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Exponential and Strong Stabilization for Inhomogeneous Semilinear Control Systems by Decomposition Method

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Abstract: In this work, we study, in a Hilbert state space, the stabilization problem of inhomogeneous semilinear control systems; the existence and uniqueness of solutions of the system are proved by the semigroup theory. The paper also gives a feedback control and sufficient conditions for exponential and strong stabilization using the decomposition method. Finally, an application to the heat equations is provided.

Keywords: *stability of control systems; stabilization of systems by feedback; heat equation.*

Mathematics Subject Classification (2010): 93-XX; 34-XX.

1 Introduction

Semilinear systems are special types of nonlinear systems. They are a transition class between linear and nonlinear systems and thus represent a wide range for modeling the dynamic behavior of various real-world phenomena. Stability is one of the most important concepts in dynamical systems theory, particularly semi-linear systems. This problem remains a major concern in the work of mathematicians and engineers. In this work, we study the stabilization of the inhomogeneous semi-linear system described by the equation

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + v(t)(Ny(t) + c), \\ y(0) = y_0, \in H, \end{cases} \tag{1}$$

where

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1. The state space is an infinite-dimensional Hilbert space H with the inner product $\langle .,. \rangle$ and the corresponding norm $\|.\|$;
2. $v(t)$ is a scalar-valued control;
3. A is an unbounded operator with the domain $D(A) \subset H$, it generates a semigroup of contractions $(S(t))_{t\geq 0}$ on H;
4. N is a nonlinear operator from H into H, which is locally Lipschitz and sequentially continuous operator such that $N(0) = 0$; since $N(.)$ is locally Lipschitz, there is $L > 0$ such that for all $z, y \in H$ satisfying $0 < \|y\| \leqslant \|z\| \leqslant R$, we have $\|Nz - Ny\| \leqslant L\|z - y\|$;
5. $c \neq 0$ is a fixed vector in H.

Remark 1.1 If N is linear, the system(1) is bilinear, and if N is not linear, the system (1) is semilinear; if $c = 0$, the system is homogeneous, and if $c \neq 0$, the system is inhomogeneous.

One of the most important concepts in systems theory is stability; we study the possibility of finding feedback $u(y(t))$ as "regular" as possible such that the system is stable; this stability can be strong, weak, or exponential. The study of the stability of homogeneous bilinear and semilinear systems has been considered in many works, and different results have been developed in finite and infinite dimensional cases, see J. Ball, M. Slemrod [1], M. Ouzahra, A. Tsouli and A. Boutoulout [2], M. Ouzahra [3], A. Benzaza and M. Ouzahra [4], E. Zerrik and M. Ouzahra [5], H. Bounit, and Hammouri [6], A. El Alami and M. Chqondi [7].

However, only a few works study the case of inhomogeneous systems; the stability of such systems has been studied in the bilinear case by Z. Hamidi and M. Ouzahra [8], who proved the necessary and sufficient conditions for weak and strong partial stabilization of inhomogeneous bilinear system by the control

$$v(t) = -\rho \frac{\langle y(t), Ny(t) + c\rangle}{|\langle y(t), Ny(t) + c\rangle| + 1}, \quad \forall t > 0, \tag{2}$$

where $\rho > 0$ is the gain control.

In this work, an exponential and strong stabilization result has been established using the same feedback control (2), provided that the following observation assumption is verified:

$$\exists \delta, T > 0 \quad \text{such that} \quad \int_0^T |\langle NS(s)y(t) + c, S(s)y(t)\rangle| \mathrm{d}s \geq \delta \|y(t)\|, \quad \forall y \in H.$$

This paper is organized as follows. In Section 2, we choose a control that ensures the stabilization of our system; in Section 3, we show the existence and uniqueness of the solution in the semilinear inhomogeneous case; in Section 4, we present an appropriate decomposition of the state space H and the system (1) via the spectral properties of the operator A. We apply this approach to study the exponential stabilization problem of the type (1); in Section 5, we look at the strong stabilization problem using the chosen control. In the last section, we give illustrations through examples governed by a heat equation.

2 Choice of Control

Let (1) be as given in the Introduction. Here, the state space is a Hilbert space H with the inner product $\langle .,. \rangle$ and corresponding norm $\|.\|$, $y(t)$ is the state, and $u(t)$ is a scalar valued control. The problem of stabilization consists of choosing a feedback control $u(t)$ such that the solution of the resulting feedback system satisfies in some sense $y(t) \to 0$ as $\to +\infty$. If we formally compute the time rate of change of the "energy", we get the following:

$$\begin{aligned} \frac{d}{dt}\|y(t)\|^2 &= 2 < y(t); \frac{d}{dt}y(t) > \\ &= 2 < y(t); Ay(t) > +2u(t) < y(t), Ny(t) + c > \end{aligned}$$

which implies, since $S(t)$ is a semigroup of contractions,

$$\frac{d}{dt}\|y(t)\|^2 \leq 2u(t) < y(t); Ny(t) + c >, \ \forall t \in [0, T].$$

Then, to make the energy nonincreasing, an obvious choice of the feedback control (though not the only one) is $v(t) = -\rho \frac{\langle y(t), Ny(t)+c\rangle}{|\langle y(t), Ny(t)+c\rangle|+1}$; $\big(\forall t > 0, \rho > 0\big)$ since this control yields the "dissipating energy inequality"

$$\frac{d}{dt}\|y(t)\|^2 \leq -2\frac{(\langle Ny(t) + c, y(t)\rangle)^2}{|\langle y(t), Ny(t) + c\rangle| + 1}; \forall t \in [0, T]. \tag{3}$$

3 Well-Possedness

Let us consider the closed loop-system

$$\begin{cases} \frac{dy(t)}{dt} = Ay(t) + f(y(t)), & \forall t > 0, \\ y(0) = y_0 \in H, \end{cases} \tag{4}$$

where $f(y(t)) = v(t)\big(Ny(t) + c\big)$, $\forall y \in H$, and $v(t)$ is the control given in (2).

We set $\phi(s) = \frac{s}{|s|+1}$ for all $s \in \mathbb{R}$ so that $f(y(t)) = -\rho\phi(< y; Ny + c >)(Ny + c)$. In this section, we aim to study the existence, uniqueness, and regularity of the solution to the system (1).

We start our study with the well-posedness result.

Proposition 3.1 *Assume that A is the infinitesimal generator of a linear C^0 semi-group of contractions on a Hilbert space H and let N be locally Lipschitz and sequentially continuous operator; then the system (1) admits a unique global mild solution $y(t)$ defined on the infinite interval $[0, +\infty[$, which is given by the following variation of constants formula:*

$$y(t) = S(t)y_0 + \int_0^t v(\tau)(Ny(\tau) + c)S(t - \tau)d\tau.$$

Moreover, we have the following estimate:

$$\|z(t)\|^2 - \|z(s)\|^2 \leqslant -2\rho \int_s^t \frac{\langle y(\tau); Ny(\tau) + c\rangle^2}{|\langle y(\tau), Ny(\tau) + c\rangle| + 1} d\tau.$$

In particular, we have $\|y(t)\| \leq \|y_0\|$. $\forall t \geq 0$.

Proof. To establish the existence and uniqueness of the solution of (4), let us show that the function f is locally Lipschitz for all $y, z \in H$. Let $x, y, z \in H$ with $t \in [0, T]$ and $R > 0$ such that $\|x-y\|, \|x-z\| \leq R$. Without loss of generality, we can take $x = 0$. Since N is locally Lipschitz, there is $L_R(N) > 0$ such that for all $z, y \in H$, satisfying $0 < \|z\| \leqslant \|y\| \leqslant R$, we have $\|Nz - Ny\| \leqslant L_R(N)\|z - y\|$.

$$\begin{aligned}\text{We have } f(t,y) - f(t,z) &= \big(\rho\phi(< z; Nz + c >)(Nz + c) - \rho\phi(< y; Ny + c >)(Ny + c)\big) \\ &= \rho\big(\phi(< z; Nz + c >) - \phi(< y; Ny + c >\,)\big(Nz + c\big) \\ &+ \phi(< y; Ny + c > \big(Nz - Ny\big).\end{aligned}$$

By making use of the function $\phi(s) = \frac{s}{|s|+1}$, we have $|\phi(s) - \phi(r)| \leq |s - r|$ and $|\phi(s)| \leq |s|$ for all $(s, r) \in \mathbb{R}^2$ since $N()$ is locally Lipschitz, and when using Schwartz's inequality, it follows that

$$\begin{aligned}\|f(t,y) - f(t,z)\| &\leq \rho\Big(| < z; Nz + c > - < y; Ny + c > |\|Nz + c\|\Big) \\ &+ \rho|\phi(< y; Ny + c >)|\|Nz - Ny\|.\end{aligned}$$

$$\begin{aligned}\text{We have } \quad | < z; Nz + c >)- < y; Ny + c > | &= | < z - y; Nz + c >)+ < y; Nz - Ny > | \\ &\leq \Bigg(\|c\| + 2L_R(N)\|y_0\|\Bigg)\|z - y\|\end{aligned}$$

$$\begin{aligned}\text{and } \quad |\phi(< y; Ny + c >)|\|Nz - Ny\| &\leq | < y; Ny + c > |L_R(N)\|z - y\| \\ &\leq \|y_0\|\Bigg(\|Ny\| + \|c\|\Bigg)|L_R(N)\|z - y\| \\ &\leq \|y_0\|L_R(N)\Bigg(L_R(N)\|y_0\| + \|c\|\Bigg)\|z - y\|.\end{aligned}$$

$$\begin{aligned}\text{So, } \|f(t,y) - f(t,z)\| &\leq \rho\Bigg(L_R(N)\|y_0\| + \|c\|\Bigg)\Bigg(\|y_0\||L_R(N) + \|c\| + 2L_R(N)\|y_0\|\Bigg)\|z - y\| \\ &\leq \rho\Bigg(L_R(N)\|y_0\| + \|c\|\Bigg)\Bigg(\|c\| + 3L_R(N)\|y_0\|\Bigg)\|z - y\| \\ &\leq \mathcal{M}_{(\|y_0\|;c)}\|z - y\|.\end{aligned}$$

$$\text{where } \quad \mathcal{M}_{(\|y_0\|;c)} = \rho\Bigg(L_R(N)\|y_0\| + \|c\|\Bigg)\Bigg(\|c\| + 3L_R(N)\|y_0\|\Bigg).$$

Remark 3.1 For c=0, we obtain the constant $\mathcal{M}_{(\|y_0\|;0)} = 3\rho\Bigg(L_R(N)\|y_0\|\Bigg)^2$ which is strictly smaller than that found in the homogeneous case in [9].

So $f(t; y(t))$ satisfies a local Lipschitz condition in y, uniformly in t on bounded intervals. Thus we may apply Theorem 1.4 [10] (p.185), to obtain that there is a $t_{\max} \leq \infty$ such that (8) has a unique mild solution y on $[0, t_{\max}[$, which is given by the following variation of constants formula:

$$\begin{aligned}y(t) &= S(t)y_0 + \int_0^t f(y(\tau))S(t - \tau)d\tau \\ &= S(t)y_0 - \rho\int_0^t \frac{\langle y(\tau), Ny(\tau) + c\rangle}{|\langle y(\tau), Ny(\tau) + c\rangle| + 1}(Ny(\tau) + c)S(t - \tau)d\tau.\end{aligned}$$

To show that $t_{\max} = +\infty$, it is sufficient to prove that for each $T > 0$, the mild solution $y(t)$ is bounded by a constant independent of T. To do this, we discuss two cases:
(i) If the initial value $y_0 \in D(A)$, then the function $w(t) := \frac{1}{2}\|y(t)\|^2$ is continuously differentiable and we can write for all $t \geq 0$, the following: since this control yields the "dissipating energy inequality",

$$\frac{d}{dt}\|y(t)\|^2 \leq -2\rho \frac{(\langle y(t); Ny(t) + c\rangle)^2}{|\langle y(t), Ny(t) + c\rangle| + 1}; \forall t \in [0, T]. \tag{5}$$

When integrating the last inequality over the interval [s,t], it follows that

$$\|y(t)\|^2 - \|y(s)\|^2 \leq -2\rho \int_s^t \frac{\langle y(\tau); Ny(\tau) + c\rangle^2}{|\langle y(\tau), Ny(\tau) + c\rangle| + 1} d\tau, \forall t \geq s \geq 0. \tag{6}$$

It follows that $\|y(t)\| \leq \|y_0\|, \forall t \geq 0$.
ii) Let $y_0 \in H$ and consider a sequence $(y_0^n)_n$ of elements in H converging to y_0. For each $T > 0$, let $y(t)$ and $y^n(t)$ be the mild solutions of (S) associated, respectively, to the initial values y_0 and y_0^n. Then one can prove that for each $t \in [0, T]$, the sequence $(y^n(t))_n$ converges in H to $y(t)$, see [11].

So, if $y_0 \notin D(A)$, then we can find a sequence $(y_0^n)_n$ of elements in $D(A)$ converging to y_0 in H (because $\overline{D(A)} = H$).

$\forall t \in [0, T]$ and $\forall n \in \mathbb{N}$, we know from i) that $\|y_0^n(t)\| \leq \|y_0^n\|$.
Now, we conclude that $\|y(t)\| \leq \|y_0\|$ for all $t \in [0, T]$.

$\|z(t)\| \leqslant \|z_0\|$, $\forall t \in [0, t_{\max}[$.

Hence $t_{\max} = +\infty$ and from (6), we have $\|y(\tau)\|^2 - \|y(t)\|^2 \geq 2\rho \int_\tau^t \frac{\langle y(s); Ny(s)+c\rangle^2}{|\langle y(t), Ny(t)+c\rangle|+1} ds$ for all $0 \leq \tau \leq t$. This completes the proof.

4 Exponential Stabilisation

4.1 Decomposition of the state space and the system

Let $\delta > 0$ be fixed in advance. We suppose that the spectrum $\sigma(A)$ of A can be decomposed into $\sigma_u(A)$ and $\sigma_s(A)$

$$\text{such that } \sigma_u(A) = \sigma(A) \cap \{\lambda : \operatorname{Re}\lambda \geqslant -\delta\}, \quad \sigma_s(A) = \sigma(A) \cap \{\lambda : \operatorname{Re}\lambda < -\delta\}.$$

Then $\sigma(A) = \sigma_u(A) \cup \sigma_s(A)$ such that $\sigma_u(A)$ can be separated from $\sigma_s(A)$ by a simple and closed curve C.

It has been shown in [12](p. 178) and [13] that the operator A may be decomposed according to the decomposition:

$$H = H_u \oplus H_s \tag{7}$$

meaning $P\mathcal{D}(A) \subset \mathcal{D}(A); AH_s \subset H_s$, $AH_u \subset H_u$ (invariance of H_s and H_u under A), where $H_u = P_uH$ and $H_s = P_sH$ with P_u being the projection operator

$$P_u = \frac{1}{2\pi i} \int_C (\lambda I - A)^{-1} d\lambda \text{ and } P_s = I - P_u.$$

Then the operator A can be decomposed as

$$A = A_u + A_s \tag{8}$$

with $A_u = P_u A$ and $A_s = P_s A$. Here, A_s and A_u are the restrictions of A on H_s and H_u, respectively.

We consider N_s and N_u, the restrictions of the operator N on H_s and H_u, respectively, such that

$(\mathbf{H}_1)$: $NH_u \subseteq H_u$.

$(\mathbf{H}_2)$: $NH_s \subseteq H_s$.

In the sequel, we suppose that the operator A may be decomposed according to the decomposition (8). Under the hypotheses $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, the system (1) can be decomposed into the two following systems:

$$\begin{cases} \frac{dy_u(t)}{dt} = A_u y_u(t) + v_u(t)(N_u y_u(t) + c_u), \quad \forall t > 0, \\ y_u(0) = (y_{u_0}) \in H_u, \end{cases} \tag{9}$$

and

$$\begin{cases} \frac{dy_s(t)}{dt} = A_s y_s(t) + v_s(t)(N_s y_s(t) + c_s), \quad \forall t > 0, \\ y_s(0) = (y_{s_0}) \in H_s, \end{cases} \tag{10}$$

where y_u and y_s are the components of the solution $y \in H$ on H_u and H_s, respectively. By [13], the semigroup $S(t)$ generated by A also commutes with P_u and P_s, and induces a C_0-semigroup $S_u(t)$ (resp. $S_s(t)$) on H_u (resp. H_s).

We further suppose that A_u generates a C_0-semigroup of contractions $S_u(t)$, and A_u generates a C_0-semigroup of contractions $S_u(t)$. If A_s satisfies the following spectrum-determined growth assumption:
$\lim_{t\to+\infty} \frac{\ln\|S_s(t)\|}{t} = \sup \operatorname{Re}(\sigma(A_s))$, then $\exists \eta, K_\eta > 0$ such that $\|S_s(t)\| \leqslant K_\eta e^{-\eta t}, t \geqslant 0$.

The aim of what follows is to study the problem of weak and strong stabilization of (1) via the properties of the systems (10) and (9). We begin with the component $y_s(t)$ of the solution $y(t)$ of the system (10).

4.2 Exponential stabilization of the component $y_s(t)$

Theorem 4.1 *Let A generate a C_0-semigroup $S(t)$, and suppose that the following conditions hold:*

1. *The operator A may be decomposed according to the decomposition (8).*

2. *A_s satisfies the following spectrum-determined growth assumption:*

$$\exists \eta, K_\eta > 0 \text{ such that: } \quad \|S_s(t)\| \leqslant K_\eta e^{-\eta t}, \quad t \geqslant 0.$$

Then if $\rho < \dfrac{\eta}{K_\eta \left(L\|y_s(0)\| + \|c_s\|\right)^2}$, *the feedback*

$$v_u(t) = -\rho \frac{\langle y_u(t), N_u y_u(t) + c_u\rangle}{|\langle y_u(t), N_s y(t) + c_u\rangle| + 1}, \ \forall t > 0, \tag{11}$$

exponentially stabilizes the system (10).
More precisely, $\exists \beta_s > 0$ such that $\|y_s(t)\| \leq K_\eta \|y_s(0)\| e^{-\beta_s t}$, $\forall t > 0$.

Proof. Using Proposition 3.1, we deduce that the system (10) admits a unique global mild solution given by

$$\begin{cases} y_s(t) = S_s(t)y_s(0) + \int_0^t S_s(t-\tau)v_s(\tau)\left(N_s y_s(\tau) + c_s\right)d\tau, t \geq 0, \\ y_s(0) \in H_s. \end{cases}$$

So, $\|y_s(t)\| \leq \|S_s(t)\|\|y_s(0)\| + \displaystyle\int_0^t |v_s(\tau)|\, \|S_s(t-\tau)\|\big(\|N_s y_s(\tau)\| + \|c_s\|\,\big)d\tau.$

We have $\|S_s(t)\| \leqslant K_\eta e^{-\eta t}$. So,

$$\begin{aligned} \|y_s(t)\| &\leq K_\eta e^{-\eta t}\|y_s(0)\| + \int_0^t |v_s(\tau)|\, K_\eta e^{-\eta(t-\tau)}\big(\|N_s y_s(\tau)\| + \|c_s\|\big)d\tau \\ &\leq K_\eta e^{-\eta t}\|y_s(0)\| + \Big(L_s\|y_s(0)\| + \|c_s\|\Big)\int_0^t |v_s(\tau)|\, K_\eta e^{-\eta(t-\tau)}d\tau \end{aligned}$$

$\Big(L_s$ is a Lipschitz constant of N_u in the ball $\mathcal{B}_{(0,\|z_0\|)}\Big)$.

The feedback (11) is a bounded function in time and is uniformly bounded with respect to the initial states, and we have

$$\begin{aligned} |v_u(t)| &\leq \rho|\left\langle y_u(t), Ny_u(t) + c_u\right\rangle| \quad \forall t > 0, \\ &\leq \rho\|y_u(t)\|\Big(L_u\|y_u(0)\| + \|c_u\|\Big); \\ &\Big(L_u \text{ is a Lipschitz constant of } N_u \text{ in the ball } \mathcal{B}_{(0,\|z_0\|)}\Big) \end{aligned}$$

So,

$$\begin{aligned} \|y_s(t)\| &\leq K_\eta e^{-\eta t}\|y_s(0)\| \\ &\quad + \rho K_\eta\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)\Big(L_s\|y_s(0)\| + \|c_s\|\,\Big)\int_0^t \|y_s(\tau)\|e^{-\eta(t-\tau)}d\tau \\ &\leq K_\eta e^{-\eta t}\|y_s(0)\| + \mathcal{A}e^{-\eta t}\int_0^t \|y_s(\tau)\|e^{\tau}d\tau, \end{aligned}$$

where $\mathcal{A} = \rho K_\eta\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)$.

So, $\|y_s(t)\|\, e^{\eta t} \leq K_\eta\|y_s(0)\| + \mathcal{A}\int_0^t \|y_s(\tau)\|e^{\tau}d\tau$.

By using Gronwall's inequality, we have

$$\begin{aligned} \|y_s(t)\|\, e^{\eta t} &\leq K\|y_s(0)\|\exp\left(\int_0^t \mathcal{A}\mathrm{d}s\right) \\ &\leq K_\eta\|y_s(0)\|e^{\mathcal{A}t}. \end{aligned}$$

So, $\|y_s(t)\| \leq K_\eta\|y_s(0)\|e^{(\mathcal{A}-\eta)t}$.

We set: $\beta_s = \eta - \mathcal{A} = \eta - \rho K\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)$ if $\rho < \dfrac{\eta}{K_\eta\Big(L_u\|y_s(0)\|+\|c_s\|\Big)\Big(L_u\|y_s(0)\|+\|c_s\|\Big)}$.

Then $\rho K_\eta\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big)\Big(L_u\|y_s(0)\| + \|c_s\|\,\Big) < \eta$, so $\beta_s > 0$.

Finally, $\|y_s(t)\| \leq K_\eta\|y_s(0)\|e^{-\beta_s t}$, $K_\eta, \beta_s > 0$. This completes the proof of the theorem.

Now let us study the component $y_u(t)$ of the solution $y(t)$ of the system (10).

4.3 Decay estimate and exponential stabilization of the component $y_u(t)$

Lemma 4.1 *Let A_u generate a semigroup $S_u(t)$ of contractions on H_u and let N_u be locally Lipschitz. Then the system (9) controlled by (2) possesses a unique mild solution $y_u(t) \in H_u$ for each $y_u(0) \in H_u$ which satisfies, when $t \to +\infty$,*

$$\int_0^T |\langle S_u(\tau)y_u(t), N_u S_u(\tau)y_u(t) + c_u\rangle| d\tau = \mathcal{O}\left(\sqrt{\int_t^{t+T} \frac{|\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle|^2}{1 + |\langle N_u y_u(\tau) + c_u, y_u(\tau)\rangle|} d\tau}\right). \quad (12)$$

Proof. Using Proposition 3.1, we deduce that the system (9) admits a unique global mild solution given by the following formula of variation of the constants:

$$y_u(t) = S_u(t)y_u(0) - \rho \int_0^t \frac{\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle}{|\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle| + 1} (N_u y_u(\tau) + c_u) S_u(t-\tau) d\tau,$$

and using the fact that $S_u(t)$ is a semigroup of contractions, and Schwartz's inequality, for all $t \in [0,T]$, we have

$$\|y_u(t) - S_u(t)y_u(0)\| \leq \rho\sqrt{T}\Big(L\,\|y_u(0)\| + \|c_u\|\Big) \left(\int_0^T \frac{|\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle|^2}{1 + |\langle y_u(\tau), N_u y_u(\tau) + c_u\rangle|} d\tau\right)^{\frac{1}{2}}. \quad (13)$$

From the relation

$$\begin{aligned}\langle N_u S_u(t)y_0 + c_u, S_u(t)y_0\rangle &= \langle N_u S_u(t)y_0, S_u(t)y_u(0) - y_u(t)\rangle + \langle c_u, S_u(t)y_u(0) - y_u(t)\rangle \\ &\quad + \langle N_u S_u(t)y_0 - N_u y_u(t), y_u(t)\rangle + \langle N_u y_u(t) + c, y_u(t)\rangle,\end{aligned}$$

when using $\|y_u(t)\| \leq \|y_u(0)\|, \forall t \in [0, t_{\max}[$, the fact that $S_u(t)$ is a semigroup of contraction, N_u is locally Lipschitz, and Schwartz's inequality, it comes

$$|\langle N_u S_u(s)y_u(0) + c_u, S_u(s)y_u(0)\rangle| \leq \Big(2L_u\|y_u(0)\| + \|c_u\|\Big)\|y_u t) - S_u(t)y_u(0)\| + |\langle N_u y_u(s) + c_u, y_u(s)\rangle|.$$

Using (13),

$$\begin{aligned}|\langle N_u S_u(s)y_u(0) + c_u, S_u(s)y_u(0)\rangle| &\leq \mathcal{C}_{(\|y_u(0)\|;c_u)} \left(\int_0^t \frac{|\langle y_u(s), N_u y_u(s) + c_u\rangle|^2}{1 + |\langle y_u(s), N_u y_u(s) + c_u\rangle|} ds\right)^{\frac{1}{2}} \\ &\quad + |\langle N_u y_u(s) + c_u, y_u(s)\rangle|,\end{aligned} \quad (14)$$

where $\mathcal{C}_{(\|y_u(0)\|;c_u)} = \rho\sqrt{T}\left(2L_u\|y_0\| + \|c_u\|\right)\left(L_u\|y_0\| + \|c_u\|\right)$. Replacing y_0 by $y_u(t)$ in (14) and using the fact that $\|y_u(t)\| \leq \|y_u(0)\|\,\forall t \geq 0$, we get

$$\begin{aligned}|\langle N_u S_u(s)y_u(t) + c_u, S_u(s)y_u(t)\rangle| &\leq \mathcal{C}_{(\|y_0\|;c_u)} \left(\int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|^2}{1 + |\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|} ds\right)^{\frac{1}{2}} \\ &\quad + |\langle N_u y_u(s+t) + c_u, y_u(s+t)\rangle|.\end{aligned}$$

Integrating the last inequality over the interval $[0,T]$ and using the semigroup property of the solution $y(t)$ and Schwartz's inequality, remarking that the mappings $x \mapsto C_x = (2L_u x + \|c\|)(L_u x + \|c_u\|)$ are increasing $\left(C_{\|y_u(t)\|} \leq C_{\|y_0\|}\right)$, we get

$$\begin{aligned}\int_0^T |\langle N_u S_u(s)y_u(t) + c_u, S_u(s)y_u(t)\rangle| ds &\leq T^3 \mathcal{C}_{(\|y_0\|;c_u)} \left(\int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|^2}{1 + |\langle y_u(s+t), N_u y(s+t) + c_u\rangle|} ds\right)^{\frac{1}{2}} \\ &\quad + \int_0^T |\langle N_u y_u(s+t) + c_u, y_u(s+t)\rangle| ds.\end{aligned}$$

Since
$\int_0^T |\langle N_u y_u(s+t) + c_u, y_u(s+t)\rangle| ds \leq \left(1 + L_u\|y_0\|^2 + \|c_u\|\|y_0\|\right) \int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|^2}{1 + |\langle y_u(s+t), N_u y_u(s+t) + c_u\rangle|} ds,$

by Schwartz's inequality, we get

$\int_0^T |\langle N_u y_u(s+t)+c_u, y_u(s+t)\rangle| ds \le T\left(1+L_u\|y_0\|^2+\|c_u\|\|y_0\|\right)\left(\int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t)+c_u\rangle|^2}{1+|\langle y_u(s+t), N_u y_u(s+t)+c_u\rangle|} ds\right)^{\frac{1}{2}}.$

We deduce that

$$\int_0^T |\langle N_u S_u(s) y_u(t)+c_u, S_u(s) y_u(t)\rangle| ds \le \mathcal{M}\left(\int_0^T \frac{|\langle y_u(s+t), N_u y_u(s+t)+c_u\rangle|^2}{1+|\langle y_u(s+t), N_u y_u(s+t)+c_u\rangle|} ds\right)^{\frac{1}{2}},$$

where $\mathcal{M} = \left(\rho T^{\frac{3}{2}} \mathcal{C}_{(\|y_0\|;c_u)}\right) + T(1+L_u\|y_0\|^2+\|c_u\|\|y_0\|)$. This gives the estimate (12).

Remark 4.1 For c=0, we obtain the same content found in the homogeneous case, see [2], $\mathcal{M} = \rho T^{\frac{3}{2}}\left(2L_u\|y_u(0)\|\right)\left(L(N_u)\|y_u(0)\|\right) + T(1+L_u\|y_u(0)\|^2)$.

Theorem 4.2 *Let A generate a C_0-semigroup $S_u(t)$, and suppose that the following conditions hold:*
1. $S_u(t)$ is a contraction semigroup;
2. there exist $\delta, T > 0$ such that

$$\int_0^T |\langle N_u S_u(s) y_u(t)+c_u, S(s) y_u(t)\rangle| ds \ge \delta \|y_u(t)\|, \quad \forall y_u \in H_u. \tag{15}$$

Then the feedback (5) exponentially stabilizes the system (6).
More precisely, there exists $\beta_u > 0$ such that $\|y_u(t)\| \le e^{-\beta_u}\|y_0\| e^{-\frac{\beta_u}{T}(t)}$ $\forall t > 0$.

Proof. Integrate now the following inequality over the interval $[kT, (k+1)T]$, for $k \in \mathbb{N}$ and $T > 0$,

$$\frac{d}{dt}\|y_u(t)\|^2 \le -2\rho \frac{(\langle y_u(t); N_u y(t)+c_u\rangle)^2}{|\langle y_u(t), N_u y(t)+c_u\rangle|+1}; \forall t \in [0,T].$$

We get $\|y_u((k+1)T)\|^2 - \|y_u(kT)\|^2 \le -2\rho \int_{kT}^{(k+1)T} \frac{|\langle y(\tau), N_u y(\tau)+c_u\rangle|^2}{|\langle y_u(\tau), N_u y_u(\tau)+c_u\rangle|+1} \, \mathrm{d}\tau$.
Using now the estimate (12), we deduce that

$$\|y_u((k+1)T)\|^2 - \|y_u(kT)\|^2 \le \frac{-2\rho}{\mathcal{M}}\left(\int_0^T |\langle S_u(\tau) y_u(t), N_u S_u(\tau) y_u(t)+c_u\rangle| \, \mathrm{d}\tau\right)^2.$$

According to the inequality (15), we have

$$\|y_u((k+1)T)\|^2 - \|y_u(kT)\|^2 \le \frac{-2\rho\delta^2}{\mathcal{M}}\|y_u(kT)\|^2. \tag{16}$$

Letting $s_k = \|y(kT)\|^2$, $k \in \mathbb{N}$, the inequality (16) can be written as

$$s_{k+1} - s_k \le \frac{-2\rho\delta^2}{\mathcal{M}} s_k, \quad \forall k \ge 0,$$

$$s_{k+1} \le C s_k, \quad \forall k \ge 0,$$

where $C = \left(1 - \frac{2\rho}{\mathcal{M}}\delta^2\right) < 1$, which gives $s_k \le e^{-k\ln\frac{1}{C}} s_0$.

So, $\|y_u(t)\| \le e^{-\frac{\ln(\frac{1}{C})}{2}}\|y_0\| e^{-\frac{\ln(\frac{1}{C})}{2T}t}$ for all $t \ge 0$, $\|y_u(t)\| \le e^{-\beta_u}\|y_0\| e^{-\frac{\beta_u}{T}t}$ for all $t \ge 0$, where $\beta_u = \frac{\ln(\frac{1}{C})}{2} > 0$.

4.4 Exponential stabilization

Theorem 4.3 *Suppose that the assumptions of both Theorems 4.1 and 4.2 are verified. Then the feedback (2) exponentially stabilizes the system (1). More precisely, there exist* $\beta > 0$ *and* $\alpha > 0$ *such that* $\|y(t)\| \leq \alpha e^{-\beta t}, \forall t \geq 0$.

Proof. Using Proposition 3.1, we deduce that the system (1) admits a unique global mild solution $y(t)$; according to the decomposition (8), we have $y(t) = y_u(t) + y_s(t)$.
It follows from Theorems 4.1 and 4.2 that

$$\begin{aligned} y(t) &\leqslant e^{-\beta_u}\|y_0\|e^{-\frac{\beta_u}{T}t} + K_\eta\|y_s(0)\|e^{-\beta_s t} \\ &\leqslant 2(e^{-\beta_u}\ \|y_0\| + K_\eta\|y_s(0)\|)e^{-min(e^{-\frac{\beta_u}{T}},\beta_s)t}. \end{aligned}$$

So, $\|y(t)\| \leq \alpha e^{-\beta t},\ \forall t \geq 0$, where $\alpha = 2(e^{-\beta_u}\ \|y_0\| + K_\eta\|y_s(0)\|)$ and $\beta = min(\frac{\beta_u}{T}, \beta_s)$.

5 Strong Stabilisation

Theorem 5.1 *Let* A *generate a semigroup* $S(t)$ *of contractions on* H. *Suppose that*
(i) N *is locally Lipschitz;*
(ii) $\exists \delta, T > 0$ *such that*

$$\int_0^T |\langle NS(s)y(t) + c, S(s)y(t)\rangle| \mathrm{d}s \geq \delta\|y(t)\|^2, \quad \forall y \in H. \tag{17}$$

Then the feedback (2) strongly stabilises the system (1) with the following decay estimate:

$$\|y(t)\| = O\left(t^{-\frac{1}{2}}\right) \text{ as } t \to +\infty.$$

Proof. If $H = H_u$ is of finite dimension, then we retrieve the result of Theorem 4.2. In the case $\dim H_u = +\infty$, following the techniques used in the proof of Lemma 4.1, we can obtain the following estimate when $t \to +\infty$:

$$\int_0^T |\langle S(\tau)y(t), NS(\tau)y(t) + c\rangle| d\tau = \mathcal{O}\left(\sqrt{\int_t^{t+T} \frac{|\langle y(\tau), Ny(\tau) + c\rangle|^2}{1 + |\langle Ny(\tau) + c, y(\tau)\rangle|} d\tau}\right). \tag{18}$$

Integrating now the inequality $\frac{d}{dt}\|y(t)\|^2 \leq -2\rho\frac{(\langle y(t);Ny(t)+c\rangle)^2}{|\langle y(t),Ny(t)+c\rangle|+1}; \forall t \in [0, T]$, over the interval $[kT, (k+1)T]$, for $k \in \mathbb{N}$ and $T > 0$, we get

$$\|y((k+1)T)\|^2 - \|y(kT)\|^2 \leq -2\rho \int_{kT}^{(k+1)T} \frac{|\langle y(\tau), Ny(\tau) + c\rangle|^2}{|\langle y(\tau), Ny(\tau) + c\rangle| + 1}\ \mathrm{d}\tau.$$

Using now the estimate (18), we deduce that

$$\|y((k+1)T)\|^2 - \|y(kT)\|^2 \leq \frac{-2\rho}{\mathcal{M}}\left(\int_0^T |\langle S(\tau)y, NS(\tau)y + c\rangle| \mathrm{d}\tau\right)^2.$$

From (17), we have $\|y((k+1)T)\|^2 - \|y(kT)\|^2 \leq \frac{-2\rho\delta^2}{\mathcal{M}}\|y(kT)\|^4$.
Letting $s_k = \|y(kT)\|^2, k \in \mathbb{N}$, the last inequality can be written as

$$s_{k+1} \leq s_k - \frac{2\rho\delta^2}{\mathcal{M}}s_k^2, \quad \forall k \geq 0.$$

Using the fact that $t \mapsto \|y(t)\|$ is a decreasing function on $[0, +\infty[$, we get

$$s_{k+1} \leq s_k - \frac{2\rho\delta^2}{\mathcal{M}} s_{k+1}^2, \quad \forall k \geq 0.$$

The last inequality can be written as follows: $s_{k+1} \leq s_k - Cs_{k+1}^2, \quad \forall k \geq 0$, where $C = \frac{2\rho\delta^2}{\mathcal{M}} > 0$. Now, to obtain the decay rate for solutions of (1), we recall the following lemma, see [14] and [15].

Lemma 5.1 *Let the sequence of non-negative real numbers s_k, $k = 0, 1, 2, \ldots$, satisfy $s_{k+1} \leqslant s_k - C(k+1)^r s_{k+1}^2$, where C is a positive real number and r is a non-negative integer. Then there exists a positive number $M = M(M, r, u_0)$ such that $s_k \leqslant \frac{M}{(k+1)^{r+1}}$, $k = 0, 1, 2, 3, \ldots$.*

So, from the lemma and for $r = 0$, there exists a positive constant K (depending on C) such that $s_k \leq \frac{M}{k+1}$, so $\|y(kT)\|^2 \leq \frac{M}{k+1}$. For $k = E(\frac{t}{T})$, $\big(E\left(\frac{t}{T}\right)$ designed the integer part of $\frac{t}{T}\big)$, we obtain

$$\|y(E(\frac{t}{T})T)\|^2 \leq \frac{M}{E(\frac{t}{T}) + 1}.$$

Using the fact that $E(\frac{t}{T})T \leq t$ and $t \mapsto \|y(t)\|$ is a decreasing function on $[0, +\infty[$, we get $\|y(t)\|^2 \leq \frac{TM}{t}$. So, $\|y(t)\| = \mathcal{O}\left(t^{-\frac{1}{2}}\right)$ as $t \to +\infty$.

