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Generalized Bessel-Riesz Operator on Morrey Spaces with Different Measures

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Abstract: This study's primary area of interest is the generalized operators, which are defined with doubling measures by generalized Bessel-Riesz kernels with various measures in Morrey spaces. In terms of Bessel decay, the kernel satisfies a few key requirements. To prove that the integral operators are bounded, we will make use of Young, Holder, and Minkȯwski inequalities and a doubling measure. Additionally, we look into the relationship, we discover that the norm of these operators will be similarly constrained by the relationship between the elements of the kernel and the integral operators based on the norm of each kernel, although according to several measures. Additionally, we investigate the boundedness of pointwise multiplier operators in Morrey spaces using generalized fractional integrals and the generalized Bessel-Riesz operator.

Keywords: *generalized Bessel-Riesz operators; doubling measure; fractional integral; Morrey spaces.*

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1 Introduction

This paper extends our recent findings in paper [1] by investigating the boundedness of Bessel-Riesz operators by a generalized kernel defined with doubling measures in Morrey spaces with various measures. Some basic requirements are being met by the operator's kernel in relation to Bessel decay. We will use the Young, Holder, and Minkȯwski inequalities and a doubling measure to demonstrate that the integral operators are bounded.

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In Morrey spaces, we also focus on the boundedness of pointwise multiplier operators. Here,

$$\|f : L^p(\mathbb{R}^n)\| := \left[\int_{\mathbb{R}^n} |f(t)|^p \, dt\right]^{1/p}$$

is defined, as a function $1 \leq p < \infty$, and a group of f in a way that $\|f : L^p(\mathbb{R}^n)\| < \infty$. The definition of the Bessel-Riesz operator is

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{R}^n} G_{\alpha,\gamma}(|x-y|) f(y) \, dy$$

for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, and $1 \leq p < \infty$ for each instance of $G_{\alpha,\gamma} : (0,\infty) \to (0,\infty)$, with

$$G_{\alpha,\gamma}(t) := \frac{t^{\alpha-n}}{[1+t]^{\gamma}}, \quad 0 < \alpha < n, \ \gamma \geq 0.$$

The $G_{\alpha,\gamma}$, the Bessel-Riesz kernel, is referred to in this context. Bessel-Riesz operators are derived from Schrödinger's equation. Schrödinger's equation, a partial differential linear equation, (see [2] for some results of differential equations of order $1 < \alpha \leq 2$, in a Banach space) is used to explain a quantum field system's wave function or state function. Schrödinger's equation is the quantum physics equivalent of Newton's law. In 1999 Kazuhiro Kurata et al. [3] conducted a research on the boundedness of integral operators with Lebesgue and generalized Morrey spaces. They then used this knowledge to estimate the Schrödinger operator $L_2 = -\triangle + V(x) + W(x)$ with nonnegative $V \in (RH)_\infty$ (reverse Hölders class) and small perturbed potentials W on Morrey spaces. Idris et al. [Theorem 6, [4]] reported the boundedness of Bessel-Riesz operators on Morrey spaces in 2016. They achieved outcomes for the boundedness of fractional integral operators that were comparable to the results by F. Chiarenza [5]. The boundedness of fractional integral operators when constructed on quasi-metric measure spaces was discussed by Eridani et al. in their study [6]. Even for Euclidean spaces, the research team's findings were novel. The boundedness of these operators for Euclidean spaces on Lebesgue and Morrey spaces was also demonstrated by Idris et al. in their paper [4], which also looked into the weighted boundedness of generalized Morrey spaces. Euclidean spaces are the most straightforward illustration of measure metric spaces. The weighted boundedness of generalized Morrey spaces has been studied by Kurata et al. [3]. We will use the Young, Holder, and Minköwski inequalities and a doubling measure to demonstrate the boundedness of these operators on Morrey spaces in Euclidean contexts. Additionally, we shall see that the generalized Bessel-Riesz operators' norm is constrained by the kernels' norm. Bessel-Riesz operators on Lebesgue spaces in measure metric spaces are bounded, which is a simple argument of the Young inequality, it was demonstrated by Saba et al. [7] using the Young inequality. Our entry into the subsequent Morrey space phenomenon is the second outcome since the generalized Bessel-Riesz operator is bounded on Lebesgue spaces [1]. The ideal constant in the Young inequality is 1, and it is used throughout the research. However, we still do not know what the optimum constant in Morrey spaces is at this point. As a result, we focus on generalized Bessel-Riesz operators in Morrey spaces in this study. We will also discuss the circumstance in which the measure meets the doubling requirement. For the relevant inequalities that were derived in [8], [9], the resulting criteria are both necessary and sufficient. When W is a scalar operator, Kurata et al. [3] have demonstrated that $W. \ I_{\alpha,\gamma}$ is bounded on generalized Morrey spaces. We will also work on this operator for boundedness with the generalized operator on Morrey spaces. See citation [3] for the examples of using the aforementioned operators in settings involving Euclidean spaces.

