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Denumerably Many Positive Radial Solutions to Iterative System of Nonlinear Elliptic Equations on the Exterior of a Ball

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Abstract: In this paper, by an application of Krasnoselskii's fixed point theorem, we establish the existence of denumerably many positive radial solutions to the iterative system of nonlinear elliptic equations of the form

$$\begin{aligned} \Delta \mathtt{u}_{\mathtt{j}} + \mathtt{P}(|\mathtt{x}|)\mathtt{g}_{\mathtt{j}}(\mathtt{u}_{\mathtt{j}+1}) = 0 \ \text{ in } \ \mathbb{R}^N \backslash \mathscr{B}_{r_0}, \\ \mathtt{u}_{\mathtt{j}} = 0 \ \text{ on } \ |\mathtt{x}| = r_0, \\ \mathtt{u}_{\mathtt{j}} \to 0 \ \text{ as } \ |\mathtt{x}| \to +\infty, \end{aligned}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, $\Delta \mathtt{u} = \mathtt{div}(\nabla \mathtt{u})$, $N > 2$, $r_0 > 0$, $\mathscr{B}_{r_0} = \{\mathtt{u} \in \mathbb{R}^N | \ |\mathtt{u}| < r_0\}$, $\mathtt{P} = \prod_{i=1}^{n} \mathtt{P}_i$, each $\mathtt{P}_i : (r_0, +\infty) \to (0, +\infty)$ is continuous, $r^{N-1}\mathtt{P}$ is integrable and may have singularities, and $\mathtt{g}_{\mathtt{j}} : [0, +\infty) \to \mathbb{R}$ is continuous.

Keywords: *nonlinear elliptic systems; exterior of a ball; positive radial solution; Krasnoselskii's fixed point theorem.*

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1 Introduction

The study of nonlinear elliptic system of equations

$$\left.\begin{aligned} \Delta \mathtt{u_j} + \mathtt{g_j}(\mathtt{u_{j+1}}) = 0 \ \text{ in } \ \Omega, \\ \mathtt{u_j} = 0 \ \text{ on } \ \partial\Omega, \end{aligned}\right\} \tag{1}$$

where $\mathtt{j} \in \{1,2,3,\cdots,\ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, and Ω is a bounded domain in $\mathbb{R}^N$, has an important applications in population dynamics, combustion theory and chemical reactor theory. For the recent literature on the existence, multiplicity and uniqueness of positive solutions for (1), see [3–5, 8, 9, 11, 12] and references therein.

In [2], Akdim, Rhoudaf and Salmani established the existence of entropy solutions for anisotropic elliptic equations of the form

$$\mathtt{Au} + \sum_{i=1}^{n} \mathtt{g}_i(\mathtt{x},\mathtt{u},\nabla\mathtt{u}) = f,$$

where $A\mathtt{u}$ is a Leray-Lions anisotropic operator. In [1], Aberqi, Bennouna and Elmassoudi established the existence results for the following nonlinear elliptic equations with some measure data in Musielak-Orlicz spaces:

$$\mathtt{Au} + \mathtt{K}(\mathtt{x},\mathtt{u},\nabla\mathtt{u}) = \mu.$$

In [6], Dong and Wei established the existence of radial solutions for the following nonlinear elliptic equations with gradient terms in annular domains:

$$\begin{aligned} -\Delta\mathtt{u} = \mathtt{g}\big(|\mathtt{x}|,\mathtt{u},\frac{\mathtt{x}}{|\mathtt{x}|}\cdot\nabla\mathtt{u}\big) \ \text{ in } \ \Omega_a^b, \\ \mathtt{u} = 0 \ \text{ on } \ \partial\Omega_a^b, \end{aligned}$$

by using Schauder's fixed point theorem and the contraction mapping theorem. In [10], R. Kajikiya and E. Ko established the existence of positive radial solutions for a semipositone elliptic equation of the form

$$\begin{aligned} -\Delta\mathtt{u} = \lambda\mathtt{g}(\mathtt{u}) \ \text{ in } \ \Omega, \\ \mathtt{u} = 0 \ \text{ on } \ \partial\Omega, \end{aligned}$$

where Ω is a ball or an annulus in $\mathbb{R}^N$. Recently, Son and Wang [13] have studied positive radial solutions for the nonlinear elliptic systems of the form

$$\begin{aligned} \Delta\mathtt{u_j} + \lambda\mathtt{K_j}(|\mathtt{x}|)\mathtt{g_j}(\mathtt{u_{j+1}}) = 0 \text{ in } \Omega_{\mathtt{E}}, \\ \mathtt{u_j} = 0 \text{ on } |\mathtt{x}| = r_0, \\ \mathtt{u_j} \to 0 \text{ as } |\mathtt{x}| \to +\infty, \end{aligned}$$