6 Applications

Example 6.1 One-dimensional heat equation.
Let us consider the following semilinear heat equation:

$$\begin{cases} \frac{\partial y(x,t)}{\partial t} = \frac{\partial^2 y(x,t)}{\partial x^2} + v(y(t))(Ny(t) + c), & x \in (0,1), t > 0, \\ \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0, & \forall t > 0, \end{cases} \tag{19}$$

where $y(t)$ is the temperature profile at time t. Here we take the state space $H = L^2(0,1)$ and the operator A is defined by

$$Ay = \frac{\partial^2 y}{\partial x^2} \text{ with } \mathcal{D}(A) = \left\{ y \in H^2(0,1) \mid \frac{\partial y(0,t)}{\partial x} = \frac{\partial y(1,t)}{\partial x} = 0 \right\}.$$

The domain of A gives the homogeneous Neumann boundary condition imposed at the ends of the bar, which requires specifying how the heat flows out of the bar and means that both ends are insulated. The control $v(y(t))$ is defined by

$$v(y(t)) = -\rho \frac{\langle y(t), Ny(t) + c \rangle}{|\langle y(t), Ny(t) + c \rangle| + 1}, \quad \forall t \geq 0. \tag{20}$$

The operator of control N is defined by $Ny = \frac{1}{1+\|y\|} \sum_{j=1}^{+\infty} \alpha_j \langle y, \varphi_j \rangle \varphi_j, \alpha_j \geq 0, \forall j \geq 1$, such that $\sum_{j=1}^{+\infty} \alpha_j^2 < \infty$; N is a nonlinear sequentially continuous and locally Lipschitz operator such that $N(0) = 0$.

The spectrum of A is given by the simple eigenvalues $\lambda_j = -\pi^2(j-1)^2, j \in \mathbb{N}^*$, and eigenfunctions $\varphi_1(x) = 1$ and $\varphi_j(x) = \sqrt{2}\cos((j-1)\pi x)$ for all $j \geq 2$. Then the subspace H_u is the one-dimensional space spanned by the eigenfunction φ_1, and we have $S_u(t)y_u = \langle y_u, \varphi_1\rangle \varphi_1$, so $S_u(t) = I_{H_u}$ (the identity) and hence $(S_u(t))_{t\geq 0}$ is a semigroup of isometries.

N_u is the restriction of the operator N on H_u defined by

$$\begin{aligned} N_u y_u(t) &= \tfrac{\alpha_1 y_u(t)}{1+\|y_u(t)\|} \\ &= \tfrac{\alpha_1}{1+|\langle y_u(t),\varphi_1\rangle|}\langle y_u(t), \varphi_1\rangle \varphi_1, \end{aligned}$$

N_u is a nonlinear sequentially continuous and locally Lipschitz operator,

$$\begin{aligned} \|N_u y_u(t) - N_u z_u(t)\| &= \|\tfrac{\alpha_1 y_u(t)}{1+\|y_u(t)\|} - \tfrac{\alpha_1 z_u(t)}{1+\|z_u(t)\|}\| \\ &\leqslant \tfrac{\alpha_1 y_u(t)}{1+\|y_u(t)\|} - \tfrac{\alpha_1 z_u(t)}{1+\|y_u(t)\|} + \tfrac{\alpha_1 z_u(t)}{1+\|y_u(t)\|} - \tfrac{\alpha_1 z_u(t)}{1+\|z_u(t)\|} \\ &\leqslant \|\tfrac{\alpha_1}{1+\|y_u(t)\|}(y_u(t) - z_u(t))\| + \|\alpha_1 z_u(t)\big(\tfrac{1}{1+\|y_u(t)} - \tfrac{1}{1+\|z_u(t)\|}\big)\| \\ &\leqslant L_u\|y_u(t) - z_u(t)\|, \text{ where } L_u = |\alpha_1|. \end{aligned}$$

Here we can see that $NH^u \subset H^u$. We have

$$\begin{aligned} < NS_u(\tau)y_u(t) + c_u, S_u(\tau)y_u(t) >_H | &= | < N_u y_u(t), y_u(t) > + < c_u, y_u(t) > | \\ &= \left(\tfrac{\alpha_1\|y_u(t)\|}{1+\|y_u(t)\|} + \|c_u\|\right)\|y_u(t)\| \end{aligned}$$

if $\alpha_1 > 0$, we have $\left(\frac{\alpha_1\|y_u(t)\|}{1+\|y_u(t)\|} + \|c_u\|\right)\|y_u(t)\| > \|c_u\|\|y_u(t)\|$, then
$| < NS_u(\tau)y_u(t) + c_u, S_u(\tau)y_u(t) > | > \|c_u\|\|y_u(t)\|$, so
$\int_0^T | < NS_u(\tau)y_u(t) + c_u, S_u(\tau)y_u(t) > |d\tau > \delta\|y_u(t)\|$ with $\delta = T\|c_u\|$.

We can see that (15) holds, and the assumptions of Theorem 4.1 are verified.
For $j \geq 2$, the subspace H_s is spanned by the eigenfunctions $(\varphi_j)_{j\geq 2}$; N_s is the restriction of the operator N on H_s defined by $N_s y_s = \frac{1}{1+\|y_s\|}\sum_{j=2}^{+\infty}\alpha_j \langle y_s, \varphi_j\rangle \varphi_j$.

We have $S_s(t) = \sum_{j=2}^{\infty} e^{-\pi^2(j-1)^2 t}$. Since $j-2 \geq 0$, one has $e^{-2\pi^2 j(j-2)t} \leqslant 1$, then $\|S_s(t)\| \leq \mathrm{e}^{-\pi^2 t}$, $\forall t \geq 0$.

So, A_s satisfies the spectrum-determined growth assumption.

$$\text{If } \rho < \frac{\pi^2}{(L_s\|y_s(0)\| + \|c_s\|)(L_u\|y_s(0)\| + \|c_s\|)}, \text{ where } L_u = \alpha_1 \text{ and } L_s = \sum_{j=2}^{+\infty}\alpha_j^2,$$

then the assumptions of Theorem 4.2 are verified.

Finally, the assumptions of both Theorems 4.1 and 4.2 are verified. Then by applying Theorem 4.3, we deduce that (19) is exponentially stabilizable by the control

$$v(y(t)) = -\rho\frac{(\alpha_1 + |c_u|)y_u(t)^2 + |c_u|y_u(t)}{(1 + \alpha_1 + |c_u|)y_u(t)^2 + (1 + |c_u|)y_u(t)}, \quad \forall t \geq 0.$$

7 Conclusion

In this work, the sets of necessary and sufficient conditions for the exponential stabilization of inhomogeneous semilinear systems are given. The stabilizing controls may be chosen bounded with respect to time and initial states and can be applied to systems subject to constraints on the control input. Though the exponential stabilization of bounded operators enables us to discuss various stabilization problems, the present study does not cover other situations. This is the case of exponential stabilization of unbounded operators in a Banach space. Also, the issue of unbounded operator control is of great interest.

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Harvesting Strategies in the Migratory Prey-Predator Model with a Crowley-Martin Type Response Function and Constant Efforts

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Abstract: This paper deals with the dynamics of prey and predator populations in the permitted and prohibited areas of harvesting with a Crowley-Martin response function. The predator can migrate easily into both areas. The prey and predator populations in the permitted area are harvested with constant efforts. The existence and stability of the interior equilibrium point are studied. The stable interior equilibrium point is connected with maximum profit. The stability of the interior equilibrium point is analysed locally using the linearization method and eigenvalues. Due to the complexity, the simulation is carried out using the relevant parameter values to determine the existence of a stable interior equilibrium point and profit function. From simulation, there exists an ordered pair of harvesting efforts that gives a stable interior equilibrium point and also maximizes the profit function. Harvesting in prey and predator populations in the permitted area can prevent the populations from extinction and also provide maximum sustainable profit. The trajectories of prey and predator populations are plotted to visualize the dynamical behaviour for a given span of time. The surface of profit function is also plotted to view the maximum profit.

Keywords: *prey-predator; harvesting effort; migration; stability; Crowley-Martin; maximum profit.*

Mathematics Subject Classification (2010): 34Dxx, 70Kxx, 93-XX.

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1 Introduction

Mathematical modelling has been applied in various fields of studies including biology, ecology, epidemiology, economics, and many other fields, see [1], [2], [3], [4]. The model aims to explain the real phenomena from the mathematical aspect [5] and is also used to make prediction for the future, see [6]. A dynamical population is one of the research objects in the modelling. The growth rates of populations and their interaction are still a concern of researchers, including the dynamics of prey-predator populations living at the same area. The dynamics of populations were not only affected by their growth rate and interaction but also by some other factors like the death rate, competition, predation response, harvesting, and migration, see [7]. The dynamical population analysis not only studies and predicts sustainability but also considers social and economic aspects, see [8].

In the prey-predator model, one thing that is very important is the form of interaction between the prey and the predator, known as the predation function. Some of the predation functions often used in prey-predator models are of the Holling type, Holling-Tanner type, Mechaelis-Menten type, and Leslie-Gower type, see [5], [9]. The Beddington-DeAngelis type as another type of functional response is used as a control to stabilize the interaction of the prey and the predator, see [10]. The use of these types of functional response is dependent on the characteristic of the prey and the predator. The Crowley-Martin response function is influenced by the predator density, catch rate, handling time, and the magnitude of disturbance among predators, see [11]. The prey-predator model with the Crowley-Martin response function has been applied for many purposes, see [12].

The Crowley-Martin response function was also applied to predict the dynamics of a phytoplankton-zooplankton system [13]. There are prey-predator models which consider two identical areas and populations can migrate to these areas. Some of the population models are useful, for example, a fish population model in fisheries management, when the populations are harvested in various ways and policies. There is a policy in the fisheries management where the population in an area is prohibited from being harvested while in the other area it is permitted. Several policies in harvesting include selective harvesting, harvesting with constant quotas, harvesting with constant effort. Harvesting activities in population dynamics have economic consequences. The populations are not only managed to be sustainable but also strived to provide the maximum benefit. In some prey-predator models, only the prey populations are harvested or only the predator populations are harvested, see [14]. There are also other researchers who considered both prey and predator populations to be harvested, see [15].

In this paper, we consider a prey-predator model with the Crowley-Martin response function in an ecosystem which is divided into two areas, namely, an area where fishing is permitted and other area where fishing is prohibited. The prey population can migrate into both areas. The modeled populations are the population of butini fish (*Glossogobius matanensis*) as the prey and the population of nila fish (*Oreochromis nilotichus*) as the predator. The butini is an endemic and native fish found in several lakes of East Luwu district, South Sulawesi province, Indonesia, see [16]. In this model, the nila fish as the predator is divided into two compartments according to where the fish is located. Both populations are allowed to be harvested in the permitted area. The model formed is a system of nonlinear differential equations and the constant harvesting efforts are used for both populations. The local stability is analyzed using the linearization method. Maximum profit is evaluated over a certain range of effort values. The surface of the profit function is given to visualize the maximum profit.

2 Material and Method

This research involves a prey-predator population model following the Crowley-Martin response function. The populations considered in this study are butini fish and nila fish that live in several lakes in East Luwu district. Both sets of fish can only be harvested in the permitted area. The location of the two sets of fish is divided into the permitted and prohibited areas for fishing, where the nila fish population can migrate into the two areas. The populations are divided into three compartments, namely, butini fish, nila fish that live in the prohibited area to be harvested, and the nila fish that live in the permitted area to be harvested. The growth models of the three compartments are expressed in the form of an autonomous system of nonlinear differential equations.

The interior equilibrium point of the model is confirmed and then stability analysis is carried out using the linearization method and checking the eigenvalues of the Jacobian matrix resulting from the linearized model around the interior equilibrium point. The butini fish and nila fish are harvested in the permitted area with constant harvesting efforts. In order to get the profit function which is the consequence of fishing activity, the cost function and revenue function should be defined. The profit function (π) is given by $\pi = TR - TC$, based on the total revenue function (TR), $TR = p_1E_1B^* + p_2E_2M^*$ and the total cost function (TC), $TC = c_1E_1 + c_2E_2$. The parameter E_i represents the harvesting efforts, p_i represents the price of fish catch per unit, and c_i represents the cost of fishing activities, where $i = 1, 2$.

The prey-predator population model is a nonlinear system and the interior equilibrium point cannot be stated explicitly. In order to perform the analysis, the parameter values of the model were used being partially obtained from data collection for the fish populations. Some of the relevant parameter values are obtained from various references and some other are assumed. The various ordered pairs of the harvesting efforts are taken within a range of values to get the interior equilibrium points. Therefore, stability of the equilibrium points and profit value are determined. From the simulation, we determine the ordered pair of efforts that gives the stable interior equilibrium point and maximize the profit.

3 Results and Discussion

3.1 Predator-prey population model

The dynamics of the predator and prey population with the Crowley-Martin response function is expressed in the form of a system of nonlinear differential equations. The environment in which the population lives is divided into two areas, the permitted and prohibited areas for harvesting. The predator population is divided into two compartments, depending on where the predator live. The predator can migrate between the two areas. The prey population is assumed to follow logistic growth. The predator and prey populations are harvested in the permitted area with constant harvesting efforts. The interaction between the prey and predator populations is shown in the following Figure 1. The growth rates of the prey-predator population with their interaction are stated in the system of nonlinear differential equations.

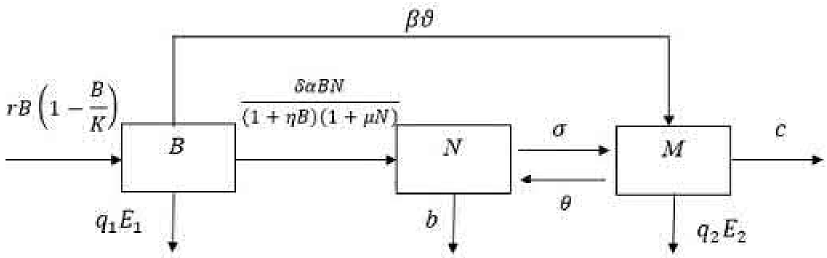

Figure 1: Interaction Diagram for the Prey and Predator Populations.

$$\frac{dB}{dt} = rB(1-\frac{B}{K}) - \frac{\alpha NB}{(1+\eta B)(1+\mu N)} - \beta MB - q_1E_1B, \tag{1}$$

$$\frac{dN}{dt} = \frac{\delta\alpha NB}{(1+\eta B)(1+\mu N)} - bN - \sigma N + \theta M, \tag{2}$$

$$\frac{dM}{dt} = \vartheta MB - cM + \sigma N - \theta M - q_2E_2M. \tag{3}$$

The symbol B is the size of the butini fish population as the prey, N and M state the size of the predator population in the permitted and prohibited area at time t, respectively. All parameters of the model are assumed to be positive. The description, meaning, and units of the parameters can be found in the related references, see [5]. For simplicity, we take $q_1 = q_2 = 1, r_1 = r - E_1, r_2 = b + \sigma$, and $r_3 = c + \theta + E_2$. Thus, the model (1, 2, 3) is rewritten as

$$\frac{dB}{dt} = r_1B(1-\frac{B}{K}) - \frac{\alpha NB}{(1+\eta B)(1+\mu N)} - \beta MB, \tag{4}$$

$$\frac{dN}{dt} = \frac{\delta\alpha NB}{(1+\eta B)(1+\mu N)} - r_2N + \theta M, \tag{5}$$

$$\frac{dM}{dt} = \vartheta MB - r_3M + \sigma N. \tag{6}$$

3.2 Equilibrium points and stability analysis

The possible non negative equilibrium points for the model (4, 5, 6) are $T_1 = (0,0,0)$, $T_2 = (K,0,0)$, $T_3 = (\omega, \frac{\omega\alpha_1+\alpha_2}{\mu\beta_1\alpha_2}, \frac{\sigma\omega\alpha_1-\alpha_2}{\mu\beta_1\beta_2\alpha_2})$, where ω are the roots of the equation $\delta\eta\mu r_1r_2\vartheta Z^5 + (-K\delta\eta\mu r_1r_2\vartheta^2 + \sigma\delta\eta\mu r_1\theta\vartheta - 2\delta\eta\mu r_1r_2r_3\vartheta + \delta\mu r_1r_2\vartheta^2)Z^4 + (-K\sigma\delta\eta\mu r_1\theta\vartheta + 2K\delta\eta\mu r_1r_2r_3\vartheta - K\alpha\beta\sigma\delta^2\vartheta + K\beta\sigma\delta\eta r_2 - K\delta\mu r_1r_2\vartheta^2 - \sigma\delta\eta\mu r_1r_3\vartheta + \delta\eta\mu r_1r_2r_3^2 - 2\delta\mu r_1r_2r_3\vartheta)Z^3 + (K\sigma\delta\eta\mu r_1r_3\theta - K\delta\eta\mu r_1r_2r_3^2 + K\alpha\beta\delta^2r_3 - K\beta\sigma^2\delta\eta\theta - K\beta\sigma\delta\eta r_1r_2 - K\delta\mu r_1r_2r_3^2 + \delta\mu r_1r_2r_3^2 - Kr_3^2\vartheta^2)Z^2 + (K\sigma\delta\mu r_1r_3\theta - K\delta\mu r_1r_2r_3^2 - K\alpha\sigma\delta r_3\theta + K\alpha\delta r_3^2r_2 + K\beta\sigma^2\delta\theta - K\beta\sigma\delta r_2r_3 - K\sigma^2\eta\theta^2 + 2K\sigma\eta\mu r_2r_3\theta - K\eta r_2^2r_3^2 - 2K\delta r_2\theta\vartheta + 2Kr_2^2r_3\vartheta)Z - K\delta^2\theta^2 + 2K\delta r_2r_3\theta - Kr_2^2r_3^2$, $\alpha_1 = \omega\eta r_2\vartheta - \omega\alpha\delta\vartheta - \alpha\delta r_3 + \sigma\eta\theta - \eta r_2r_3$, $\alpha_2 = \omega r_2\vartheta + \sigma\theta - r_2r_3$, $\beta_1 = \eta\omega + 1$, and $\beta_2 = \omega\vartheta - r_3$.

We focus to analyze the equilibrium point T_3 which is located in the first octant when $\omega > 0, \alpha_1 + \alpha_2 > 0$, and $\sigma\omega\alpha_1 > \alpha_2 > 0$. Because of the complexity of the system, we just consider the local stability of the interior equilibrium point.

3.3 Bionomic equilibrium and maximum profit

The bionomic equilibrium point is a condition where $\frac{dB}{dt} = \frac{dM}{dt} = \frac{dN}{dt} = 0$ and $\pi = 0$. The only interior equilibrium point satisfies the condition $T_3 = (\omega, \frac{\omega\alpha_1+\alpha_2}{\mu\beta_1\alpha_2}, \frac{\sigma\omega\alpha_1-\alpha_2}{\mu\beta_1\beta_2\alpha_2})$ which can be written in terms of E_1 and E_2 so that it becomes $T_3 = (B^*, N^*, M^*) = (\omega(r-E_1), \frac{\omega\alpha_1+\alpha_2(b+\sigma)}{\mu\beta_1\alpha_2}, \frac{\sigma\omega\alpha_1-\alpha_2(c+\theta+E_2)}{\mu\beta_1\beta_2\alpha_2})$. The total revenue function (TR) obtained from harvesting of the populations B and M evaluated at the equilibrium point T_3 is given by $TR(B^*, N^*, M^*) = TR(B^*)+TR(M^*) = p_1E_1B^*+p_2E_2M^*$. After substituting the values of B^* and M^* in the state of equilibrium, we get

$$TR = p_1\omega rE_1 - p_1\omega E_1^2 + \frac{p_2(\sigma\omega\alpha_1-\alpha_2)(c+\theta)E_2}{\mu\beta_1\beta_2\alpha_2} + \frac{p_2(\sigma\omega\alpha_1-\alpha_2)E_2^2}{\mu\beta_1\beta_2\alpha_2}.$$

The total cost function (TC) can be expressed as $C = c_1E_1 + c_2E_2$. Furthermore, the profit function (π) is given as

$$\pi = (p_1\omega r-c_1)E_1-p_1\omega E_1^2+\frac{(p_2(\sigma\omega\alpha_1-\alpha_2)(c+\theta)-\mu\beta_1\beta_2\alpha_2c_3)E_2}{\mu\beta_1\beta_2\alpha_2}+\frac{p_2(\sigma\omega\alpha_1-\alpha_2)E_2^2}{\mu\beta_1\beta_2\alpha_2}. \tag{7}$$

The profit function (7) now depends on the efforts E_1, E_2, and the parameter ω which is a positive root of the polynomial of degree five and cannot be written explicitly. The value of ω also depends on the efforts E_1 and E_2. As a standard procedure to get the maximum value of profit, we need to get the stationary points via the first partial derivative with respect to E_1 and E_2. Since ω cannot be stated in terms of E_1 and E_2, we evaluate the value of profit by taking various values of ordered pairs (E_1, E_2) and determine the interior equilibrium point T_3 and its stability by showing the eigenvalues. The eigenvalues are related to the Jacobian matrix evaluated at the equilibrium point T_3. Furthermore, the profit function at each value of ordered pairs (E_1, E_2) can be determined. In this study, we restrict the value of efforts as $0 \leq E_1, E_2 \leq E_{max}$, and $E_{max} = 1$. The ordered pair (E_1, E_2) to be considered is the ordered pair that gives the interior equilibrium point T_3 and is stable.

3.4 Simulation

In order to simulate the profil function, we set the values of parameters for the model equations (4), (5, (6) as follows: $K = 100, r = 0.7, \alpha = 0.3, \eta = 0.01, \mu = 0.01, \beta = 0.1, \vartheta = 0.01, \delta = 0.03, \sigma = 0.25, \theta = 0.25, b = 0.2$, and $c = 0.1$, see [5,17]. The values of these parameters are partly based on data collection. The interior equilibrium point T_3 with various ordered pairs of (E_1, E_2) are given in Table 1.

The various ordered pairs of the efforts $(E1, E2)$ give the interior equilibrium point T_3 and the stability is determined by inspection of the real part of eigenvalues. The equilibrium point is asymptoticaly stable when the real parts of eigenvalues are negative. The ordered pairs of efforts and eigenvalues of the interior equilibrium point T_3 are given in Table 2.

In order to simulate and determine the profit, we set the values of parameters related to the total revenue and total cost, namely, $p_1 = 3.5, p_2 = 1.3, c_1 = 0.5$ and $c_2 = 0.3$ in appropriate units. Together with the various values of ordered pairs of efforts, we determine the profit evaluated at the equilibrium point T_3 following the formula $\pi(E_1, E_2) = p_1B^*E_1 + p_2M^*E_2 - (c_1E_1 + c_2E_2)$. The ordered pairs of efforts and profit are given in Table 3.

Table 1: Ordered Pairs of Efforts and Interior Equilibrium Point T_3.

$E1/E2$	0	0.1	0.2	0.3
0	15.94, 1.52, 1.99	23.01, 1.52, 1.73	29.69, 1.51, 1.48	35.97, 1.47, 1.27
0.1	15.93, 1.26, 1.65	22.98, 1.23, 1.40	29.65, 1.19, 1.18	35.91, 1.14, 0.98
0.2	15.92, 1.00, 1.31	22.95, 0.95, 1.08	29.61, 0.89, 0.87	35.85, 0.81, 0.70
0.23	15.91, 0.92, 1.21	22.95, 0.87, 0.98	29.60, 0.80, 0.78	35.83, 0.72, 0.61
0.235	15.91, 0.91, 1.19	22.94, 0.85, 0.97	29.59, 0.78, 0.77	35.82, 0.70, 0.60
0.3	15.91, 0.74, 0.97	22.93, 0.67, 0.76	29.57, 0.58, 0.57	35.78, 0.48, 0.42
0.4	15.89, 0.48, 0.63	22.91, 0.39, 0.44	29.53, 0.28, 0.27	35.72, 0.16, 0.14
0.5	15.88, 0.22, 0.29	22.88, 0.11, 0.12	-	-

$E1/E2$	0.4	0.5	0.6	0.7
0	41.81, 1.43, 1.08	47.21, 1.38, 0.91	52.11, 1.32, 0.77	56.54, 1.26, 0.65
0.1	41.72, 1.08, 0.81	47.07, 1.01, 0.66	51.94, 0.93, 0.54	56.33, 0.85, 0.43
0.2	41.63, 0.73, 0.54	46.95, 0.63, 0.41	51.78, 0.54, 0.31	56.12, 0.44, 0.23
0.23	41.61, 0.62, 0.46	46.91, 0.52, 0.34	51.73, 0.42, 0.24	56.06, 0.32, 0.16
0.235	41.60, 0.61, 0.45	46.90, 0.51, 0.33	51.72, 0.40, 0.23	56.05, 0.29, 0.15
0.3	41.54, 0.38, 0.28	46.83, 0.26, 0.17	51.61, 0.15, 0.08	55.92, 0.03, 0.01
0.4	41.45, 0.03, 0.02	-	-	-

$E1/E2$	0.8	0.9	1
0	60.51, 1.19, 0.55	64.05, 1.13, 0.46	67.18, 1.07, 0.39
0.1	60.25, 0.76, 0.35	63.74, 0.69, 0.28	66.82, 0.61, 0.22
0.2	60.01, 0.34, 0.15	63.44, 0.24, 0.10	66.47, 0.16, 0.05
0.23	59.92, 0.21, 0.09	63.34, 0.12, 0.04	66.36, 0.02, 0.009
0.235	59.91, 0.19, 0.08	63.33, 0.09, 0.03	66.34, 0.002, 0.0009

Table 2: Ordered Pairs of Efforts and Eigenvalues of Interior Equilibrium Point T_3.

E_1/E_2	0	0.1	0.2	0.3
0	-0.02 ± 0.28I, -0.53	-0.03 ± 0.31I, -0.52	-0.05 ± 0.32I, -0.52	-0.07 ± 0.31I, -0.52
0.1	-0.02 ± 0.25I, -0.53	-0.04 ± 0.27I, -0.52	-0.06 ± 0.28I, -0.52	-0.08 ± 0.27I, -0.52
0.2	-0.03 ± 0.22I, -0.53	-0.05 ± 0.24I, -0.51	-0.07 ± 0.24I, -0.51	-0.09 ± 0.22I, -0.52
0.23	-0.03 ± 0.21I, -0.51	-0.05 ± 0.23I, -0.51	-0.07 ± 0.22I, -0.51	-0.09 ± 0.21I, -0.52
0.235	-0.03±0.21I, -0.53	-0.05±0.22I, -0.51	-0.07±0.22I, -0.51	-0.09±0.21I, -0.52
0.3	-0.03±0.19I, -0.52	-0.05± 0.21I, -0.52	-0.08± 0.18I, -0.51	-0.08±0.18I, -0.51
0.4	-0.04±0.15I, -0.52	-0.06±0.14I, -0.51	-0.09±0.11I, -0.51	-0.15, -0.08, -0.51
0.5	-0.05±0.09I, -0.52	-0.07±0.04I, -0.51	-	-

Table 3 shows that maximum profit is reached when the efforts $(E_1, E_2) = (0.235, 1)$ with $\pi_{max} = 217.05$. The profit becomes maximum when the predator population in the permitted area is harvested at the maximum level of efforts and the prey population is harvested at the level 0.235. The maximum profit occurs at the top of the surface of the profit function, as shown in Figure 2.

For the model without harvesting, we get an interior equilibrium point at the level

E_1/E_2	0.4	0.5	0.6	0.7
0	-0.09±0.31I, -0.54	-0.11±0.29I, -0.56	-0.13±0.27I, -0.58	-0.15±0.24I, -0.61
0.1	-0.11±0.26I, -0.53	-0.12±0.23I, -0.55	-0.14±0.21I, -0.58	-0.16±0.17I, -0.61
0.2	-0.07±0.24I, -0.51	-0.14±0.16I, -0.55	-0.16±0.11I, -0.58	-0.23, -0.12, -0.61
0.23	-0.12±0.17I, -0.53	-0.14±0.13I, -0.55	-0.01±0.05I, -0.58	-0.29, -0.07, -0.61
0.235	-0.12±0.17I, -0.53	-0.14±0.12I, -0.55	-0.16±0.03I, -0.58	-0.31, -0.06, -0.61
0.3	-0.13±0.11I, -0.52	-0.21, -0.09, -0.54	-0.31, -0.03, -0.57	-0.38, -0.006, -0.6
0.4	-0.27, -0.009, -0.52	-	-	-

E_1/E_2	0.8	0.9	1
0	-0.17±0.21I, -0.65,	-0.19±0.18I, -0.71,	-0.21±0.14I, -0.75
0.1	-0.18±0.12I, -0.65,	-0.20±0.04I, -0.70,	-0.32, -0.11, -0.76
0.2	-0.33, -0.06, -0.66	-0.39, -0.04, -0.71	-0.43, -0.02, -0.77
0.23	-0.36, -0.04, -0.66	-0.42, -0.02, -0.71	-0.45, -0.003, -0.77
0.235	-0.37, -0.03, -0.66	-0.42, -0.01, -0.72	-0.46, -0.0003, -0.77

Table 3: Ordered Pairs of Efforts and Profit.

$E1/E2$	0	0.2	0.4	0.6	0.8	1
0	0	54.62	103.91	142.36	171.36	193.88
0.1	3.29	60.67	112.26	152.66	183.20	206.97
0.2	5.472	64.64	117.70	159.27	190.74	215.27
0.23	5.907	65.43	118.79	160.54	191.96	216.84
0.235	5.971	65.55	118.94	160.72	192.01	217.05
0.3	6.532	66.53	120.21	162.23	-	-
0.4	6.476	66.362	119.82	-	-	-
0.5	5.308	-	-	-	-	-

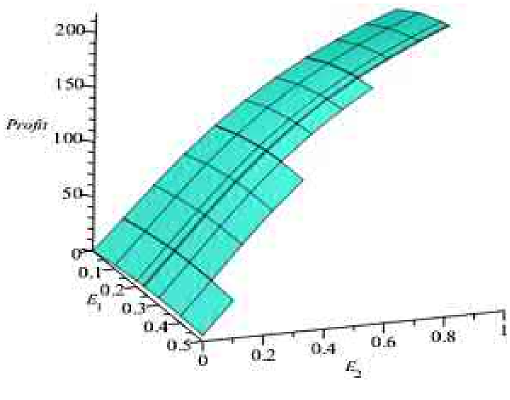

Figure 2: Surface of the Profit Function.

(15.94, 1.52, 1.99) and the related eigenvalues -0.02±0.28I, -0.53. This means that the prey (B) and the predators (M and N) will live sustainably. From Tables 1, 2 and 3, as the efforts of harvesting increase, the equilibrium points are still in the first octant and also remain stable but there are changes in the type of stability of the equilibrium point which are indicated by all the eigenvalues having negative real values. In addition, the value of the profit function also increases. When the prey and the predator are harvested at the level $(E_1, E_2) = (0.23, 1)$, the ordered pair of efforts gives an interior equilibrium point at the level (66.36, 0.02, 0.009), the eigenvalues -0.46, -0.0003, -0.77, and the profit at the level 216.84. The dynamics of the solution curve of the prey (B) and the predators (M and N) with the initial population $B(0) = 66.36, N(0) = 0.02$, and $M(0) = 0.009$ are shown in Figures 3, 4, 5.

Figures 3, 4, 5 show that with a given initial value of the prey and the predator

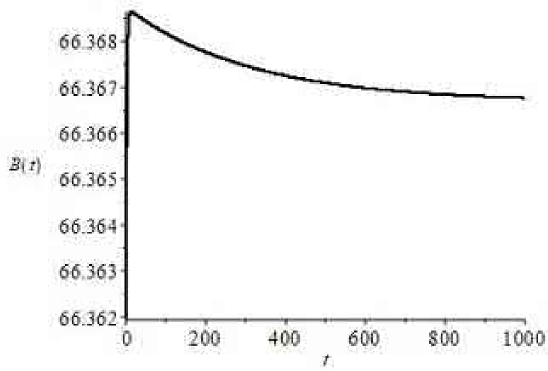

Figure 3: Solution Curve of Prey Population (B) with $t \in [0, 1000]$.

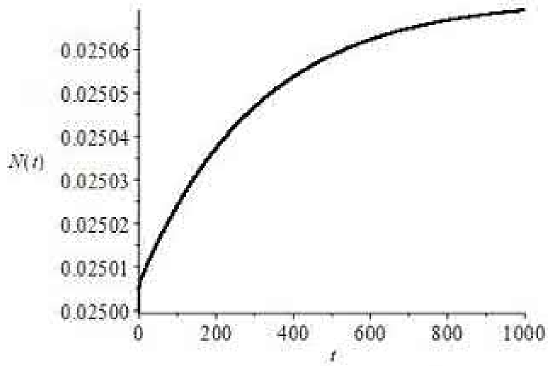

Figure 4: Solution Curve of Predator Population (N) with $t \in [0, 1000]$.

populations, there is initially little oscillatory motion. This is caused by the nonlinear term in the model and then the trajectories of the populations move monotonously toward the equilibrium point. It takes a long time to reach the equilibrium state. The ordered pair of efforts provides a stable interior equilibrium point and also almost maximizes the profit, the maximum profit is at the level 217.05.

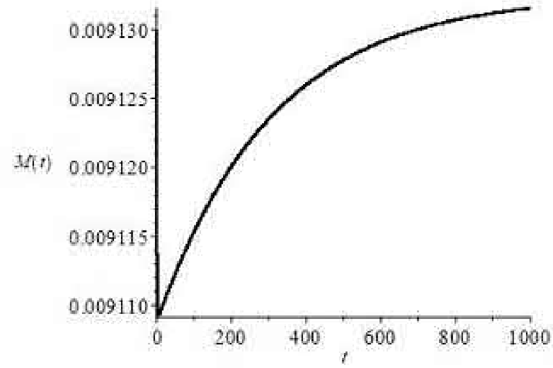

Figure 5: Solution Curve of Predator Population (M) with $t \in [0, 1000]$.

4 Conclusion

The model for growth of butini fish as the prey and nila fish as the predator in the permitted and prohibited areas for harvesting with the Crowley-Martin type response function and migration possibly has an interior equilibrium point. The prey and predator populations in the permitted area are harvested with constant efforts. The interior equilibrium point cannot be stated explicitly because of complexity of the nonlinear model. In order to get the maximum value of the profit function, several ordered pairs of harvesting efforts are evaluated to obtain a stable interior equilibrium point. Using the suitable parameter values and harvesting efforts, we get an ordered pairs of efforts that give a stable interior equilibrium point and maximize the profit.

The analysis and simulation show that if the level of harvesting effort for the prey population is increased, the equilibrium state for the prey and predator populations will decrease. This is because of the more prey populations are harvested, the lower number of prey populations exists. This has a consequence for predators having difficulty to get food, which results in the number of predator also decreasing. On the other hand, if the level of harvesting effort in the predator population is increased, this condition will result in a decrease in the number of predator population both in the permitted and prohibited areas for harvesting. This causes the prey population tending to increase because the number of predator population decreases. Harvesting with constant efforts for the prey and predator populations in the permitted area can obviously increase the number of the prey population, give maximum profit, and the populations also remain sustainable.

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Estimation of Forefinger Motion with Multi-DOF Using Advanced Kalman Filter

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Abstract: Data from the World Health Organization (WHO) of the year 2016 recorded that stroke cases were ranked second as a non-communicable disease that causes death, and the third leading cause of disability worldwide. Stroke can cause disability or weakness on one side of the body, including the upper limbs such as the fingers being difficult to move, so rehabilitation is required to restore the function of the hand. A finger arm robot is one solution to help accelerate the rehabilitation process specifically for finger movements. One of the efforts to develop a finger robot is finger motion estimation. It is started with the inverse kinematic modeling of the finger arm robot with 3 joints matching the structure of a human finger. One reliable estimation method frequently used is the Advanced Kalman Filter method. In this paper, the Advanced Kalman Filter is divided into two methods, that is, the Ensemble Kalman Filter (EnKF) and the Ensemble Kalman Filter Square Root (EnKF-SR). The focus of this paper is to estimate the fingers, especially the index finger of the left hand, using the EnKF and Square Root EnKF (SR-EnKF) methods. And, the simulation results show that both methods reached an accuracy of 99% when 400 ensembles were generated on a semicircular path by the EnKf-SR with lower error.

Keywords: *finger arm robot; EnKF; SR-EnKF; finger motion estimation.*

Mathematics Subject Classification (2010): 93E10, 62F10.

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1 Introduction

Data from the World Health Organization (WHO) of 2016 recorded that stroke cases were ranked second as a non-communicable disease that causes death and ranked third as a leading cause of disability worldwide. A stroke can cause weakness in one part or side of the body (hemiparesis), due to which a stroke sufferer finds it difficult to move and use parts of the respective side of the body [1]. A stroke can cause disability or weakness on one side of the body, including the upper limbs such as the fingers that are difficult to move, so rehabilitation is needed to restore the function of the hand. Hands and fingers are the most important and complex body parts that humans have. Their muscles can carry out any movement as the human brain commands, without having to control it one by one.

Robotics technology is currently developing rapidly along with advances in science and technology to assist medical rehabilitation, one of which is post-stroke rehabilitation, especially the rehabilitation of finger movements [2]. This is also due to the human desire to help each other in accelerating the recovery of post-stroke patients. The manufacture of technology in the form of robots can be inspired by the phenomena of living things, among others, by referring to the basic principles of movement of the human body. For instance, the way humans walk, talk, hold objects and others. The aim of medical rehabilitation is to maximize functional independence and ability of a patient to continue his or her pre-illness way of living or roles and to improve quality of his or her life.

A finger is one part of the human body, having an important role in human body movement to do various activities [2]. A human has a total of ten fingers functioning to hold objects. The working principles of the human finger are then used as the basis of developing a finger robot designed to hold objects.