2 Doubling Conditions

We take into account $\rho : (0,\infty) \to (0,\infty)$ and define (DC) as a collection of ρ such that

$$\frac{1}{2} \le \frac{s}{t} \le 2 \;\Rightarrow\; \frac{1}{C} \le \frac{\rho(s)}{\rho(t)} \le C$$

for some $C \ge 1$. If $\rho \in$(DC), then we have

$$\rho(R) \sim \int_R^{2R} \frac{\rho(t)}{t}\,dt, \quad \text{or} \quad C^*\rho(R) \le \int_R^{2R} \frac{\rho(t)}{t}\,dt \le C^{**}\rho(R)$$

for some $C^{**} \ge C^* > 0$. Additionally, if $\rho \in$ (DC), then $G_{\rho,\gamma} \in$ (DC), where

$$G_{\rho,\gamma}(t) := \frac{\rho(t)}{t^n[1+t]^\gamma}, \quad t > 0.$$

Assume that μ is a random measure on $\mathbb{R}^n$. We consider a group of measures that meet the growth condition, denoted by (GC). Now, since $B(a,R) := \{x \in \mathbb{R}^n : |x-a| < R\}$ in $\mathbb{R}^n$ exists, we define $\mu \in$ (GC) if and only if $C_1 > 0$ such that

$$\mu(B(a,R)) \le C_1 R^n$$

exists for all open balls. See [6] for more details on a doubling measure.

Using the definitions provided above, we attempt to estimate

$$\|G_{\rho,\gamma} : L^s(\mu)\| := \left(\int_{\mathbb{R}^n} |G_{\rho,\gamma}(|x|)|^s\,d\mu(x)\right)^{1/s}, \quad s \ge 1.$$

For $1 \le s < \infty$ and $R > 0$, we take into account the following:

$$\begin{aligned}
&\int_{|x|<R} \frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}\,d\mu(x) = \sum_{k=-\infty}^{-1} \int_{2^kR\le|x|<2^{k+1}R} \frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}\,d\mu(x)\\
&\le C \sum_{k=-\infty}^{-1} \frac{\rho(2^kR)^s \mu(B(0,2^kR))}{(2^kR)^{sn}[1+2^kR]^{s\gamma}} \le C \sum_{k=-\infty}^{-1} \frac{\rho(2^kR)^s}{(2^kR)^{(s-1)n}}\\
&\le C \sum_{k=-\infty}^{-1} \int_{2^kR}^{2^{k+1}R} \frac{\rho(t)^s}{t^{(s-1)n+1}}\,dt = C \int_0^R \frac{\rho(t)^s}{t^{(s-1)n+1}}\,dt, \quad \text{and also}
\end{aligned}$$

$$\begin{aligned}
&\int_{R\le|x|} \frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}\,d\mu(x) = \sum_{k=1}^{\infty} \int_{2^kR\le|x|<2^{k+1}R} \frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}\,d\mu(x)\\
&\le C \sum_{k=0}^{\infty} \frac{\rho(2^kR)^s \mu(B(0,2^kR))}{(2^kR)^{sn}[1+2^kR]^{s\gamma}} \le C \sum_{k=0}^{\infty} \frac{\rho(2^kR)^s}{(2^kR)^{(s-1)n+s\gamma}}\\
&\le C \sum_{k=0}^{\infty} \int_{2^kR}^{2^{k+1}R} \frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}\,dt = C \int_R^{\infty} \frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}\,dt,
\end{aligned}$$