where $\mathtt{j} \in \{1,2,3,\cdots,\ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, $\lambda > 0$, $N > 2$, $r_0 > 0$, and $\Omega_{\mathtt{E}}$ is an exterior of a ball, and established existence, multiplicity and uniqueness results for various nonlinearities in $\mathtt{g_j}$. Inspired by the aforementioned works, in this paper, we apply Krasnoselskii's fixed point theorem to derive necessary conditions for the existence of denumerably many positive radial solutions of the following iterative system of nonlinear elliptic equations in the exterior of a ball:

$$\left.\begin{aligned} \Delta\mathtt{u_j} + \mathtt{P}(|\mathtt{x}|)\mathtt{g_j}(\mathtt{u_{j+1}}) = 0 \text{ in } \mathbb{R}^N\backslash\mathscr{B}_{r_0}, \\ \mathtt{u_j} = 0 \text{ on } |\mathtt{x}| = r_0, \\ \mathtt{u_j} \to 0 \text{ as } |\mathtt{x}| \to +\infty, \end{aligned}\right\} \tag{2}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, $\Delta\mathtt{u} = \mathtt{div}(\nabla\mathtt{u})$, $N > 2$, $r_0 > 0$, $\mathscr{B}_{r_0} = \{\mathtt{u} \in \mathbb{R}^N \mid |\mathtt{u}| < r_0\}$, $\mathtt{P} = \prod_{i=1}^{n} \mathtt{P}_i$, each $\mathtt{P}_i : (r_0, +\infty) \to (0, +\infty)$ is continuous, $r^{N-1}\mathtt{P}$ is integrable and may have singularities, and $\mathtt{g}_\mathtt{j} : [0, +\infty) \to \mathbb{R}$ is continuous.

The study of positive radial solutions to (2) reduces to the study of positive solutions to the following iterative system of two-point boundary value problems:

$$\left.\begin{aligned} \mathtt{u}_\mathtt{j}''(\tau) + \mathtt{Q}(\tau)\mathtt{g}_\mathtt{j}\big(\mathtt{u}_{\mathtt{j}+1}(\tau)\big) = 0, \ \tau \in (0, 1), \\ \mathtt{u}_\mathtt{j}(0) = 0, \ u_\mathtt{j}(1) = 0, \end{aligned}\right\} \tag{3}$$

where $\mathtt{j} \in \{1, 2, 3, \cdot \cdot \cdot, \ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, and $\mathtt{Q}(\tau) = \frac{r_0^2}{(N-2)^2}\tau^{\frac{2(N-1)}{2-N}} \prod_{i=1}^{n} \mathtt{Q}_i(\tau)$, $\mathtt{Q}_i(\tau) = \mathtt{P}_i(r_0\tau^{\frac{1}{2-N}})$ by a Kelvin type transformation through the change of variables $r = |\mathtt{x}|$ and $\tau = \left(\frac{r}{r_0}\right)^{2-N}$. Here, $\mathtt{Q}_i$ may have singularities on $[0, 1]$. Thus, for each $i \in \{1, 2, 3, \cdots, n\}$, we assume that the following conditions hold throughout the paper:

($\mathcal{H}_1$) $\mathtt{Q}_i \in L^{\mathtt{p}_i}[0, 1], (\mathtt{p}_i \geq 1)$ and may have denumerably many singularities on $(0, 1/2)$.

($\mathcal{H}_2$) There exists a sequence $\{\tau_k\}_{k=1}^{\infty}$ such that $0 < \tau_{k+1} < \tau_k < \dfrac{1}{2}$, $k \in \mathbb{N}$,

$$\lim_{k\to\infty} \tau_k = \tau^* < \frac{1}{2}, \ \lim_{\tau\to\tau_k} \mathtt{Q}_i(\tau) = +\infty, \ k \in \mathbb{N}, \ i = 1, 2, 3, \cdots, n,$$

and each $\mathtt{Q}_i(\tau)$ does not vanish identically on any subinterval of $[0, 1]$. Moreover, there exists $\mathtt{Q}_i^* > 0$ such that

$$\mathtt{Q}_i^* < \mathtt{Q}_i(\tau) < \infty \ \text{ a.e. on } \ [0, 1].$$

The rest of the paper is organized in the following fashion. In Section 2, we convert the boundary value problem (3) into the equivalent integral equation which involves the kernel. Also, we estimate bounds for the kernel which are useful in our main results. In Section 3, we develop a criteria for the existence of denumerably many positive radial solutions for (2) by applying Krasnoselskii's cone fixed point theorem in a Banach space. Finally, as an application, an example is given to demonstrate our results.

2 Kernel and Its Bounds

In this section, we constructed a kernel to the homogeneous boundary value problem corresponding to (3) and established certain lemmas for the bounds of the kernel.

