of bounded linear operators $\{R(t)\}_{t\ge 0}$ on X.

A one-parameter family of bounded linear operator $\{R(t)\}_{t\ge0}$ on X is called a resolvent operator of (10) if the following conditions are verified.
(i) Function $R(\cdot):[0,\infty)\to\mathcal{L}(X)$ is strongly continuous and $R(0)x = x$ for all $x\in X$.
(ii) For $x\in D(A)$, $R(\cdot)\in\mathcal{C}([0,+\infty);D(A))\cap\mathcal{C}^1([0,+\infty);X)$, and

$$\begin{aligned} \frac{dR(t)x}{dt} &= AR(t)x + \int_0^t K(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)K(s)xds, \quad for \quad t\ge 0. \end{aligned}$$

(iii) There exist constants $M>0$, β such that $\|R(t)\|\le M.e^{\beta t}$ for every $t\ge 0$.

An $\mathcal{F}_t$-adapted càdlàg stochastic process $x : [0,T] \to \mathbb{H}$ is called a mild solution of (9) if for each $u \in L_2^{\mathcal{F}}(J,U)$ and for arbitrary $t \in [0,T]$,$\mathbf{P}\{\omega : \int_J \|x(s)\|_{\mathbb{H}}^2 ds < +\infty\} = 1$, it satisfies the integral equation

$$\begin{aligned} x(t) =& R(t)[\varphi(0) - G(0,\varphi)] + G(t, x(t-\tau)) \\ &+ \int_0^t R(t-s)[F(s,x(s),x(s-\tau)) + Bu(s)]ds \\ &+ \int_0^t R(t-s)\sigma(s,x(s),x(s-\tau))dW(s) \\ &+ \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)) \\ &+ \int_0^t \int_{\mathbf{Z}} R(t-s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz) \end{aligned}$$

for any $x_0(\cdot) = \varphi(\cdot) \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H})$.

By implementing appropriate conditions on the functions, one can easily show that by adapting and employing the techniques used in Theorem 3.1, the stochastic control system (9) is exactly controllable on $[0,T]$.

Remark 3.3 We now consider the non-autonomous versions of systems (1) and (9), where the operator A is replaced by $\{A(t) : t \in [0,T]\}$. To proceed to prove the exact controllability results in a similar manner as employed in the above theorem, an evolution system $\{U(t,s) : 0 \le s \le t \le T\}$ and a resolvent family $\{R(t,s) : 0 \le s \le t \le T\}$ are guaranteed to exist. Conditions guaranteeing the existence of $U(t,s)$ and $R(t,s)$ can be found in [22] and [11], respectively. Therefore, the above theorem can be extended to the time-dependent case by making suitable modifications involving the use of the properties of the time-dependent evolution system and the time-dependent resolvent family in the above arguments.

4 Conclusions

This paper focuses on establishing the exact controllability for impulsive neutral stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces through the application of one of the most important results of the analysis and considers the main source of the metric fixed point theory known as the "Banach Contraction Principle". In the future, we aim to expand this study to the approximate controllability for impulsive neutral second-order stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces.

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