where a constant, not necessarily the same one, is denoted by the letter $C > 0$ at all time. At this stage, we define

$$\|G_{\rho,\,\gamma} : L^s(\mu)\| = \sup R > 0\left(\int_0^R \frac{\rho(t)^s}{t^{(s-1)n+1}}\,dt + \int_R^\infty \frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}\,dt\right)^{1/s}$$

for $1 \le s < \infty,\ \mu \in$ (GC) and $\rho \in$ (DC).

3 Morrey Spaces with the Generalized Bessel-Riesz Operator

We begin this section with the definition that is given below. Take into account $0 < \lambda < 1 \le p < \infty,\ B := B(a,R)$, and $\mu(B(a,R)) \sim R^n$. If and only if

$$\|f : L^{p,\lambda}(\nu,\mu)\| = \sup_B\left(\frac{1}{\mu(B)^\lambda}\int_B |f(y)|^p\,d\nu(y)\right)^{1/p} < \infty$$

holds, then we define $f \in L^{p,\lambda}(\nu,\mu)$. We define

$$f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^{\mathrm{C}}}$$

for each $f \in L^{p,\lambda}(\nu,\mu)$ and $\tilde{B} := B(a,2R)$, given that

$$\|f_1 : L^p(\nu)\| = \left[\int_{\tilde{B}} |f(y)|^p\,d\nu(y)\right]^{1/p} \le \mu(\tilde{B})^{\lambda/p}\|f : L^{p,\lambda}(\nu,\mu)\| < \infty.$$

For each B, we arrive at the following estimation when $f_1 \in L^p(\nu)$:

$$\begin{aligned}\left[\int_B |I_{\rho,\gamma}f_1(y)|^p\,d\mu(y)\right]^{1/p} &\le \|I_{\rho,\gamma}f_1 : L^p(\mu)\|\\ &\le C\,\|G_{\rho,\gamma} : L^1(\mu)\|\cdot\|f_1 : L^p(\nu)\|\\ &\le C\,\mu(B)^{\lambda/p}\|G_{\rho,\gamma} : L^1(\mu)\|\cdot\|f : L^{p,\lambda}(\nu,\mu)\|,\end{aligned}$$

and of course, we will come to the following estimation:

$$\left[\frac{1}{\mu(B)^\lambda}\int_B |I_{\rho,\gamma}f_1(y)|^p\,d\mu(y)\right]^{1/p} \le C\,\|G_{\rho,\gamma} : L^1(\mu)\|\cdot\|f : L^{p,\lambda}(\nu,\mu)\|.$$

Contrarily, we have the estimation given below for each $x \in B$:

$$\begin{aligned}|I_{\rho,\gamma}f_2(x)| &\le \int_{\tilde{B}} I_{\rho,\gamma}f_2(x)d\mu(x)\\ &\le \int_{|x-y|\ge R} I_{\rho,\gamma}f_2(x)d\mu(x)\\ &= \sum_{k=0}^\infty \int_{2^kR\le|x-y|<2^{k+1}R} I_{\rho,\gamma}f_2(x)d\mu(x)\\ &\le \sum_{k=0}^\infty \frac{\rho(2^kR)}{(2^kR)^{n+\gamma}}\int_{|x-y|<2^{k+1}R} |f(y)|\,d\nu(y)\\ &\le C\,\|f : L^{p,\lambda}(\nu,\mu)\|\sum_{k=0}^\infty \frac{\rho(2^kR)\nu(B(x,2^{k+1}R))^{1-1/p}}{(2^kR)^{n-[n\lambda/p]+\gamma}}.\end{aligned}$$