Lemma 2.1 *Let $y \in \mathcal{C}[0, 1]$. Then the boundary value problem*

$$\left.\begin{aligned} \mathtt{u}_1''(\tau) + \mathtt{Q}(\tau)y(\tau) = 0, \ \tau \in (0, 1), \\ \mathtt{u}_1(0) = 0, \ u_1(1) = 0, \end{aligned}\right\} \tag{4}$$

has a unique solution

$$\mathtt{u}_1(\tau) = \int_0^1 \aleph(\tau, s)\mathtt{Q}(s)y(s)ds, \tag{5}$$

where

$$\aleph(\tau, s) = \begin{cases} s(1-\tau), & 0 \leq s \leq \tau \leq 1, \\ \tau(1-s), & 0 \leq \tau \leq s \leq 1. \end{cases}$$

Lemma 2.2 *The kernel $\aleph(\tau, s)$ has the following properties:*

(i) $\aleph(\tau, s)$ *is nonnegative and continuous on* $[0,1]\times[0,1]$,

(ii) $\aleph(\tau, s) \le \aleph(s, s)$ *for* $t, \tau \in [0,1]$,

(iii) there exists $\beta \in (0, \frac{1}{2})$ *such that* $\beta\aleph(s, s) \le \aleph(\tau, s)$ *for* $\tau \in [\beta, 1-\beta], s \in [0,1]$.

Proof. From the definition of kernel $\aleph(\tau, s)$, it is clear that (i) and (ii) hold. To prove (iii), let $\tau \in [\beta, 1-\beta]$ and $s \le \tau$, then

$$\frac{\aleph(\tau, s)}{\aleph(s, s)} = \frac{s(1-\tau)}{s(1-s)} \ge 1-\tau \ge \beta,$$

and for $\tau \le s$, we have

$$\frac{\aleph(\tau, s)}{\aleph(s, s)} = \frac{\tau(1-s)}{s(1-s)} \ge \tau \ge \beta.$$

This completes the proof.

From Lemma 2.1, we note that an ℓ-tuple $(\mathtt{u}_1, \mathtt{u}_2, \cdots, \mathtt{u}_\ell)$ is a solution of the boundary value problem (3) if and only if

$$\mathtt{u}_1(\tau) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right.$$
$$\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1.$$

In general,

$$\mathtt{u}_\mathtt{j}(\tau) = \int_0^1 \aleph(\tau, s)\mathtt{Q}(s)\mathtt{g}_\mathtt{j}\big(\mathtt{u}_{\mathtt{j}+1}(s)\big)ds, \ \ \mathtt{j} = 1, 2, 3, \cdots, \ell,$$
$$\mathtt{u}_1(\tau) = \mathtt{u}_{\ell+1}(\tau).$$

We denote the Banach space $\mathcal{C}([0,1], \mathbb{R})$ by $\mathscr{B}$ with the norm $\|\mathtt{u}\| = \max_{\tau\in[0,1]} |\mathtt{u}(\tau)|$. For $\beta \in (0, 1/2)$, the cone $\mathcal{P}_\beta \subset \mathscr{B}$ is defined by

$$\mathcal{P}_\beta = \Big\{\mathtt{u} \in \mathscr{B} : \mathtt{u}(\tau) \ge 0, \min_{\tau\in[\beta, 1-\beta]} \mathtt{u}(\tau) \ge \beta\|\mathtt{u}\|\Big\}.$$

For any $\mathtt{u}_1 \in \mathcal{P}_\beta$, define an operator $\Omega : \mathcal{P}_\beta \to \mathscr{B}$ by

$$(\Omega\mathtt{u}_1)(\tau) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right.$$
$$\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1.$$

Lemma 2.3 *For each* $\beta \in (0, 1/2)$, $\Omega(\mathcal{P}_\beta) \subset \mathcal{P}_\beta$ *and* $\Omega : \mathcal{P}_\beta \to \mathcal{P}_\beta$ *is completely continuous.*