A finger robot is one solution to help accelerate the rehabilitation process specifically for finger movements. One of the efforts to develop the finger robot is finger motion estimation [3]. One reliable estimation method frequently used is the Advanced Kalman Filter method. In this paper, the Advanced Kalman Filter method is divided into two methods, that is, the Ensemble Kalman Filter (EnKF) and the Ensemble Kalman Filter Square Root (EnKF-SR) ones. These EnKF and EnkF-SR methods are very reliable for both linear and nonlinear models [4], [5], [6]. The EnKF method was frequently used for estimating the motion and position of AUV [7], ASV [8]- [9] and missiles [10]. And in this paper, the finger estimation is carried out, particularly for the index finger of the left hand by using the EnKF and Square Root EnKF (SR-EnKF) methods on a semi-circular path, and the simulation results produce comparison of the accuracy of one motion estimation method and that of the other [11].

2 Inverse Kinematic Modeling of Finger Motion with 3 Joints

The following is a modeling analysis of the 3-joint finger arm robot [12]. Figure 1 shows the 3-joint arm robot using x and y coordinates in its working area. Just like the 2-joint arm robot, the 3-joint arm robot uses forward kinematics as an equation analysis [5].

The angle Ψ is the angle of the direction of the third part toward the X-axis, as in equation

$$\Psi = (\theta_1 + \theta_2 + \theta_3). \tag{1}$$

Figure 2 shows that the equations for the projection of link 1, link 2 and link 3 toward the x- and y-axis can be obtained by the analysis and combination of equations to locate

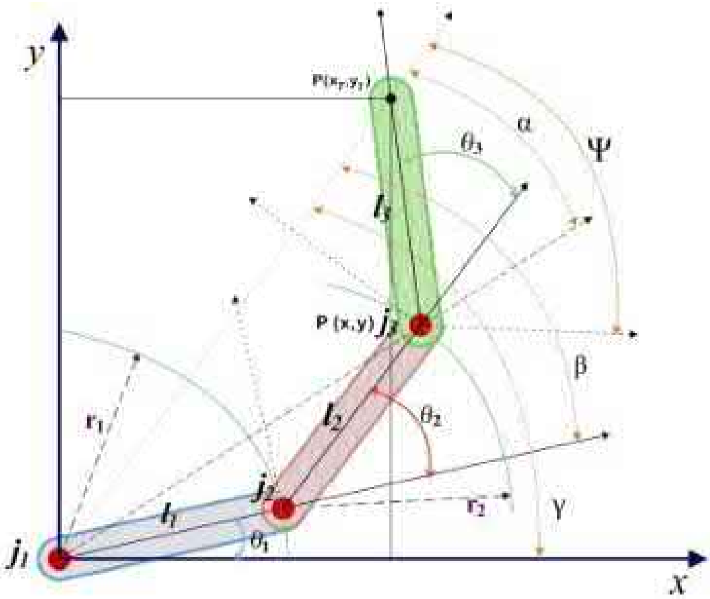

Figure 1: Configuration of a Finger Arm Robot with 3 Joints.

the coordinate points of X_T and Y_T as follows:

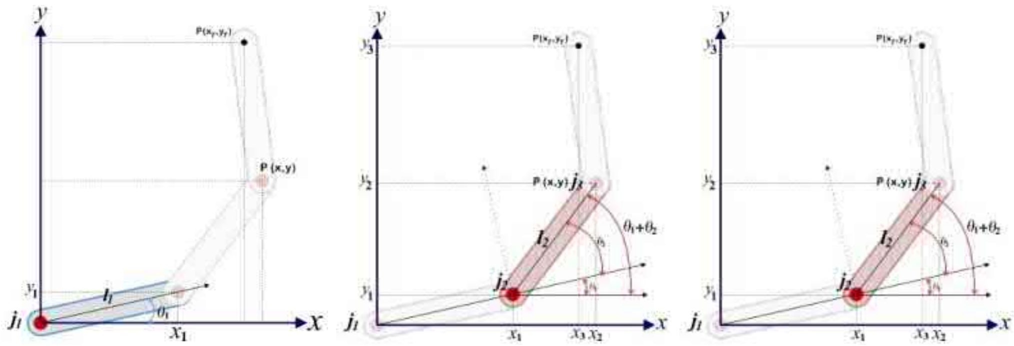

Figure 2: The First part, Second part and Third part of a Finger Arm Robot with 3 DOF.

$$
\begin{aligned}
x &= x_1 + x_2 + x_3, \\
x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1 + \theta_2) + l_3 \cos(\theta_1 + \theta_2 + \theta_3),
\end{aligned} \tag{2}
$$

and

$$
\begin{aligned}
y &= y_1 + y_2 + y_3, \\
y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1 + \theta_2) + l_3 \sin(\theta_1 + \theta_2 + \theta_3),
\end{aligned} \tag{3}
$$

so that

$$\begin{aligned} x_T &= l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3), \\ y_T &= l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) + l_3 \sin(\theta_1+\theta_2+\theta_3). \end{aligned} \tag{4}$$

Simplify by using the trigonometric identity formula [5]:

$$\begin{aligned} x_T &= x - l_3 \cos(\theta_1+\theta_2+\theta_3), \\ y_T &= y - l_3 \sin(\theta_1+\theta_2+\theta_3). \end{aligned} \tag{5}$$

The formula for the forward kinematic equation of 3 joint is

$$\begin{aligned} x &= l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos(\theta_1+\theta_2+\theta_3), \\ y &= l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) + l_3 \sin(\theta_1+\theta_2+\theta_3). \end{aligned} \tag{6}$$

For the inverse kinematics, if the coordinates of $P(x_T, y_T)$ and $P(x, y)$ are known, then θ_1 and θ_2 can be obtained by using the same equation as that applied to the two joint arm robot:

$$\begin{aligned} \theta_2 &= \cos^{-1}\left(\frac{x^2+y^2-l_1^2-l_2^2}{2l_1 l_2}\right), \\ \theta_1 &= \tan^{-1}\left(\frac{y(l_1+l_2\cos\theta_2) - x l_2 \sin\theta_2}{x(l_1+l_2\cos\theta_2) + y l_2 \sin\theta_2}\right). \end{aligned} \tag{7}$$

The angle $\Psi = (\theta_1+\theta_2+\theta_3)$ can be obtained by using $P(x_T, y_T)$ and $P(x, y)$ inserted into equations (5) and (6) so that θ_3 can be found.

By substituting $P(x_T, y_T)$ and $P(x, y)$ into equation (5), we get

$$l_3 \cos\Psi = 0, \tag{8}$$

whereas, by substituting $P(x_T, y_T)$ and $P(x, y)$ into equation (6), it becomes

$$\begin{aligned} y_T &= x - l_3 \sin\Psi, \\ l_1 \sin\theta_1 + l_2 \sin(\theta_1+\theta_2) + l_3 \sin\Psi &= l_1 \cos\theta_1 + l_2 \cos(\theta_1+\theta_2) + l_3 \cos\Psi - l_3 \sin\Psi, \\ 2l_3 \sin\Psi - l_3 \cos\Psi &= l_1(\cos\theta_1 - \sin\theta_1) + l_2(\cos(\theta_1+\theta_2) - \\ &\quad \sin(\theta_1+\theta_2)). \end{aligned} \tag{9}$$

Since in equation (8), $l_3 \cos\Psi = 0$, we obtain

$$\begin{aligned} \Psi &= \theta_1+\theta_2+\theta_3 \\ &= \sin^{-1}\left(\frac{l_1(\cos\theta_1 - \sin\theta_1) + l_2(\cos(\theta_1+\theta_2) - \sin(\theta_1+\theta_2))}{2l_3}\right). \end{aligned} \tag{10}$$

Below is the picture of a finger arm robot and its finger parts.

3 Ensemble Kalman Filter and EnKF Square Root Algorithm

The Ensemble Kalman Filter and the Square Root Ensemble Kalman Filter (SR-EnKF) algorithms are summarized in Table 1.

Figure 3: The Arm Robot Image with a Focus on the Finger Arm Robot.

4 Simulation Results

This study started with the inverse kinematic modeling of a finger arm robot with 3 joints that matches the structure of the human fingers. In this section, two choices of the number of ensembles, that is, 300 ensembles and 400 ensembles were tested on a semi-circle track. Such a choice of track was because in a semi-circle, all the joints of a finger can move optimally.

In this trajectory, the diameter used is about 7.5 cm. This is due to the fact that the finger length of most people in Indonesia ranges from 7.5 to 8.2 cm. So, with a semi-circular movement having a diameter of about 7.5 cm, physical exercises for the index finger can be carried out thoroughly. And, the simulation results can be seen in Figures 4-7.

Figure 4 shows the simulation results by the EnKF and EnKF-SR methods using 300 ensembles and a time of 400 seconds. Figure 4 a) shows the forefinger movement, moving up to 8 cm of the X-axis, by both estimation methods with a small error of 0.1 for the EnKF method and 0.09 for the EnkF-SR method. Figure 4 b) shows the forefinger movement on the Y-axis is of only 2.5 cm, and the EnKF and EnKF-SR methods have sufficient accuracy.

Figure 5 shows the results of the simulation by the EnKF and EnKF-SR methods, producing a movement resembling a semi-circle with a diameter of $\sqrt{(8^2 + 2.5^2)} = \sqrt{70.25} =$ 8.3 cm, so overall if in terms of the diameter of about 7.5 cm, when using 300 ensembles, it has an error of about 10%.

Figure 6 shows the simulation results by the EnKF and EnKF-SR methods using 400 ensembles and a time of 400 seconds. Figure 6a) shows the forefinger movement, moving up to 6.8 cm of the X-axis, and by both estimation methods, having a small error of 0.09% for the EnKF method and 0.08% for the EnkF-SR method. Figure 6b) shows that the forefinger movement on the Y-axis is of only 3 cm, and the EnKF and EnKF-SR

EnKF	**EnKF-SR**
System Model and Measurement Model	
$x_{k+1} = f(u_k, x_k) + w_k, w_k \sim N(0, Q_k)$ $z_k = Hx_k + v_k, v_k \sim N(0, R_k)$	$x_{k+1} = f(u_k, x_k) + w_k, w_k \sim N(0, Q_k)$ $z_k = Hx_k + v_k, v_k \sim N(0, R_k)$
Initialization	
Generate N *ensemble* in accordance with initial estimate $\overline{x}_0$ $x_{0,i} = [x_{0,1}\ x_{0,2}\ x_{0,3}\ \ldots\ x_{0,Ne}]$ Determine initial value : $\widehat{x}_0 = \frac{1}{N_e} \sum_{i=1}^{N} X_{0,i}$	Generate N *ensemble* in accordance with initial estimate $\overline{x}_0$ $x_{0,i} = [x_{0,1}\ x_{0,2}\ x_{0,3}\ \ldots\ x_{0,Ne}]$ Initial Mean *Ensemble* : $\overline{x}_{0,i} = x_{0,i} 1_N$ *Ensemble* initial error : $\widetilde{x}_{0,i} = x_{0,i} - \overline{x}_{0,i} = x_{0,i}(I - 1_N)$
Prediction Stage	
$\widehat{x}^-_{k,i} = f(\widehat{x}_{k-1,i}, u_{k-1,i}) + w_{k,i}$ with $w_{k,i} \sim N(0, Q_k)$ Estimate : $\widehat{x}^-_k = \frac{1}{N_e} \sum_{i=1}^{N} \widehat{x}^-_{k,i}$ Covariance error : $P^-_k = \frac{1}{N_e - 1} \sum_{i=1}^{N} (\widehat{x}^-_{k,i} - \widehat{x}^-_k)(\widehat{x}^-_{k,i} - \widehat{x}^-_k)^T$	$\widehat{x}^-_{k,i} = f(\widehat{x}_{k-1,i}, u_{k-1,i}) + w_{k,i}$ of which $w_{k,i} \sim N(0, Q_k)$ *Ensemble Mean* : $\overline{x}^-_{k,i} = \widehat{x}^-_{k,i} 1_N$ *Ensemble Error* : $\widetilde{x}^-_{k,i} = \widehat{x}^-_{k,i} - \overline{x}^-_{k,i} = \widehat{x}^-_{k,i}(I - 1_N)$
Correction Stage	
$z_{k,i} = z_k + v_{k,i}$ with $v_{k,i} \sim N(0, R_k)$ Kalman gain : $K_k = P^-_k H^T (H P^-_k H^T + R_k)^{-1}$ Estimate : $\widehat{x}_{k,i} = \widehat{x}^-_{k,i} + K_k(z_{k,i} - H\widehat{x}^-_{k,i})$ $\widehat{x}_k = \frac{1}{N_e} \sum_{i=1}^{N} \widehat{x}_{k,i}$	$z_{k,i} = z_k + v_{k,i}$ of which $v_{k,i} \sim N(0, R_k)$ $S_k = H\widetilde{x}^-_{k,i}, E_k = (v_1, v_2, ..., v_N)$, and $C_k = S_k S_k^T + E_k E_k^T$ *Ensemble Mean* : $\bar{x}_{k,i} = \bar{x}^-_{k,i} + \widetilde{x}^-_{k,i} S_k^T C_k^{-1} (\bar{z}_{k,i} - H\bar{x}^-_{k,i})$ Square root schema: - decompose eigenvalue of $C_k = U_k \Lambda_k U_k^T$ - compute matrices $M_k = \Lambda_k^{-1/2} U_k^T S_k^-$ - determine SVD from $M_k = Y_k L_k V_k^T$ *Ensemble Error* : $\widetilde{x}_{k,i} = \widetilde{x}^-_{k,i} V_k (I - L_k^T L_k)^{1/2}$ *Ensemble Estimate* : $\widehat{x}_{k,i} = \widetilde{x}_{k,i} + \bar{x}_{k,i}$

Table 1: EnKF and EnKF-SR Algorithms [13, 14].

methods have high accuracy.

Figure 7 shows the results of the simulation using the EnKF and EnKF-SR methods resulting in a movement resembling a semi-circle with a diameter of $\sqrt{(6.8^2 + 3^2)} = \sqrt{46.24 + 9} = \sqrt{55.24} = 7.43$ cm, so overall if viewed in terms of a diameter of about 7.5 cm, when using 400 ensembles, it has an error of about 0.09. In Table 2, it can be seen that the EnKF-SR method is more accurate than EnKF because there is a factor square root in the correction stage. Viewed in comparison of the numbers of ensembles, the generating of 400 ensembles is more accurate than that of 300 ensembles. When the

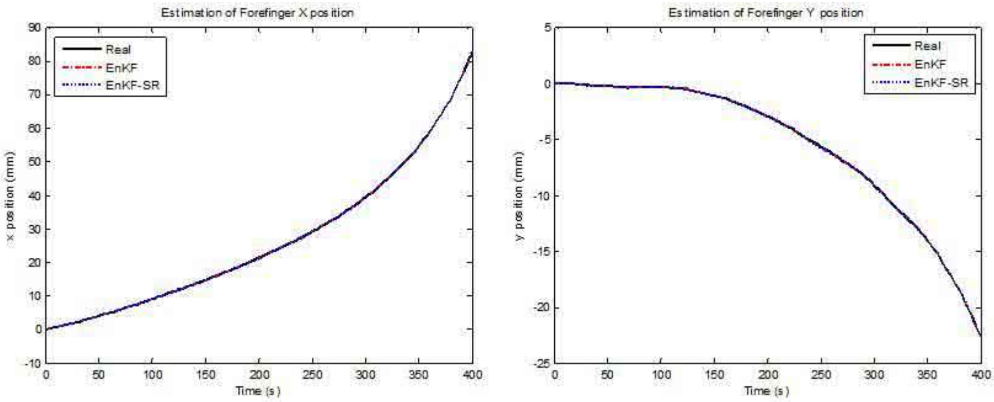

Figure 4: Estimation of Forefinger Motion using 300 Ensembles: a) X position, b) Y Position.

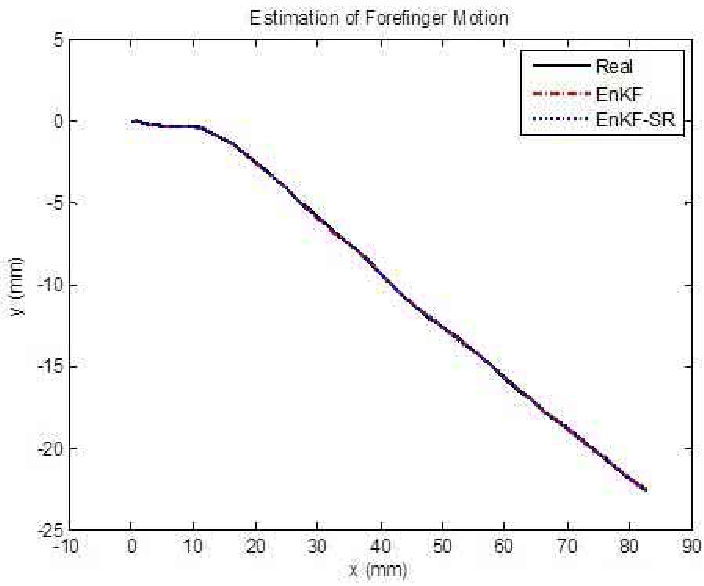

Figure 5: Estimation of Forefinger Motion in XY-Plane using 300 Ensembles.

index finger size of Indonesian people is generally around 7.5 cm (as the diameter) as a reference for the half-track trajectory, then the error for XY motion using 300 ensembles is about 10%, but that for XY motion using 400 ensembles is about 0.09%.

5 Conclusion

Based on the simulation results and the analysis above, it can be concluded that the EnKF and EnKF-SR methods were effective to estimate the movement of the index finger, especially for the finger size of the Indonesian people, with an accuracy of about

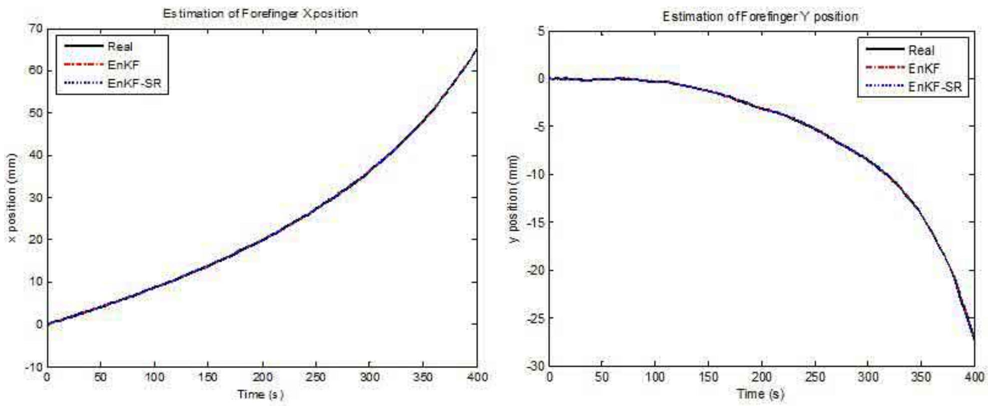

Figure 6: Estimation of Forefinger Motion using 400 Ensembles: a) X position, b) Y Position.

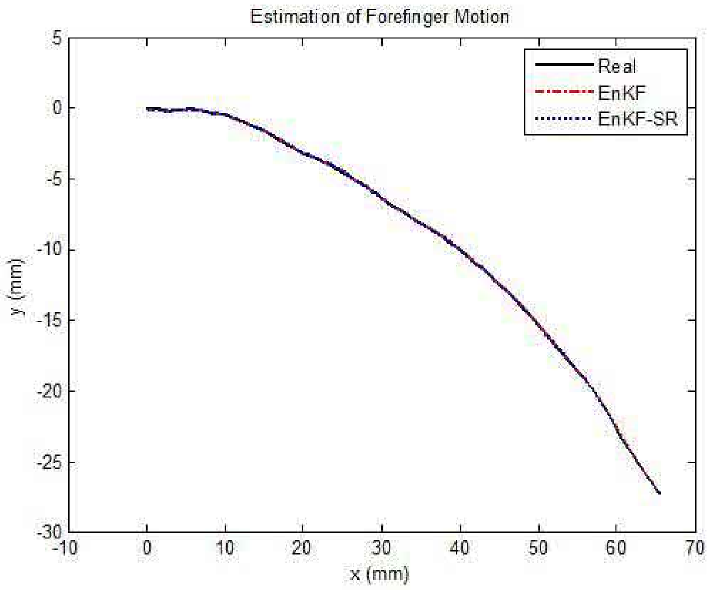

Figure 7: Estimation of Forefinger Motion in XY-Plane using 400 Ensembles.

99% and an error of 0.09%, and an error of 10% if using 300 ensembles. The error was obtained after comparing to the average finger size of the Indonesian people, about 7.5 as the diameter of the movement forming a semicircular path.

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	300 Ensembles		400 Ensembles	
	EnKF	**EnKF-SR**	**EnKF**	**EnKF-SR**
X Motion	0.1%	0.09%	0.094%	**0.091%**
Y Motion	0.12%	0.095%	0.09%	**0.087**
XY Motion (compare with real simulation)	0.13%	0.1%	0.093%	**0.09%**
XY Motion (compare with the real (average) finger size of Indonesian people)	10.2%	10%	0.092%	**0.09%**

Table 2: The Value of Motion Error by the EnKF and EnKF-SR Based on 400 Iterations.

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A Note on the Controllability of Stochastic Partial Differential Equations Driven by Lévy Noise

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Abstract: This paper discusses the exact controllability for impulsive neutral stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces. Under the Lipschitz conditions, the linear growth conditions are weakened and under the condition that the corresponding linear system is exactly controllable, a new set of sufficient conditions is derived by using a fixed point approach without imposing a severe compactness condition on the semigroup.

Keywords: *exact controllability; neutral stochastic partial differential equations; impulse; delay; Lévy noise.*

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1 Introduction

Exact controllability is one of the fundamental concepts in mathematical control theory, it plays an important role in both deterministic and stochastic control systems. It is well known that the controllability of deterministic systems is widely used in many fields of science and technology (for instance, see [4, 7, 21, 26, 28]). Stochastic control theory is a stochastic generalization of classic control theory. The theory of controllability of differential equations in infinite dimensional spaces has been extensively studied in the literature, and the details can be found in various papers and monographs, see [3, 16, 29] and the references therein. Besides white noise or stochastic perturbation, many systems, for example, predator-prey systems, arising from realistic models depend heavily on the histories or impulsive effect [10, 12, 13, 17, 20, 24, 28]. Therefore, there is a real need to discuss impulsive neutral partial differential systems with delays. Tai and Lun [25]

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studied the exact controllability of fractional impulsive neutral infinite delay evolution integrodifferential systems in Banach spaces. However, in order to establish the results, the conditions presented in [25] state that the resolvent operator associated with the linear part is compact and the controllability operator is also compact, thus the induced inverse does not exist in the infinite dimensional state space. To relax that restriction, in this paper, we will study the exact controllability of stochastic nonlinear systems together with the condition that the compactness of the semigroup $S(t)$ is not assumed. Besides the environmental noise, sometimes, we have to consider the impulsive effects, which exist in many evolution processes, because the impulsive effects may bring an abrupt change at certain moments of time (see, e.g. [28]). Moreover, in most research on nonlinear stochastic systems, the control function $u_\alpha(t,x)$ is always constructed by its corresponding linear systems and the stochastic maximum principle [1], however, the stochastic maximum principle is not available in impulsive stochastic systems as a result of its linear form. Therefore, there is a real need to discuss impulsive differential control systems with memory (delay).

On the other hand, in recent years, stochastic partial differential equations with Poisson jumps have gained much attention since Poisson jumps not only exist widely but also can be used to study many phenomena in the real life. Therefore, it is necessary to consider the Poisson jumps into the stochastic systems. To be more precise, in [6], Cui et al. investigated the exponential stability of mild solutions to neutral stochastic partial differential equations with delays and Poisson jumps by using the Banach fixed point principle. Bao et al. [5] studied the existence, uniqueness and some sufficient conditions for stability in the distribution of mild solutions to stochastic partial differential delay equations with Poisson jumps. More recently, by using the successive approximations method, Yin and Xiao [27] considered the controllability of a stochastic partial equation driven by a Poisson random measure. For more details about the stochastic partial differential equations with Poisson jumps, we refer the reader to the monographs [2, 14, 15, 23] and the references therein.

However, to the best of our knowledge, the exact controllability problem for impulsive neutral stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces has not been investigated yet. Motivated by the above works, in this paper, we will study the exact controllability problem for impulsive neutral stochastic delay partial differential equations driven by Lévy noise, which are natural generalizations of controllability concepts well known in the theory of infinite dimensional deterministic control systems. More precisely, we consider the following form:

$$\begin{cases} d[x(t)-G(t,x(t-\tau))] = A[x(t)-G(t,x(t-\tau))]dt \\ \quad +\big[F(t,x(t),x(t-\tau))+Bu(t)\big]dt+\sigma(t,x(t),x(t-\tau))dW(t) \\ \quad +\int_{\mathbf{Z}} L(t,x(t),x(t-\tau),z)\widetilde{N}(dt,dz), t_k \neq t \in J := [0,T], \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k \in \{1,2,...,m\}, \\ x(t) = \varphi(t) \in \mathcal{C}_\tau = \mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H}), \quad -\tau \le t \le 0, \quad \tau > 0, \end{cases} \tag{1}$$

where $x(\cdot)$ is a stochastic process taking values in a real separable Hilbert space $\mathbb{H}$; $A : D(A) \subset \mathbb{H} \to \mathbb{H}$ is the infinitesimal generator of a strongly continuous semigroup of the bounded linear operators $S(t)$, $t \ge 0$ in $\mathbb{H}$. Assume that the mappings $G : J\times\mathbb{H} \to \mathbb{H}$, $F : J\times\mathbb{H}\times\mathbb{H} \to \mathbb{H}$, $\sigma : J\times\mathbb{H}\times\mathbb{H} \to \mathcal{L}^0_2$, $L : J\times\mathbb{H}\times\mathbb{H}\times\mathbf{Z} \to \mathbb{H}$ are the Borel measurable functions and $I_k : \mathbb{H} \to \mathbb{H}$, $k = 1,2,...,m$ are continuous functions. The control function $u(\cdot)$ takes values in $L^{\mathcal{F}}_2(J,U)$ of admissible control functions for a separable Hilbert space

U and B is a bounded linear operator from U into $\mathbb{H}$. Furthermore, let $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$ be prefixed points, and $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ represent the jump of the function x at time t_k with I_k determining the size of the jump, where $x(t_k^+)$ and $x(t_k^-)$ represent the right and left-hand limits of $x(t)$ at $t = t_k$, respectively. Let $\varphi(t) : [-\tau, 0] \to \mathbb{H}$ be càdlàg independent of the Wiener process W and the Poisson point process $p(\cdot)$ with $\mathbf{E}[\sup_{-\tau \le s \le 0} \|\varphi\|_{\mathbb{H}}^2] < \infty$.

The structure of this paper is as follows. In Section 2, we briefly present some basic notations, preliminaries and assumptions. The main results in Section 3 are devoted to the study of the exact controllability for the system (1) and supplied with their proofs.

2 Preliminaries

Let $(\mathbb{H}, \|\cdot\|_{\mathbb{H}}, \langle \cdot, \cdot \rangle_{\mathbb{H}})$ and $(\mathbb{K}, \|\cdot\|_{\mathbb{K}}, \langle \cdot, \cdot \rangle_{\mathbb{K}})$ denote two real separable Hilbert space, with their vector norms and their inner products, respectively. We denote by $\mathcal{L}(\mathbb{K}; \mathbb{H})$ the set of all linear bounded operators from $\mathbb{K}$ into $\mathbb{H}$, which is equipped with the usual operator norm $\|\cdot\|$. In this paper, we use the symbol $\|\cdot\|$ to denote the norms of operators regardless of the spaces potentially involved when no confusion possibly arises. Let $(\Omega, \mathcal{F}, \mathbb{F} = \{\mathcal{F}_t\}_{t \ge 0}, \mathbf{P})$ be a complete filtered probability space satisfying the usual condition (i.e., it is right-continuous and $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets). Let $W = (W(t))_{t \ge 0}$ be a Q-Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P})$ with the covariance operator Q such that $Tr(Q) < \infty$. We assume that there exist a complete orthonormal system $\{e_k\}_{k \ge 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $Qe_k = \lambda_k e_k, k = 1, 2, ...$, and a sequence of independent Brownian motions $\{\beta_k\}_{k \ge 1}$ such that

$$\langle W(t), e \rangle_{\mathbb{K}} = \sum_{k=1}^{\infty} \sqrt{\lambda_k} \langle e_k, e \rangle_{\mathbb{K}} \beta_k(t), \quad e \in \mathbb{K}, t \ge 0.$$

Let $\mathcal{L}_2^0 = \mathcal{L}_2(Q^{\frac{1}{2}}\mathbb{K}; \mathbb{H})$ be the space of all Hilbert-Schmidt operators from $Q^{\frac{1}{2}}\mathbb{K}$ into $\mathbb{H}$ with the inner product

$$\langle \Psi, \phi \rangle_{\mathcal{L}_2^0} = Tr[\Psi Q \phi^*],$$

where ϕ^* is the adjoint of the operator ϕ.

Let $p = p(t)$, $t \in D_p$ (the domain of $p(t)$), be a stationary $\mathcal{F}_t$-Poisson point process taking its value in a measurable space $(\mathbf{Z}, \mathcal{B}(\mathbf{Z}))$ with a σ-finite intensity measure $\lambda(dz)$. We will denote by $N(dt, dz)$ the Poisson counting measure of p such that

$$N(t, \mathbf{Z}) = \sum_{s \in D_p, s \le t} \mathbb{I}_{\mathbf{Z}}(p(s))$$

for any measurable set $\mathbf{Z} \in \mathcal{B}(\mathbb{K} - \{0\})$, which denotes the Borel σ-field of $(\mathbb{K} - \{0\})$. Let

$$\widetilde{N}(dt, dz) := N(dt, dz) - \lambda(dz)dt$$

be the compensated Poisson measure that is independent of $W(t)$.

Let $\tau > 0$ and $\mathcal{C} := \mathcal{C}([-\tau, 0]; \mathbb{H})$ denote the family of all right-continuous functions with left-hand limits (càdlàg) from $[-\tau, 0]$ to $\mathbb{H}$. The space $\mathcal{C}$ is assumed to be equipped with the norm

$$\|\varsigma\|_{\mathcal{C}} := \sup_{-\tau \le t \le 0} \|\varsigma(t)\|_{\mathbb{H}}, \quad \varsigma(t) \in \mathcal{C}.$$

We also assume that $\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H})$ denotes the family of all almost surely bounded, $\mathcal{F}_0$-measurable, $\mathcal{C}([-\tau,0];\mathbb{H})$-valued random variables. For all $t\geq 0$,

$$x_t=\{x(t+\theta):-\tau\leq\theta\leq 0\}$$

is regarded as the $\mathcal{C}([-\tau,0];\mathbb{H})$-valued stochastic process. Further, we consider the Banach space $\boldsymbol{B}_T$ of all $\mathbb{H}$-valued $\mathcal{F}_t$-adapted càdlàg process $x(t)$ defined on $[0,T]$ with

$$x(t)=\varphi(t),\quad t\in[-\tau,0]$$

such that

$$\|x\|^2_{\boldsymbol{B}_T}:=\mathbf{E}[\sup_{0\leq t\leq T}\|x(t)\|^2_{\mathbb{H}}]<\infty.$$

Next, let us recall the definition of a mild solution for (1).

Definition 2.1 An $\mathcal{F}_t$-adapted càdlàg stochastic process $x:J\to\mathbb{H}$ is called a mild solution of (1) if for each $u\in L_2^{\mathcal{F}}(J,U)$ and for arbitrary $t\in J$, $\mathbf{P}\{\omega:\int_J\|x(s)\|^2_{\mathbb{H}}ds<+\infty\}=1$, it satisfies the integral equation

$$\begin{aligned}x(t)=&S(t)[\varphi(0)-G(0,\varphi)]+G(t,x(t-\tau))\\&+\int_0^t S(t-s)[F(s,x(s),x(s-\tau))+Bu(s)]ds\\&+\int_0^t S(t-s)\sigma(s,x(s),x(s-\tau))dW(s)+\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k^-))\\&+\int_0^t\int_{\mathbf{Z}}S(t-s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz)\end{aligned}\tag{2}$$

for any $x_0(\cdot)=\varphi(\cdot)\in\mathcal{C}_\tau$.

Consider the following linear stochastic system of the form:

$$\begin{cases}dx(t)=[Ax(t)+Bu(t)]dt+\sigma(t)dW(t),\quad t\in J,\\x(0)=x_0.\end{cases}\tag{3}$$

It is convenient to introduce the relevant operators and the basic controllability condition.

(i) The operator $L_0^T\in\mathcal{L}(L_2^{\mathcal{F}}(J,\mathbb{H}),L_2(\Omega,\mathcal{F}_T,\mathbb{H}))$ is defined by

$$L_0^Tu=\int_0^T S(T-s)Bu(s)ds,$$

where $L_2^{\mathcal{F}}(J,\mathbb{H})$ is the space of all $\mathcal{F}_t$-adapted, H-valued measurable square integrable processes on $J\times\Omega$. Clearly, the adjoint $(L_0^T)^*:L_2(\Omega,\mathcal{F}_T,\mathbb{H})\to L_2^{\mathcal{F}}(J,\mathbb{H})$ is defined by

$$[(L_0^T)^*z](t)=B^*S^*(T-t)\mathbf{E}\{z\mid\mathcal{F}_t\}.$$

(ii) The controllability operator Π_0^T associated with (3) is defined by

$$\Pi_0^T\{\cdot\}=L_0^T(L_0^T)^*\{\cdot\}=\int_0^T S(T-t)BB^*S^*(T-t)\mathbf{E}\{\cdot\mid\mathcal{F}_t\}dt$$

and belongs to $\mathcal{L}(L_2(\Omega,\mathcal{F}_T,\mathbb{H}),L_2(\Omega,\mathcal{F}_T,\mathbb{H}))$, and the controllability operator $\Gamma_s^T\in\mathcal{L}(\mathbb{H},\mathbb{H})$ is

$$\Gamma_s^T=\int_s^T S(T-t)BB^*S^*(T-t)dt,\quad 0\le s<t.$$

Lemma 2.1 *([19]) The linear stochastic system (3) is exactly controllable on J iff there exists a $\gamma>0$ such that*

$$\mathbf{E}\langle\Pi_0^Tx,x\rangle\ge\gamma\mathbf{E}\|x\|^2,\quad\forall x\in L_2(\Omega,\mathcal{F}_T,\mathbb{H}).$$

Then

$$\mathbf{E}\|(\Pi_0^T)^{-1}\|^2\le\frac{1}{\gamma}.$$

Let $x(t;\varphi,u)$ denote the state value of the system (1) at time t corresponding to the control $u\in L_2^{\mathcal{F}}(J,U)$ and the initial value φ. In particular, the state of system (1) at $t=T$, $x(T;\varphi,u)$, is called the terminal state with the control u.

$$\mathcal{R}_T:=\mathcal{R}(T,\varphi)=\{x(T;\varphi,u):u(\cdot)\in L_2^{\mathcal{F}}(J,U)\}$$

is called the reachable set of the system (1).

Definition 2.2 The stochastic system (1) is said to be exactly controllable on the interval J if

$$\mathcal{R}_T=L_2(\Omega,\mathcal{F}_T,\mathbb{H}).$$

To prove our main results, we list the following basic assumptions of this paper.
(**H**1) A is the infinitesimal generator of a contraction C_0-semigroup $S(t)$, $t\ge0$, in $\mathbb{H}$.
(**H**2) There exists a positive constant C_0 such that for all $t\in J$, $x,y\in\mathbb{H}$,

$$\|G(t,x)-G(t,y)\|_{\mathbb{H}}^2\le C_0\big(\|x-y\|_{\mathbb{H}}^2\big).$$

(**H**3) There exist a positive constant C_1 such that for all $t\in J$, $x_1,y_1,x_2,y_2\in\mathbb{H}$,

$$\begin{aligned}\|F(t,x_1,y_1)-F(t,x_2,y_2)\|_{\mathbb{H}}^2&\vee\|\sigma(t,x_1,y_1)-\sigma(t,x_2,y_2)\|_{\mathcal{L}_2^0}^2\\&\vee\int_{\mathbf{Z}}\|L(t,x_1,y_1,z)-L(t,x_2,y_2,z)\|_{\mathbb{H}}^2\lambda(dz)\\&\le C_1\big(\|x_1-x_2\|_{\mathbb{H}}^2+\|y_1-y_2\|_{\mathbb{H}}^2\big).\end{aligned}$$

(**H**4) There exists a positive constant C_2 such that for all $t\in J$, $x_1,y_1,x_2,y_2\in\mathbb{H}$,

$$\int_{\mathbf{Z}}\|L(t,x_1,y_1,z)-L(t,x_2,y_2,z)\|_{\mathbb{H}}^4\lambda(dz)\le C_2\big(\|x_1-x_2\|_{\mathbb{H}}^4+\|y_1-y_2\|_{\mathbb{H}}^4\big).$$

(**H**5) There exists some positive constants Q_k, $k=1,2,...,m$ such that for all $t\in J$, $x,y\in\mathbb{H}$,

$$\|I_k(x)-I_k(y)\|_{\mathbb{H}}^2\le Q_k\|x-y\|_{\mathbb{H}}^2.$$

(**H**6) For all $t \in J$, there exists a positive constant M such that

$$\|G(t,0)\|^2_{\mathbb{H}} \vee \|F(t,0,0)\|^2_{\mathbb{H}} \vee \|\sigma(t,0,0)\|^2_{\mathbb{H}} \vee \|I_k(0)\|^2_{\mathbb{H}}$$
$$\vee \int_{\mathbf{Z}} \|L(t,0,0,z)\|^2_{\mathbb{H}} \lambda(dz) \vee \int_{\mathbf{Z}} \|L(t,0,0,z)\|^4_{\mathbb{H}} \lambda(dz) \le M.$$

(**H**7) The linear stochastic system (3) is exactly controllable.