Considering that $\nu \in$ (GC),

$$\begin{aligned}|I_{\rho,\gamma} f_2(x)| &\le C\,\|f : L^{p,\lambda}(\nu,\mu)\| \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^{[n/p]-[n\lambda/p]+\gamma}} \\ &\le C\,\|f : L^{p,\lambda}(\nu,\mu)\| \int_R^{\infty} \frac{\rho(t)}{t^{1+\gamma+[n(1-\lambda)/p]}}\,dt \\ &\le C\,R^{n(\lambda-1)/p} \|f : L^{p,\lambda}(\nu,\mu)\| \int_R^{\infty} \frac{\rho(t)}{t^{1+\gamma}}\,dt \\ &\le C\,R^{n(\lambda-1)/p} \|f : L^{p,\lambda}(\nu,\mu)\| \cdot \|G_{\rho,\gamma} : L^1(\mu)\|.\end{aligned}$$

And ultimately, we shall have that for any open ball B, we come to

$$\left[\frac{1}{\mu(B)} \int_B |I_{\rho,\gamma} f_2(x)|^p \, d\mu(x)\right]^{1/p} \le C\,R^{n(\lambda-1)/p} \|f : L^{p,\lambda}(\nu,\mu)\| \cdot \|G_{\rho,\gamma} : L^1(\mu)\|,$$

$$\text{or } \left[\frac{1}{\mu(B)^{\lambda}} \int_B |I_{\rho,\gamma} f_2(x)|^p \, d\mu(x)\right]^{1/p} \le C\,\|f : L^{p,\lambda}(\nu,\mu)\| \cdot \|G_{\rho,\gamma} : L^1(\mu)\|.$$

Corollary 3.1 *Let us say there are $\nu \in$ (GC) and $1 < p < \infty$. In the case when both $f \in L^{p,\lambda}(\nu,\mu)$ and $G_{\rho,\gamma} \in L^1(\mu)$ are true, one has $I_{\rho,\gamma} f \in L^{p,\lambda}(\mu)$. In addition, there is $C > 0$ such that*

$$\|I_{\rho,\gamma} f : L^{p,\lambda}(\mu)\| \le C\,\|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|f : L^{p,\lambda}(\nu,\mu)\|.$$

3.1 Minkö wski's inequality

Before stating our key findings on the boundedness of $I_{\rho,\gamma}$, we take into consideration the following simple finding [10].

Lemma 3.1 *Assume we are given $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. When considering any measure ν and μ on $\mathbb{R}^n$,*

$$\left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} G(x-y)\,d\nu(y)\right|^p d\mu(x)\right]^{1/p} \le \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |G(x-y)|^p \, d\mu(x)\right]^{1/p} d\nu(y).$$

4 Main Results

Theorem 4.1 *Let ν be any measure on $\mathbb{R}^n$ and $\mu \in$ (GC).*
If there is $C_s > 0$ such that $f \in L^{1,\lambda}(\nu)$ and $G_{\rho,\gamma} \in L^s(\mu)$, then

$$\|I_{\rho,\gamma} f : L^{s,\lambda}(\mu)\| \le C_s \|G_{\rho,\gamma} : L^s(\mu)\| \cdot \|f : L^{1,\lambda}(\nu)\|, \quad s \ge 1.$$

Proof. According to Minkö wski's inequality, with $1 \le s < \infty$, we have

$$\begin{aligned}\left[\frac{1}{\mu(B)^{\lambda}} \int_B |I_{\rho,\gamma} f(x)|^s \, d\mu(x)\right]^{1/s} &= \left(\int_B \left|\int_B \frac{1}{\mu(B)^{\lambda}} G_{\rho,\gamma}(|x-y|) f(y)\,d\nu(y)\right|^s d\mu(x)\right)^{1/s} \\ &\le \int_B \left(\int_B \frac{1}{\mu(B)^{\lambda}} |G_{\rho,\gamma}(|x-y|) f(y)|^s \, d\mu(x)\right)^{1/s} d\nu(y) \\ &\le \int_B \left(\int_B |G_{\rho,\gamma}(|x-y|)|^s \, d\mu(x)\right)^{1/s} \left|\frac{1}{\mu(B)^{\lambda}} f(y)\right| d\nu(y) \\ &\le C\,\|G_{\rho,\gamma} : L^s(\mu)\| \cdot \|f : L^{1,\lambda}(\nu)\|.\end{aligned}$$

So

$$\left[\frac{1}{\mu(B)^{\lambda}}\int_B |I_{\rho,\gamma}f(x)|^s\,d\mu(x)\right]^{1/s} \le \sup_{x\in B}\|I_{\rho,\gamma}f(x):L^{s,\lambda}\| \quad (1)$$

$$\le C\,\|G_{\rho,\gamma}:L^s(\mu)\|\cdot\|f:L^{1,\lambda}(\nu)\|. \quad \blacksquare \quad (2)$$

In accordance with the aforementioned Theorem 4.1, we additionally have the following.