Proof. Let $\beta \in (0, 1/2)$. Since $\mathtt{g_j}(\mathtt{u_{j+1}}(\tau))$ is nonnegative for $\tau \in [0, 1]$, $\mathtt{u}_1 \in \mathcal{P}_\beta$. Since $\aleph(\tau, s)$ is nonnegative for all $\tau, s \in [0, 1]$, it follows that $\Omega(\mathtt{u}_1(\tau)) \geq 0$ for all $\tau \in [0, 1]$, $\mathtt{u}_1 \in \mathcal{P}_\beta$. Now, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\min_{\tau\in[\beta,1-\beta]} (\Omega \mathtt{u}_1)(\tau) \\
&= \min_{\tau\in[\beta,1-\beta]} \left\{ \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1 \left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2 \left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right.\right. \\
&\qquad\qquad \left.\left.\left. \mathtt{g}_{\ell-1} \left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big) ds_\ell \right] \cdots \right] ds_3 \right] ds_2 \right] ds_1 \Bigg\} \\
&\geq \beta \int_0^1 \aleph(s_1, s_1)\mathtt{Q}(s_1)\mathtt{g}_1 \left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2 \left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right. \\
&\qquad\qquad \mathtt{g}_{\ell-1} \left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big) ds_\ell \right] \cdots \Bigg] ds_3 \Bigg] ds_2 \Bigg] ds_1 \\
&\geq \beta \left\{ \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1 \left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2 \left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right.\right. \\
&\qquad\qquad \mathtt{g}_{\ell-1} \left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big) ds_\ell \right] \cdots \Bigg] ds_3 \Bigg] ds_2 \Bigg] ds_1 \Bigg\} \\
&\geq \beta \max_{\tau\in[0,1]} |\Omega \mathtt{u}_1(\tau)|.
\end{aligned}$$

Thus $\Omega(\mathcal{P}_\beta) \subset \mathcal{P}_\beta$. Therefore, the operator Ω is completely continuous by standard methods and by the Arzela-Ascoli theorem.

3 Denumerably Many Positive Radial Solutions

In this section, we establish the existence of denumerably many positive radial solutions for the system (2) by utilizing the following theorems.

Theorem 3.1 *[7] Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and Λ_1, Λ_2 be open sets with $0 \in \Lambda_1, \overline{\Lambda}_1 \subset \Lambda_2$. Let $\mathcal{T} : \mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1) \to \mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{T}\mathtt{u}\| \leq \|\mathtt{u}\|$, $\mathtt{u} \in \mathcal{E} \cap \partial\Lambda_1$, and $\|\mathcal{T}\mathtt{u}\| \geq \|\mathtt{u}\|$, $\mathtt{u} \in \mathcal{E} \cap \partial\Lambda_2$, or

(b) $\|\mathcal{T}\mathtt{u}\| \geq \|\mathtt{u}\|$, $\mathtt{u} \in \mathcal{E} \cap \partial\Lambda_1$, and $\|\mathcal{T}\mathtt{u}\| \leq \|\mathtt{u}\|$, $\mathtt{u} \in \mathcal{E} \cap \partial\Lambda_2$.

Then $\mathcal{T}$ has a fixed point in $\mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1)$.

Theorem 3.2 *(Hölder's) Let $f \in L^{\mathtt{p}_i}[0, 1]$ with $\mathtt{p}_i > 1$, for $i = 1, 2, \cdots, n$ and $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1$. Then $\prod_{i=1}^{n} f_i \in L^1[0, 1]$ and $\|\prod_{i=1}^n f_i\|_1 \leq \prod_{i=1}^n \|f_i\|_{\mathtt{p}_i}$. Further, if $f \in L^1[0, 1]$ and $g \in L^\infty[0, 1]$, then $fg \in L^1[0, 1]$ and $\|fg\|_1 \leq \|f\|_1 \|g\|_\infty$.*

Consider the following three possible cases for $\mathtt{P_j} \in L^{\mathtt{p}_i}[0, 1]$:

$$\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1, \quad \sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} = 1, \quad \sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} > 1.$$

Firstly, we seek denumerably many positive radial solutions for the case $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1.$

Theorem 3.3 *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold, let* $\{\beta_k\}_{k=1}^{\infty}$ *be a sequence with* $\tau_{k+1} < \beta_k < \tau_k$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{S}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{R}_{k+1} < \beta_k \mathtt{S}_k < \mathtt{S}_k < \mathfrak{N}\mathtt{S}_k < \mathtt{R}_k, \ k \in \mathbb{N},$$

where

$$\mathfrak{N} = \max\left\{ \left[\beta_1 \frac{r_0^2}{(N-2)^2} \prod_{i=1}^{n} \mathtt{Q}_i^* \int_{\beta_1}^{1-\beta_1} \aleph(s,s) s^{\frac{2(N-1)}{2-N}} ds\right]^{-1}, \ 1\right\}.$$

Further, assume that $\mathtt{g}_\mathtt{j}$ *satisfies*

$(\mathcal{A}_1)$ $\mathtt{g}_\mathtt{j}(\mathtt{u}(\tau)) \le \mathtt{M}_1 \mathtt{R}_k$ *for all* $\tau \in [0,1]$, $0 \le \mathtt{u} \le \mathtt{R}_k$,
where

$$\mathtt{M}_1 < \left[\frac{r_0^2}{(N-2)^2} \|\aleph\|_q \prod_{i=1}^{n} \|\mathtt{Q}_i\|_{p_i}\right]^{-1}, \quad \aleph(s) = \aleph(s,s) s^{\frac{2(N-1)}{2-N}},$$

$(\mathcal{A}_2)$ $\mathtt{g}_\mathtt{j}(\mathtt{u}(\tau)) \ge \mathfrak{N}\mathtt{S}_k$ *for all* $\tau \in [\beta_k, 1-\beta_k]$, $\beta_k \mathtt{S}_k \le \mathtt{u} \le \mathtt{S}_k$.