We now note that for the proof of our main results, we need the following lemmas.

Lemma 2.2 *(see [8], Proposition 7.3). Suppose that $\Phi(t)$, $t \ge 0$, is a $\mathcal{L}^0_2$-valued predictable process and let $W^{\Phi}_A = \int_0^t S(t-s)\Phi(s)dW(s)$, $t \in [0,T]$. Then for any arbitrary $p > 2$, there exists a constant $C(p,T) > 0$ such that*

$$\mathbf{E} \sup_{t\in[0,T]} \|W^{\Phi}_A\|^p_{\mathbb{H}} \le \overline{C} \sup_{t\in[0,T]} \|S(t)\|^p \mathbf{E} \int_0^T \|\Phi(s)\|^p_{\mathcal{L}^0_2} ds,$$

where $\overline{C} = C(p,T)$. Moreover, if $\mathbf{E} \int_0^T \|\Phi(s)\|^p ds < +\infty$, then there exists a continuous version of the process $\{W^{\Phi}_A\}_{t\ge 0}$. If $(S(t))_{t\ge 0}$ is a contraction semigroup, then the above result is true for $p \ge 2$.

Lemma 2.3 *(see [18], Lemma 2.2). Let the space $M^p_{\nu}([0,T] \times \Omega \times (\mathbb{K} - \{0\}), \mathbb{H})$, $p \ge 2$, denote the set of all random process $J(x,y)$ with values in $\mathbb{H}$, predictable with respect to $\{\mathcal{F}_t\}$ such that*

$$\mathbf{E}\Big(\int_0^T \int_{\mathbf{Z}} \|J(t,y)\|^p_{\mathbb{H}} \nu(dy)dt\Big) < \infty.$$

Suppose $J \in M^2_{\nu}([0,T] \times \Omega \times (\mathbb{K} - \{0\}), \mathbb{H}) \cap M^4_{\nu}([0,T] \times \Omega \times (\mathbb{K} - \{0\}), \mathbb{H})$. Then for any $t \in [0,T]$,

$$\mathbf{E}\Big[\sup_{\theta\in[0,t]} \Big\| \int_0^{\theta} \int_{\mathbf{Z}} S(\theta - s) J(s,y) \widetilde{N}(ds,dy) \Big\|^2_{\mathbb{H}}\Big] \le C \Bigg\{ \mathbf{E}\Big(\int_0^t \int_{\mathbf{Z}} \|J(t,y)\|^2_{\mathbb{H}} \nu(dy)dt\Big) + \mathbf{E}\Big(\int_0^t \int_{\mathbf{Z}} \|J(t,y)\|^4_{\mathbb{H}} \nu(dy)dt\Big)^{\frac{1}{2}} \Bigg\}$$

for some constant $C = C(T) > 0$, dependent on $T > 0$.

3 Main Results

In this section, we shall investigate the exact controllability for impulsive neutral stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces.

The main result of this section is the following theorem.

Theorem 3.1 *Let the assumptions* (**H1**)$-$(**H7**) *hold. Then the system (1) is exactly controllable on J provided that*

$$\begin{cases} \Big(1 + \frac{30T^2C_B^2}{\gamma}\Big)\Big[6(C_0 + 2T^2C_1 + 2T\overline{C}C_1 + m\sum_{k=1}^m Q_k) \\ +\frac{1}{2} + (2CC_1 + C^2C_2)T\Big] < 1, \ \textit{where } \|B\|^2 = C_B. \end{cases} \tag{4}$$

Proof. Using the assumptions, for an arbitrary function $x(\cdot)$, choose the feedback control function

$$
\begin{aligned}
u_x^T(t) =& B^* S^*(T-t)\mathbf{E}\Big\{(\Pi_0^T)^{-1}\big[x_T - S(T)[\varphi(0) - G(0,\varphi)] - G(t, x(t-\tau)) \\
& - \int_0^T S(T-s)F(s, x(s), x(s-\tau))ds \\
& - \int_0^T S(T-s)\sigma(s, x(s), x(s-\tau))dW(s) \\
& - \int_0^T \int_{\mathbf{Z}} S(T-s)L(s, x(s), x(s-\tau), z)\widetilde{N}(ds, dz) \\
& - \sum_{0<t_k<T} S(T-t_k)I_k(x(t_k^-))\big] \mid \mathcal{F}_t\Big\}.
\end{aligned} \tag{5}
$$

We transform (1) into a fixed point problem. Consider the operator $\Pi : \mathbf{B}_T \to \mathbf{B}_T$ defined by

$$
\begin{aligned}
\Pi(x)(t) =& S(t)[\varphi(0) - G(0,\varphi)] + G(t, x(t-\tau)) \\
& + \int_0^t S(t-s)[F(s, x(s), x(s-\tau)) + Bu_x^T(s)]ds \\
& + \int_0^t S(t-s)\sigma(s, x(s), x(s-\tau))dW(s) \\
& + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)) \\
& + \int_0^t \int_{\mathbf{Z}} S(t-s)L(s, x(s), x(s-\tau), z)\widetilde{N}(ds, dz).
\end{aligned} \tag{6}
$$

In what follows, we shall show that when using the control $u_x^T(\cdot)$, the operator Π has a fixed point, which is then a mild solution for system (1).

By our assumptions, Hölder's inequality, Lemma 2.1, Lemma 2.2, Lemma 2.3 and the basic inequality $\Big(\sum_{i=1}^n x_i\Big)^p \le n^{(p-1)\vee 0} \sum_{i=1}^n x_i^p$, $p > 0$, we obtain that for $x \in \mathbf{B}_T$,

$$
\begin{aligned}
\|\Pi(x)(t)\|_{\mathbf{B}_T}^2 \le 7\Big\{& \mathbf{E}\big(\sup_{t\in J} \|S(t)[\varphi(0) - G(0,\varphi)]\|^2\big) \\
& + \mathbf{E}\big(\sup_{t\in J} \|G(t, x(t-\tau))\|^2\big) \\
& + \mathbf{E}\big(\sup_{t\in J} \Big\| \int_0^t S(t-s)F(s, x(s), x(s-\tau))ds\Big\|^2\big)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathbf{E}\big(\sup_{t\in J}\Big\|\int_0^t S(t-s)Bu_x^T(s)ds\Big\|^2\big)\\
&+\mathbf{E}\big(\sup_{t\in J}\Big\|\int_0^t S(t-s)\sigma(s,x(s),x(s-\tau))dW(s)\Big\|^2\big)\\
&+\mathbf{E}\big(\sup_{t\in J}\Big\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\Big\|^2\big)\\
&+\mathbf{E}\big(\sup_{t\in J}\Big\|\int_0^t\int_{\mathbf{Z}} S(t-s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz)\Big\|^2\big)\Bigg\}\\
\le\ & 7\Bigg\{\Big[2(1+C_0)\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+2\big[M+C_0(\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+\|x\|_{\mathbf{B}_T}^2)\big]\\
&+2T\big[MT+C_1(\tau\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+2T\|x\|_{\mathbf{B}_T}^2)\big]\\
&+2\overline{C}\big[MT+C_1(\tau\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+2T\|x\|_{\mathbf{B}_T}^2)\big]+2m\sum_{k=1}^m Q_k(M+\|x\|_{\mathbf{B}_T}^2)\\
&+8C\big[MT+C_1(\tau\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+2T\|x\|_{\mathbf{B}_T}^2)\big]\\
&+8C\big[MT+\sqrt{C_2}(\sqrt{\tau}\mathbf{E}\|\varphi\|_{\mathcal{C}}^2+\sqrt{2T}\|x\|_{\mathbf{B}_T}^2)\big]\Big]\\
&\times\Big(1+\frac{7T^2C_B^2}{\gamma}\Big)+\frac{7T^2C_B^2}{\gamma}\mathbf{E}\|x_T\|_{\mathbb{H}}^2\Bigg\}<\infty.
\end{aligned}
$$

Thus, Π maps $\mathbf{B}_T$ into itself.

Now, we shall prove that Π is a contraction mapping in $\mathbf{B}_T$. For any $x,y\in\mathbf{B}_T$, in the same ways as above, we can get

$$
\begin{aligned}
&\|\Pi(x)(t)-\Pi(y)(t)\|_{\mathbf{B}_T}^2\\
&\le\Big[6(C_0+2T^2C_1+2T\overline{C}C_1+m\sum_{k=1}^m Q_k)+\frac{1}{2}+(2CC_1+C^2C_2)T\Big]\|x-y\|_{\mathbf{B}_T}^2\\
&\quad+6T^2C_B\mathbf{E}\big(\sup_{t\in J}\|u_x^T-u_y^T\|_{\mathbb{H}}^2\big)\\
&\le\Big(1+\frac{30T^2C_B^2}{\gamma}\Big)\Big[6(C_0+2T^2C_1+2T\overline{C}C_1+m\sum_{k=1}^m Q_k)\\
&\quad+\frac{1}{2}+(2CC_1+C^2C_2)T\Big]\|x-y\|_{\mathbf{B}_T}^2.
\end{aligned}
$$

By assumption (4), we conclude that Π is a contraction mapping on $\mathbf{B}_T$. On the other hand, by the Banach fixed point theorem, there exists a unique fixed point $x(\cdot)\in\mathbf{B}_T$ such that $(\Pi x)(t)=x(t)$. This fixed point is then the mild solution of the system (1). Thus, the system (1) is exactly controllable on J. The proof of Theorem 3.1 is complete.

Remark 3.1 From the assumptions $(\mathbf{H1})-(\mathbf{H6})$, for every $u(\cdot)\in U$, the system (1) has a unique solution in $\mathbf{B}_T$.

Now let us consider a special case for the system (1). If $G\equiv 0$, $m\equiv 0$, then the system (1) becomes the following stochastic delay partial differential equations driven by

Lévy noise:

$$\begin{cases} dx(t) = Ax(t)dt + \big[F(t,x(t),x(t-\tau)) + Bu(t)\big]dt + \sigma(t,x(t),x(t-\tau))dW(t) \\ \qquad\qquad + \int_{\mathbf{Z}} L(t,x(t),x(t-\tau),z)\widetilde{N}(dt,dz), \quad t\in[0,T], \\ x(t) = \varphi(t) \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H}), \quad -\tau\le t\le 0, \quad \tau>0. \end{cases} \tag{7}$$

Corollary 3.1 *Suppose that the assumptions* (**H1**)*,* (**H3**)*,* (**H4**)*,* (**H6**)*,* (**H7**) *hold. Then the system (10) is exactly controllable on J provided that*

$$\Big(1+\frac{12T^2C_B^2}{\gamma}\Big)\Big[\frac{1}{2}+8(T^2C_1+T\overline{C}C_1)+(2CC_1+C^2C_2)T\Big]<1. \tag{8}$$

Remark 3.2 As we all know, the mathematical formulation of many physical phenomena contains integro-differential equations arisen in various applications such as viscoelasticity, heat equations, fluid dynamics, chemical kinetics and so on. More recently, M.A. Diop et al. [9, 10] studied the asymptotic stability of neutral impulsive stochastic partial integro-differential equations with delays and Poisson jumps in Hilbert spaces. In this remark, we consider the exact controllability for impulsive neutral stochastic delay partial integro-differential equations driven by Lévy noise in the form

$$\begin{cases} d[x(t)-G(t,x(t-\tau))] = A[x(t)-G(t,x(t-\tau))]dt \\ \qquad\qquad + \int_0^t K(t-s)[x(s)-G(s,x(s-\tau))]dsdt \\ \qquad\qquad + \big[F(t,x(t),x(t-\tau))+Bu(t)\big]dt \\ \qquad\qquad + \sigma(t,x(t),x(t-\tau))dW(t) \\ \qquad\qquad + \int_{\mathbf{Z}} L(t,x(t),x(t-\tau),z)\widetilde{N}(dt,dz), \quad t_k\ne t\in[0,T], \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad k\in\{1,2,...,m\}, \\ x(t)=\varphi(t)\in\mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H}), \quad -\tau\le t\le 0, \end{cases} \tag{9}$$

where $K(t)$, $t\ge 0$, is a closed linear operator defined on a common domain which is dense in a Banach space X.

Further, we assume that the integro-differential abstract Cauchy problem

$$\frac{dx(t)}{dt} = Ax(t) + \int_0^t K(t-s)x(s)ds, \quad x(0)=x_0\in X, \tag{10}$$

has an associated resolvent operator of bounded linear operators $\{R(t)\}_{t\ge0}$ on X.

A one-parameter family of bounded linear operator $\{R(t)\}_{t\ge0}$ on X is called a resolvent operator of (10) if the following conditions are verified.
(i) Function $R(\cdot):[0,\infty)\to\mathcal{L}(X)$ is strongly continuous and $R(0)x=x$ for all $x\in X$.
(ii) For $x\in D(A)$, $R(\cdot)\in\mathcal{C}([0,+\infty);D(A))\cap\mathcal{C}^1([0,+\infty);X)$, and

$$\begin{aligned} \frac{dR(t)x}{dt} &= AR(t)x+\int_0^t K(t-s)R(s)xds \\ &= R(t)Ax+\int_0^t R(t-s)K(s)xds, \quad for \quad t\ge 0. \end{aligned}$$

(iii) There exist constants $M>0$, β such that $\|R(t)\|\le M.e^{\beta t}$ for every $t\ge0$.

An $\mathcal{F}_t$-adapted càdlàg stochastic process $x : [0,T] \to \mathbb{H}$ is called a mild solution of (9) if for each $u \in L_2^{\mathcal{F}}(J,U)$ and for arbitrary $t \in [0,T]$,$\mathbf{P}\{\omega : \int_J \|x(s)\|_{\mathbb{H}}^2 ds < +\infty\} = 1$, it satisfies the integral equation

$$\begin{aligned}
x(t) =&R(t)[\varphi(0) - G(0,\varphi)] + G(t, x(t-\tau)) \\
&+ \int_0^t R(t-s)[F(s,x(s),x(s-\tau)) + Bu(s)]ds \\
&+ \int_0^t R(t-s)\sigma(s,x(s),x(s-\tau))dW(s) \\
&+ \sum_{0<t_k<t} R(t-t_k)I_k(x(t_k^-)) \\
&+ \int_0^t \int_{\mathbf{Z}} R(t-s)L(s,x(s),x(s-\tau),z)\widetilde{N}(ds,dz)
\end{aligned}$$

for any $x_0(\cdot) = \varphi(\cdot) \in \mathcal{C}^b_{\mathcal{F}_0}([-\tau,0];\mathbb{H})$.

By implementing appropriate conditions on the functions, one can easily show that by adapting and employing the techniques used in Theorem 3.1, the stochastic control system (9) is exactly controllable on $[0,T]$.

Remark 3.3 We now consider the non-autonomous versions of systems (1) and (9), where the operator A is replaced by $\{A(t) : t \in [0,T]\}$. To proceed to prove the exact controllability results in a similar manner as employed in the above theorem, an evolution system $\{U(t,s) : 0 \leq s \leq t \leq T\}$ and a resolvent family $\{R(t,s) : 0 \leq s \leq t \leq T\}$ are guaranteed to exist. Conditions guaranteeing the existence of $U(t,s)$ and $R(t,s)$ can be found in [22] and [11], respectively. Therefore, the above theorem can be extended to the time-dependent case by making suitable modifications involving the use of the properties of the time-dependent evolution system and the time-dependent resolvent family in the above arguments.

4 Conclusions

This paper focuses on establishing the exact controllability for impulsive neutral stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces through the application of one of the most important results of the analysis and considers the main source of the metric fixed point theory known as the "Banach Contraction Principle". In the future, we aim to expand this study to the approximate controllability for impulsive neutral second-order stochastic delay partial differential equations driven by Lévy noise in Hilbert spaces.

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Asymmetric Duffing Oscillator: Jump Manifold and Border Set

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Abstract: The jump phenomenon, present in the forced asymmetric Duffing oscillator, is studied using the known steady-state asymptotic solution. The main result consists in construction of a new mathematical object – a jump manifold – encoding global information about all possible jumps. The jump manifold is computed for the forced asymmetric Duffing oscillator, and several examples of jumps are calculated, showing the advantages of the method.

Keywords: *metamorphoses of amplitude-frequency curves; jump phenomenon.*

Mathematics Subject Classification (2010): 34C05, 34C25, 34E05, 37G35, 70K30.

1 Introduction

In this work, we study steady-state dynamics of the forced asymmetric Duffing oscillator governed by the equation

$$\ddot{y} + 2\zeta\dot{y} + \gamma y^3 = F_0 + F\cos(\Omega t)\,, \tag{1}$$

which has a single equilibrium position and a corresponding one-well potential [1], where ζ, γ, F_0, F are parameters and Ω is the angular frequency of the periodic force. This dynamical system in particular and Duffing-type equations in general, which can be used to describe pendulums, vibration absorbers, beams, cables, micromechanical structures, and electrical circuits, have a long history [2]. The equation of motion (1) can describe several nonlinear phenomena such as various nonlinear resonances, symmetry breaking, chaotic dynamics, period-doubling route to chaos, multistability and fractal dependence on initial conditions, and jumps [1–6].

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Our work aims to research the jump phenomenon using the implicit function machinery. Kovacic and Brennan described and investigated jumps for the system (1) in their interesting study [1]. Recently, Kalmár-Nagy and Balachandran [6] applied a differential condition to detect vertical tangents, characteristic of the jump phenomenon.

A standard approach to nonlinear equations of the form (1) is based on asymptotic methods [7, 8]. More exactly, in the case of Eq.(1), approximate nonlinear resonances $1:1$ are computed in the form

$$y(t) = A_0 + A_1 \cos(\Omega t + \theta), \tag{2a}$$

$$f_i(A_0,\ A_1,\ \Omega;\ \underline{c}) = 0, \quad i = 1, 2, 3, \tag{2b}$$

where A_0, A_1 and Ω fulfill nonlinear implicit amplitude-frequency response algebraic equations (2b) and $\underline{c} = (\gamma,\ \zeta,\ F,\ F_0)$ [1,5,9].

We have proposed in our earlier papers an analysis of differential properties of solutions of the implicit amplitude-frequency response equations, see [10] and references therein. It turns out that bifurcations of dynamics, such as hysteresis and jump phenomenon, are related to the appearance/disappearance of branches of solutions, as well as more complicated bifurcations such as, for example, the creation/destruction of solutions follow from the changes of differential properties of solutions of the equations (2b), induced by a change of the parameters $\underline{c}$. Analytical methods, permitting the prediction of metamorphoses of solutions, are of great help in numerical simulation. Our formalism applies also to several models of coupled oscillators [11, 12], see also [13].

The novelty of this work consists in defining, in the differential geometry formulation, the jump manifold encoding global information about all possible jumps. Our formalism generalizes the differential condition of Kalmár-Nagy and Balachandran [6] for an arbitrary implicit amplitude-frequency response function.

In the next section, we describe the steady-state solution (2a) [1,5,9], given by implicit equations (2b). Working in the framework developed in our earlier papers, see [10] and references therein, we compute the jump manifold in Section 3 (see Eq.(9a) and Table 2) which contains information about all possible jumps – this is the main achievement of this work. In Section 4, we compare the analytical predictions with the numerical solutions of Eq.(1). We summarize our results in the last Section 5.

2 The Steady-State Solution

The steady-state solution of Eq.(1) of the form (2a) was computed in Refs. [1, 5, 9] with the following implicit amplitude-frequency response equations (2b):

$$-A_1\Omega^2 + 3\gamma A_0^2 A_1 + \frac{3}{4}\gamma A_1^3 - F\cos\theta = 0, \tag{3a}$$

$$-2\zeta A_1 \Omega - F\sin\theta = 0, \tag{3b}$$

$$\gamma A_0^3 + \frac{3}{2}\gamma A_0 A_1^2 - F_0 = 0. \tag{3c}$$

Eliminating θ from Eqs.(3a), (3b), we get two implicit equations for $A_0,\ A_1$, and Ω:

$$A_1^2\left(-\Omega^2 + 3\gamma A_0^2 + \frac{3}{4}\gamma A_1^2\right)^2 + 4\Omega^2\zeta^2 A_1^2 = F^2, \tag{4a}$$

$$\gamma A_0^3 + \frac{3}{2}\gamma A_1^2 A_0 - F_0 = 0. \tag{4b}$$

Computing A_1^2 from Eq.(4b) and substituting into (4a), we obtain finally one implicit equation for A_0, Ω:

$$f(\Omega, A_0; \gamma, \zeta, F, F_0) = \sum\nolimits_{k=0}^{9} c_k A_0^k = 0, \tag{5}$$

where the coefficients c_k are given in Table 1.

Table 1: Coefficients c_k of the polynomial (5).

$c_9 = 25\gamma^3$	$c_4 = 16\Omega^2\gamma F_0$
$c_8 = 0$	$c_3 = -9\gamma F_0^2 + 6\gamma F^2$
$c_7 = -20\Omega^2\gamma^2$	$c_2 = -4F_0\Omega^4 - 16\zeta^2\Omega^2 F_0$
$c_6 = -15\gamma^2 F_0$	$c_1 = 4\Omega^2 F_0^2$
$c_5 = -15\gamma^2 F_0$	$c_0 = -F_0^3$

We can also obtain an implicit equation for A_1, Ω. We solve the cubic equation (4b) for A_0 computing the only one real root:

$$A_0 = -\frac{A_1^2}{2Y} + Y, \quad Y \equiv \sqrt[3]{\sqrt{\frac{1}{8}A_1^6 + \frac{1}{4\gamma^2}F_0^2} + \frac{1}{2\gamma}F_0}. \tag{6}$$

Two other roots are indeed complex since the discriminant of Eq.(4b), $D = -4p^3 - 27q^2$, $p = \frac{3}{2}A_1^2$,$q = -\frac{F_0}{\gamma}$, $\gamma \neq 0$, is negative.

Then we substitute A_0, Y from Eq.(6) into Eq.(4a), obtaining finally a complicated but useful implicit equation for A_1, Ω:

$$g(\Omega, A_1; \gamma, \zeta, F, F_0) = A_1^2 \left(3\gamma A_0^2 + \tfrac{3}{4}\gamma A_1^2 - \Omega^2\right)^2 + 4\Omega^2\zeta^2 A_1^2 - F^2 = 0, \tag{7}$$

where A_0 and Y are defined in (6).

3 Jump Phenomenon

3.1 Jump conditions and jump manifold

Jump conditions in the implicit function setting read [10]

$$f(\Omega, A_0; \gamma, \zeta, F, F_0) = 0, \tag{8a}$$
$$\frac{\partial f(\Omega, A_0; \gamma, \zeta, F, F_0)}{\partial A_0} = 0, \tag{8b}$$

where equation (8b) is the condition for a vertical tangency.

Solving equations (8), we obtain

$$J(A_0; \gamma, \zeta, F, F_0) = \sum\nolimits_{k=0}^{21} a_k A_0^k = 0, \tag{9a}$$
$$\Omega^2 = \frac{\left(-50\gamma^4\right)A_0^{12} + 95\gamma^3 F_0 A_0^9 + \left(6F^2\gamma^2 - 39\gamma^2 F_0^2\right)A_0^6 + \left(3F^2\gamma F_0 - 7\gamma F_0^3\right)A_0^3 + F_0^4}{2A_0\left(F_0 - 10\gamma A_0^3\right)\left(F_0 - \gamma A_0^3\right)^2}, \tag{9b}$$

the non-zero coefficients a_k of the polynomial $J(A_0)$ are given in Table 2.

The polynomial $J(A_0)$, complicated as it is, encodes global information about all possible jumps. We shall thus refer to equation (9a), which defines an implicit function

Table 2: Non-zero coefficients of polynomial (9a).

$a_{21} = 4000\gamma^7\zeta^2$	$a_9 = 3248\gamma^3\zeta^2F_0^4 - 72F^2\gamma^3\zeta^2F_0^2$
$a_{18} = -16\,000\gamma^6\zeta^2F_0$	$a_8 = 36F^4\gamma^3F_0 - 978F^2\gamma^3F_0^3$
$a_{17} = 600F^2\gamma^6$	$a_6 = 528\gamma^2\zeta^2F_0^5 - 240F^2\gamma^2\zeta^2F_0^3$
$a_{15} = 23\,880\gamma^5\zeta^2F_0^2 - 480F^2\gamma^5\zeta^2$	$a_5 = 9F^4\gamma^2F_0^2 + 138F^2\gamma^2F_0^4$
$a_{14} = -1920F^2\gamma^5F_0$	$a_3 = 24F^2\gamma\zeta^2F_0^4 - 152\gamma\zeta^2F_0^6$
$a_{12} = 768F^2\gamma^4\zeta^2F_0 - 15\,512\gamma^4\zeta^2F_0^3$	$a_2 = -6F^2\gamma F_0^5$
$a_{11} = 36F^4\gamma^4 + 2166F^2\gamma^4F_0^2$	$a_0 = 8\zeta^2F_0^7$

of variables A_0, γ, ζ, F, F_0, as a *jump manifold equation.* Thus, the jump manifold $\mathcal{J}\left(A_0, \gamma, \zeta, F, F_0\right)$:

$$\mathcal{J}\left(A_0; \gamma, \zeta, F, F_0\right) = \left\{(A_0, \gamma, \zeta, F, F_0) : \quad J\left(A_0; \gamma, \zeta, F, F_0\right) = 0\right\}, \tag{10}$$

belongs to a $5D$ space. It is purposeful to introduce the projection of the jump manifold onto the parameter space:

$$\mathcal{J}_\perp = \left\{(\gamma, \zeta, F, F_0) : \text{there is a real } A_0 \text{ such that } J\left(A_0; \gamma, \zeta, F, F_0\right) = 0\right\}. \tag{11}$$

In other words, for any set of parameters γ, ζ, F, F_0 belonging to $\mathcal{J}_\perp$, there is a jump in the dynamical system (1) and all jumps occur for the parameters belonging to $\mathcal{J}_\perp$.

We shall consider $2D$ and $3D$ projections, plotting $\mathcal{J}\left(A_0; \gamma_*, \zeta_*, F_*, F_0\right)$ and $\mathcal{J}\left(A_0; \gamma_*, \zeta_*, F, F_0\right)$, respectively, where the parameters γ_*, ζ_*, F_* or γ_*, ζ_* are fixed.

3.1.1 $2D$ projection, $J\left(A_0; \gamma_*, \zeta_*, F_*, F_0\right) = 0$

The global picture of the jump manifold $\mathcal{J}\left(A_0; \gamma_*, \zeta_*, F_*, F_0\right)$, where $\gamma_* = 0.0783$, $\zeta_* = 0.025$, $F_* = 0.1$ and A_0, F_0 are variable, is shown in Fig.1. We have chosen the values of γ, ζ, F such as in [1] for the sake of comparison.

All points lying on the blue curve (jump manifold) correspond to jumps (vertical tangents). Moreover, there are four critical points dividing Fig.1 into parts and referred to as the *border points*: $F_0^{(1)} = 0$, $F_0^{(2)} = 0.0920$, $F_0^{(3)} = 0.7385$, $F_0^{(4)} = 6.5321$, *where the number of jumps changes*, it is defined and computed in Subsection 3.2. More precisely, these critical points are where the red dashed vertical lines are *tangent* to the blue jump manifold.

Indeed, for $F_0 \in \left(F_0^{(1)}, F_0^{(2)}\right)$, there are two jumps; for $F_0 \in \left(F_0^{(2)}, F_0^{(3)}\right)$, there are four; for $F_0 \in \left(F_0^{(3)}, F_0^{(4)}\right)$, there are two, and there are no jumps for $F_0 > F_0^{(4)}$.

For example, in Fig.2 below, the case $F_0 = 0.4$ is shown. More exactly, the implicit function $A_1(\Omega)$, computed with the help of Eq.(7), is plotted for $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, and $F_0 = 0.4$. The red dots, denoting vertical tangents, correspond to the red dots in Fig.1. These points can be easily computed from Eqs.(8), (4).

Indeed, solving equations (8) for $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, $F_0 = 0.4$, we get four real solutions Ω, A_0 shown in the first two columns in Table 3. Then, for the above values of Ω, we solve equations (4) obtaining the same four values of A_0 and the corresponding values of A_1 listed in the third column of Table 3.

In Fig.3, the bifurcation diagram is shown for the set of parameters listed in Fig.2, where y is a numerical solution of Eq.(1). Note that branches a-b, c-d, e-f in Fig.3 correspond to analogous branches in Fig.2.

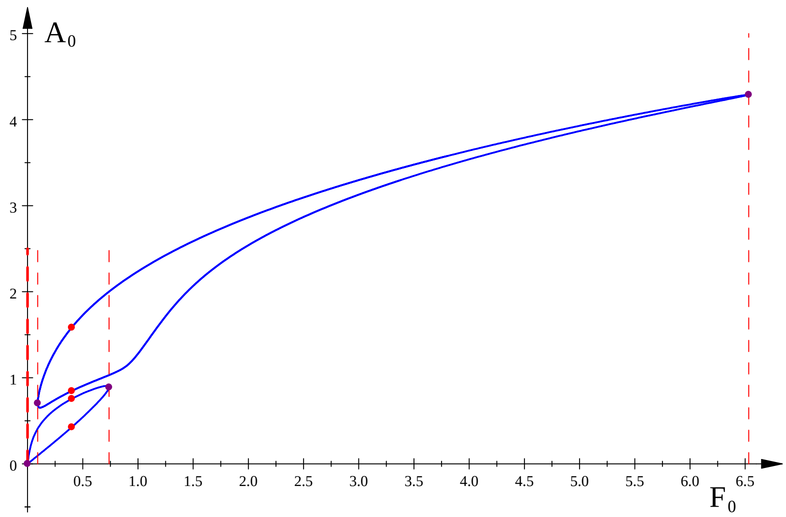

Figure 1: Jump manifold $\mathcal{J}(A_0;\gamma_*,\zeta_*,F_*,F_0)$, $\gamma_*=0.0783$, $\zeta_*=0.025$, $F_*=0.1$ (blue) and four border points (purple dots) – points of contact between $\mathcal{J}$ and the vertical red lines.

Table 3: Solutions of Eqs.(8) and (4).

Ω	A_0	A_1
0.576 122 891	0.846 633 527	1. 882 759 746
0.643 209 846	0.755 260 872	2. 032 001 367
0.690 545 624	1. 583 776 750	0.691 474 188
0.711 882 658	0.425 889 574	2. 806 379 023

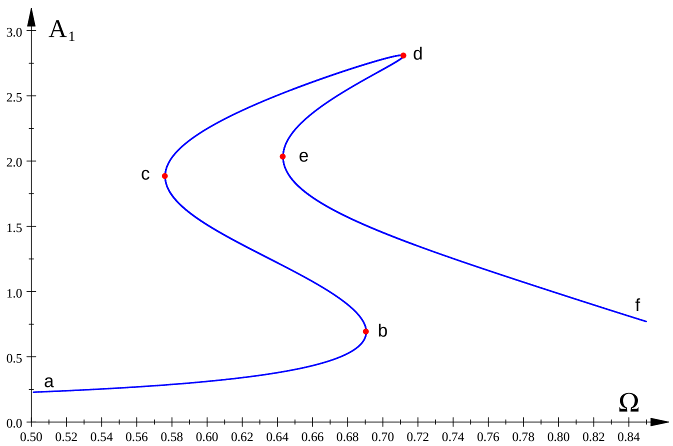

Figure 2: Amplitude-frequency response curve $A_1(\Omega)$, $\gamma=0.0783$, $\zeta=0.025$, $F=0.1$, $F_0=0.4$. Stable branches: a-b, c-d, e-f.

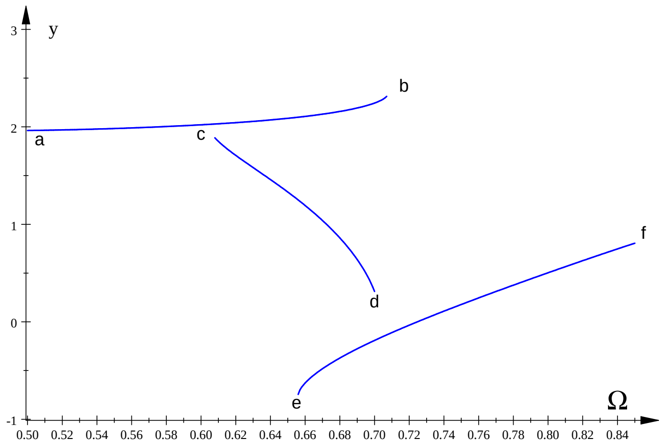

Figure 3: Bifurcation diagram, $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, $F_0 = 0.4$.

3.1.2 $3D$ projection, $J(A_0; \gamma_*, \zeta_*, F, F_0) = 0$

We now fix two parameters only, for example, $\gamma_* = 0.0783$, $\delta_* = 0.025$, and plot the jump manifold $\mathcal{J}(A_0; \gamma_*, \zeta_*, F, F_0)$ as a $3D$ surface, see Fig.4.

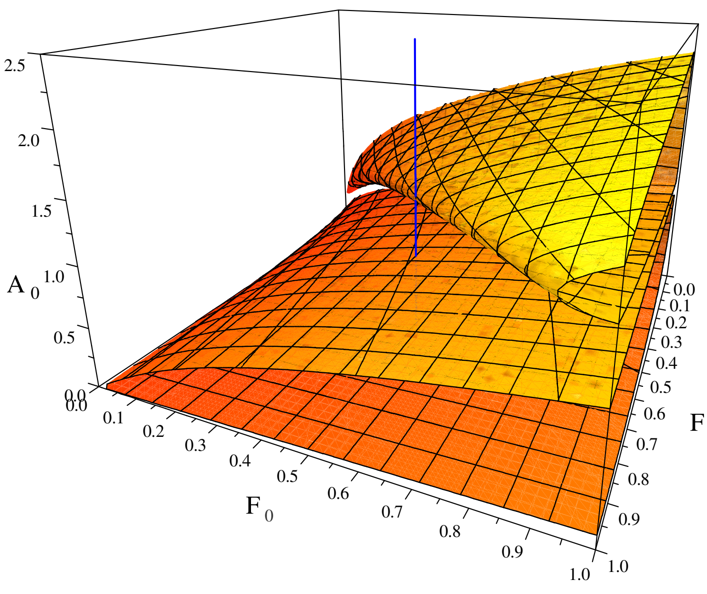

Figure 4: Jump manifold $\mathcal{J}(A_0; \gamma_*, \zeta_*, F, F_0)$, $\gamma_* = 0.0783$, $\zeta_* = 0.025$.

Next, we compute one border point. For the sake of example, we choose $F_0 = 0.5$ ($\gamma_* = 0.0783, \delta_* = 0.025$) and compute the corresponding border point as $F = 0.544\,860$, $A_0 = 1.238\,340$ as explained in the next subsection.

The blue vertical line, $(0.544\,860,\ 0.5,\ A_0)$ with A_0 variable, touches the upper lobe of the jump manifold exactly at the border point $(0.544\,860,\ 0.5,\ 1.238\,340)$.

3.2 Border sets

We shall now determine the condition for the border set: the set of points in the parameter space $(\gamma,\ \zeta,\ F_0,\ F)$ is such that the number of vertical tangents changes at these points. The mathematical condition for the border set is that the polynomial $J\left(A_0\right)$ given in Eq.(9a) and Table 2 has multiple roots.

The qualitative behavior of the polynomial equation $J\left(A_0\right)$ can be seen in Figs.1 and 4, where $2D$ and $3D$ projections of the implicit function $J\left(A_0;\gamma,\zeta,F,F_0\right)=0$ are shown. To find the parameter values for which the polynomial $J\left(A_0;\gamma,\zeta,F,F_0\right)$ has multiple roots, we demand that the resultant of $J\left(A_0\right)$ and its derivative $J'\left(A_0\right)=\frac{d}{dA_0}J\left(A_0\right)$ is zero [15]:

$$R\left(J,J';\gamma,\zeta,F,F_0\right)=0. \tag{12}$$

The resultant of the polynomial $J\left(A_0\right)$ and its derivative $J'\left(A_0\right)=\sum_{k=0}^{20}b_kA_0^k$ is a determinant of the $(m+n)\times(m+n)$ Sylvester matrix, $n=21$, $m=20$,

$$R\left(J,J';\gamma,\zeta,F,F_0\right)=\det\begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \dots & 0 & 0 & 0\\ 0 & a_n & a_{n-1} & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & a_1 & a_0 & 0\\ 0 & 0 & 0 & \dots & a_2 & a_1 & a_0\\ b_m & b_{m-1} & b_{m-2} & \dots & 0 & 0 & 0\\ 0 & b_m & b_{m-1} & \dots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \dots & b_1 & b_0 & 0\\ 0 & 0 & 0 & \dots & b_2 & b_1 & b_0 \end{pmatrix} \tag{13}$$

and is an enormously complicated polynomial in variables $\gamma,\ \zeta,\ F,\ F_0$. However, if we fix three parameters, say $\gamma,\ \zeta,\ F$, then the equation $R\left(J,J';F_0\right)=0$ can be solved numerically and thus the critical values of F_0 can be computed.