Corollary 4.1 *Consider the case when ν is any measure on $\mathbb{R}^n$ and $\mu\in$ (GC). There exists $C>0$ such that if $f\in L^{1,\lambda}(\nu)$ and $G_{\rho,\gamma}\in L^1(\mu)$, then*

$$\|I_{\rho,\gamma}f:L^{1,\lambda}(\mu)\|\le C\|G_{\rho,\gamma}:L^1(\mu)\|\cdot\|f:L^{1,\lambda}(\nu)\|.$$

Next, we come to a particular case of Young's inequality.
Suppose

$$\frac{1}{s}=\frac{1}{p}+\frac{1}{q}-1,\quad \text{or}\quad 1=\frac{1}{s}+1-\frac{1}{p}+1-\frac{1}{q}=\frac{1}{s}+\frac{1}{p'}+\frac{1}{q'}.$$

Note that

$$p=q'\left(1-\frac{p}{s}\right),\quad q=p'\left(1-\frac{q}{s}\right).$$

Theorem 4.2 *Assume $\mu,\nu\in$ (GC). If $f\in L^p(\nu)$ and $G_{\rho,\gamma}\in L^q(\mu)$, then there exists $C>0$ so that*

$$\|I_{\rho,\gamma}f:L^{s,\lambda}(\mu)\|\le C\|G_{\rho,\gamma}:L^q(\mu)\|\cdot\|f:L^{p,\lambda}(\nu)\|,\quad \frac{1}{s}=\frac{1}{p}+\frac{1}{q}-1.$$

***Proof*.** With Hölder's inequality, we start with the following:

$$\begin{aligned}
&\left[\frac{1}{\mu(B)^{\lambda}}\int_B |I_{\rho,\gamma}f(x)|^s\,d\mu(x)\right]^{1/s}\\
&\le \int_B \frac{1}{\mu(B)^{\lambda}}|f(y)|^{p/s+(1-p/s)}|G_{\rho,\gamma}(|x-y|)|^{q/s+(1-q/s)}\,d\nu(y)\\
&\le \left[\int_B |G_{\rho,\gamma}(|x-y|)|^q|f(y)|^p\,d\nu(y)\right]^{1/s}\\
&\left[\int_B \frac{1}{\mu(B)^{\lambda}}|f(y)|^{q'(1-p/s)}\,d\nu(y)\right]^{1/q'}\\
&\times\left[\int_B |G_{\rho,\gamma}(|x-y|)|^{p'(1-q/s)}\,d\nu(y)\right]^{1/p'}\\
&= \left[\int_B |G_{\rho,\gamma}(|x-y|)|^q|f(y)|^p\,d\nu(y)\right]^{1/s}\\
&\left[\int_B \frac{1}{\mu(B)^{\lambda}}|f(y)|^{p}\,d\nu(y)\right]^{1/q'}\\
&\times\left[\int_B |G_{\rho,\gamma}(|x-y|)|^{q}\,d\nu(y)\right]^{1/p'}.
\end{aligned}$$