The iterative system (2) *has denumerably many radial solutions* $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]}, \cdots, \mathtt{u}_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\mathtt{u}_\mathtt{j}^{[k]}(\tau) \ge 0$ *on* $(0,1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. Consider the sequences $\{\Lambda_{1,k}\}_{k=1}^{\infty}$ and $\{\Lambda_{2,k}\}_{k=1}^{\infty}$ of the open subsets of $\mathscr{B}$ defined by

$$\Lambda_{1,k} = \{\mathtt{u} \in \mathscr{B} : \|\mathtt{u}\| < \mathtt{R}_k\}, \ \Lambda_{2,k} = \{\mathtt{u} \in \mathscr{B} : \|\mathtt{u}\| < \mathtt{S}_k\}.$$

Let $\{\beta_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $\tau^* < \tau_{k+1} < \beta_k < \tau_k < \frac{1}{2}$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone $\mathcal{P}_{\beta_k}$ by

$$\mathcal{P}_{\beta_k} = \left\{\mathtt{u} \in \mathscr{B} : \mathtt{u}(\tau) \ge 0 \ \text{ and } \min_{\tau \in [\beta_k, 1-\beta_k]} \mathtt{u}(t) \ge \beta_k \|\mathtt{u}(\tau)\|\right\}.$$

Let $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{1,k}$. Then $\mathtt{u}_1(s) \le \mathtt{R}_k = \|\mathtt{u}_1\|$ for all $s \in [0,1]$. By $(\mathcal{A}_1)$ and $0 < s_{\ell-1} < 1$, we have

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell &\le \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell \\ &\le \mathtt{M}_1\mathtt{R}_k \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)ds_\ell \\ &\le \mathtt{M}_1\mathtt{R}_k \frac{r_0^2}{(N-2)^2} \int_0^1 \aleph(s_\ell, s_\ell) s_\ell^{\frac{2(N-1)}{2-N}} \prod_{i=1}^{n} \mathtt{Q}_i(s_\ell)ds_\ell.\end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} + \frac{1}{\mathtt{q}} = 1$. By the first part of Theorem 3.2, we have

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell &\le \mathtt{M}_1\mathtt{R}_k \frac{r_0^2}{(N-2)^2} \|\aleph\|_q \prod_{i=1}^{n} \|\mathtt{Q}_i\|_{p_i} \\ &\le \mathtt{R}_k.\end{aligned}$$

It follows, in a similar manner, for $0 < s_{\ell-2} < 1$,

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-2}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]ds_{\ell-1}\\ \leq \int_0^1 \aleph(s_{\ell-1}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})\mathtt{g}_{\ell-1}(\mathtt{R}_k)ds_{\ell-1}\\ \leq \mathtt{M}_1\mathtt{R}_k\int_0^1 \aleph(s_{\ell-1}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})ds_{\ell-1}\\ \leq \mathtt{M}_1\mathtt{R}_k\frac{r_0^2}{(N-2)^2}\|\aleph\|_q\prod_{i=1}^n\|\mathtt{Q}_i\|_{p_i}\\ \leq \mathtt{R}_k.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\mathtt{u}_1)(t) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots\right.\right.\\ \left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1\\ \leq R_k.\end{aligned}$$

Since $\mathtt{R}_k = \|\mathtt{u}_1\|$ for $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{1,k}$, we get

$$\|\Omega\mathtt{u}_1\| \leq \|\mathtt{u}_1\|. \tag{6}$$

Let $\tau \in [\beta_k, 1-\beta_k]$. Then $\mathtt{S}_k = \|\mathtt{u}_1\| \geq \mathtt{u}_1(t) \geq \min_{\tau\in[\beta_k,1-\beta_k]} \mathtt{u}_1(t) \geq \beta_k\,\|\mathtt{u}_1\| \geq \beta_k\mathtt{S}_k$. By $(\mathcal{A}_2)$ and for $s_{\ell-1} \in [\beta_k, 1-\beta_k]$, we have