For example, we have solved equation (12), $R\left(J,J';\gamma_*,\zeta_*,F_*,F_0\right)=0$, for $\gamma_*=0.0783$, $\zeta_*=0.025$, $F_*=0.1$, obtaining the following real positive solutions: $F_0^{(1)}=0$, $F_0^{(2)}=0.092\,075$, $F_0^{(3)}=0.738\,510$, $F_0^{(4)}=6.532\,092$. In Figs.5, the border amplitudes $A_1\left(\Omega\right)$ are shown for $\gamma=0.0783$, $\zeta=0.025$, $F=0.1$ and $F_0^{(2)}$, $F_0^{(3)}$, $F_0^{(4)}$, with critical points marked with blue crosses. At these points, jumps just appear/disappear – there is a metamorphosis of the amplitude-frequency response function. For example, the function $A_1\left(\Omega\right)$ has no jumps for $F_0>6.532\,092$, and two jumps appear for $F_0<6.532\,092$, see Fig.8.4e plotted for $F_0=0.95$ in [1]. Vertical tangents at these points are also plotted with dashed lines. Blue dots denote extant points of jumps.

We have also solved equation (12), $R\left(J,J';\gamma_*,\zeta_*,F,F_{0*}\right)=0$, for $\gamma_*=0.0783$, $\zeta_*=0.025$, $F_{0*}=0.5$, obtaining real positive solutions: $F^{(1)}=0$, $F^{(2)}=0.026\,998\,9$, $F^{(3)}=0.077\,925\,6$, $F^{(4)}=0.544\,859\,5$. Next, for $\gamma=0.0783$, $\zeta=0.025$, $F_0=0.5$, and $F=0.544\,859\,5$, we have computed from Eqs.(8) the border value $A_0=1.238\,340$, see the blue vertical line in Fig.4.

3.3 Number of solutions of Eq.(5) for a given value of Ω

There are also other qualitative changes in the amplitudes $A_1\left(\Omega\right)$ controlled by the parameters. For example, the number of solutions of Eq.(5) for a given value of Ω may

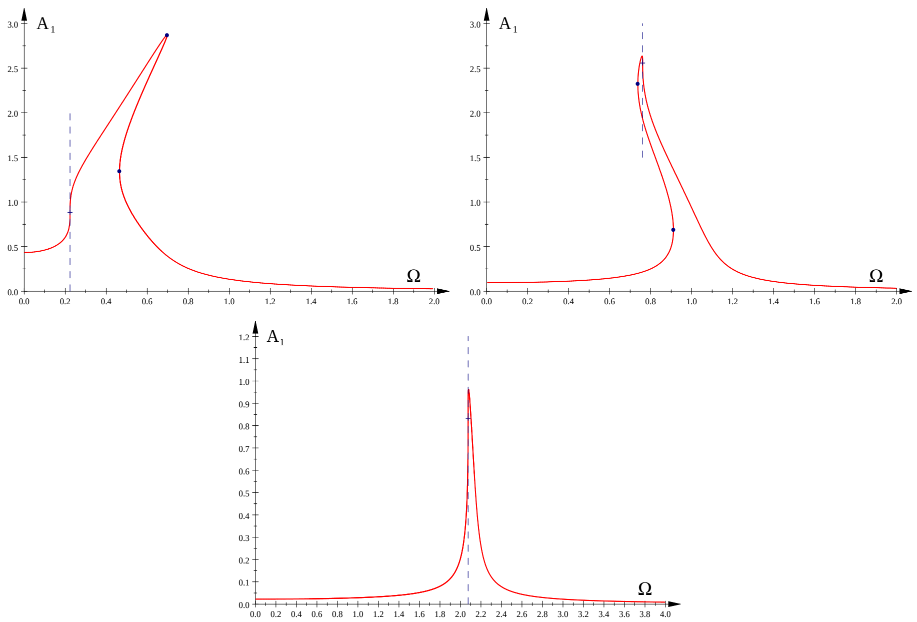

Figure 5: Amplitude-frequency response curves: $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, $F_0^{(2)} = 0.092$ (top left), $F_0^{(3)} = 0.7385$ (top right), $F_0^{(4)} = 6.532$ (bottom).

change. This happens when two vertical tangents appear at the same value of Ω.

To find a value of F_0 for which this occurs, we have to find a double root Ω of equations (8). For example, let $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$. Now, solving Eqs.(8) numerically for several values of F_0, we easily find that for $F_0 = 0.301\,007$, there is indeed a double root: $\Omega = 0.597\,114$, $A_0 = 0.679\,284$ and $\Omega = 0.597\,114$, $A_0 = 1.\,411\,787$. There is another similar case: for $F_0 = 0.429\,166$, there is a double root: $\Omega = 0.714\,419$, $A_0 = 0.459\,118$ and $\Omega = 0.714\,419$, $A_0 = 1.\,628\,271$, see Fig.6 as well as the related Fig.9.

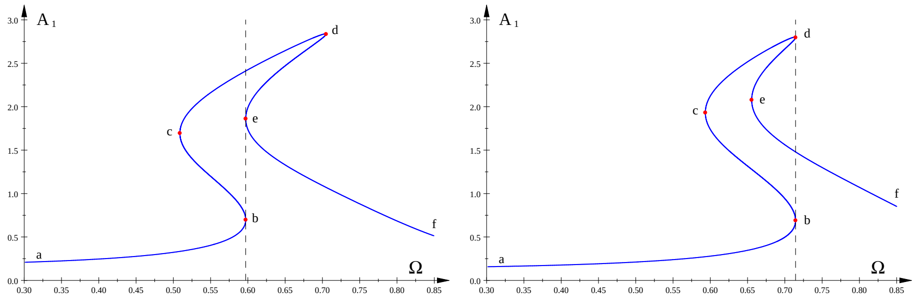

Figure 6: Amplitude-frequency response curves $A_1(\Omega)$: $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, $F_0 = 0.301$ (left), $F_0 = 0.429$ (right). Stable branches: a-b, c-d, e-f.

Therefore, for $F_0 \in (0.301,\ 0.429)$, equation (5) has five solutions for some values of Ω (three stable, two unstable), see, for example, Figs.2, 3, where $F_0 = 0.4$.

4 Numerical Verification and Analysis of the Results

We start with the verification of our results for the border sets obtained in Section 3, comparing them with numerical computations carried out for the equation (1). Consider, for example, the top right figure in Fig.5. In Fig.7, we show magnification of this critical curve with a vertical tangency on the red curve and two curves: just before (green) and just after (blue) the formation of the vertical tangency.

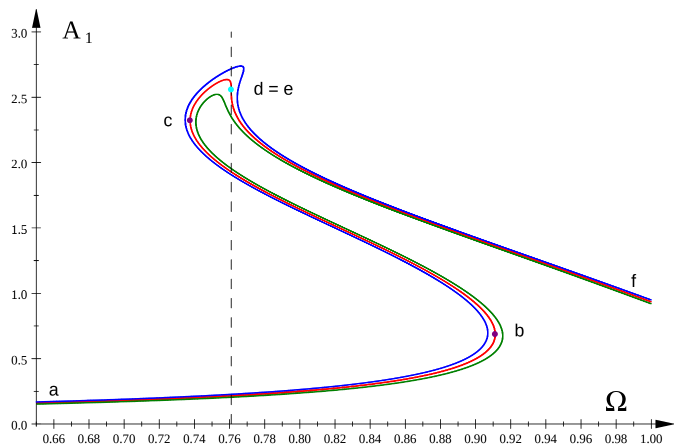

Figure 7: Implicit curves $A_1(\Omega)$: $\gamma = 0.0783$, $\zeta = 0.025$, $F_0 = 0.738\,510$ and $F = 0.095$ (green), $F = 0.1$ (red), $F = 0.105$ (blue). Critical vertical tangency on the red curve is at $(\Omega,\ A_1) = (0.761,\ 2.558)$ (light blue-green dot).

Recall that we have solved equation (12) for $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, obtaining four real positive solutions: $F_0^{(1)} = 0$, $F_0^{(2)} = 0.092\,075$, $F_0^{(3)} = 0.738\,510$, $F_0^{(4)} = 6.\,532\,092$. Curves in Fig.7 have been plotted for $\gamma = 0.0783$, $\zeta = 0.025$, $F_0^{(3)} = 0.738\,510$ and $F = 0.095$ (green), $F = 0.1$ (red), $F = 0.105$ (blue). We have decided to plot curves $A_1(\Omega)$ for the variable F since the shapes of these curves are very sensitive to this parameter.

When we pass from the green to the red curve, we note the formation of vertical tangency on the red curve. Stable branches on the red curve are: a-b, c-d=e, and d=e-f.

In Fig.8, we show the bifurcation diagrams computed by solving numerically Eq.(1) for the values of the parameters γ, ζ, F_0 such as in Fig.7 and $F = 0.114$ (green), $F = 0.116$ (blue), respectively.

These two bifurcation diagrams correspond qualitatively to the green and blue curves $A_1(\Omega)$ in Fig.7. The main difference between these two plots is a discontinuity of the blue curve corresponding to the creation of the jump phenomenon at $\Omega = 0.785$.

Note that the discontinuity appears in the interval $F \in (0.114,\ 0.116)$ while the analytically predicted value is $F = 0.1$. This discrepancy can be attributed to the error of the asymptotic method used to compute the solution (2a).

We now discuss the results obtained in Subsection 3.3. In Fig.9, two bifurcation diagrams are shown, corresponding to the amplitude-frequency curves shown in Fig.6. Two Figures 6 were computed for $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$ and $F_0 = 0.301$ and $F_0 = 0.429$ can be set together with Figures 8.4b, 8.4c, 8.4d from Ref. [1], computed

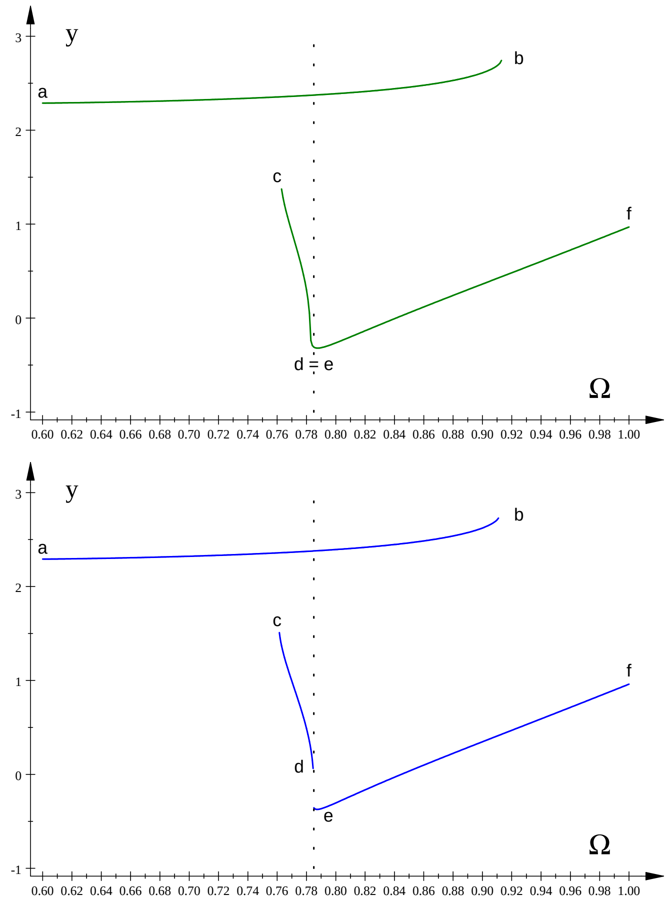

Figure 8: Bifurcation diagrams: $F = 0.114$ (green), $F = 0.116$ (blue), the values of the parameters γ, ζ, F_0 are the same as in Fig.7. Discontinuity is at $\Omega = 0.785$.

for the same values of γ, ζ, F and for $F_0 = 0.2$, $F_0 = 0.4$, $F_0 = 0.5$, respectively. The sequence of the amplitude-frequency curves plotted for $F_0 = 0.2,\ 0.301,\ 0.4,\ 0.429,\ 0.5$ shows the metamorphoses of these curves.

In Figures 6, there are two different jumps for the same value of Ω. Indeed, in the bifurcation diagrams shown in Fig.9, two different branches of the solution of Eq.(1) end or begin at the same value of Ω (these places are denoted in Fig.6 and Fig.9 as "b" and "e").

It follows that three stable solutions of Eq.(1) are in the interval $F_0 \in (0.284,\ 0.395)$, $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, while analytical prediction was $F_0 \in (0.301,\ 0.429)$ (see the end of Subsection 3.3). This discrepancy is again due to the unavoidable errors of the asymptotic method.

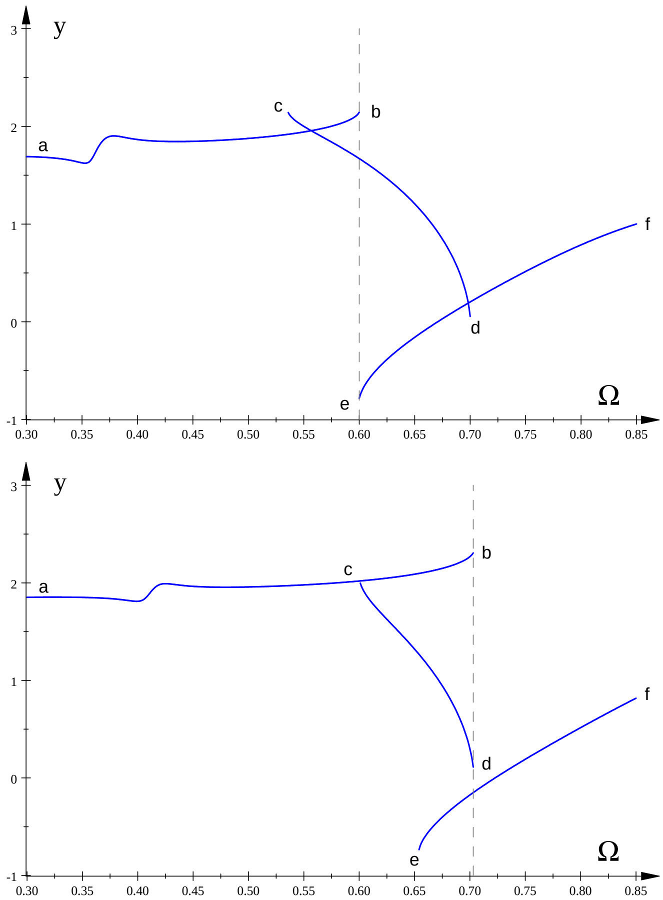

Figure 9: Bifurcation diagrams: $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, $F_0 = 0.284$ (top), $F_0 = 0.395$ (bottom).

5 Summary

Working in the implicit function framework [10], we have computed the jump manifold, cf. (10) and Table 2, including information about all jumps in the dynamical system (1).

Our work on the asymmetric Duffing oscillator is a supplementation and amplification of the results obtained by Kovacic and Brennan [1]. The sequence of Figures 8.4 (a) –(e), computed in [1] for $\gamma = 0.0783$, $\zeta = 0.025$, $F = 0.1$, and $F_0 = 0.01, 0.2, 0.4, 0.5, 0.95$, respectively, can be appended with Figs.5 and 6 computed for $F_0 = 0.092, 0.7385, 6.532$, and $F_0 = 0.301, 0.429$. The sequence of metamorphoses of the curve $A_1(\Omega)$ consists of the plots computed for $F_0 = \mathbf{0.01}, 0.092, \mathbf{0.2}, 0.301, \mathbf{0.4}, 0.429, \mathbf{0.5}, 0.7385, \mathbf{0.95}, 6.532$, where the numbers highlighted in bold correspond to Figs.8.4 (a)–(e) plotted in [1].

We show in Section 4 how a jump phenomenon arises in the dynamical system (1) and how it can be predicted based on a solution of Eq.(3), see Figs.7, 8. In short, the dynamical signature of the appearance of the jump phenomenon consists in a rupture of a stable branch, see Fig.8. Jumps are created at a border set, see Eqs.(12), (13). We have computed the values of parameter F_0, at which a change of multi-stability occurs.

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Generalized Bessel-Riesz Operator on Morrey Spaces with Different Measures

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Abstract: This study's primary area of interest is the generalized operators, which are defined with doubling measures by generalized Bessel-Riesz kernels with various measures in Morrey spaces. In terms of Bessel decay, the kernel satisfies a few key requirements. To prove that the integral operators are bounded, we will make use of Young, Holder, and Minköwski inequalities and a doubling measure. Additionally, we look into the relationship, we discover that the norm of these operators will be similarly constrained by the relationship between the elements of the kernel and the integral operators based on the norm of each kernel, although according to several measures. Additionally, we investigate the boundedness of pointwise multiplier operators in Morrey spaces using generalized fractional integrals and the generalized Bessel-Riesz operator.

Keywords: *generalized Bessel-Riesz operators; doubling measure; fractional integral; Morrey spaces.*

Mathematics Subject Classification (2010): 45P05, 70K99, 93B28, 47A30.

1 Introduction

This paper extends our recent findings in paper [1] by investigating the boundedness of Bessel-Riesz operators by a generalized kernel defined with doubling measures in Morrey spaces with various measures. Some basic requirements are being met by the operator's kernel in relation to Bessel decay. We will use the Young, Holder, and Minköwski inequalities and a doubling measure to demonstrate that the integral operators are bounded.

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In Morrey spaces, we also focus on the boundedness of pointwise multiplier operators. Here,

$$\|f : L^p(\mathbb{R}^n)\| := \left[\int_{\mathbb{R}^n} |f(t)|^p \, dt\right]^{1/p}$$

is defined, as a function $1 \le p < \infty$, and a group of f in a way that $\|f : L^p(\mathbb{R}^n)\| < \infty$. The definition of the Bessel-Riesz operator is

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{R}^n} G_{\alpha,\gamma}(|x-y|) f(y) \, dy$$

for every $f \in L^p_{\text{loc}}(\mathbb{R}^n)$, and $1 \le p < \infty$ for each instance of $G_{\alpha,\gamma} : (0,\infty) \to (0,\infty)$, with

$$G_{\alpha,\gamma}(t) := \frac{t^{\alpha-n}}{[1+t]^{\gamma}}, \quad 0 < \alpha < n, \ \gamma \ge 0.$$

The $G_{\alpha,\gamma}$, the Bessel-Riesz kernel, is referred to in this context. Bessel-Riesz operators are derived from Schrödinger's equation. Schrödinger's equation, a partial differential linear equation, (see [2] for some results of differential equations of order $1 < \alpha \le 2$, in a Banach space) is used to explain a quantum field system's wave function or state function. Schrödinger's equation is the quantum physics equivalent of Newton's law. In 1999 Kazuhiro Kurata et al. [3] conducted a research on the boundedness of integral operators with Lebesgue and generalized Morrey spaces. They then used this knowledge to estimate the Schrödinger operator $L_2 = -\triangle + V(x) + W(x)$ with nonnegative $V \in (RH)_\infty$ (reverse Hölders class) and small perturbed potentials W on Morrey spaces. Idris et al. [Theorem 6, [4]] reported the boundedness of Bessel-Riesz operators on Morrey spaces in 2016. They achieved outcomes for the boundedness of fractional integral operators that were comparable to the results by F. Chiarenza [5]. The boundedness of fractional integral operators when constructed on quasi-metric measure spaces was discussed by Eridani et al. in their study [6]. Even for Euclidean spaces, the research team's findings were novel. The boundedness of these operators for Euclidean spaces on Lebesgue and Morrey spaces was also demonstrated by Idris et al. in their paper [4], which also looked into the weighted boundedness of generalized Morrey spaces. Euclidean spaces are the most straightforward illustration of measure metric spaces. The weighted boundedness of generalized Morrey spaces has been studied by Kurata et al. [3]. We will use the Young, Holder, and Minköwski inequalities and a doubling measure to demonstrate the boundedness of these operators on Morrey spaces in Euclidean contexts. Additionally, we shall see that the generalized Bessel-Riesz operators' norm is constrained by the kernels' norm. Bessel-Riesz operators on Lebesgue spaces in measure metric spaces are bounded, which is a simple argument of the Young inequality, it was demonstrated by Saba et al. [7] using the Young inequality. Our entry into the subsequent Morrey space phenomenon is the second outcome since the generalized Bessel-Riesz operator is bounded on Lebesgue spaces [1]. The ideal constant in the Young inequality is 1, and it is used throughout the research. However, we still do not know what the optimum constant in Morrey spaces is at this point. As a result, we focus on generalized Bessel-Riesz operators in Morrey spaces in this study. We will also discuss the circumstance in which the measure meets the doubling requirement. For the relevant inequalities that were derived in [8], [9], the resulting criteria are both necessary and sufficient. When W is a scalar operator, Kurata et al. [3] have demonstrated that $W.\ I_{\alpha,\gamma}$ is bounded on generalized Morrey spaces. We will also work on this operator for boundedness with the generalized operator on Morrey spaces. See citation [3] for the examples of using the aforementioned operators in settings involving Euclidean spaces.

2 Doubling Conditions

We take into account $\rho : (0, \infty) \to (0, \infty)$ and define (DC) as a collection of ρ such that

$$\frac{1}{2} \leq \frac{s}{t} \leq 2 \;\Rightarrow\; \frac{1}{C} \leq \frac{\rho(s)}{\rho(t)} \leq C$$

for some $C \geq 1$. If $\rho \in$(DC), then we have

$$\rho(R) \sim \int_R^{2R} \frac{\rho(t)}{t}\, dt, \quad \text{or} \quad C^* \rho(R) \leq \int_R^{2R} \frac{\rho(t)}{t}\, dt \leq C^{**} \rho(R)$$

for some $C^{**} \geq C^* > 0$. Additionally, if $\rho \in$ (DC), then $G_{\rho,\gamma} \in$ (DC), where

$$G_{\rho,\gamma}(t) := \frac{\rho(t)}{t^n [1+t]^\gamma}, \quad t > 0.$$

Assume that μ is a random measure on $\mathbb{R}^n$. We consider a group of measures that meet the growth condition, denoted by (GC). Now, since $B(a, R) := \{x \in \mathbb{R}^n : |x - a| < R\}$ in $\mathbb{R}^n$ exists, we define $\mu \in$ (GC) if and only if $C_1 > 0$ such that

$$\mu(B(a, R)) \leq C_1 R^n$$

exists for all open balls. See [6] for more details on a doubling measure.

Using the definitions provided above, we attempt to estimate

$$\|G_{\rho,\gamma} : L^s(\mu)\| := \left(\int_{\mathbb{R}^n} |G_{\rho,\gamma}(|x|)|^s \, d\mu(x) \right)^{1/s}, \quad s \geq 1.$$

For $1 \leq s < \infty$ and $R > 0$, we take into account the following:

$$\begin{aligned}
&\int_{|x|<R} \frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}\, d\mu(x) = \sum_{k=-\infty}^{-1} \int_{2^k R \leq |x| < 2^{k+1} R} \frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}\, d\mu(x) \\
&\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^k R)^s \mu(B(0, 2^k R))}{(2^k R)^{sn}[1+2^k R]^{s\gamma}} \leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^k R)^s}{(2^k R)^{(s-1)n}} \\
&\leq C \sum_{k=-\infty}^{-1} \int_{2^k R}^{2^{k+1} R} \frac{\rho(t)^s}{t^{(s-1)n+1}}\, dt = C \int_0^R \frac{\rho(t)^s}{t^{(s-1)n+1}}\, dt, \quad \text{and also}
\end{aligned}$$

$$\begin{aligned}
&\int_{R \leq |x|} \frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}\, d\mu(x) = \sum_{k=1}^{\infty} \int_{2^k R \leq |x| < 2^{k+1} R} \frac{\rho(|x|)^s}{|x|^{sn}[1+|x|]^{s\gamma}}\, d\mu(x) \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)^s \mu(B(0, 2^k R))}{(2^k R)^{sn}[1+2^k R]^{s\gamma}} \leq C \sum_{k=0}^{\infty} \frac{\rho(2^k R)^s}{(2^k R)^{(s-1)n+s\gamma}} \\
&\leq C \sum_{k=0}^{\infty} \int_{2^k R}^{2^{k+1} R} \frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}\, dt = C \int_R^{\infty} \frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}\, dt,
\end{aligned}$$

where a constant, not necessarily the same one, is denoted by the letter $C > 0$ at all time. At this stage, we define

$$\|G_{\rho,\gamma} : L^s(\mu)\| = \sup R > 0 \left(\int_0^R \frac{\rho(t)^s}{t^{(s-1)n+1}}\, dt + \int_R^\infty \frac{\rho(t)^s}{t^{(s-1)n+s\gamma+1}}\, dt \right)^{1/s}$$

for $1 \le s < \infty,\ \mu \in$ (GC) and $\rho \in$ (DC).

3 Morrey Spaces with the Generalized Bessel-Riesz Operator

We begin this section with the definition that is given below. Take into account $0 < \lambda < 1 \le p < \infty,\ B := B(a,R)$, and $\mu(B(a,R)) \sim R^n$. If and only if

$$\|f : L^{p,\lambda}(\nu,\mu)\| = \sup_B \left(\frac{1}{\mu(B)^\lambda} \int_B |f(y)|^p \, d\nu(y) \right)^{1/p} < \infty$$

holds, then we define $f \in L^{p,\lambda}(\nu,\mu)$. We define

$$f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^{\mathrm{C}}}$$

for each $f \in L^{p,\lambda}(\nu,\mu)$ and $\tilde{B} := B(a, 2R)$, given that

$$\|f_1 : L^p(\nu)\| = \left[\int_{\tilde{B}} |f(y)|^p \, d\nu(y) \right]^{1/p} \le \mu(\tilde{B})^{\lambda/p} \|f : L^{p,\lambda}(\nu,\mu)\| < \infty.$$

For each B, we arrive at the following estimation when $f_1 \in L^p(\nu)$:

$$\begin{aligned} \left[\int_B |I_{\rho,\gamma} f_1(y)|^p \, d\mu(y) \right]^{1/p} &\le \|I_{\rho,\gamma} f_1 : L^p(\mu)\| \\ &\le C \, \|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|f_1 : L^p(\nu)\| \\ &\le C \, \mu(B)^{\lambda/p} \|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|f : L^{p,\lambda}(\nu,\mu)\|, \end{aligned}$$

and of course, we will come to the following estimation:

$$\left[\frac{1}{\mu(B)^\lambda} \int_B |I_{\rho,\gamma} f_1(y)|^p \, d\mu(y) \right]^{1/p} \le C \, \|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|f : L^{p,\lambda}(\nu,\mu)\|.$$

Contrarily, we have the estimation given below for each $x \in B$:

$$\begin{aligned} |I_{\rho,\gamma} f_2(x)| &\le \int_{\tilde{B}} I_{\rho,\gamma} f_2(x) d\mu(x) \\ &\le \int_{|x-y| \ge R} I_{\rho,\gamma} f_2(x) d\mu(x) \\ &= \sum_{k=0}^\infty \int_{2^k R \le |x-y| < 2^{k+1} R} I_{\rho,\gamma} f_2(x) d\mu(x) \\ &\le \sum_{k=0}^\infty \frac{\rho(2^k R)}{(2^k R)^{n+\gamma}} \int_{|x-y| < 2^{k+1} R} |f(y)| \, d\nu(y) \\ &\le C \, \|f : L^{p,\lambda}(\nu,\mu)\| \sum_{k=0}^\infty \frac{\rho(2^k R) \nu(B(x, 2^{k+1} R))^{1-1/p}}{(2^k R)^{n - [n\lambda/p] + \gamma}}. \end{aligned}$$

Considering that $\nu \in$ (GC),

$$|I_{\rho,\gamma} f_2(x)| \le C \, \|f : L^{p,\lambda}(\nu,\mu)\| \sum_{k=0}^{\infty} \frac{\rho(2^k R)}{(2^k R)^{[n/p]-[n\lambda/p]+\gamma}}$$
$$\le C \, \|f : L^{p,\lambda}(\nu,\mu)\| \int_R^{\infty} \frac{\rho(t)}{t^{1+\gamma+[n(1-\lambda)/p]}} \, dt$$
$$\le C \, R^{n(\lambda-1)/p} \|f : L^{p,\lambda}(\nu,\mu)\| \int_R^{\infty} \frac{\rho(t)}{t^{1+\gamma}} \, dt$$
$$\le C \, R^{n(\lambda-1)/p} \|f : L^{p,\lambda}(\nu,\mu)\| \cdot \|G_{\rho,\gamma} : L^1(\mu)\|.$$

And ultimately, we shall have that for any open ball B, we come to

$$\left[\frac{1}{\mu(B)} \int_B |I_{\rho,\gamma} f_2(x)|^p \, d\mu(x)\right]^{1/p} \le C \, R^{n(\lambda-1)/p} \|f : L^{p,\lambda}(\nu,\mu)\| \cdot \|G_{\rho,\gamma} : L^1(\mu)\|,$$

$$\text{or } \left[\frac{1}{\mu(B)^{\lambda}} \int_B |I_{\rho,\gamma} f_2(x)|^p \, d\mu(x)\right]^{1/p} \le C \, \|f : L^{p,\lambda}(\nu,\mu)\| \cdot \|G_{\rho,\gamma} : L^1(\mu)\|.$$

Corollary 3.1 *Let us say there are $\nu \in$ (GC) and $1 < p < \infty$. In the case when both $f \in L^{p,\lambda}(\nu,\mu)$ and $G_{\rho,\gamma} \in L^1(\mu)$ are true, one has $I_{\rho,\gamma} f \in L^{p,\lambda}(\mu)$. In addition, there is $C > 0$ such that*

$$\|I_{\rho,\gamma} f : L^{p,\lambda}(\mu)\| \le C \, \|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|f : L^{p,\lambda}(\nu,\mu)\|.$$

3.1 Minkӧwski's inequality

Before stating our key findings on the boundedness of $I_{\rho,\gamma}$, we take into consideration the following simple finding [10].

Lemma 3.1 *Assume we are given $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$. When considering any measure ν and μ on $\mathbb{R}^n$,*

$$\left[\int_{\mathbb{R}^n} \left|\int_{\mathbb{R}^n} G(x-y) \, d\nu(y)\right|^p d\mu(x)\right]^{1/p} \le \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n} |G(x-y)|^p \, d\mu(x)\right]^{1/p} d\nu(y).$$

4 Main Results

Theorem 4.1 *Let ν be any measure on $\mathbb{R}^n$ and $\mu \in$ (GC).*
If there is $C_s > 0$ such that $f \in L^{1,\lambda}(\nu)$ and $G_{\rho,\gamma} \in L^s(\mu)$, then

$$\|I_{\rho,\gamma} f : L^{s,\lambda}(\mu)\| \le C_s \|G_{\rho,\gamma} : L^s(\mu)\| \cdot \|f : L^{1,\lambda}(\nu)\|, \quad s \ge 1.$$

Proof. According to Minkӧwski's inequality, with $1 \le s < \infty$, we have

$$\left[\frac{1}{\mu(B)^{\lambda}} \int_B |I_{\rho,\gamma} f(x)|^s \, d\mu(x)\right]^{1/s} = \left(\int_B \left|\int_B \frac{1}{\mu(B)^{\lambda}} G_{\rho,\gamma}(|x-y|) f(y) \, d\nu(y)\right|^s d\mu(x)\right)^{1/s}$$
$$\le \int_B \left(\int_B \frac{1}{\mu(B)^{\lambda}} |G_{\rho,\gamma}(|x-y|) f(y)|^s \, d\mu(x)\right)^{1/s} d\nu(y)$$
$$\le \int_B \left(\int_B |G_{\rho,\gamma}(|x-y|)|^s \, d\mu(x)\right)^{1/s} |\frac{1}{\mu(B)^{\lambda}} f(y)| \, d\nu(y)$$
$$\le C \, \|G_{\rho,\gamma} : L^s(\mu)\| \cdot \|f : L^{1,\lambda}(\nu)\|.$$

So

$$\left[\frac{1}{\mu(B)^{\lambda}}\int_B |I_{\rho,\gamma}f(x)|^s\,d\mu(x)\right]^{1/s} \le \sup_{x\in B}\|I_{\rho,\gamma}f(x) : L^{s,\lambda}\| \tag{1}$$

$$\le C\,\|G_{\rho,\gamma} : L^s(\mu)\|\cdot\|f : L^{1,\lambda}(\nu)\|. \quad \blacksquare \tag{2}$$

In accordance with the aforementioned Theorem 4.1, we additionally have the following.

Corollary 4.1 *Consider the case when ν is any measure on $\mathbb{R}^n$ and $\mu \in$ (GC). There exists $C > 0$ such that if $f \in L^{1,\lambda}(\nu)$ and $G_{\rho,\gamma} \in L^1(\mu)$, then*

$$\|I_{\rho,\gamma}f : L^{1,\lambda}(\mu)\| \le C\|G_{\rho,\gamma} : L^1(\mu)\|\cdot\|f : L^{1,\lambda}(\nu)\|.$$

Next, we come to a particular case of Young's inequality.

Suppose

$$\frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1, \quad \text{or} \quad 1 = \frac{1}{s} + 1 - \frac{1}{p} + 1 - \frac{1}{q} = \frac{1}{s} + \frac{1}{p'} + \frac{1}{q'}.$$

Note that

$$p = q'\left(1 - \frac{p}{s}\right), \quad q = p'\left(1 - \frac{q}{s}\right).$$

Theorem 4.2 *Assume $\mu, \nu \in$ (GC). If $f \in L^p(\nu)$ and $G_{\rho,\gamma} \in L^q(\mu)$, then there exists $C > 0$ so that*

$$\|I_{\rho,\gamma}f : L^{s,\lambda}(\mu)\| \le C\|G_{\rho,\gamma} : L^q(\mu)\|\cdot\|f : L^{p,\lambda}(\nu)\|, \quad \frac{1}{s} = \frac{1}{p} + \frac{1}{q} - 1.$$

Proof. With Hölder's inequality, we start with the following:

$$\begin{aligned}
&\left[\frac{1}{\mu(B)^{\lambda}}\int_B |I_{\rho,\gamma}f(x)|^s\,d\mu(x)\right]^{1/s}\\
&\le \int_B \frac{1}{\mu(B)^{\lambda}}|f(y)|^{p/s+(1-p/s)}|G_{\rho,\gamma}(|x-y|)|^{q/s+(1-q/s)}\,d\nu(y)\\
&\le \left[\int_B |G_{\rho,\gamma}(|x-y|)|^q|f(y)|^p\,d\nu(y)\right]^{1/s}\\
&\left[\int_B \frac{1}{\mu(B)^{\lambda}}|f(y)|^{q'(1-p/s)}\,d\nu(y)\right]^{1/q'}\\
&\times\left[\int_B |G_{\rho,\gamma}(|x-y|)|^{p'(1-q/s)}\,d\nu(y)\right]^{1/p'}\\
&= \left[\int_B |G_{\rho,\gamma}(|x-y|)|^q|f(y)|^p\,d\nu(y)\right]^{1/s}\\
&\left[\int_B \frac{1}{\mu(B)^{\lambda}}|f(y)|^p\,d\nu(y)\right]^{1/q'}\\
&\times\left[\int_B |G_{\rho,\gamma}(|x-y|)|^q\,d\nu(y)\right]^{1/p'}.
\end{aligned}$$

The right-hand side will now be estimated for every $f \in L^{p,\lambda}(\nu,\mu)$ and $\tilde{B} := B(a,2R)$, we define

$$f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^{C}}.$$

$$\text{Since } \|f_1 : L^p(\nu)\| = \left[\int_{\tilde{B}} |f(y)|^p \, d\nu(y)\right]^{1/p} \leq \mu(\tilde{B})^{\lambda/p}\|f : L^{p,\lambda}(\nu,\mu)\| < \infty,$$

we have $f_1 \in L^p(\nu)$, and for every B, we come to the following estimation:

$$\begin{aligned}
\int_B |G_{\rho,\gamma}(|x-y|)|^q |f_1(y)|^p \, d\nu(y) &\leq \||G_{\rho,\gamma}(|x-y|)|^q f_1^p : L^p(\nu)\| \\
&\leq C \, \|G_{\rho,\gamma}(|x-y|) : L^q(\mu)\|^q \cdot \|f_1 : L^p(\nu)\|^p \\
&\leq C \, \mu(B)^{\lambda/p} \|G_{\rho,\gamma}(|x-y|) : L^q(\mu)\|^q \cdot \|f : L^{p,\lambda}(\nu,\mu)\|^p \\
&\leq \|G_{\rho,\gamma}(|x-y|) : L^q(\mu)\|^q \cdot \|f : L^{p,\lambda}(\nu,\mu)\|^p.
\end{aligned}$$

On the other hand, for every $x \in B$, we have the following estimation:

$$\begin{aligned}
\int_B |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) &\leq \int_{\tilde{B}} |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) \\
&\leq \int_{|x-y|\geq R} |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) \\
&= \sum_{k=0}^{\infty} \int_{2^k R \leq |x-y| < 2^{k+1}R} |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) \\
&\leq \sum_{k=0}^{\infty} \left(\frac{\rho(2^k R)}{(2^k R)^{n+\gamma}}\right)^q \int_{|x-y|<2^{k+1}R} |f(y)|^p \, d\nu(y) \\
&\leq C \, \|f : L^{p,\lambda}(\nu,\mu)\| \sum_{k=0}^{\infty} \left(\frac{\rho(2^k R)\nu(B(x,2^{k+1}R))^{1-1/p}}{(2^k R)^{n-[n\lambda/p]+\gamma}}\right)^q.
\end{aligned}$$

Since $\nu \in (GC)$, we have

$$\begin{aligned}
\int_B |G_{\rho,\gamma}(|x-y|)|^q |f_2(y)|^p \, d\nu(y) &\leq C \, \|f : L^{p,\lambda}(\nu,\mu)\|^p \sum_{k=0}^{\infty} \left(\frac{\rho(2^k R)}{(2^k R)^{[n/p]-[n\lambda/p]+\gamma}}\right)^q \\
&\leq C \, \|f : L^{p,\lambda}(\nu,\mu)\|^p \int_R^{\infty} \left(\frac{\rho(t)}{t^{1+\gamma+[n(1-\lambda)/p]}}\right)^q dt \\
&\leq C \, R^{n(\lambda-1)q/p} \|f : L^{p,\lambda}(\nu,\mu)\|^p \int_R^{\infty} \left(\frac{\rho(t)}{t^{1+\gamma}}\right)^q dt \\
&\leq C \, R^{n(\lambda-1)q/p} \|f : L^{p,\lambda}(\nu,\mu)\|^p \cdot \|G_{\rho,\gamma} : L^q(\mu)\|^q \\
&\leq C \|f : L^{p,\lambda}(\nu,\mu)\|^p \cdot \|G_{\rho,\gamma} : L^q(\mu)\|^q,
\end{aligned}$$

it implies

$$\int_B |G_{\rho,\gamma}(|x-y|)|^q |f(y)|^p \, d\nu(y) \leq C \|f : L^{p,\lambda}(\nu,\mu)\|^p \cdot \|G_{\rho,\gamma} : L^q(\mu)\|^q. \tag{3}$$

Now we want to estimate the right-hand side, especially when for $x \in B$ and $R > 0$, we will have

$$\int_B |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) = \int_{|x-y|<R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) + \int_{|x-y|\geq R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y).$$

We start with

$$\begin{aligned}
\int_{|x-y|<R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) &= \sum_{k=-\infty}^{-1} \int_{2^kR\leq|x-y|<2^{k+1}R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) \\
&\sim C \sum_{k=-\infty}^{-1} G_{\rho,\gamma}(2^kR)^q \nu(B(x,2^{k+1}R)) \\
&\leq C \sum_{k=-\infty}^{-1} \frac{\rho(2^kR)^q}{(2^kR)^{(q-1)n}} \\
&\leq C \int_0^R \frac{\rho(t)^q}{t^{(q-1)n+1}} \, dt \\
&\leq C\|G_{\rho,\gamma} : L^q(\mu)\|^q,
\end{aligned}$$

and also

$$\begin{aligned}
\int_{|x-y|\geq R} |G_{\rho,\gamma}(|x-y|)|^q \, d\nu(y) &\sim C \sum_{k=0}^{\infty} G_{\rho,\gamma}(2^kR)^q \nu(B(x,2^{k+1}R)) \\
&\leq C \sum_{k=0}^{\infty} \frac{\rho(2^kR)^q}{(2^kR)^{nq-n+q\gamma}} \\
&\leq C \int_R^{\infty} \frac{\rho(t)^q}{t^{n(q-1)+q\gamma+1}} \, dt \\
&\leq C\|G_{\rho,\gamma} : L^q(\mu)\|^q.
\end{aligned}$$

Up to now, for every $x \in B$, we already have

$$\begin{aligned}
\left[\frac{1}{\mu(B)^\lambda} \int_B |I_{\rho,\gamma}f(x)|^s \, d\mu(x)\right]^{1/s} &\leq C\|G_{\rho,\gamma} : L^q(\mu)\|^{qs/p'} \cdot \|f : L^{p,\lambda}(\nu)\|^{sp/q'} \\
&\cdot \left[\int_B |G_{\rho,\gamma}(|x-y|)|^q |f(y)|^p \, d\nu(y)\right]^{1/s}.
\end{aligned}$$

By using (1) and (3), finally, we have

$$\begin{aligned}
&\|I_{\rho,\gamma}f(x) : L^{s,\lambda}\|^s \leq C\,\|G_{\rho,\gamma} : L^q(\mu)\|^{qs/p'} \cdot \|f : L^{p,\lambda}(\nu)\|^{sp/q'} \\
&\leq C\|f : L^{p,\lambda}(\nu,\mu)\|^p \cdot \|G_{\rho,\gamma} : L^q(\mu)\|^q, \\
&\leq C\,\|G_{\rho,\gamma} : L^q(\mu)\|^{q+qs/p'} \cdot \|f : L^{p,\lambda}(\nu)\|^{p+sp/q'}.
\end{aligned}$$

So

$$q + \frac{qs}{p'} = s = p + \frac{sp}{q'}.$$

■

Consequently, we also have the following result.