The right-hand side will now be estimated for every $f \in L^{p,\lambda}(\nu,\mu)$ and $\tilde{B} := B(a, 2R)$, we define

$$f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^C}.$$

$$\text{Since } \|f_1 : L^p(\nu)\| = \left[\int_{\tilde{B}} |f(y)|^p \, d\nu(y)\right]^{1/p} \leq \mu(\tilde{B})^{\lambda/p} \|f : L^{p,\lambda}(\nu,\mu)\| < \infty,$$

we have $f_1 \in L^p(\nu)$, and for every B, we come to the following estimation:

$$\begin{aligned}
\int_B |G_{\rho,\gamma}(|x-y|)|^q |f_1(y)|^p \, d\nu(y) &\leq \||G_{\rho,\gamma}(|x-y|)|^q f_1^p : L^p(\nu)\| \\
&\leq C \, \|G_{\rho,\gamma}(|x-y|) : L^q(\mu)\|^q \cdot \|f_1 : L^p(\nu)\|^p \\
&\leq C \, \mu(B)^{\lambda/p} \|G_{\rho,\gamma}(|x-y|) : L^q(\mu)\|^q \cdot \|f : L^{p,\lambda}(\nu,\mu)\|^p \\
&\leq \|G_{\rho,\gamma}(|x-y|) : L^q(\mu)\|^q \cdot \|f : L^{p,\lambda}(\nu,\mu)\|^p.
\end{aligned}$$

On the other hand, for every $x \in B$, we have the following estimation:

$$\begin{aligned}
\int_B |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) &\leq \int_{\tilde{B}} |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) \\
&\leq \int_{|x-y|\geq R} |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) \\
&= \sum_{k=0}^{\infty} \int_{2^k R \leq |x-y| < 2^{k+1} R} |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) \\
&\leq \sum_{k=0}^{\infty} (\frac{\rho(2^k R)}{(2^k R)^{n+\gamma}})^q \int_{|x-y|<2^{k+1}R} |f(y)|^p \, d\nu(y) \\
&\leq C \, \|f : L^{p,\lambda}(\nu,\mu)\| \sum_{k=0}^{\infty} (\frac{\rho(2^k R) \nu(B(x, 2^{k+1} R))^{1-1/p}}{(2^k R)^{n-[n\lambda/p]+\gamma}})^q.
\end{aligned}$$

Since $\nu \in (GC)$, we have

$$\begin{aligned}
\int_B |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) &\leq C \, \|f : L^{p,\lambda}(\nu,\mu)\|^p \sum_{k=0}^{\infty} (\frac{\rho(2^k R)}{(2^k R)^{[n/p]-[n\lambda/p]+\gamma}})^q \\
&\leq C \, \|f : L^{p,\lambda}(\nu,\mu)\|^p \int_R^{\infty} (\frac{\rho(t)}{t^{1+\gamma+[n(1-\lambda)/p]}})^q \, dt \\
&\leq C \, R^{n(\lambda-1)q/p} \|f : L^{p,\lambda}(\nu,\mu)\|^p \int_R^{\infty} (\frac{\rho(t)}{t^{1+\gamma}})^q \, dt \\
&\leq C \, R^{n(\lambda-1)q/p} \|f : L^{p,\lambda}(\nu,\mu)\|^p \cdot \|G_{\rho,\gamma} : L^q(\mu)\|^q \\
&\leq C \|f : L^{p,\lambda}(\nu,\mu)\|^p \cdot \|G_{\rho,\gamma} : L^q(\mu)\|^q,
\end{aligned}$$

it implies

$$\int_B |G_{\rho,\gamma}(|x-y|)|^q |f(y)|^p \, d\nu(y) \leq C \|f : L^{p,\lambda}(\nu,\mu)\|^p \cdot \|G_{\rho,\gamma} : L^q(\mu)\|^q. \tag{3}$$

Now we want to estimate the right-hand side, especially when for $x \in B$ and $R > 0$, we will have

$$\int_B |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) = \int_{|x-y|<R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) + \int_{|x-y|\geq R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y).$$

We start with

$$\begin{aligned}
\int_{|x-y|<R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) &= \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x-y| < 2^{k+1} R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) \\
&\sim C \sum_{k=-\infty}^{-1} G_{\rho,\gamma}(2^k R)^q \nu(B(x, 2^{k+1} R)) \\
&\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^k R)^q}{(2^k R)^{(q-1)n}} \\
&\leq C \int_0^R \frac{\rho(t)^q}{t^{(q-1)n+1}} \, dt \\
&\leq C \|G_{\rho,\gamma} : L^q(\mu)\|^q,
\end{aligned}$$

and also

$$\begin{aligned}
\int_{|x-y|\geq R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) &\sim C \sum_{k=0}^{\infty} G_{\rho,\gamma}(2^k R)^q \nu(B(x, 2^{k+1} R)) \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)^q}{(2^k R)^{nq-n+q\gamma}} \\
&\leq C \int_R^{\infty} \frac{\rho(t)^q}{t^{n(q-1)+q\gamma+1}} \, dt \\
&\leq C \|G_{\rho,\gamma} : L^q(\mu)\|^q.
\end{aligned}$$