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell &\geq \int_{\beta_k}^{1-\beta_k} \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\\ &\geq \mathfrak{N}\mathtt{S}_k\int_{\beta_k}^{1-\beta_k} \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)ds_\ell\\ &\geq \mathfrak{N}\mathtt{S}_k\beta_1\int_{\beta_1}^{1-\beta_1} \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)ds_\ell\\ &\geq \mathfrak{N}\mathtt{S}_k\beta_1\frac{r_0^2}{(N-2)^2}\int_{\beta_1}^{1-\beta_1} \aleph(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}}\prod_{i=1}^n\mathtt{Q}_i(s_\ell)ds_\ell\\ &\geq \mathfrak{N}\mathtt{S}_k\beta_1\frac{r_0^2}{(N-2)^2}\prod_{i=1}^n\mathtt{Q}_i^*\int_{\beta_1}^{1-\beta_1} \aleph(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}}ds_\ell\\ &\geq \mathtt{S}_k.\end{aligned}$$

Continuing with the bootstrapping argument, we get

$$
\begin{aligned}
(\Omega \mathtt{u}_1)(\tau) = \int_0^1 &\aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots\right.\right.\\
&\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1\\
\geq \mathtt{S}_k&.
\end{aligned}
$$

Thus, if $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{2,k}$, then

$$\|\Omega \mathtt{u}_1\| \geq \|\mathtt{u}_1\|. \tag{7}$$

It is evident that $0 \in \Lambda_{2,k} \subset \overline{\Lambda}_{2,k} \subset \Lambda_{1,k}$. From (6),(7), it follows from Theorem 3.1 that the operator Ω has a fixed point $\mathtt{u}_1^{[k]} \in \mathcal{P}_{\beta_k} \cap \big(\overline{\Lambda}_{1,k}\backslash\Lambda_{2,k}\big)$ such that $\mathtt{u}_1^{[k]}(t) \geq 0$ on $(0,1)$, and $k \in \mathbb{N}$. Next, setting $\mathtt{u}_{\ell+1} = \mathtt{u}_1$, we obtain denumerably many positive solutions $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]}, \cdots, \mathtt{u}_\ell^{[k]})\}_{k=1}^\infty$ of (3) given iteratively by

$$
\begin{aligned}
\mathtt{u}_\mathtt{j}(\tau) &= \int_0^1 \aleph(\tau, s)\mathtt{Q}(s)\mathtt{g}_\mathtt{j}(\mathtt{u}_{\mathtt{j}+1}(s))ds,\ \mathtt{j} = 1, 2, \cdots, \ell-1, \ell,\\
\mathtt{u}_{\ell+1}(\tau) &= \mathtt{u}_1(\tau).
\end{aligned}
$$

The proof is completed.

For $\sum_{i=1}^n \mathtt{p}_i = 1$, we have the following theorem.

Theorem 3.4 *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold, let* $\{\beta_k\}_{k=1}^\infty$ *be a sequence with* $\tau_{k+1} < \beta_k < \tau_k$. *Let* $\{\mathtt{R}_k\}_{k=1}^\infty$ *and* $\{\mathtt{S}_k\}_{k=1}^\infty$ *be such that*

$$\mathtt{R}_{k+1} < \beta_k \mathtt{S}_k < \mathtt{S}_k < \mathfrak{N}\mathtt{S}_k < \mathtt{R}_k,\ \ k \in \mathbb{N}.$$

Further, assume that $\mathtt{g}_\mathtt{j}$ *satisfies* $(\mathcal{A}_2)$ *and*

$(\mathcal{A}_3)$ $\mathtt{g}_\iota(\mathtt{u}(\tau)) \leq \mathtt{M}_2\mathtt{R}_k$ *for all* $0 \leq \mathtt{u}(\tau) \leq \mathtt{R}_k$, $\tau \in [0,1]$, *where*

$$\mathtt{M}_2 < \min\left\{\left[\frac{r_0^2}{(N-2)^2}\|\aleph\|_\infty \prod_{i=1}^n \|\mathtt{Q}_i\|_{\mathtt{p}_i}\right]^{-1},\ \mathfrak{N}\right\}.$$

The iterative system (2) *has denumerably many radial solutions* $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]}, \cdots, \mathtt{u}_\ell^{[k]})\}_{k=1}^\infty$ *such that* $\mathtt{u}_\mathtt{j}^{[k]}(\tau) \geq 0$ *on* $(0,1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. Let $\Lambda_{1,k}$ be as in the proof of Theorem 3.3 and let $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{2,k}$. Again, $\mathtt{u}_1(\tau) \leq \mathtt{R}_k = \|\mathtt{u}_1\|$ for all $\tau_1 \in [0,1]$. By $(\mathcal{A}_3)$ and $0 < \tau_{\ell-1} < 1$, we have

$$
\begin{aligned}
\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell &\leq \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\\
&\leq \mathtt{M}_1\mathtt{R}_k \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)ds_\ell\\
&\leq \mathtt{M}_1\mathtt{R}_k \frac{r_0^2}{(N-2)^2}\int_0^1 \aleph(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}}\prod_{i=1}^n \mathtt{Q}_i(s_\ell)ds_\ell\\
&\leq \mathtt{M}_1\mathtt{R}_k \frac{r_0^2}{(N-2)^2}\|\aleph\|_\infty \prod_{i=1}^n \|\mathtt{Q}_i\|_{\mathtt{p}_i}\\
&\leq \mathtt{R}_k.
\end{aligned}
$$