Corollary 4.2 *Consider $\mu, \nu \in$ (GC). If $f \in L^{s,\lambda}(\nu)$ and $G_{\rho,\gamma} \in L^1(\mu)$, now there is $C > 0$, and thus*

$$\|I_{\rho,\gamma} f : L^{s,\lambda}(\mu)\| \le C\|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|f : L^{s,\lambda}(\nu)\|, \quad s \ge 1.$$

5 Pointwise Multiplier Operators

Say that $1 < p < \infty$ and $1 = 1/p + 1/p'$ with $f \in L^{p,\lambda}(\mu)$ and $g \in L^{p',\lambda}(\mu)$, respectively. The Hölder inequality will then result in $\|f \cdot g : L^{1,\lambda}(\mu)\| \le \|f : L^{p,\lambda}(\mu)\| \cdot \|g : L^{p',\lambda}(\mu)\|$.

We will examine a pointwise multiplier operator W by

$$W : f \mapsto W \cdot f, \quad \text{with} \quad [W \cdot f](x) := W(x) \cdot f(x), \quad x \in \mathbb{R}^n.$$

Therefore, based on the Hölder inequality, if $W \in L^{p',\lambda}(\mu)$, then W is a bounded operator from $L^{p,\lambda}(\mu)$ to $L^{1,\lambda}(\mu)$, with $\|W \cdot f : L^{1,\lambda}(\mu)\| \le \|W : L^{p',\lambda}(\mu)\| \cdot \|f : L^{p,\lambda}(\mu)\|$, $1 < p < \infty$.

The following is another illustration. Assume we take a look at a fractional integral operator I_α, and we define

$$W \cdot I_\alpha : f \mapsto W \cdot I_\alpha f, \quad \text{with} \quad [W \cdot I_\alpha f](x) := W(x) \cdot I_\alpha f(x), \quad x \in \mathbb{R}^n.$$

Reiterating the previous point, given that $1 < p < n/\alpha$ and $1/q + \alpha/n = 1/p$ are both affected by the Hölder inequality, we get

$$\|W \cdot I_\alpha f : L^{p,\lambda}\| \le \|W : L^{n/\alpha,\lambda}\| \cdot \|I_\alpha f : L^{q,\lambda}\| \le C\|W : L^{n/\alpha,\lambda}\| \cdot \|f : L^{p,\lambda}\|.$$

In other words, if $W \in L^{n/\alpha,\lambda}$, then $W \cdot I_\alpha : L^{p,\lambda} \to L^{p,\lambda}$ is a bounded operator.

From our primary findings, we also have the following.

Corollary 5.1 *Assume ν is any measure on $\mathbb{R}^n$ and $\mu \in$ (GC). If $f \in L^{1,\lambda}(\nu)$, $W \in L^{s',\lambda}(\mu)$ and $G_{\rho,\gamma} \in L^s(\mu)$ are true, then*

$$W \cdot I_{\rho,\gamma} : L^{1,\lambda}(\nu) \to L^{1,\lambda}(\mu)$$

is a bounded operator. To put it another way, for any $s \in [1, \infty)$, there exists $C_s > 0$ such that

$$\|W \cdot I_{\rho,\gamma} f : L^{1,\lambda}(\mu)\| \le C_s\|W : L^{s',\lambda}(\mu)\| \cdot \|G_{\rho,\gamma} : L^s(\mu)\| \cdot \|f : L^{1,\lambda}(\nu)\|.$$

Corollary 5.2 *Suppose $\mu, \nu \in$ (GC).*
If $f \in L^{s,\lambda}(\nu)$, $W \in L^{s',\lambda}(\mu)$, and $G_{\rho,\gamma} \in L^{p',\lambda}(\mu)$, then

$$W \cdot I_{\rho,\gamma} : L^{s,\lambda}(\nu) \to L^{1,\lambda}(\mu)$$

is a bounded operator. That is, for every $s \in [1, \infty)$, there exists $C_s > 0$ such that

$$\|W \cdot I_{\rho,\gamma} f : L^{1,\lambda}(\mu)\| \le C\|G_{\rho,\gamma} : L^1(\mu)\| \cdot \|W : L^{p',\lambda}(\mu)\| \cdot \|f : L^{s,\lambda}(\nu)\|.$$

The next result is our last corollary.

Corollary 5.3 *Suppose $1/s + 1/q' = 1/p$, and $\mu, \nu \in$ (GC).*
If $f \in L^{p,\lambda}(\nu)$, $W \in L^{q',\lambda}(\mu)$ and $G_{\rho,\gamma} \in L^q(\mu)$, then

$$W \cdot I_{\rho,\gamma} : L^{p,\lambda}(\nu) \to L^{p,\lambda}(\mu)$$

is a bounded operator. That is, there exists $C > 0$ such that

$$\|W \cdot I_{\rho,\gamma} f : L^{p,\lambda}(\mu)\| \le C\|G_{\rho,\gamma} : L^q(\mu)\| \cdot \|W : L^{q',\lambda}(\mu)\| \cdot \|f : L^{p,\lambda}(\nu)\|.$$

6 Conclusions

As a result of this investigation, we have learned that generalized Bessel-Riesz operators defined with doubling measures in Morrey spaces with various measures are working toward boundedness. Regarding Bessel decay, the kernel of the operators satisfies a few essential characteristics. To prove that the integral operators are bounded, we used the Young, Hölder, and Minköwski inequalities and a doubling measure. The norm of these generalized operators is similarly bounded by the norm of their respective kernels, but with different measures, according to our investigation of the relationship between the kernel's parameters and generalized integral operators. The Bessel-Riesz kernel is used in studying the behavior of the solution of a Schrödinger type equation [3] which is related to quantum mechanics. In future we will consider it for the generalized Bessel-Riesz kernel. [11] investigated a new discrete chaotic system with rational fraction including the symmetry. Furthermore, symmetry properties of a nonlinear two-dimensional space-fractional diffusion equation with the Riesz potential of the order $\alpha \in (0, 1)$ will be further considered.

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8 Availability Statement

While the results of the research are being commercialized, the data that were used to support the findings of this study are currently under embargo. After the publication of this article, requests for data will be taken into consideration by the corresponding author.

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Tsunami Wave Simulation in the Presense of a Barrier

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Abstract: A tsunami is a series of waves that are generally caused by a vertical change in the seabed due to an earthquake beneath or on the seabed. Tsunamis usually strike coastal areas and result in damage to the shoreline, it can destroy buildings and roads and even take the lives of those who are in the area. One way to reduce the impact of a tsunami is to know the dangers of a tsunami, including natural signs. So, in this paper, it is shown by numerical simulation using the finite difference method, namely, by adding a barrier to the shallow water wave equation. The simulation results obtained in the presence of a barrier, show that the Tsunami waves are split due to hitting the barrier and experience a reduction in wave strength.

Keywords: *tsunami; shallow water equation; finite difference method.*

Mathematics Subject Classification (2010): 65L12, 76M20.

1 Introduction

Indonesia as a country that is located between three world plates, namely, the Eurasian, Indoaustralian and Pacific plates, has a high potential for natural disasters [1, 2]. These plates have a high seismic activity, which causes the emergence of many natural disasters [3,4], one of which is the occurrence of earthquakes as a primary impact of seismic activity and tsunamis as a secondary impact [2].

Tsunamis are one of the most dangerous natural disasters and damage the area around the coast [5]. Tsunamis arise as a result of displacement of large volumes of water due to earthquakes, volcanic eruptions, landslides or other phenomena that occur above or below the seabed [6]. The sea waves are not dangerous if their height does not exceed

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1 meter. However, it becomes disastrous when the wave energy is concentrated and the wavelength is greater than the depth of the sea. When the wave enters the shallow water zone, the wave velocity at the foreshore decreases sharply and the wave height increases tenfold [5].

The impact of the tsunami waves causes huge losses to humans, both in terms of the large number of casualties and the large losses in the economic field. Coastal areas affected by tsunami waves require a long time and expensive economic resources to restore [7]. Recorded in the last two decades, there have been 12 tsunamis out of 252 earthquakes with a total loss of 79.5 trillion rupiah [8]. Based on this, to minimize losses that will occur, disaster mitigation is needed.

Based on Law of the Republic of Indonesia Number 24 of 2007 concerning disaster management, mitigation is defined as a series of efforts to reduce disaster risk, both through physical development and awareness and increased capacity to face disaster threats. The embodiment of examples of mitigation activities is spatial planning, development arrangements, arrangements for infrastructure development and building layout, as well as the provision of education, counseling and training, both conventional and modern. In Indonesia, orientation in minimizing disaster risk is more towards emergency or curative handling and has not yet led to preventive aspects [9]. Therefore, it is necessary to improve the understanding of physical, engineering and social factors related to disaster mitigation implementation.

According to the initial joint rapid assessment report by the Central BMKG, BNPB, National Media, regional PUSDALOPS and community responses in the evaluation of the tsunami early warning system in the event of the Aceh earthquake and tsunami on April 11, 2012, no one really knew when the tsunami hit an area and how big was the strength of the tsunami until the time was so critical, it quickly passed and the earthquake was felt until the impact of the first wave [11]. The danger from tsunami waves is so unpredictable, sudden and extraordinary, it is almost impossible to avoid it [5]. However, other countries experiencing similar disasters have carried out several related studies in minimizing disaster risk by constructing tsunami barriers using analytical research methods and numerical modeling approaches [5].

The shallow water equation is commonly used in describing fluid problems that are based on physical conservation. With increased computation capabilities and refinement of the numerical aspects related to boundary conditions, it is possible to overcome the inherent limitations of the classical depth mean model. Shallow water equation models have been widely applied in atmospheric flows, storms, water flows around the pier, tsunami prediction, and so on.

Research on fluid computing has been widely studied, including the Airway Pressure Valve [12], Solitary Wave [13], Navier-Stokes Equation [14], Shallow Water Equation [15], [16]. The shallow water equations can be solved using the finite difference method. The method used to solve partial differential equations is the finite difference method [17,18].

In this paper, the concept of a tsunami barrier as an obstacle is presented as an attempt to deal with a complex breakwater configuration. The breakwater is studied to identify the hydrodynamic induced tsunami.

2 Research Method

2.1 Finite difference method

The Finite Difference Method is a method used in solving Partial Differential Equations (PDP) which can be used to approach the Taylor series [19]. The differential equation is an estimate of the value δ at the calculation points $U_{1,1}, U_{1,2}, \ldots, U_{i,j}, \ldots$ for the estimation it can be done by substituting the derivative of the partial differential equation using the difference estimate up to [20].

The first derivative can be calculated using the forward difference approach with $\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t}$ of a differential equation based on the Taylor series which can be written as below:

$$\frac{\partial u(x,y,t)}{\partial x} = \frac{1}{\triangle x}\left(u(x+\triangle x,y,t)\right) - u(x,y,t), \tag{1}$$

$$\frac{\partial u(x,y,t)}{\partial y} = \frac{1}{\triangle y}\left(u(x,y+\triangle y,t)\right) - u(x,y,t), \tag{2}$$

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{1}{\triangle t}\left(u(x,y,t+\triangle t)\right) - u(x,y,t). \tag{3}$$

If the approach is backwards, the first derivatives of $\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t}$ can be written as below:

$$\frac{\partial u(x,y,t)}{\partial x} = \frac{1}{\triangle x}\left(u(x+\triangle x,y,t)\right) - u(x-\triangle x,y,t), \tag{4}$$

$$\frac{\partial u(x,y,t)}{\partial y} = \frac{1}{\triangle y}\left(u(x+\triangle x,y,t)\right) - u(x,y-\triangle y,t), \tag{5}$$

$$\frac{\partial u(x,y,t)}{\partial t} = \frac{1}{\triangle t}\left(u(x+\triangle x,y,t)\right) - u(x,y,t-\triangle t). \tag{6}$$

Second order center order is obtained:

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\triangle x^2}\left(u(x+\triangle x,y,t) - 2u(x,y,t) - u(x-\triangle x,y,t)\right), \tag{7}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\triangle y^2}\left(u(x,y+\triangle y,t) - 2u(x,y,t) - u(x,y-\triangle y,t)\right), \tag{8}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{1}{\triangle t^2}\left(u(x,y,t+\triangle t) - 2u(x,y,t) - u(x,y,t-\triangle t)\right). \tag{9}$$

When using the subscript index i which is used to express the discrete point x, as well as the subscript index j which is used in expressing the discrete point y, and also the subscript index n used in expressing the discrete point t, it can be written as the equation below:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\cup_{i+1,j}^{n} - 2\cup_{i,j}^{n} + \cup_{i-1,j}^{n}}{\partial x^2}, \tag{10}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\cup_{i,j+1}^{n} - 2\cup_{i,j}^{n} + \cup_{i-1,j}^{n}}{\partial y^2}, \tag{11}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\cup_{i,j}^{n+1} - 2\cup_{i,j}^{n} + \cup_{i-1,j}^{n-1}}{\partial t^2}. \tag{12}$$

2.2 Shallow water equation

The shallow water equation is generally used to model a surface wave of water that is influenced by gravity, for example, a wave flow on the surface of the seashore, lake, river, or on a smaller domain such as the water surface in a bathtub [16].

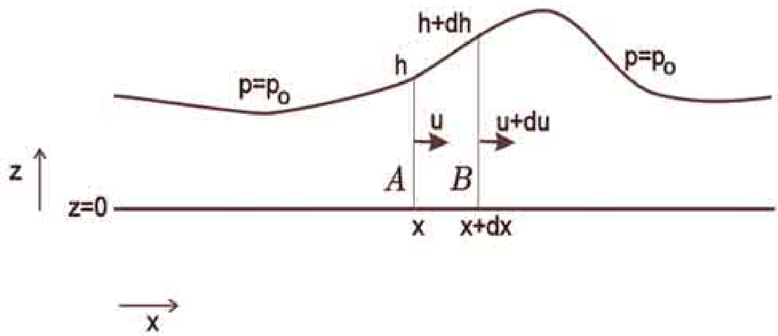

Figure 1: Shallow Water System Illustration [20].

The shallow water equation will take effect when the wavelength is greater than the depth. For a one-dimensional problem, the shallow water equation is stated as follows:

$$h_t + (uh)_x = 0, \tag{13}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x = 0. \tag{14}$$

Meanwhile, for a two-dimensional problem, the shallow water equation is stated as follows:

$$h_t + (hu)_x + (hv)_y = 0, \tag{15}$$

$$(hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y = 0, \tag{16}$$

$$(hu)_t + (huv)_x + \left(hu^2 + \frac{1}{2}gh^2\right)_y = 0, \tag{17}$$

where g is the Earth's gravitational constant, h is the height of the sea surface, (u, v) is the vector of the velocity of the water flow, and hu and hv are the momentum in two directions [16].

3 Result and Discussion

3.1 Tsunami wave analysis

In this study, the model used is a model of the shallow water equation to simulate the movement of a tsunami wave that has resistance by using the finite difference method. The shallow water equation comes from the mass conservation equation and the conservation of linear momentum (Navier-Stokes equation).

The mass equation in volume is defined as follows:

$$\begin{aligned}\frac{\partial m}{\partial t} &= \rho u(x)h(x) - \rho u(x+dx)h(x+dx)\\ &= -\rho\frac{\partial(uh)}{\partial x}dx,\end{aligned}$$

assuming $m = \rho h\, dx$, we obtain

$$\begin{aligned}\frac{\partial m}{\partial t} &= -\rho\frac{\partial(uh)}{\partial x}dx,\\ \frac{\partial \rho h}{\partial t}dx &= -\rho\frac{\partial(uh)}{\partial x}dx,\\ \frac{\partial h}{\partial t} &= -\frac{\partial(uh)}{\partial x},\\ \frac{\partial h}{\partial t}+\frac{\partial(uh)}{\partial x} &= 0,\\ h_t + (uh)_x &= 0. \qquad (18)\end{aligned}$$

Based on Newton's laws of motion applied to volume, we define

$$F = m\frac{du}{dt} = -\rho g h\frac{\partial h}{\partial x}dx$$

by applying $m = \rho h dx$, we then obtain

$$\begin{aligned}m\frac{du}{dt} &= -\rho g h\frac{\partial h}{\partial x}dx,\\ \rho h dx\frac{du}{dt} &= -\rho g h\frac{\partial h}{\partial x}dx,\\ \frac{du}{dt} &= -g\frac{\partial h}{\partial x}.\end{aligned}$$

Then the chain rule is applied to $\frac{d}{dt}$, we obtain

$$\begin{aligned}\frac{du}{dt} &= \frac{\partial u}{\partial t}+\frac{dx}{dt}\frac{\partial u}{\partial x},\\ &= \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}.\end{aligned}$$

So the equation is expressed by

$$\begin{aligned}\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x} &= -g\frac{\partial h}{\partial x},\\ \left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)dx &= \left(-g\frac{\partial h}{\partial x}\right)dx,\\ h\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right) &= -\frac{1}{2}gh\frac{\partial h}{\partial x},\\ h\frac{\partial u}{\partial t}+hu\frac{\partial u}{\partial x} &= -\frac{1}{2}gh\frac{\partial h}{\partial x},\\ h\frac{\partial u}{\partial t}+hu\frac{\partial u}{\partial x}+\frac{1}{2}gh\frac{\partial h}{\partial x} &= 0,\\ (hu)_t + \left(hu^2+\frac{1}{2}gh^2\right)_x &= 0. \qquad (19)\end{aligned}$$

For the problems adopted in this study, the shallow water equation does not have a Coriolis force. Thus, equation (15) is based on equation (18), while equation (16) and equation (17) are based on equation (19).

3.2 Discretization of tsunami wave time

Based on the shallow water equation which is approached by the finite difference method, equations 4, 5 and 6 are substituted in equations 16 and 17 in order to obtain

$$\begin{aligned}
u(i,j,n+1) &= \frac{(u(i+1,j,n)+u(i-1,j,n)+u(i,j+1,n)+u(i,j-1,n))}{4} \\
&-\frac{1}{2}\frac{dt}{dx}\left(\frac{u(i+1,j,n)^2}{2}-\frac{u(i-1,j,n)2}{2}\right) \\
&-\frac{1}{2}\frac{dt}{dy}v(i,j,n)\left(u(i,j+1,n)-u(i,j-1,n)\right) \\
&-\frac{1}{2}g\frac{dt}{dx}\left(h(i+1,j,n)-h(i-1,j,n)\right), \\
v(i,j,n+1) &= \frac{(v(i+1,j,n)+v(i-1,j,n)+v(i,j+1,n)+v(i,j-1,n))}{4} \\
&-\frac{1}{2}\frac{dt}{dy}\left(\frac{v(i,j+1,n)^2}{2}-\frac{v(i,j+1,n)2}{2}\right) \\
&-\frac{1}{2}\frac{dt}{dx}u(i,j,n)\left(v(i+1,j,n)-v(i-1,j,n)\right) \\
&-\frac{1}{2}g\frac{dt}{dy}\left(h(i,j+1,n)-h(i,j-1,n)\right),
\end{aligned}$$

$$\begin{aligned}
h(i,j,n+1) &= \frac{(h(i+1,j,n)+h(i-1,j,n)+h(i,j+1,n)+h(i,j-1,n))}{4} \\
&-\frac{1}{2}\frac{dt}{dx}u(i,j,n)\left(h(i+1,j,n)-b(i+1,j)\right) \\
&-\left(h(i-1,j,n)-b(i-1,j)\right) \\
&-\frac{1}{2}\frac{dt}{dy}v(i,j,n)\left(h(i,j+1,n)-b(i,j+1)\right) \\
&-\left(h(i,j-1,n)-b(i,j-1)\right) \\
&-\frac{1}{2}\frac{dt}{dx}\left(h(i,j,n)-b(i,j)\right)\left(u(i+1,j,n)-u(i-1,j,n)\right) \\
&-\frac{1}{2}\frac{dt}{dy}\left(h(i,j,n)-b(i,j)\right)\left(v(i,j+1,n)-v(i,j-1,n)\right)
\end{aligned}$$

with the following boundary conditions:

$$\begin{aligned}
u(1,i,n+1) &= \frac{10}{4}u(2,i,n+1)-2u(3,i,n+1)+\frac{1}{2}u(4,i,n+1), \\
v(1,i,n+1) &= \frac{10}{4}v(2,i,n+1)-2v(3,i,n+1)+\frac{1}{2}v(4,i,n+1), \\
h(1,i,n+1) &= \frac{10}{4}h(2,i,n+1)-2h(3,i,n+1)+\frac{1}{2}h(4,i,n+1).
\end{aligned}$$

The initial simulation of the formation of a wave ($t = 1$) is shown in Figure 2. At $t = 1$, it is assumed to be the time when the tsunami first appearance on the surface after a shift in the earth's plate under the sea. So its first appears on the surface is very large at the starting point of its appearance, then it travels towards the mainland.

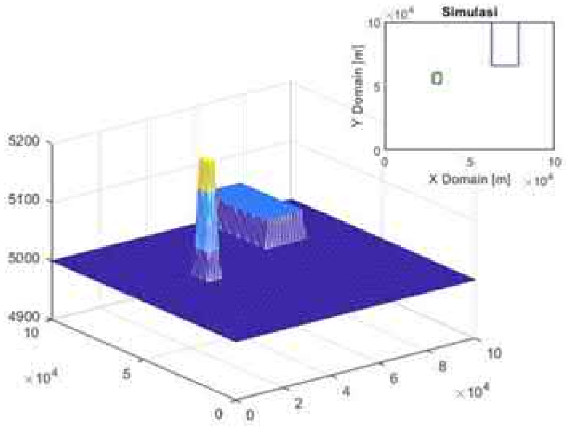

Figure 2: Tsunami Wave Simulation at t = 1.

Figure 2 shows simulation of a tsunami wave that appears to have a height that exceeds the height of the barrier by a height of 8 meters above sea level. Then, at the next time, the tsunami waves will move around, one of which being towards the shallow water where a barrier has been built. Based on the simulation, the tsunami wave will experience a collision with the barrier in 35 seconds since the initial appearance of the tsunami.

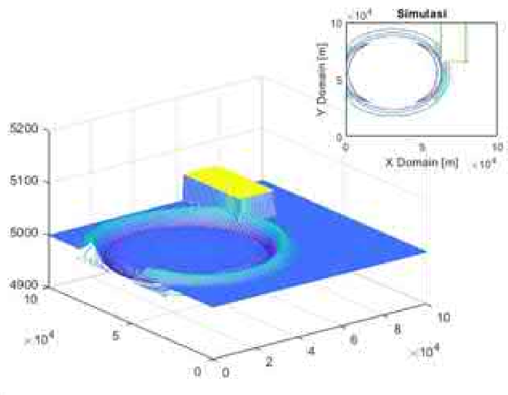

Figure 3: Tsunami Wave Simulation at t = 40.

The collision of the wave with the barrier is shown in Figure 3. As a result of the collision with the tsunami wave barrier, it breaks and can be suppressed. The damping by the barrier means that the area behind the barrier is not directly affected by the tsunami. The tsunami waves hit the area behind the barrier due to the impact of the tsunami from the area that is not protected by the barrier. The impact is not as high as that of a tsunami wave without a barrier.

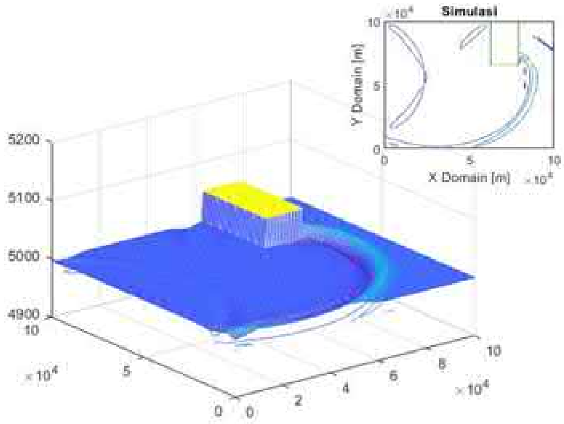

Figure 4: Tsunami Wave Simulation at t = 65.

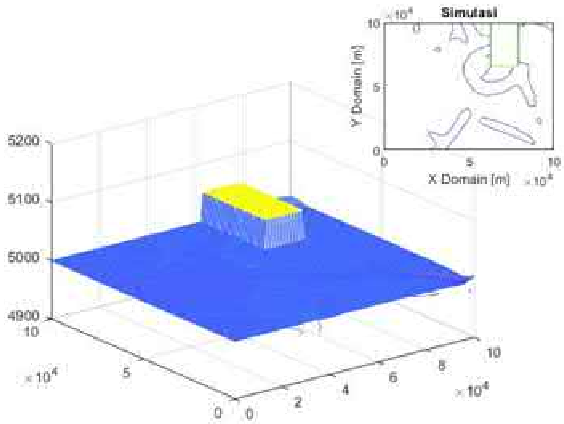

Figure 5: Tsunami Wave Simulation at t = 100.

Figure 4 shows the condition of the tsunami waves behind the damping barrier. The damping process is carried out continuously by the barrier that has been formed, until the waves calm down again. This condition is shown in Figure 5.

4 Conclusion

Numerical solutions and simulations have been carried out for the shallow water equation to represent a tsunami wave with the construction of a barrier. It is known that the construction of an obstacle can break up tsunami waves and can reduce the strength of the waves. Thus, if it is realized, it can really be an effort to reduce the impact of the tsunami disaster.

Acknowledgment

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Analysis of Dengue Disease Transmission Model with General Incidence Functions

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Abstract: In this work, we propose a non-linear system of differential equations that models the dynamics of transmission of dengue fever. Then, we perform a stability analysis of this model. In particular, we prove that when the threshold of the model called the basic reproduction ratio is less than unity, the disease-free equilibrium is globally asymptotically stable. Furthermore, when this value is greater than unity, under suitable conditions, the endemic equilibrium is globally asymptotically stable. Some numerical simulations are provided to illustrate the obtained theoretical results. We also propose a global sensitivity analysis of the basic reproduction ratio.

Keywords: *dengue; general incidence function; mathematical analysis; basic reproduction number; Lyapunov function; stability analysis; sensitivity.*

Mathematics Subject Classification (2010): 70K75, 93C15, 34D23.

1 Introduction

Mathematical modelling and numerical simulation are important decision tools that can be used to study and control human and animal diseases [1, 2]. However, to tackle real situations, the resulting models need to be adapted to each specific disease and its biological characteristics [3].

From a general point of view, mathematical models are used to predict the behaviour of a disease in a particular population [4, 5]. In particular, they help to determine if the disease under consideration will be endemic (i.e., it remains active in the population)

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or not (i.e., it disappears). In this work, we introduce and study a particular mathematical model to estimate the dynamic of the so-called dengue fever disease in a human population.

Dengue disease is a common arboviral disease in tropical regions and the mediterranean. It is transmitted to humans by the bite of *Aedes mosquitoes*. Four serotypes have been recognized, they are denoted by DEN-I, DEN-II, DEN-III, and DEN-IV. These viruses are carried by two kinds of mosquitoes referred to as *Aedes aegypti* and *Aedes albopictus* which spread the disease through their bite. However, *Aedes aegypti* has been the principal vector of dengue virus transmission. Another interesting fact is the shift of patients phenomena when dengue fever previously attacked children of primary school age, but now everybody is vulnerable to the fever [6]. Dengue viruses can infect only a restricted number of vertebrates but it is an essentially human disease. Infection for any dengue serotype produces permanent immunity to it, but apparently only temporary cross immunity to other serotypes. Therefore, individuals that live in dengue endemic areas can have more than one infection of dengue disease. It is considered that human population growth and the dramatic redistribution of the human population in the urban centers of developing countries have contributed to the introduction and enhancement of dengue fever [7].

Mathematical models and methods of non-linear dynamic are used in comparing, planning, implementing, evaluating and optimizing various detection, prevention, therapy and control programmes [8, 9]. Thus, mathematical models are a useful tool to better understand the mechanisms that allow the spread of a dengue epidemic and then to increase the efficiency of the vector control strategy. There are a number of mathematical expert models for dengue fever which involve differential equations. In general, they use compartmental dynamic such as susceptible, infected, removed (SIR) and susceptible, exposed, infected, removed ($SEIR$). In [10], the authors formulated stochastic models for dengue in the presence of Wolbachia. This research aims to measure the effectiveness of the Wolbachia intervention to reduce dengue transmission. It determines the proportion of reduction in the basic reproduction number and also the probability of extinction. Putri *et al.* (see [11]) proposed the study where the aim is to forecast and analyze the spread of COVID-19 outbreak in Indonesia by applying machine learning and hybrid approaches. Abdelhamid Zaghdani (see [9]) formulated a modified $SEIR$ mathematical model for the coronavirus infected disease-2019 (COVID-19). The author computed the basic reproduction number ($\mathcal{R}_0$) and proposed a qualitative analysis of the local and global stability of the equilibrium points.

In this paper, our aim is to study the dengue epidemic model presented and studied in [5] by J. J. Tewa *et al.* with the law of mass action as the incidence functions. The similar model was presented and studied in [12]. It appears that the incidence function form is determinative in the study of the model system. Then, changing the form of the incidence function can potentially change the behaviour of the system. In this work, we study a coupling model (Humans and Vectors) with two general incidence functions given by f, g. From the analysis of the global stability of the equilibrium points, we used the same technique as in Guiro *et al.* [13]. We find conditions on the incidence function to get the stability of the model.

This paper is organized as follows. In Section 2, we describe the mathematical model which is studied in the paper. In Section 3, we give the equilibrium points, the basic reproduction number, we define also a positive invariant and attractive set, which will be used in the studies of the stability of equilibrium points. In Section 4, we study the stability of equilibrium points. Section 6 contains the numerical result and comments. Section 7 is devoted to the analysis of global sensitivity of the parameters in the basic reproduction number $\mathcal{R}_0$. We end by the conclusion.

2 Description of the Model

In this section, we recall the model studied in [5] by J. J. Tewa *et al.*, which is given with a particular incidence function as follows:

$$\begin{cases} \dot{S}_H = \mu_H N_H - \dfrac{\beta_H b}{N_H + m} S_H I_V - \mu_H S_H, \\ \dot{I}_H = \dfrac{\beta_H b}{N_H + m} S_H I_V - (\mu_H + \gamma_H) I_H, \\ \dot{R}_H = \gamma_H I_H - \mu_H R_H, \\ \dot{S}_V = A - \dfrac{\beta_V b}{N_H + m} S_V I_H - \mu_V S_V, \\ \dot{I}_V = \dfrac{\beta_V b}{N_H + m} S_V I_H - \mu_V I_V. \end{cases} \tag{1}$$

The model above is described as follows: the human and vector populations are divided into classes or states containing susceptible, infective and immune individuals. At time t, there are the susceptible humans (S_H) and the infectious humans (I_H), we assume that the infectious humans recover (or get treated) at a constant rate γ_H, $\mu_H + \gamma_H$ is the total exit of the infectious humans, R_H are the immune humans, S_V are the susceptible mosquitoes and I_V are the infectious mosquitoes. The mosquito population does not have an immune class since their infectious period ends with their death. Let $N_H = S_H + I_H + R_H$ and $N_V = S_V + I_V$ be, respectively, the total human and vector population at time t. Total death in the mosquito population occurs at a rate $\mu_V N_V$, where μ_V is the per capita mortality rate of mosquitoes. In this model, it is assumed that the human population has constant size with the birth and death rate constant number equal to μ_H. Also, for the mosquito population, it is assumed a constant recruitment rate A, independent of the actual number of adult mosquitoes. It is admitted that the flow from the susceptible to the infectious class, for each species, depends on the biting rate of the mosquitoes, the transmission probabilities, as well as the number of infectious and susceptible of each species.

Let b denote the biting rate of mosquitoes, which is the average number of bites per mosquito per day. m denotes the number of alternative hosts available as blood sources, then the probability that a mosquito chooses a human individual as a host is given by $\dfrac{N_H}{N_H + m}$. Thus, it is admitted that a human receives $b\dfrac{N_V}{N_H}\dfrac{N_H}{N_H + m}$ bites per unit of time, and a mosquito takes $\dfrac{bN_H}{N_H + m}$ human blood meals per unit of time. Then, the infection rates per susceptible human and susceptible vector are given by

$$\beta_H b \frac{N_V}{N_H} \frac{N_H}{N_H + m} \frac{I_V}{N_V} = \frac{\beta_H b}{N_H + m} I_V,$$

$$\beta_V b \frac{N_H}{N_H + m} \frac{I_H}{N_H} = \frac{\beta_V b}{N_H + m} I_H,$$

respectively. Here β_H is the transmission probability from a vector to a human and β_V is the transmission probability from a human to a vector.

The aim of our work is to generalize the model (1) with the incidence function as the general incidence functions f and g. The interaction between the human population and the vector population is given by the following diagram, see Figure 1:

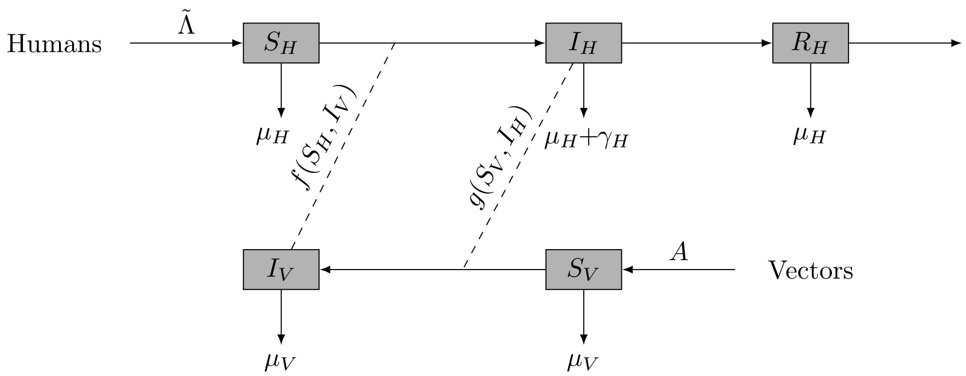


Figure 1: Transfer diagram for the mathematical model of dengue.