Up to now, for every $x \in B$, we already have

$$\begin{aligned}
\left[\frac{1}{\mu(B)^\lambda} \int_B |I_{\rho,\gamma} f(x)|^s \, d\mu(x) \right]^{1/s} &\leq C \|G_{\rho,\gamma} : L^q(\mu)\|^{qs/p'} \cdot \|f : L^{p,\lambda}(\nu)\|^{sp/q'} \\
&\cdot \left[\int_B |G_{\rho,\gamma}(|x-y|)|^q |f(y)|^p \, d\nu(y) \right]^{1/s}.
\end{aligned}$$

By using (1) and (3), finally, we have

$$\begin{aligned}
&\|I_{\rho,\gamma} f(x) : L^{s,\lambda}\|^s \leq C \, \|G_{\rho,\gamma} : L^q(\mu)\|^{qs/p'} \cdot \|f : L^{p,\lambda}(\nu)\|^{sp/q'} \\
&\leq C \|f : L^{p,\lambda}(\nu,\mu)\|^p \cdot \|G_{\rho,\gamma} : L^q(\mu)\|^q, \\
&\leq C \, \|G_{\rho,\gamma} : L^q(\mu)\|^{q+qs/p'} \cdot \|f : L^{p,\lambda}(\nu)\|^{p+sp/q'}.
\end{aligned}$$

So

$$q + \frac{qs}{p'} = s = p + \frac{sp}{q'}.$$

■

Consequently, we also have the following result.

Corollary 4.2 *Consider* $\mu, \nu \in$ (GC). *If* $f \in L^{s,\lambda}(\nu)$ *and* $G_{\rho,\gamma} \in L^1(\mu)$, *now there is* $C > 0$, *and thus*

$$\|I_{\rho,\gamma} f : L^{s,\lambda}(\mu)\| \le C\|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|f : L^{s,\lambda}(\nu)\|, \quad s \ge 1.$$

5 Pointwise Multiplier Operators

Say that $1 < p < \infty$ and $1 = 1/p + 1/p'$ with $f \in L^{p,\lambda}(\mu)$ and $g \in L^{p',\lambda}(\mu)$, respectively. The Hölder inequality will then result in $\|f \cdot g : L^{1,\lambda}(\mu)\| \le \|f : L^{p,\lambda}(\mu)\| \cdot \|g : L^{p',\lambda}(\mu)\|$.

We will examine a pointwise multiplier operator W by

$$W : f \mapsto W \cdot f, \quad \text{with} \quad [W \cdot f](x) := W(x) \cdot f(x), \quad x \in \mathbb{R}^n.$$

Therefore, based on the Hölder inequality, if $W \in L^{p',\lambda}(\mu)$, then W is a bounded operator from $L^{p,\lambda}(\mu)$ to $L^{1,\lambda}(\mu)$, with $\|W \cdot f : L^{1,\lambda}(\mu)\| \le \|W : L^{p',\lambda}(\mu)\| \cdot \|f : L^{p,\lambda}(\mu)\|$, $1 < p < \infty$.

The following is another illustration. Assume we take a look at a fractional integral operator I_α, and we define

$$W \cdot I_\alpha : f \mapsto W \cdot I_\alpha f, \quad \text{with} \quad [W \cdot I_\alpha f](x) := W(x) \cdot I_\alpha f(x), \quad x \in \mathbb{R}^n.$$

Reiterating the previous point, given that $1 < p < n/\alpha$ and $1/q + \alpha/n = 1/p$ are both affected by the Hölder inequality, we get

$$\|W \cdot I_\alpha f : L^{p,\lambda}\| \le \|W : L^{n/\alpha,\lambda}\| \cdot \|I_\alpha f : L^{q,\lambda}\| \le C\|W : L^{n/\alpha,\lambda}\| \cdot \|f : L^{p,\lambda}\|.$$

In other words, if $W \in L^{n/\alpha,\lambda}$, then $W \cdot I_\alpha : L^{p,\lambda} \to L^{p,\lambda}$ is a bounded operator.