Continuing with this bootstrapping argument, we get

$$(\Omega \mathtt{u}_1)(t) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right.$$
$$\left.\left. \mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right] \cdots \right] ds_3\right] ds_2\right] ds_1$$
$$\leq R_k.$$

Thus, $\|\Omega \mathtt{u}_1\| \leq \|\mathtt{u}_1\|$ for $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{1,k}$. Now define $\Lambda_{2,k} = \{\mathtt{u} \in \mathscr{B} : \|\mathtt{u}\| < \mathtt{S}_k\}$. Let $\mathtt{u}_1 \in \mathcal{P}_{\Omega_k} \cap \partial\Lambda_{2,k}$ and let $s_{\ell-1} \in [\beta_k,\, 1-\beta_k]$. Then the argument leading to (7) can be applied to the present case. Hence, the theorem is proved.

Finally, we deal with the case $\sum_{i=1}^{n} \mathtt{p}_i > 1$.

Theorem 3.5 *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold, let* $\{\beta_k\}_{k=1}^{\infty}$ *be a sequence with* $\tau_{k+1} < \beta_k < \tau_k$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{S}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{R}_{k+1} < \beta_k \mathtt{S}_k < \mathtt{S}_k < \mathfrak{N}\mathtt{S}_k < \mathtt{R}_k,\ \ k \in \mathbb{N}.$$

Further, assume that $\mathtt{g}_\mathtt{j}$ *satisfies* $(\mathcal{A}_2)$ *and*
$(\mathcal{A}_4)$ $\mathtt{g}_\iota(\mathtt{u}(\tau)) \leq \mathtt{M}_3\mathtt{R}_k$ *for all* $0 \leq \mathtt{u}(\tau) \leq \mathtt{R}_k$, $\tau \in [0,1]$, *where*

$$\mathtt{M}_3 < \min\left\{ \left[\frac{r_0^2}{(N-2)^2}\|\aleph\|_\infty \prod_{i=1}^{n}\|\mathtt{Q}_i\|_1\right]^{-1},\ \mathfrak{N}\right\}.$$

The iterative system (2) *has denumerably many radial solutions* $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]}, \cdots, \mathtt{u}_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\mathtt{u}_\mathtt{j}^{[k]}(\tau) \geq 0$ *on* $(0,1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.3.

4 Applications

Example 4.1 *Consider the following fractional order boundary value problem:*

$$\left.\begin{array}{c} \Delta \mathtt{u}_\mathtt{j} + \mathtt{P}(|\mathtt{x}|)\mathtt{g}_\mathtt{j}(\mathtt{u}_{\mathtt{j}+1}) = 0 \ \textit{in}\ \mathbb{R}^3 \backslash \mathscr{B}_1, \\ \mathtt{u}_\mathtt{j} = 0 \ \textit{on}\ |\mathtt{x}| = 1, \\ \mathtt{u}_\mathtt{j} \to 0 \ \textit{as}\ |\mathtt{x}| \to +\infty, \end{array}\right\} \tag{8}$$

where $\mathtt{j} \in \{1, 2\}$, $\mathtt{u}_3 = \mathtt{u}_1$, $\mathtt{Q}(\tau) = \frac{1}{\tau^4}\prod_{i=1}^{2}\mathtt{Q}_i(\tau)$, $\mathtt{Q}_i(\tau) = \mathtt{P}_i(\frac{1}{\tau})$, in which

$$\mathtt{P}_1(t) = \frac{1}{|t-4|^{\frac{1}{2}}} \qquad \text{and} \qquad \mathtt{P}_2(t) = \frac{1}{|t-3|^{\frac{1}{2}}},$$

$$\mathtt{g}_\mathtt{j}(\mathtt{u}) = \begin{cases} 5\times 10^{-14}, & \mathtt{u}\in(10^{-4},+\infty),\\ \frac{30\times10^{-(4k+2)}-5\times10^{-4k-10}}{10^{-(4k+2)}-10^{-4k}}(\mathtt{u}-10^{-4k})+5\times10^{-4k-10}, & \\ & \mathtt{u}\in\left[10^{-(4k+2)},10^{-4k}\right],\\ 30\times10^{-(4k+2)}, \quad \mathtt{u}\in\left(\frac{1}{5}\times10^{-(4k+2)},10^{-(4k+2)}\right), & \\ \frac{30\times10^{-(4k+2)}-5\times10^{-(4k+14)}}{\frac{1}{5}\times10^{-(4k+2)}-10^{-(4k+4)}}(\mathtt{u}-10^{-(4k+4)})+5\times10^{-(4k+14)}, & \\ & \mathtt{u}\in\left(10^{-(4k+4)},\frac{1}{5}\times10^{-(4k+2)}\right], \end{cases}$$