Then, according to Figure 1, we have the following system of five differential equations:

$$\begin{cases} \dot{S}_H = \tilde{\Lambda} - f(S_H, I_V) - \mu_H S_H, \\ \dot{I}_H = f(S_H, I_V) - (\mu_H + \gamma_H) I_H, \\ \dot{R}_H = \gamma_H I_H - \mu_H R_H, \\ \dot{S}_V = A - g(S_V, I_H) - \mu_V S_V, \\ \dot{I}_V = g(S_V, I_H) - \mu_V I_V. \end{cases} \tag{2}$$

In the system (2), we use the same constant and the same subdivision of the human population and the vector population as described in the system (1).

Since R_H does not appear in the first and second equations of system (2), it is sufficient to analyse the behavior of solutions of the following system:

$$\begin{cases} \dot{S}_H = \tilde{\Lambda} - f(S_H, I_V) - \mu_H S_H, \\ \dot{I}_H = f(S_H, I_V) - (\mu_H + \gamma_H) I_H, \\ \dot{S}_V = A - g(S_V, I_H) - \mu_V S_V, \\ \dot{I}_V = g(S_V, I_H) - \mu_V I_V. \end{cases} \tag{3}$$

We assume that the functions f and g satisfy the following hypotheses:
H1 f and g are non-negative C^1 functions in the non-negative orthant.
H2 For all $(S_H, I_H, R_H, S_V, I_V) \in \mathbb{R}^5_+$, $f(S, 0) = f(0, I) = 0$ and $g(S, 0) = g(0, I) = 0$. Also, we denote by f_1, g_1 and f_2, g_2 the partial derivative of f and g with respect to S and I.

Remark 2.1 f and g are two incidence functions which explain the contact between two species. Therefore, **H2** is a natural assumption which means that if there is no infected in the human and vector populations, then the incidence functions are equal to zero. The incidence functions are also equal to zero when there is no susceptible in the human and vector populations.

3 Basic Properties and Basic Reproduction number

In this section, we study the basic properties of the solution of system (3) and also, we compute the basic reproduction number associated to the system (3).

Proposition 3.1 *The positive orthant*

$$\{(S_H, I_H, S_V, I_V) \in \mathbb{R}^4, S_H \geq 0, I_H \geq 0, S_V \geq 0, I_V \geq 0\}$$

is positively invariant for system (3)*.*

To prove Proposition 3.1, we need the following lemma.

Lemma 3.1 *[14]: Let $L : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a differentiable function, and let $a \in \mathbb{R}$. Let $X(x)$ be the vector field, and let G be the closed set $G = \{x \in \mathbb{R}^n : \ L(x) \leq a\}$ such that $\nabla L(x) \neq 0$ for all $x \in L^{-1}(a) = \{x \in \mathbb{R}^n, L(x) = a\}$. If $< X(x), \nabla L(x) > \leq 0$ for all $x \in L^{-1}(a)$, then the set G is positively invariant.*

Proof of Proposition 3.1 Let $x = (S_H, I_H, S_V, I_V)$. Now, we have to prove that $\{S_H \geq 0\}$ is positively invariant.

Let $L(x) = -S_H$. L is differentiable and $\nabla L(x) = (-1, 0, 0, 0) \neq 0_{\mathbb{R}^5}$ for all $x \in L(x)^{-1}(0) = \{x \in \mathbb{R}^4 / L(x) = 0\}$. The vector field on $\{S_H = 0\}$ is

$$X(x) = \begin{pmatrix} \tilde{\Lambda} \\ -(\mu_H + \gamma_H) I_H \\ A - g(S_V, I_H) - \mu_V S_V \\ g(S_V, I_H) - \mu_V I_V \end{pmatrix}.$$

Then $< X(x), \nabla L(x) >= -\tilde{\Lambda} < 0$. This proves that $\{S_H \geq 0\}$ is positively invariant. Similarly, we prove that $\{I_H \geq 0\}$, $\{R_H \geq 0\}$, $\{S_V \geq 0\}$, $\{I_V \geq 0\}$ are positively invariant. Then $\{(S_H, I_H, S_V, I_V) \in \mathbb{R}^5, S_H \geq 0, I_H \geq 0, S_V \geq 0, I_V \geq 0\}$ is positively invariant for system (3). □

Therefore, the model is mathematically well posed and epidemiologically reasonable since all the variables remain non-negative for all $t > 0$.

Proposition 3.2 *Let (S_H, I_H, S_V, I_V) be the solution of system* (3) *with the initial condition $(S_{0H}, I_{0H}, S_{0V}, I_{0V})$ and the compact set*

$$\mathcal{D} = \left\{ (S_H, I_H, S_V, I_V) \in \mathbb{R}^4_+, W_1 \leq N_H + \epsilon, W_2 \leq \frac{A}{\mu_V} + \epsilon, for\ \epsilon > 0 \right\} \tag{4}$$

with $W_1 = S_H + I_H$ and $W_2 = S_V + I_V$. Then, under the flow described by (3)*, $\mathcal{D}$ is a positively invariant set that attracts all solutions in $\mathbb{R}^4_+$.*

Proof. By adding the first two equations of system (3), we have

$$\begin{aligned}
\frac{dS_H}{dt}+\frac{dI_H}{dt} &= \tilde{\Lambda}-\mu_H S_H-(\mu_H+\gamma_H)I_H,\\
\frac{d(S_H+I_H)}{dt} &\le \tilde{\Lambda}-\mu_H(S_H+I_H),\\
\frac{dW_1}{dt} &\le \tilde{\Lambda}-\mu_H W_1,\\
\frac{dW_1}{dt}+\mu_H W_1 &\le \tilde{\Lambda}+\epsilon. \qquad (5)
\end{aligned}$$

According to [15], from inequation (5), we have

$$W_1(t)\le \frac{\tilde{\Lambda}}{\mu_H}+\frac{\epsilon}{\mu_H}+(W_1(0)-\frac{\tilde{\Lambda}}{\mu_H}-\frac{\epsilon}{\mu_H})e^{-\mu_H t}, \qquad (6)$$

where $W_1(0)=S_{0H}+I_{0H}$. Thus, when $t\longrightarrow+\infty$, $W_1(t)\le\frac{\tilde{\Lambda}}{\mu_H}+\frac{\epsilon}{\mu_H}$. Similarly, we prove that $W_2(t)\le\frac{A}{\mu_V}+\frac{\epsilon}{\mu_V}$, where $W_1(0)$ and $W_2(0)$ are, respectively, the initial conditions of $W_1(t)$ and $W_2(t)$. Thus, as $t\longrightarrow\infty$, $0\le(W_1(t),W_2(t))\le(N_H+\frac{\epsilon}{\mu_H},\frac{A}{\mu_V}+\frac{\epsilon}{\mu_V})$ and one can conclude that $\mathcal{D}$ is an attractive set. □

Let $E=(S_H,I_H,S_V,I_V)$ be an equilibrium point of (3). Thus, we have

$$\begin{cases}
\tilde{\Lambda}-f(S_H,I_V)-\mu_H S_H=0,\\
f(S_H,I_V)-(\mu_H+\gamma_H)I_H=0,\\
A-g(S_V,I_H)-\mu_V S_V=0,\\
g(S_V,I_H)-\mu_V I_V=0.
\end{cases} \qquad (7)$$

By adding the first two and the last two equations (7), we get

$$S_H=\frac{\tilde{\Lambda}-(\mu_H+\gamma_H)I_H}{\mu_H}\ ,\ S_V=\frac{A-\mu_V I_V}{\mu_V},$$

and

$$E=\left(\frac{\tilde{\Lambda}-(\mu_H+\gamma_H)I_H}{\mu_H},I_H,\frac{A-\mu_V I_V}{\mu_V},I_V\right).$$

Hence, the disease-free equilibrium and the endemic equilibrium of (3) are given by

$$E_0=(S_H^0,I_H^0,S_V^0,I_V^0)=(\frac{\tilde{\Lambda}}{\mu_H},0,\frac{A}{\mu_V},0)$$

and

$$E^*=(S_H^*,I_H^*,S_V^*,I_V^*)=\left(\frac{\tilde{\Lambda}-(\mu_H+\gamma_H)I_H^*}{\mu_H},I_H^*,\frac{A-\mu_V I_V^*}{\mu_V},I_V^*\right).$$

Here, I_H^* and I_V^* are the design infected human and infected mosquito at endemic period.

The reproduction number of model (3) is obtained by creating the next generation matrix and funding the maximum eigenvalues of that matrix [16]. The reproduction number of that model is given by

$$\mathcal{R}_0=\sqrt{\frac{f_2(S_H^0,0)g_2(S_V^0,0)}{\mu_V(\mu_H+\gamma_H)}}.$$

Theorem 3.1 *If $\mathcal{R}_0 > 1$, then the model* (3) *has only a unique endemic equilibrium* E^*.

Proof. Let us define the function $\psi(I_H, I_V) = (\psi_1(I_H, I_V); \psi_2(I_H, I_V))$, where

$$\psi_1(I_H, I_V) = f(S_H^0 - I_H, I_V) - (\mu_H + \gamma_H)I_H$$

and

$$\psi_2(I_H, I_V) = g(S_V^0 - I_V, I_H) - \mu_V I_V.$$

Hence, it follows that any solution of the equation $\psi = 0$ in the set $(0, S_H^0) \times (0, S_V^0)$ corresponds to an equilibrium, with $S_H, I_H, S_V, I_V > 0$. Since **H2** holds, one has $\psi(0,0) = 0$ and $\psi(S_H^0, S_V^0) \leq 0$. Then the sufficient condition for the equation $\psi = 0$ to have a solution in $(0, S_H^0) \times (0, S_V^0)$ is that ψ is increasing at 0. This implies that an endemic equilibrium exits if

$$\nabla\psi(0,0) > 0, \tag{8}$$

where

$$\begin{aligned}
\nabla\psi(0,0) &= (\nabla\psi_1(0,0), \nabla\psi_2(0,0)) \\
&= (-f_1(H_s^0, 0) - \mu_H + \gamma_H + f_2(S_H^0, 0), -g_1(S_V^0, 0) - \mu_V + g_2(S_s^0, 0)).
\end{aligned}$$

Note that $f_1(S_H^0, 0) = g_1(S_V^0, 0) = 0$. Then inequality (8) is equivalent to

$$f_2(S_H^0, 0) > \mu_H + \gamma_H, \text{ and } g_2(S_V^0, 0) > \mu_V,$$

which give

$$\alpha\gamma f_2(S_H^0, 0) g_2(S_V^0, 0) > (\mu_H + \gamma_H)\mu_V.$$

That is,

$$\mathcal{R}_0 = \frac{f_2(S_H^0, 0) g_2(S_V^0, 0)}{(\mu_H + \gamma_H)\mu_V} > 1.$$

Then system (3) has a unique endemic equilibrium given by
$E^* = \left(\dfrac{\tilde{\Lambda} - (\mu_H + \gamma_H) I_H^*}{\mu_H}, I_H^*, \dfrac{A - \mu_V I_V^*}{\mu_V}, I_V^* \right)$. The proof is completed. □

4 Stability of Equilibrium

In this section, we analyze the stability of the diseases-free equilibrium E_0 and the endemic equilibrium E^*.
H3 For all $(S_H, I_H, S_V, I_V) \in \mathbb{R}_+^4$,

$$f(S_H, I_V) \leq f_2(S_H^0, 0) I_V \text{ and } g(S_V, I_H) \leq g_2(S_V^0, 0) I_H.$$

H4 $1 < \dfrac{f_2(S_H^0, 0)}{\mu_H + \gamma_H}$ and $1 < \dfrac{g_2(S_V^0, 0)}{\mu_V}$.

Remark 4.1 The assumptions **H3** and **H4** are the technical assumptions which are also used to have the global stability of the diseases-free equilibrium E_0. Biologically, the assumption **H3** allows for the control of the infection speed.

Theorem 4.1 *Assume that* **H3** *and* **H4** *hold, then if* $\mathcal{R}_0 \leq 1$, *the diseases-free equilibrium* E_0 *is globally asymptotically stable on* $\mathcal{D}$.

Proof. Let us consider the candidate Lyapunov function

$$V = \mu_V I_H + (\mu_H + \gamma_H) I_V.$$

By differentiating V with respect to time, we have

$$\begin{aligned}
\dot{V} &= \mu_V \dot{I}_H + (\mu_H + \gamma_H)\dot{I}_V \\
&= \mu_V f(S_H, I_V) - \mu_V(\mu_H + \gamma_H) I_H + (\mu_H + \gamma_H) g(S_V, I_H) - \mu_V(\mu_H + \gamma_H) I_V \\
&= \mu_V f(S_H, I_V) + (\mu_H + \gamma_H) g(S_V, I_H) - \mu_V(\mu_H + \gamma_H)(I_H + I_V).
\end{aligned}$$

By using the assumption **H3**, we get

$$\dot{V} \leq f_2(S_H^0, 0)\mu_V I_V + g_2(S_V^0, 0)(\mu_H + \gamma_H) I_H - \mu_V(\mu_H + \gamma_H)(I_H + I_V).$$

By adding and subtracting $f_2(S_H^0, 0) g_2(S_V^0, 0)(I_H + I_V)$ in the inequality above, we have

$$\begin{aligned}
\dot{V} \leq\ & f_2(S_H^0, 0) g_2(S_V^0, 0)(I_H + I_V) + f_2(S_H^0, 0) I_V [\mu_V - g_2(S_V^0, 0)] \\
& + g_2(S_V^0, 0) I_H [(\mu_H + \gamma_H) - f_2(S_H^0, 0)] - \mu_V(\mu_H + \gamma_H)(I_H + I_V).
\end{aligned}$$

By using the assumption **H4**, we obtain

$$\begin{aligned}
\dot{V} &\leq \mu_V(\mu_H + \gamma)(I_H + I_V)\left(\frac{f_2(S_H^0, 0) g_2(S_V^0, 0)}{\mu_V(\mu_H + \gamma)} - 1\right) \\
&\leq \mu_V(\mu_H + \gamma_H)(I_H + I_V)(\mathcal{R}_0^2 - 1).
\end{aligned}$$

Since $\mathcal{R}_0 \leq 1$, we have $\dot{V} \leq 0$, with equality only if $I_H = 0$ and $I_V = 0$. According to LaSalle's extension to Lyapunov method's [17], the limit set of each solution is contained in the largest invariant set, for which $I_H = 0$ and $I_V = 0$, which is the singleton $\{E_0\}$. Thus, the unique disease-free equilibrium E_0 is globally asymptotically stable on $\mathcal{D}$. $\square$

We assume that the functions f and g satisfy the following assumptions:

H5 For all $(S_H, I_H, S_V, I_V) \in \mathbb{R}_+^4$, $1 \leq \dfrac{f(S_H, I_V)}{f(S_H, I_V^*)} \leq \dfrac{I_V}{I_V^*}$ and $1 \leq \dfrac{g(S_V, I_H)}{g(S_V, I_H^*)} \leq \dfrac{I_H}{I_H^*}$.

H6 For all $(S_H, S_V) \in \mathbb{R}_+^2$, $Sign(S_H - S_H^*) = Sign(f(S_H, I_V^*) - f(S_H^*, I_V^*))$ and $Sign(S_V - S_V^*) = Sign(g(S_V, I_H^*) - g(S_V^*, I_H^*))$.

Remark 4.2 The assumptions **H5** and **H6** are the technical assumptions which are used in the proof of the global stability of the endemic equilibrium.

Theorem 4.2 *When* $\mathcal{R}_0 > 1$, *then the endemic equilibrium* E^* *of system* (3) *exists and is globally asymptotically stable on* $\mathcal{D}$.

Proof. At the endemic equilibrium E^* and from the system (3), we have

$$\begin{cases}
\tilde{\Lambda} = f(S_H^*, I_V^*) + \mu_H S_H^*, \\
f(S_H^*, I_V^*) = (\mu_H + \gamma_H) I_H^*, \\
A = g(S_V^*, I_H^*) + \mu_V S_V^*, \\
g(S_V^*, I_H^*) = \mu_V I_V^*.
\end{cases} \tag{9}$$

Let us define the function h on $\mathbb{R}_+$ by $h(x) = x - 1 - \ln x$. The function h is non-negative for all $x \in \mathbb{R}_+$. Let us consider the candidate Lyapunov function U defined by

$$U(t) = U_H(t) + U_V(t) \;\; \text{where } U_H(t) = U_{S_H}(t) + U_{I_H}(t) \text{ and } U_V(t) = U_{S_V}(t) + U_{I_V}(t)$$

with

$$\begin{aligned} U_{S_H} &= S_H - S_H^* - \int_{S_H^*}^{S_H} \frac{f(S_H^*, I_V^*)}{f(\chi, I_V^*)} d\chi, \;\; U_{I_H} = I_H^* h\left(\frac{I_H}{I_H^*}\right), \\ U_{S_V} &= S_V - S_V^* - \int_{S_V^*}^{S_V} \frac{g(S_V^*, I_H^*)}{g(\chi, I_H^*)} d\chi, \;\; U_{I_V} = I_V^* h\left(\frac{I_V}{I_V^*}\right). \end{aligned}$$

Now, we have to differentiate the function U with respect to time.

$$\begin{aligned} \dot{U}_{S_H} &= \left(1 - \frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)}\right) \dot{S}_H \\ &= \left(1 - \frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)}\right) (\tilde{\Lambda} - f(S_H, I_V) - \mu_H S_H). \end{aligned}$$

By using the first equation of system (9), we have

$$\begin{aligned} \dot{U}_{S_H} &= -\mu_H (S_H - S_H^*) \left(1 - \frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)}\right) \\ &\quad + f(S_H^*, I_V^*) \left[1 - \frac{f(S_H, I_V)}{f(S_H^*, I_V^*)} - \frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)} + \frac{f(S_H, I_V)}{f(S_H, I_V^*)}\right]. \end{aligned}$$

Let us calculate $\dot{U}_{I_H}$:

$$\begin{aligned} \dot{U}_{I_H} &= \left(1 - \frac{I_H^*}{I_H}\right) \dot{I}_H \\ &= \left(1 - \frac{I_H^*}{I_H}\right) (f(S_H, I_V) - (\mu_H + \gamma_H) I_H^* \frac{I_H}{I_H^*}). \end{aligned}$$

By using the second equation of system (9), we get

$$\begin{aligned} \dot{U}_{I_H} &= \left(1 - \frac{I_H^*}{I_H}\right) (f(S_H, I_V) - f(S_H^*, I_V^*) \frac{I_H}{I_H^*}) \\ &= f(S_H^*, I_V^*) \left(1 - \frac{I_H^*}{I_H}\right) \left(\frac{f(S_H, I_V)}{f(S_H^*, I_V^*)} - \frac{I_H}{I_H^*}\right) \\ &= f(S_H^*, I_V^*) \left(\frac{f(S_H, I_V)}{f(S_H^*, I_V^*)} - \frac{I_H}{I_H^*} - \frac{I_H^*}{I_H} \frac{f(S_H, I_V)}{f(S_H^*, I_V^*)} + 1\right). \end{aligned}$$

Let us now evaluate $\dot{U}_H$:

$$\begin{aligned} \dot{U}_H &= \dot{U}_{S_H} + \dot{U}_{I_H} \\ &= -\mu_H (S_H - S_H^*) \left(1 - \frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)}\right) + f(S_H^*, I_V^*) Q(S_H, I_V), \end{aligned}$$

where $Q(S_H, I_V) = 2 - \dfrac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)} + \dfrac{f(S_H, I_V)}{f(S_H, I_V^*)} - \dfrac{I_H}{I_H^*} - \dfrac{I_H^*}{I_H}\dfrac{f(S_H, I_V)}{f(S_H^*, I_V^*)}$.

By adding and subtracting $1+\ln\dfrac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)}+\ln\dfrac{f(S_H, I_V)}{f(S_H, I_V^*)}+\ln\dfrac{I_H}{I_H^*}$ to and from $Q(S_H, I_V)$, we get

$$\begin{aligned}
Q(S_H, I_V) &= \left(-\frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)} + 1 + \ln\frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)}\right) + \left(-\frac{I_H}{I_H^*} + 1 + \ln\frac{I_H}{I_H^*}\right)\\
&+ \left(\frac{f(S_H, I_V)}{f(S_H, I_V^*)} - 1 - \ln\frac{f(S_H, I_V)}{f(S_H, I_V^*)}\right)\left(-\frac{I_H^*}{I_H}\frac{f(S_H, I_V)}{f(S_H^*, I_V^*)} + 1 + \ln\frac{I_H^*}{I_H}\frac{f(S_H, I_V)}{f(S_H^*, I_V^*)}\right)\\
&= -h\left(\frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)}\right) - h\left(\frac{I_H}{I_H^*}\right) + h\left(\frac{f(S_H, I_V)}{f(S_H, I_V^*)}\right) - h\left(\frac{I_H^*}{I_H}\frac{f(S_H, I_V)}{f(S_H^*, I_V^*)}\right).
\end{aligned}$$

Let us calculate $\dot{U}_{S_V}$:

$$\begin{aligned}
\dot{U}_{S_V} &= \left(1 - \frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)}\right)\dot{S}_V\\
&= \left(1 - \frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)}\right)(A - g(S_V, I_H) - \mu_V S_V).
\end{aligned}$$

By using the third equation of system (9), we obtain

$$\begin{aligned}
\dot{U}_{S_V} &= -\mu_V(S_V - S_V^*)\left(1 - \frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)}\right)\\
&\quad + g(S_V^*, I_H^*)\left(1 - \frac{g(S_V, I_H)}{g(S_V^*, I_H^*)} - \frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)} + \frac{g(S_V, I_H)}{g(S_V, I_H^*)}\right).
\end{aligned}$$

Let us calculate $\dot{U}_{I_V}$:

$$\begin{aligned}
\dot{U}_{I_V} &= \left(1 - \frac{I_V^*}{I_V}\right)\dot{I}_V\\
&= g(S_V^*, I_H^*)\left(1 + \frac{g(S_V, I_H)}{g(S_V^*, I_H^*)} - \frac{I_V}{I_V^*} - \frac{I_V^*}{I_V}\frac{g(S_V, I_H)}{g(S_V^*, I_H^*)}\right).
\end{aligned}$$

Let us now evaluate $\dot{U}_V$:

$$\begin{aligned}
\dot{U}_V &= \dot{U}_{S_V} + \dot{U}_{I_V}\\
&= -\mu_V(S_V - S_V^*)\left(1 - \frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)}\right) + g(S_V^*, I_H^*)\Psi(S_V, I_H),
\end{aligned}$$

where $\Psi(S_V, I_H) = 2 - \dfrac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)} + \dfrac{g(S_V, I_H)}{g(S_V, I_H^*)} - \dfrac{I_V}{I_V^*} - \dfrac{I_V^*}{I_V}\dfrac{g(S_V, I_H)}{g(S_V^*, I_H^*)}$.

By adding and subtracting $1 + \ln\dfrac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)} + \ln\dfrac{g(S_V, I_H)}{g(S_V, I_H^*)} + \ln\dfrac{I_V}{I_V^*}$ to and from

$\Psi(S_V, I_H)$, we have

$$\begin{aligned}\Psi(S_V, I_H) &= \left(-\frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)} + 1 + \ln\frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)}\right) + \left(-\frac{I_V}{I_V^*} + 1 + \ln\frac{I_V}{I_V^*}\right)\\ &\quad + \left(\frac{g(S_V, I_H)}{g(S_V, I_H^*)} - 1 - \ln\frac{g(S_V, I_H)}{g(S_V, I_H^*)}\right)\\ &\quad + \left(-\frac{I_V^*}{I_V}\frac{g(S_V, I_H)}{g(S_V^*, I_H^*)} + 1 + \ln\frac{I_V^*}{I_V}\frac{g(S_V, I_H)}{g(S_V^*, I_H^*)}\right)\\ &= -h\left(\frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)}\right) - h\left(\frac{I_V}{I_V^*}\right) + h\left(\frac{g(S_V, I_H)}{g(S_V, I_H^*)}\right) - h\left(\frac{I_V^*}{I_V}\frac{g(S_V, I_H)}{g(S_V^*, I_H^*)}\right).\end{aligned}$$

Let $\zeta = \max\{f(S_H^*, I_V^*); g(S_V^*, I_H^*)\}$,

$$\begin{aligned}\dot{U} &\leq -\mu_H(S_H - S_H^*)\left(1 - \frac{f(S_H^*, I_V^*)}{f(S_H, I_V^*)}\right) - \mu_V(S_V - S_V^*)\left(1 - \frac{g(S_V^*, I_H^*)}{g(S_V, I_H^*)}\right)\\ &\quad + \zeta(Q(S_H, I_V) + \Psi(S_V, I_H)).\end{aligned}$$

By using the assumption **H5**, we have

$$h\left(\frac{f(S_H, I_V)}{f(S_H, I_V^*)}\right) \leq h\left(\frac{I_V}{I_V^*}\right) \text{ and } h\left(\frac{g(S_V, I_H)}{g(S_V, I_H^*)}\right) \leq h\left(\frac{I_H}{I_H^*}\right),$$

thus, from the assumption **H6**, we can see that $\dot{U} \leq 0$. In addition, we can see that $U > 0$ for all $S_H, I_H, S_V, I_V \in \mathbb{R}_+$ and $U = 0$ for $S_H = S_H^*$, $I_H = I_H^*$, $S_V = S_V^*$ and $I_V = I_V^*$. Then the equilibrium state E^* is the only positively invariant set of the system (3) contained in $\{(S_H, I_H, S_V, I_V) \in \mathbb{R}_+^4; S_H = S_H^*, I_H = I_H^*, S_V = S_V^* \text{ and } I_V = I_V^*\}$ and hence, by the asymptotic stability theorem [17], the unique endemic equilibrium state E^* is globally asymptotically stable on $\mathcal{D}$. $\square$

5 Examples of Incidence Functions

In this section, we give the examples of incidence functions for which the required hypotheses are satisfied.

1. **Mass action incidence.** These incidence functions are defined by $f(S_H, I_V) = \alpha_1 S_H I_V$ and $g(S_V, I_H) = \alpha_2 S_V I_H$, where α_1 is the positive contact rate between a susceptible human and an infectious mosquito and α_2 designs the positive contact rate between a susceptible mosquito and an infectious human. Then hypotheses (**H1**) – (**H6**) are satisfied and so the global dynamics are determined by the magnitude of the basic reproduction number $\mathcal{R}_0$.

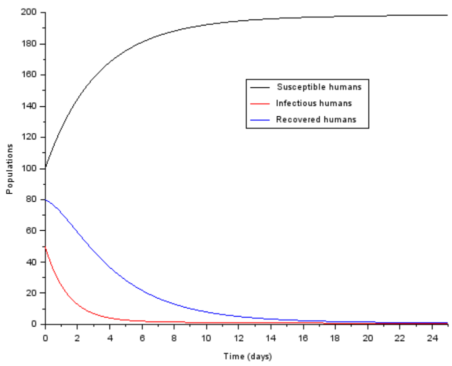
2. **Saturating incidence.** Let $f(S_H, I_V) = S_H \dfrac{I_V}{1 + c_1 I_V}$ and $g(S_V, I_H) = S_V \dfrac{I_H}{1 + c_2 I_H}$, where c_1 and c_2 are non-negative constant. Then hypotheses (**H1**) – (**H6**) are satisfied and so the global dynamics are determined by the value of $\mathcal{R}_0$.

3. **Standard incidence.** These functions are given by $f(S_H, I_V) = \dfrac{S_H I_V}{S_H + I_H}$ and $g(S_V, I_H) = \dfrac{S_V I_H}{S_V + I_V}$. Then the assumptions (**H1**) – (**H6**) are satisfied and so the global dynamics are given by the value of the basic reproduction number $\mathcal{R}_0$.

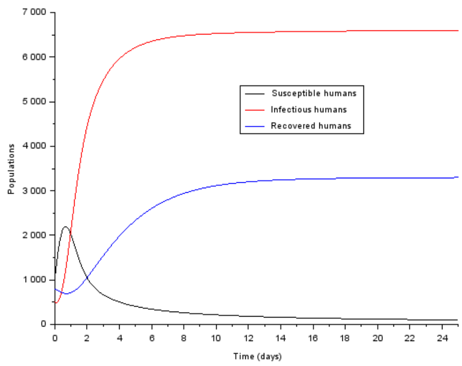
In the following paragraph, we carry out the numerical simulation taking the mass action law as the incidence function. But we specify that the dynamics remains the same as with the other incidence functions.

6 Simulation and Comments

In this section, we carry out the computation work that supports our study. In our simulation, we used the mass action as the incidence functions which are defined by $f(S_H, I_V) = \alpha_1 S_H I_V$ and $g(S_V, I_H) = \alpha_2 S_V I_H$, where α_1 and α_2 are positives constants. We present the graphics which illustrate the evolution of the different classes in two cases: when $\mathcal{R}_0 \leq 1$ and $\mathcal{R}_0 > 1$. The parameter values used in our simulation are: $\tilde{\Lambda} = 200$; $\mu_H = 0.3$; $\mu_V = 0.2$; $\gamma_H = 0.4$; $\alpha_1 = 0.0005$; $\alpha_2 = 0.0021$; $A = 100$. From these values, we have $\mathcal{R}_0 = 0.87$. When we change the values of α_1 and α_2 to $\alpha_1 = 0.001$ and $\alpha_2 = 0.21$, we get $\mathcal{R}_0 = 12.25$. The software used for the simulation is *scilab*.



(a) human population, $\mathcal{R}_0 < 1$.

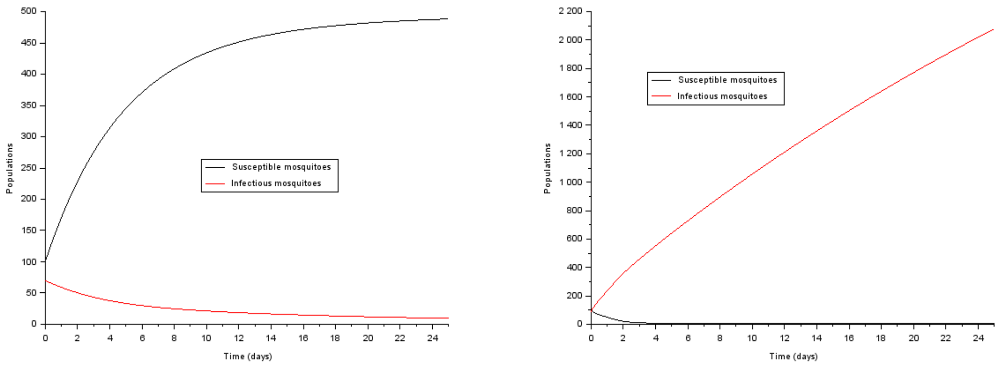


(b) human population, $\mathcal{R}_0 > 1$.

Figure 2: Dynamics of the human population for different magnitudes of $\mathcal{R}_0$. Figure 2a give the dynamic of susceptible, infectious and remove, in model (2). These curves also indicate that the disease tends to disappear. Figure 2b presents the dynamic of the same classes, these curves show us that the disease persists in the population.

7 Global Sensitivity Analysis for $\mathcal{R}_0$

In this paragraph, we use the notion of sensitivity analysis to show the importance of different parameters in the basic reproduction number $\mathcal{R}_0$. In Subsection 7.1, we define the notion of sensitivity of some parameter p of the model (2). Subsection 7.2 is devoted to the calculations of the analytical expressions of the sensitivity indices of different parameters in the basic reproduction number. In Subsection 7.3, we give a numerical representation and comments for different sensitivity indice.



(a) Vector population, $\mathcal{R}_0 < 1$. (b) Vector population, $\mathcal{R}_0 > 1$.

Figure 3: Dynamics of the vector population for different values of $\mathcal{R}_0$. Figure 3a designs the dynamic of susceptible and infectious mosquitoes, in model (2). The graphs together also show that the disease tends to disappear. Figure 3b shows the dynamic of the vector population, these curves indicate that the disease will persist in the population.

7.1 Definition

Let p be a parameter of the mathematical model (2). The parameter p is said to be sensitive if any small alteration of p causes a significant change in the solution. It is worthy to note that the parameter p is termed to be locally sensitive if the change in the value of the parameter p influences the output of the model. In the same way, global sensitivity takes into account the overall change in the model output as a result of the change in all parameter values within their respective range [18].

In computing the normalized sensitivity index ($\wp_p^{\mathcal{R}_0}$) for the basic reproduction number $\mathcal{R}_0$ for each parameter p, we use the relation given by

$$\wp_p^{\mathcal{R}_0} = \frac{\partial \mathcal{R}_0}{\partial p} \times \frac{p}{\mathcal{R}_0}. \tag{10}$$

7.2 Analytic representation of the elasticity

We use the law of mass action as an incidence function. The general incidence functions f and g are defined by the relations $f(S_H, I_V) = \tilde{\beta} S_H I_V$ and $g(S_V, I_H) = \epsilon S_V I_H$. In this case, the expression of the basic reproduction number $\mathcal{R}_0$ is given by the relation

$$\mathcal{R}_0 = \sqrt{\frac{\tilde{\beta}\epsilon\tilde{\Lambda}A}{\mu_H \mu_V^2 (\mu_H + \gamma_H)}}.$$

The sensitivity indices of different parameters are given as follows.

Using the principle given by (10), we obtain

$$\wp_{\tilde{\beta}}^{\mathcal{R}_0} = \frac{1}{2}, \quad \wp_{\epsilon}^{\mathcal{R}_0} = \frac{1}{2}, \wp_{\tilde{\Lambda}}^{\mathcal{R}_0} = \frac{1}{2}, \quad \wp_{A}^{\mathcal{R}_0} = \frac{1}{2},$$

$$\wp^{\mathcal{R}_0}_{\gamma_H} = -\frac{1}{2}\frac{\gamma_H}{\mu_H + \gamma_H}, \; \wp^{\mathcal{R}_0}_{\mu_V} = -1, \; \wp^{\mathcal{R}_0}_{\mu_H} = -\frac{2\mu_H + \gamma_H}{2}.$$

Table 1: Parameter description and elasticity value.

Parameter	Description	Elasticity index
$\tilde{\Lambda}$	Humans recruitment rate	0.5
$\tilde{\beta}$	Positive contact rate between I_V and S_H	0.5
ϵ	Positive contact rate between I_H and S_V	0.5
A	Mosquito recruitment rate	0.5
γ_H	Infectious humans who pass in R_H	-0.28
μ_V	Natural death of mosquito	-1
μ_H	Natural death of humans	-0.5

7.3 Numerical representation and comments

In this subsection, we give some numerical representation and comments for different sensitivity indices while the analytical expressions and values are obtained in Subsection 7.2. For the numerical representation, we use the R software and the graph is given in Figure 4.

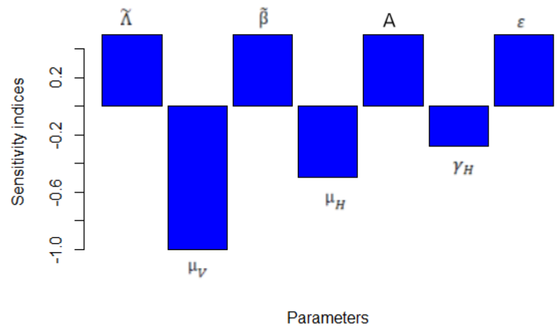


Figure 4: Global sensitivity plot.

Parameters with a positive sensitivity index indicate an increase in the transmission of dengue in the population for an increase in these values. On the other hand, parameters with a negative sensitivity index mean that an increase in these values leads to a decrease in the transmission of dengue in the population. For example, the sensitivity index of $\tilde{\Lambda}$ in $\mathcal{R}_0$ is 0.5. This implies that an increase of 1% in the value of $\tilde{\Lambda}$ leads to an increase of 0.5% in the value of $\mathcal{R}_0$. The sensitivity indices of $\tilde{\Lambda}$, $\tilde{\beta}$, ϵ and A are the same, which means that these parameters have the same impact on the secondary infection rate. In the same way, the elasticity of μ_V in $\mathcal{R}_0$ is -1 meaning that the increase of 1% in the value of mosquito mortality implies the decrease of 1% in the value of $\mathcal{R}_0$. The fact that $\wp^{\mathcal{R}_0}_{\mu_H} = -0.28$ means that 1% increase in μ_V will produce 0.28% decrease in $\mathcal{R}_0$. Also, the fact that $\wp_{\gamma_H}^{\mathcal{R}_0} = -0.5$ implies that 1% increase in γ_H will produce 0.5% decrease in the value of the basic reproduction number.

Thus, we find that the parameter μ_V, which denotes the mosquito mortality rate, is a good parameter for controlling the dynamics of dengue transmission. As it increases, the basic reproduction number $\mathcal{R}_0$ decreases more rapidly. However, it is not the only parameter whose growth leads to a decrease in the basic reproduction number.

8 Conclusion

In this paper, we have studied the dengue disease transmission model, which includes the human and vector populations with general admission incidence functions. We proved the existence of the equilibrium and its stability. When the value of the basic reproduction number $\mathcal{R}_0$ is less than unity, the disease-free equilibrium is globally asymptotically stable, in this case the disease will disappear. When $\mathcal{R}_0 > 1$, the endemic equilibrium exists and it is globally asymptotically stable, in this case the disease will persist in the population. We used the Lyapunov function to study the stability of our equilibrium points. We have also presented the numerical simulations, and the evolution of our curves corroborate with the theoretical results. We carried out a sensitivity study of the parameters in order to determine the influence of different parameters on the transmission of the disease. We notice that the parameter μ_V, which denotes the mortality rate of the mosquitoes, allows to better control the dynamics of dengue disease transmission. In our future work we will integrate the spatial distribution of the disease.