From our primary findings, we also have the following.

Corollary 5.1 *Assume* ν *is any measure on* $\mathbb{R}^n$ *and* $\mu \in$ (GC). *If* $f \in L^{1,\lambda}(\nu)$, $W \in L^{s',\lambda}(\mu)$ *and* $G_{\rho,\gamma} \in L^s(\mu)$ *are true, then*

$$W \cdot I_{\rho,\gamma} : L^{1,\lambda}(\nu) \to L^{1,\lambda}(\mu)$$

is a bounded operator. To put it another way, for any $s \in [1, \infty)$, *there exists* $C_s > 0$ *such that*

$$\|W \cdot I_{\rho,\gamma} f : L^{1,\lambda}(\mu)\| \le C_s\|W : L^{s',\lambda}(\mu)\| \cdot \|G_{\rho,\gamma} : L^s(\mu)\| \cdot \|f : L^{1,\lambda}(\nu)\|.$$

Corollary 5.2 *Suppose* $\mu, \nu \in$ (GC).
If $f \in L^{s,\lambda}(\nu)$, $W \in L^{s',\lambda}(\mu)$, *and* $G_{\rho,\gamma} \in L^{p',\lambda}(\mu)$, *then*

$$W \cdot I_{\rho,\gamma} : L^{s,\lambda}(\nu) \to L^{1,\lambda}(\mu)$$

is a bounded operator. That is, for every $s \in [1, \infty)$, *there exists* $C_s > 0$ *such that*

$$\|W \cdot I_{\rho,\gamma} f : L^{1,\lambda}(\mu)\| \le C\|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|W : L^{p',\lambda}(\mu)\| \cdot \|f : L^{s,\lambda}(\nu)\|.$$

The next result is our last corollary.

Corollary 5.3 *Suppose* $1/s + 1/q' = 1/p$, *and* $\mu, \nu \in$ (GC).
If $f \in L^{p,\lambda}(\nu)$, $W \in L^{q',\lambda}(\mu)$ *and* $G_{\rho,\gamma} \in L^q(\mu)$, *then*

$$W \cdot I_{\rho,\gamma} : L^{p,\lambda}(\nu) \to L^{p,\lambda}(\mu)$$

is a bounded operator. That is, there exists $C > 0$ *such that*

$$\|W \cdot I_{\rho,\gamma} f : L^{p,\lambda}(\mu)\| \le C\|G_{\rho,\gamma} : L^q(\mu)\| \cdot \|W : L^{q',\lambda}(\mu)\| \cdot \|f : L^{p,\lambda}(\nu)\|.$$

6 Conclusions

As a result of this investigation, we have learned that generalized Bessel-Riesz operators defined with doubling measures in Morrey spaces with various measures are working toward boundedness. Regarding Bessel decay, the kernel of the operators satisfies a few essential characteristics. To prove that the integral operators are bounded, we used the Young, Hölder, and Minköwski inequalities and a doubling measure. The norm of these generalized operators is similarly bounded by the norm of their respective kernels, but with different measures, according to our investigation of the relationship between the kernel's parameters and generalized integral operators. The Bessel-Riesz kernel is used in studying the behavior of the solution of a Schrödinger type equation [3] which is related to quantum mechanics. In future we will consider it for the generalized Bessel-Riesz kernel. [11] investigated a new discrete chaotic system with rational fraction including the symmetry. Furthermore, symmetry properties of a nonlinear two-dimensional space-fractional diffusion equation with the Riesz potential of the order $\alpha \in (0, 1)$ will be further considered.

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8 Availability Statement

While the results of the research are being commercialized, the data that were used to support the findings of this study are currently under embargo. After the publication of this article, requests for data will be taken into consideration by the corresponding author.

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