$\mathtt{j}=1,2$. Let

$$\tau_k = \frac{31}{64} - \sum_{r=1}^{k}\frac{1}{4(r+1)^4},\ \beta_k = \frac{1}{2}(\tau_k+\tau_{k+1}),\ k=1,2,3,\cdots,$$

then

$$\beta_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}$$

and

$$\tau_{k+1} < \beta_k < \tau_k,\ \beta_k > \frac{1}{5}.$$

It is easy to see

$$\tau_1 = \frac{15}{32} < \frac{1}{2},\ \tau_k - \tau_{k+1} = \frac{1}{4(k+2)^4},\ k=1,2,3,\cdots.$$

Since $\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$ and $\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}$, it follows that

$$\tau^* = \lim_{k\to\infty}\tau_k = \frac{31}{64} - \sum_{i=1}^{\infty}\frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5}.$$

Also,

$$\mathtt{P}_1,\mathtt{P}_2\in L^{\mathtt{p}}[0,1] \quad \text{and} \quad \prod_{i=1}^{2}\mathtt{Q}_i^* = \frac{1}{\sqrt{12}},$$

$$\int_{\beta_1}^{1-\beta_1}\aleph(s,s)s^{\frac{2(N-1)}{2-N}}ds = 0.2657555992,$$

$$\beta_1\frac{r_0^2}{(N-2)^2}\prod_{i=1}^{n}\mathtt{Q}_i^*\int_{\beta_1}^{1-\beta_1}\aleph(s,s)s^{\frac{2(N-1)}{2-N}}ds = 0.03584271890,$$

$$\mathfrak{N} = \max\left\{\left[\beta_1\frac{r_0^2}{(N-2)^2}\prod_{i=1}^{n}\mathtt{Q}_i^*\int_{\beta_1}^{1-\beta_1}\aleph(s,s)s^{\frac{2(N-1)}{2-N}}ds\right]^{-1},\ 1\right\} \approx 27.89966918.$$

Let $\mathtt{q} = 2, \mathtt{p}_1 = \mathtt{p}_2 = 1/4$, then

$$\mathtt{M}_1 < \left[\frac{r_0^2}{(N-2)^2} \|\aleph\|_{\mathtt{q}} \prod_{i=1}^{n} \|\mathtt{Q}_i\|_{p_i} \right]^{-1} \approx 5.95134 \times 10^{-10}.$$

So, let $\mathtt{M}_1 = 5.5 \times 10^{-10}$. In addition, if we take

$$\mathtt{R}_k = 10^{-4k},\ \mathtt{S}_k = 10^{-(4k+2)},$$

then

$$\mathtt{R}_{k+1} = 10^{-(4k+4)} < \frac{1}{5} \times 10^{-(4k+2)} < \beta \mathtt{S}_k < \mathtt{S}_k = 10^{-(4k+2)} < \mathtt{R}_k = 10^{-4k},$$

and $\mathtt{g}_1, \mathtt{g}_2$ satisfy the following growth conditions:

$$\mathtt{g}_\mathtt{j}(\mathtt{u}) \leq \mathtt{M}_1 \mathtt{R}_k = 5.5 \times 10^{-4k-10}, \quad \mathtt{u} \in \left[0, 10^{-4k}\right],$$
$$\mathtt{g}_\mathtt{j}(\mathtt{u}) \geq \mathfrak{N} \mathtt{S}_k = 27.89966918 \times 10^{-(4k+2)}, \quad \mathtt{u} \in \left[\frac{1}{5} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right].$$

Then all the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the boundary value problem (8) has denumerably many positive solutions $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]})\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \leq \|\mathtt{u}_\mathtt{j}^{[k]}\| \leq 10^{-4k}$ for each $k = 1, 2, 3, \cdots$, and $\mathtt{j} = 1, 2$.

5 Conclusion

This paper focuses on establishing the existence of denumerably many positive radial solutions to the iterative system of nonlinear elliptic equations through the application of one of the most important fixed point theorems known as "Krasnoselskii's fixed point theorem". These ease the proof of the existence of the positive solution attached to the system under study.

In the future, we aim to expand this study by adapting some techniques used to other ideas and extracting new results that show the effectiveness of this study and its effect in the midst of scientific research. The closest result we would like to prove is the establishment of the multiple and sign-changing solutions for the iterative system of nonlinear elliptic equations with critical potential and critical parameters.

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