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Denumerably Many Positive Radial Solutions to Iterative System of Nonlinear Elliptic Equations on the Exterior of a Ball

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Abstract: In this paper, by an application of Krasnoselskii's fixed point theorem, we establish the existence of denumerably many positive radial solutions to the iterative system of nonlinear elliptic equations of the form

$$\begin{gathered} \Delta \mathtt{u}_{\mathtt{j}} + \mathtt{P}(|\mathtt{x}|)\mathtt{g}_{\mathtt{j}}(\mathtt{u}_{\mathtt{j}+1}) = 0 \ \text{ in } \ \mathbb{R}^N \backslash \mathscr{B}_{r_0}, \\ \mathtt{u}_{\mathtt{j}} = 0 \ \text{ on } \ |\mathtt{x}| = r_0, \\ \mathtt{u}_{\mathtt{j}} \to 0 \ \text{ as } \ |\mathtt{x}| \to +\infty, \end{gathered}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, $\Delta \mathtt{u} = \mathtt{div}(\nabla \mathtt{u})$, $N > 2$, $r_0 > 0$, $\mathscr{B}_{r_0} = \{\mathtt{u} \in \mathbb{R}^N |\ |\mathtt{u}| < r_0\}$, $\mathtt{P} = \prod_{i=1}^{n} \mathtt{P}_i$, each $\mathtt{P}_i : (r_0, +\infty) \to (0, +\infty)$ is continuous, $r^{N-1}\mathtt{P}$ is integrable and may have singularities, and $\mathtt{g}_{\mathtt{j}} : [0, +\infty) \to \mathbb{R}$ is continuous.

Keywords: *nonlinear elliptic systems; exterior of a ball; positive radial solution; Krasnoselskii's fixed point theorem.*

Mathematics Subject Classification (2010): 35J66, 35J60, 34B18, 47H10.

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1 Introduction

The study of nonlinear elliptic system of equations

$$\left.\begin{aligned} \Delta \mathtt{u}_{\mathtt{j}} + \mathtt{g}_{\mathtt{j}}(\mathtt{u}_{\mathtt{j}+1}) = 0 \ \text{ in } \ \Omega, \\ \mathtt{u}_{\mathtt{j}} = 0 \ \text{ on } \ \partial\Omega, \end{aligned}\right\} \tag{1}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, and Ω is a bounded domain in $\mathbb{R}^N$, has an important applications in population dynamics, combustion theory and chemical reactor theory. For the recent literature on the existence, multiplicity and uniqueness of positive solutions for (1), see [3–5, 8, 9, 11, 12] and references therein.

In [2], Akdim, Rhoudaf and Salmani established the existence of entropy solutions for anisotropic elliptic equations of the form

$$\mathtt{Au} + \sum_{i=1}^{n} \mathtt{g}_i(\mathtt{x}, \mathtt{u}, \nabla \mathtt{u}) = f,$$

where $A\mathtt{u}$ is a Leray-Lions anisotropic operator. In [1], Aberqi, Bennouna and Elmassoudi established the existence results for the following nonlinear elliptic equations with some measure data in Musielak-Orlicz spaces:

$$\mathtt{Au} + \mathtt{K}(\mathtt{x}, \mathtt{u}, \nabla \mathtt{u}) = \mu.$$

In [6], Dong and Wei established the existence of radial solutions for the following nonlinear elliptic equations with gradient terms in annular domains:

$$\begin{aligned} -\Delta \mathtt{u} = \mathtt{g}\big(|\mathtt{x}|, \mathtt{u}, \frac{\mathtt{x}}{|\mathtt{x}|} \cdot \nabla \mathtt{u}\big) \ \text{ in } \ \Omega_a^b, \\ \mathtt{u} = 0 \ \text{ on } \ \partial\Omega_a^b, \end{aligned}$$

by using Schauder's fixed point theorem and the contraction mapping theorem. In [10], R. Kajikiya and E. Ko established the existence of positive radial solutions for a semipositone elliptic equation of the form

$$\begin{aligned} -\Delta \mathtt{u} = \lambda \mathtt{g}(\mathtt{u}) \ \text{ in } \ \Omega, \\ \mathtt{u} = 0 \ \text{ on } \ \partial\Omega, \end{aligned}$$

where Ω is a ball or an annulus in $\mathbb{R}^N$. Recently, Son and Wang [13] have studied positive radial solutions for the nonlinear elliptic systems of the form

$$\begin{aligned} \Delta \mathtt{u}_{\mathtt{j}} + \lambda \mathtt{K}_{\mathtt{j}}(|\mathtt{x}|)\mathtt{g}_{\mathtt{j}}(\mathtt{u}_{\mathtt{j}+1}) = 0 \text{ in } \Omega_{\mathtt{E}}, \\ \mathtt{u}_{\mathtt{j}} = 0 \text{ on } |\mathtt{x}| = r_0, \\ \mathtt{u}_{\mathtt{j}} \to 0 \text{ as } |\mathtt{x}| \to +\infty, \end{aligned}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, $\lambda > 0$, $N > 2$, $r_0 > 0$, and $\Omega_{\mathtt{E}}$ is an exterior of a ball, and established existence, multiplicity and uniqueness results for various nonlinearities in $\mathtt{g}_{\mathtt{j}}$. Inspired by the aforementioned works, in this paper, we apply Krasnoselskii's fixed point theorem to derive necessary conditions for the existence of denumerably many positive radial solutions of the following iterative system of nonlinear elliptic equations in the exterior of a ball:

$$\left.\begin{aligned} \Delta \mathtt{u}_{\mathtt{j}} + \mathtt{P}(|\mathtt{x}|)\mathtt{g}_{\mathtt{j}}(\mathtt{u}_{\mathtt{j}+1}) = 0 \text{ in } \mathbb{R}^N \backslash \mathscr{B}_{r_0}, \\ \mathtt{u}_{\mathtt{j}} = 0 \text{ on } |\mathtt{x}| = r_0, \\ \mathtt{u}_{\mathtt{j}} \to 0 \text{ as } |\mathtt{x}| \to +\infty, \end{aligned}\right\} \tag{2}$$

where $\mathtt{j} \in \{1, 2, 3, \cdots, \ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, $\Delta\mathtt{u} = \mathtt{div}(\nabla\mathtt{u})$, $N > 2$, $r_0 > 0$, $\mathscr{B}_{r_0} = \{\mathtt{u} \in \mathbb{R}^N \mid |\mathtt{u}| < r_0\}$, $\mathtt{P} = \prod_{i=1}^{n} \mathtt{P}_i$, each $\mathtt{P}_i : (r_0, +\infty) \to (0, +\infty)$ is continuous, $r^{N-1}\mathtt{P}$ is integrable and may have singularities, and $\mathtt{g}_\mathtt{j} : [0, +\infty) \to \mathbb{R}$ is continuous.

The study of positive radial solutions to (2) reduces to the study of positive solutions to the following iterative system of two-point boundary value problems:

$$\left.\begin{aligned} \mathtt{u}_\mathtt{j}''(\tau) + \mathtt{Q}(\tau)\mathtt{g}_\mathtt{j}\big(\mathtt{u}_{\mathtt{j}+1}(\tau)\big) = 0,\ \tau \in (0,1),\\ \mathtt{u}_\mathtt{j}(0) = 0,\ u_\mathtt{j}(1) = 0, \end{aligned}\right\} \tag{3}$$

where $\mathtt{j} \in \{1, 2, 3, \cdot\cdot\cdot, \ell\}$, $\mathtt{u}_1 = \mathtt{u}_{\ell+1}$, and $\mathtt{Q}(\tau) = \frac{r_0^2}{(N-2)^2}\tau^{\frac{2(N-1)}{2-N}} \prod_{i=1}^{n} \mathtt{Q}_i(\tau)$, $\mathtt{Q}_i(\tau) = \mathtt{P}_i(r_0\tau^{\frac{1}{2-N}})$ by a Kelvin type transformation through the change of variables $r = |\mathtt{x}|$ and $\tau = \left(\frac{r}{r_0}\right)^{2-N}$. Here, $\mathtt{Q}_i$ may have singularities on $[0, 1]$. Thus, for each $i \in \{1, 2, 3, \cdots, n\}$, we assume that the following conditions hold throughout the paper:

($\mathcal{H}_1$) $\mathtt{Q}_i \in L^{\mathtt{p}_i}[0, 1], (\mathtt{p}_i \geq 1)$ and may have denumerably many singularities on $(0, 1/2)$.

($\mathcal{H}_2$) There exists a sequence $\{\tau_k\}_{k=1}^{\infty}$ such that $0 < \tau_{k+1} < \tau_k < \dfrac{1}{2},\ k \in \mathbb{N}$,

$$\lim_{k\to\infty} \tau_k = \tau^* < \frac{1}{2},\ \lim_{\tau\to\tau_k} \mathtt{Q}_i(\tau) = +\infty,\ k \in \mathbb{N},\ i = 1, 2, 3, \cdots, n,$$

and each $\mathtt{Q}_i(\tau)$ does not vanish identically on any subinterval of $[0, 1]$. Moreover, there exists $\mathtt{Q}_i^* > 0$ such that

$$\mathtt{Q}_i^* < \mathtt{Q}_i(\tau) < \infty \quad \text{a.e. on} \quad [0, 1].$$

The rest of the paper is organized in the following fashion. In Section 2, we convert the boundary value problem (3) into the equivalent integral equation which involves the kernel. Also, we estimate bounds for the kernel which are useful in our main results. In Section 3, we develop a criteria for the existence of denumerably many positive radial solutions for (2) by applying Krasnoselskii's cone fixed point theorem in a Banach space. Finally, as an application, an example is given to demonstrate our results.

2 Kernel and Its Bounds

In this section, we constructed a kernel to the homogeneous boundary value problem corresponding to (3) and established certain lemmas for the bounds of the kernel.

Lemma 2.1 *Let $y \in \mathcal{C}[0, 1]$. Then the boundary value problem*

$$\left.\begin{aligned} \mathtt{u}_1''(\tau) + \mathtt{Q}(\tau)y(\tau) = 0,\ \tau \in (0,1),\\ \mathtt{u}_1(0) = 0,\ u_1(1) = 0, \end{aligned}\right\} \tag{4}$$

has a unique solution

$$\mathtt{u}_1(\tau) = \int_0^1 \aleph(\tau, s)\mathtt{Q}(s)y(s)ds, \tag{5}$$

where

$$\aleph(\tau, s) = \begin{cases} s(1-\tau), & 0 \leq s \leq \tau \leq 1, \\ \tau(1-s), & 0 \leq \tau \leq s \leq 1. \end{cases}$$

Lemma 2.2 *The kernel $\aleph(\tau, s)$ has the following properties:*

(i) $\aleph(\tau, s)$ *is nonnegative and continuous on* $[0, 1] \times [0, 1]$,

(ii) $\aleph(\tau, s) \leq \aleph(s, s)$ *for* $t, \tau \in [0, 1]$,

(iii) there exists $\beta \in (0, \frac{1}{2})$ *such that* $\beta\aleph(s, s) \leq \aleph(\tau, s)$ *for* $\tau \in [\beta, 1 - \beta], s \in [0, 1]$.

Proof. From the definition of kernel $\aleph(\tau, s)$, it is clear that (i) and (ii) hold. To prove (iii), let $\tau \in [\beta, 1 - \beta]$ and $s \leq \tau$, then

$$\frac{\aleph(\tau, s)}{\aleph(s, s)} = \frac{s(1 - \tau)}{s(1 - s)} \geq 1 - \tau \geq \beta,$$

and for $\tau \leq s$, we have

$$\frac{\aleph(\tau, s)}{\aleph(s, s)} = \frac{\tau(1 - s)}{s(1 - s)} \geq \tau \geq \beta.$$

This completes the proof.

From Lemma 2.1, we note that an ℓ-tuple $(\mathtt{u}_1, \mathtt{u}_2, \cdots, \mathtt{u}_\ell)$ is a solution of the boundary value problem (3) if and only if

$$\mathtt{u}_1(\tau) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \\ \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\Bigg] \cdots \Bigg]ds_3\Bigg]ds_2\Bigg]ds_1.$$

In general,

$$\mathtt{u}_\mathtt{j}(\tau) = \int_0^1 \aleph(\tau, s)\mathtt{Q}(s)\mathtt{g}_\mathtt{j}\big(\mathtt{u}_{\mathtt{j}+1}(s)\big)ds, \ \mathtt{j} = 1, 2, 3, \cdots, \ell,$$
$$\mathtt{u}_1(\tau) = \mathtt{u}_{\ell+1}(\tau).$$

We denote the Banach space $\mathcal{C}([0, 1], \mathbb{R})$ by $\mathscr{B}$ with the norm $\|\mathtt{u}\| = \max_{\tau\in[0,1]} |\mathtt{u}(\tau)|$. For $\beta \in (0, 1/2)$, the cone $\mathcal{P}_\beta \subset \mathscr{B}$ is defined by

$$\mathcal{P}_\beta = \Big\{\mathtt{u} \in \mathscr{B} : \mathtt{u}(\tau) \geq 0, \min_{\tau\in[\beta,\, 1-\beta]} \mathtt{u}(\tau) \geq \beta\|\mathtt{u}\|\Big\}.$$

For any $\mathtt{u}_1 \in \mathcal{P}_\beta$, define an operator $\Omega : \mathcal{P}_\beta \to \mathscr{B}$ by

$$(\Omega\mathtt{u}_1)(\tau) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \\ \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\Bigg] \cdots \Bigg]ds_3\Bigg]ds_2\Bigg]ds_1.$$

Lemma 2.3 *For each* $\beta \in (0, 1/2)$, $\Omega(\mathcal{P}_\beta) \subset \mathcal{P}_\beta$ *and* $\Omega : \mathcal{P}_\beta \to \mathcal{P}_\beta$ *is completely continuous.*

Proof. Let $\beta \in (0, 1/2)$. Since $\mathtt{g_j}(\mathtt{u_{j+1}}(\tau))$ is nonnegative for $\tau \in [0,1]$, $\mathtt{u}_1 \in \mathcal{P}_\beta$. Since $\aleph(\tau, s)$ is nonnegative for all $\tau, s \in [0,1]$, it follows that $\Omega(\mathtt{u}_1(\tau)) \geq 0$ for all $\tau \in [0,1]$, $\mathtt{u}_1 \in \mathcal{P}_\beta$. Now, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&\min_{\tau\in[\beta,1-\beta]}(\Omega\mathtt{u}_1)(\tau)\\
&= \min_{\tau\in[\beta,1-\beta]}\Bigg\{\int_0^1 \aleph(\tau,s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1,s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2,s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots\\
&\qquad\qquad \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1},s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\Bigg\}\\
&\geq \beta\int_0^1 \aleph(s_1,s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1,s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2,s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots\\
&\qquad\qquad \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1},s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\\
&\geq \beta\Bigg\{\int_0^1 \aleph(\tau,s_1)\mathtt{Q}(s_1)\mathtt{g}_1\Bigg[\int_0^1 \aleph(s_1,s_2)\mathtt{Q}(s_2)\mathtt{g}_2\Bigg[\int_0^1 \aleph(s_2,s_3)\mathtt{Q}(s_3)\mathtt{g}_4\cdots\\
&\qquad\qquad \mathtt{g}_{\ell-1}\Bigg[\int_0^1 \aleph(s_{\ell-1},s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\Bigg]\cdots\Bigg]ds_3\Bigg]ds_2\Bigg]ds_1\Bigg\}\\
&\geq \beta\max_{\tau\in[0,1]}|\Omega\mathtt{u}_1(\tau)|.
\end{aligned}$$

Thus $\Omega(\mathcal{P}_\beta) \subset \mathcal{P}_\beta$. Therefore, the operator Ω is completely continuous by standard methods and by the Arzela-Ascoli theorem.

3 Denumerably Many Positive Radial Solutions

In this section, we establish the existence of denumerably many positive radial solutions for the system (2) by utilizing the following theorems.

Theorem 3.1 *[7] Let $\mathcal{E}$ be a cone in a Banach space $\mathcal{X}$ and Λ_1, Λ_2 be open sets with $0 \in \Lambda_1, \overline{\Lambda}_1 \subset \Lambda_2$. Let $\mathcal{T} : \mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1) \to \mathcal{E}$ be a completely continuous operator such that*

(a) $\|\mathcal{T}\mathtt{u}\| \leq \|\mathtt{u}\|, \mathtt{u} \in \mathcal{E} \cap \partial\Lambda_1,$ and $\|\mathcal{T}\mathtt{u}\| \geq \|\mathtt{u}\|, \mathtt{u} \in \mathcal{E} \cap \partial\Lambda_2,$ or

(b) $\|\mathcal{T}\mathtt{u}\| \geq \|\mathtt{u}\|, \mathtt{u} \in \mathcal{E} \cap \partial\Lambda_1,$ and $\|\mathcal{T}\mathtt{u}\| \leq \|\mathtt{u}\|, \mathtt{u} \in \mathcal{E} \cap \partial\Lambda_2.$

Then $\mathcal{T}$ has a fixed point in $\mathcal{E} \cap (\overline{\Lambda}_2 \backslash \Lambda_1)$.

Theorem 3.2 *(Hölder's) Let $f \in L^{\mathtt{p}_i}[0,1]$ with $\mathtt{p}_i > 1$, for $i = 1, 2, \cdots, n$ and $\displaystyle\sum_{i=1}^{n}\frac{1}{\mathtt{p}_i} = 1$. Then $\displaystyle\prod_{i=1}^{n} f_i \in L^1[0,1]$ and $\|\prod_{i=1}^n f_i\|_1 \leq \prod_{i=1}^n \|f_i\|_{\mathtt{p}_i}$. Further, if $f \in L^1[0,1]$ and $g \in L^\infty[0,1]$, then $fg \in L^1[0,1]$ and $\|fg\|_1 \leq \|f\|_1\|g\|_\infty$.*

Consider the following three possible cases for $\mathtt{P_j} \in L^{\mathtt{p}_i}[0,1]$:

$$\sum_{i=1}^{n}\frac{1}{\mathtt{p}_i} < 1,\ \ \sum_{i=1}^{n}\frac{1}{\mathtt{p}_i} = 1,\ \ \sum_{i=1}^{n}\frac{1}{\mathtt{p}_i} > 1.$$

Firstly, we seek denumerably many positive radial solutions for the case $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} < 1$.

Theorem 3.3 *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold, let* $\{\beta_k\}_{k=1}^{\infty}$ *be a sequence with* $\tau_{k+1} < \beta_k < \tau_k$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{S}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{R}_{k+1} < \beta_k \mathtt{S}_k < \mathtt{S}_k < \mathfrak{N}\mathtt{S}_k < \mathtt{R}_k, \; k \in \mathbb{N},$$

where

$$\mathfrak{N} = \max\left\{ \left[\beta_1 \frac{r_0^2}{(N-2)^2} \prod_{i=1}^{n} \mathtt{Q}_i^* \int_{\beta_1}^{1-\beta_1} \aleph(s,s) s^{\frac{2(N-1)}{2-N}} ds \right]^{-1}, \; 1 \right\}.$$

Further, assume that $\mathtt{g}_\mathtt{j}$ *satisfies*

$(\mathcal{A}_1)$ $\mathtt{g}_\mathtt{j}(\mathtt{u}(\tau)) \leq \mathtt{M}_1 \mathtt{R}_k$ *for all* $\tau \in [0,1]$, $0 \leq \mathtt{u} \leq \mathtt{R}_k$,
where

$$\mathtt{M}_1 < \left[\frac{r_0^2}{(N-2)^2} \|\aleph\|_q \prod_{i=1}^{n} \|\mathtt{Q}_i\|_{p_i} \right]^{-1}, \quad \aleph(s) = \aleph(s,s) s^{\frac{2(N-1)}{2-N}},$$

$(\mathcal{A}_2)$ $\mathtt{g}_\mathtt{j}(\mathtt{u}(\tau)) \geq \mathfrak{N}\mathtt{S}_k$ *for all* $\tau \in [\beta_k, 1-\beta_k]$, $\beta_k \mathtt{S}_k \leq \mathtt{u} \leq \mathtt{S}_k$.

The iterative system (2) *has denumerably many radial solutions* $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]}, \cdots, \mathtt{u}_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\mathtt{u}_\mathtt{j}^{[k]}(\tau) \geq 0$ *on* $(0,1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. Consider the sequences $\{\Lambda_{1,k}\}_{k=1}^{\infty}$ and $\{\Lambda_{2,k}\}_{k=1}^{\infty}$ of the open subsets of $\mathscr{B}$ defined by

$$\Lambda_{1,k} = \{\mathtt{u} \in \mathscr{B} : \|\mathtt{u}\| < \mathtt{R}_k\}, \; \Lambda_{2,k} = \{\mathtt{u} \in \mathscr{B} : \|\mathtt{u}\| < \mathtt{S}_k\}.$$

Let $\{\beta_k\}_{k=1}^{\infty}$ be as in the hypothesis and note that $\tau^* < \tau_{k+1} < \beta_k < \tau_k < \frac{1}{2}$ for all $k \in \mathbb{N}$. For each $k \in \mathbb{N}$, define the cone $\mathcal{P}_{\beta_k}$ by

$$\mathcal{P}_{\beta_k} = \left\{ \mathtt{u} \in \mathscr{B} : \mathtt{u}(\tau) \geq 0 \;\text{ and }\; \min_{\tau \in [\beta_k, 1-\beta_k]} \mathtt{u}(t) \geq \beta_k \|\mathtt{u}(\tau)\| \right\}.$$

Let $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{1,k}$. Then $\mathtt{u}_1(s) \leq \mathtt{R}_k = \|\mathtt{u}_1\|$ for all $s \in [0,1]$. By $(\mathcal{A}_1)$ and $0 < s_{\ell-1} < 1$, we have

$$\begin{aligned} \int_0^1 \aleph(s_{\ell-1}, s_\ell) \mathtt{Q}(s_\ell) \mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big) ds_\ell &\leq \int_0^1 \aleph(s_\ell, s_\ell) \mathtt{Q}(s_\ell) \mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big) ds_\ell \\ &\leq \mathtt{M}_1 \mathtt{R}_k \int_0^1 \aleph(s_\ell, s_\ell) \mathtt{Q}(s_\ell) ds_\ell \\ &\leq \mathtt{M}_1 \mathtt{R}_k \frac{r_0^2}{(N-2)^2} \int_0^1 \aleph(s_\ell, s_\ell) s_\ell^{\frac{2(N-1)}{2-N}} \prod_{i=1}^{n} \mathtt{Q}_i(s_\ell) ds_\ell. \end{aligned}$$

There exists a $\mathtt{q} > 1$ such that $\sum_{i=1}^{n} \frac{1}{\mathtt{p}_i} + \frac{1}{\mathtt{q}} = 1$. By the first part of Theorem 3.2, we have

$$\begin{aligned} \int_0^1 \aleph(s_{\ell-1}, s_\ell) \mathtt{Q}(s_\ell) \mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big) ds_\ell &\leq \mathtt{M}_1 \mathtt{R}_k \frac{r_0^2}{(N-2)^2} \|\aleph\|_q \prod_{i=1}^{n} \|\mathtt{Q}_i\|_{p_i} \\ &\leq \mathtt{R}_k. \end{aligned}$$

It follows, in a similar manner, for $0 < s_{\ell-2} < 1$,

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-2}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]ds_{\ell-1} \\ \leq \int_0^1 \aleph(s_{\ell-1}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})\mathtt{g}_{\ell-1}(\mathtt{R}_k)ds_{\ell-1} \\ \leq \mathtt{M}_1\mathtt{R}_k \int_0^1 \aleph(s_{\ell-1}, s_{\ell-1})\mathtt{Q}(s_{\ell-1})ds_{\ell-1} \\ \leq \mathtt{M}_1\mathtt{R}_k \frac{r_0^2}{(N-2)^2}\|\aleph\|_q \prod_{i=1}^n \|\mathtt{Q}_i\|_{p_i} \\ \leq \mathtt{R}_k.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$\begin{aligned}(\Omega\mathtt{u}_1)(t) = \int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right. \\ \left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1 \\ \leq R_k.\end{aligned}$$

Since $\mathtt{R}_k = \|\mathtt{u}_1\|$ for $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{1,k}$, we get

$$\|\Omega\mathtt{u}_1\| \leq \|\mathtt{u}_1\|. \tag{6}$$

Let $\tau \in [\beta_k, 1-\beta_k]$. Then $\mathtt{S}_k = \|\mathtt{u}_1\| \geq \mathtt{u}_1(t) \geq \min_{\tau\in[\beta_k,1-\beta_k]} \mathtt{u}_1(t) \geq \beta_k\|\mathtt{u}_1\| \geq \beta_k\mathtt{S}_k$. By $(\mathcal{A}_2)$ and for $s_{\ell-1} \in [\beta_k, 1-\beta_k]$, we have

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell &\geq \int_{\beta_k}^{1-\beta_k} \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell \\ &\geq \mathfrak{N}\mathtt{S}_k \int_{\beta_k}^{1-\beta_k} \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)ds_\ell \\ &\geq \mathfrak{N}\mathtt{S}_k\beta_1 \int_{\beta_1}^{1-\beta_1} \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)ds_\ell \\ &\geq \mathfrak{N}\mathtt{S}_k\beta_1 \frac{r_0^2}{(N-2)^2}\int_{\beta_1}^{1-\beta_1} \aleph(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}} \prod_{i=1}^n \mathtt{Q}_i(s_\ell)ds_\ell \\ &\geq \mathfrak{N}\mathtt{S}_k\beta_1 \frac{r_0^2}{(N-2)^2}\prod_{i=1}^n \mathtt{Q}_i^* \int_{\beta_1}^{1-\beta_1} \aleph(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}}ds_\ell \\ &\geq \mathtt{S}_k.\end{aligned}$$

Continuing with the bootstrapping argument, we get

$$\begin{aligned}(\Omega \mathtt{u}_1)(\tau) = &\int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots\right.\right.\\ &\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1\\ \geq &\,\mathtt{S}_k.\end{aligned}$$

Thus, if $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{2,k}$, then

$$\|\Omega \mathtt{u}_1\| \geq \|\mathtt{u}_1\|. \tag{7}$$

It is evident that $0 \in \Lambda_{2,k} \subset \overline{\Lambda}_{2,k} \subset \Lambda_{1,k}$. From (6),(7), it follows from Theorem 3.1 that the operator Ω has a fixed point $\mathtt{u}_1^{[k]} \in \mathcal{P}_{\beta_k} \cap \big(\overline{\Lambda}_{1,k} \backslash \Lambda_{2,k}\big)$ such that $\mathtt{u}_1^{[k]}(t) \geq 0$ on $(0,1)$, and $k \in \mathbb{N}$. Next, setting $\mathtt{u}_{\ell+1} = \mathtt{u}_1$, we obtain denumerably many positive solutions $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]}, \cdots, \mathtt{u}_\ell^{[k]})\}_{k=1}^{\infty}$ of (3) given iteratively by

$$\begin{aligned}\mathtt{u}_\mathtt{j}(\tau) &= \int_0^1 \aleph(\tau, s)\mathtt{Q}(s)\mathtt{g}_\mathtt{j}(\mathtt{u}_{\mathtt{j}+1}(s))ds, \ \mathtt{j} = 1, 2, \cdots, \ell-1, \ell,\\ \mathtt{u}_{\ell+1}(\tau) &= \mathtt{u}_1(\tau).\end{aligned}$$

The proof is completed.

For $\sum_{i=1}^n \mathtt{p}_i = 1$, we have the following theorem.

Theorem 3.4 *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold, let* $\{\beta_k\}_{k=1}^{\infty}$ *be a sequence with* $\tau_{k+1} < \beta_k < \tau_k$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{S}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{R}_{k+1} < \beta_k \mathtt{S}_k < \mathtt{S}_k < \mathfrak{N}\mathtt{S}_k < \mathtt{R}_k, \ k \in \mathbb{N}.$$

Further, assume that $\mathtt{g}_\mathtt{j}$ *satisfies* $(\mathcal{A}_2)$ *and*
$(\mathcal{A}_3)$ $\mathtt{g}_\iota(\mathtt{u}(\tau)) \leq \mathtt{M}_2\mathtt{R}_k$ *for all* $0 \leq \mathtt{u}(\tau) \leq \mathtt{R}_k$, $\tau \in [0,1]$, *where*

$$\mathtt{M}_2 < \min\left\{\left[\frac{r_0^2}{(N-2)^2}\|\aleph\|_\infty \prod_{i=1}^n \|\mathtt{Q}_i\|_{\mathtt{p}_i}\right]^{-1}, \ \mathfrak{N}\right\}.$$

The iterative system (2) *has denumerably many radial solutions* $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]}, \cdots, \mathtt{u}_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\mathtt{u}_\mathtt{j}^{[k]}(\tau) \geq 0$ *on* $(0,1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. Let $\Lambda_{1,k}$ be as in the proof of Theorem 3.3 and let $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{2,k}$. Again, $\mathtt{u}_1(\tau) \leq \mathtt{R}_k = \|\mathtt{u}_1\|$ for all $\tau_1 \in [0,1]$. By $(\mathcal{A}_3)$ and $0 < \tau_{\ell-1} < 1$, we have

$$\begin{aligned}\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell &\leq \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\\ &\leq \mathtt{M}_1\mathtt{R}_k \int_0^1 \aleph(s_\ell, s_\ell)\mathtt{Q}(s_\ell)ds_\ell\\ &\leq \mathtt{M}_1\mathtt{R}_k \frac{r_0^2}{(N-2)^2}\int_0^1 \aleph(s_\ell, s_\ell)s_\ell^{\frac{2(N-1)}{2-N}}\prod_{i=1}^n \mathtt{Q}_i(s_\ell)ds_\ell\\ &\leq \mathtt{M}_1\mathtt{R}_k \frac{r_0^2}{(N-2)^2}\|\aleph\|_\infty \prod_{i=1}^n \|\mathtt{Q}_i\|_{\mathtt{p}_i}\\ &\leq \mathtt{R}_k.\end{aligned}$$

Continuing with this bootstrapping argument, we get

$$
\begin{aligned}
(\Omega \mathtt{u}_1)(t) = &\int_0^1 \aleph(\tau, s_1)\mathtt{Q}(s_1)\mathtt{g}_1\left[\int_0^1 \aleph(s_1, s_2)\mathtt{Q}(s_2)\mathtt{g}_2\left[\int_0^1 \aleph(s_2, s_3)\mathtt{Q}(s_3)\mathtt{g}_4 \cdots \right.\right.\\
&\left.\left.\mathtt{g}_{\ell-1}\left[\int_0^1 \aleph(s_{\ell-1}, s_\ell)\mathtt{Q}(s_\ell)\mathtt{g}_\ell\big(\mathtt{u}_1(s_\ell)\big)ds_\ell\right]\cdots\right]ds_3\right]ds_2\Bigg]ds_1\\
\leq &R_k.
\end{aligned}
$$

Thus, $\|\Omega \mathtt{u}_1\| \leq \|\mathtt{u}_1\|$ for $\mathtt{u}_1 \in \mathcal{P}_{\beta_k} \cap \partial\Lambda_{1,k}$. Now define $\Lambda_{2,k} = \{\mathtt{u} \in \mathscr{B} : \|\mathtt{u}\| < \mathtt{S}_k\}$. Let $\mathtt{u}_1 \in \mathcal{P}_{\Omega_k} \cap \partial\Lambda_{2,k}$ and let $s_{\ell-1} \in [\beta_k,\, 1-\beta_k]$. Then the argument leading to (7) can be applied to the present case. Hence, the theorem is proved.

Finally, we deal with the case $\sum_{i=1}^{n} \mathtt{p}_i > 1$.

Theorem 3.5 *Suppose* $(\mathcal{H}_1) - (\mathcal{H}_2)$ *hold, let* $\{\beta_k\}_{k=1}^{\infty}$ *be a sequence with* $\tau_{k+1} < \beta_k < \tau_k$. *Let* $\{\mathtt{R}_k\}_{k=1}^{\infty}$ *and* $\{\mathtt{S}_k\}_{k=1}^{\infty}$ *be such that*

$$\mathtt{R}_{k+1} < \beta_k \mathtt{S}_k < \mathtt{S}_k < \mathfrak{N}\mathtt{S}_k < \mathtt{R}_k,\ \ k \in \mathbb{N}.$$

Further, assume that $\mathtt{g}_\mathtt{j}$ *satisfies* $(\mathcal{A}_2)$ *and*
$(\mathcal{A}_4)$ $\mathtt{g}_\iota(\mathtt{u}(\tau)) \leq \mathtt{M}_3\mathtt{R}_k$ *for all* $0 \leq \mathtt{u}(\tau) \leq \mathtt{R}_k$, $\tau \in [0,1]$, *where*

$$\mathtt{M}_3 < \min\left\{\left[\frac{r_0^2}{(N-2)^2}\|\aleph\|_\infty \prod_{i=1}^{n}\|\mathtt{Q}_i\|_1\right]^{-1},\ \mathfrak{N}\right\}.$$

The iterative system (2) *has denumerably many radial solutions* $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]}, \cdots, \mathtt{u}_\ell^{[k]})\}_{k=1}^{\infty}$ *such that* $\mathtt{u}_\mathtt{j}^{[k]}(\tau) \geq 0$ *on* $(0,1)$, $\mathtt{j} = 1, 2, \cdots, \ell$ *and* $k \in \mathbb{N}$.

Proof. The proof is similar to the proof of Theorem 3.3.

4 Applications

Example 4.1 *Consider the following fractional order boundary value problem:*

$$
\left.\begin{aligned}
\Delta \mathtt{u}_\mathtt{j} + \mathtt{P}(|\mathtt{x}|)\mathtt{g}_\mathtt{j}(\mathtt{u}_{\mathtt{j}+1}) = 0 \ &in\ \mathbb{R}^3\backslash\mathscr{B}_1,\\
\mathtt{u}_\mathtt{j} = 0 \ &on\ |\mathtt{x}| = 1,\\
\mathtt{u}_\mathtt{j} \to 0 \ &as\ |\mathtt{x}| \to +\infty,
\end{aligned}\right\} \tag{8}
$$

where $\mathtt{j} \in \{1,2\}$, $\mathtt{u}_3 = \mathtt{u}_1$, $\mathtt{Q}(\tau) = \frac{1}{\tau^4}\prod_{i=1}^{2}\mathtt{Q}_i(\tau)$, $\mathtt{Q}_i(\tau) = \mathtt{P}_i(\frac{1}{\tau})$, in which

$$\mathtt{P}_1(t) = \frac{1}{|t-4|^{\frac{1}{2}}} \qquad \text{and} \qquad \mathtt{P}_2(t) = \frac{1}{|t-3|^{\frac{1}{2}}},$$

$$
\mathtt{g}_{\mathtt{j}}(\mathtt{u}) = \begin{cases} 5 \times 10^{-14}, & \mathtt{u} \in (10^{-4}, +\infty), \\ \frac{30\times 10^{-(4k+2)}-5\times 10^{-4k-10}}{10^{-(4k+2)}-10^{-4k}}(\mathtt{u} - 10^{-4k}) + 5 \times 10^{-4k-10}, & \\ & \mathtt{u} \in \left[10^{-(4k+2)}, 10^{-4k}\right], \\ 30 \times 10^{-(4k+2)}, \qquad \mathtt{u} \in \left(\frac{1}{5} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right), & \\ \frac{30\times 10^{-(4k+2)}-5\times 10^{-(4k+14)}}{\frac{1}{5}\times 10^{-(4k+2)}-10^{-(4k+4)}}(\mathtt{u} - 10^{-(4k+4)}) + 5 \times 10^{-(4k+14)}, & \\ & \mathtt{u} \in \left(10^{-(4k+4)}, \frac{1}{5} \times 10^{-(4k+2)}\right], \end{cases}
$$

$\mathtt{j} = 1, 2$. Let

$$
\tau_k = \frac{31}{64} - \sum_{r=1}^{k} \frac{1}{4(r+1)^4}, \ \beta_k = \frac{1}{2}(\tau_k + \tau_{k+1}), \ k = 1, 2, 3, \cdots,
$$

then

$$
\beta_1 = \frac{15}{32} - \frac{1}{648} < \frac{15}{32}
$$

and

$$
\tau_{k+1} < \beta_k < \tau_k, \ \beta_k > \frac{1}{5}.
$$

It is easy to see

$$
\tau_1 = \frac{15}{32} < \frac{1}{2}, \ \tau_k - \tau_{k+1} = \frac{1}{4(k+2)^4}, \ k = 1, 2, 3, \cdots.
$$

Since $\sum_{k=1}^{\infty} \frac{1}{k^4} = \frac{\pi^4}{90}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, it follows that

$$
\tau^* = \lim_{k\to\infty} \tau_k = \frac{31}{64} - \sum_{i=1}^{\infty} \frac{1}{4(i+1)^4} = \frac{47}{64} - \frac{\pi^4}{360} > \frac{1}{5}.
$$

Also,

$$
\mathtt{P}_1, \mathtt{P}_2 \in L^{\mathtt{p}}[0,1] \quad \text{and} \quad \prod_{i=1}^{2} \mathtt{Q}_i^* = \frac{1}{\sqrt{12}},
$$

$$
\int_{\beta_1}^{1-\beta_1} \aleph(s,s) s^{\frac{2(N-1)}{2-N}} ds = 0.2657555992,
$$

$$
\beta_1 \frac{r_0^2}{(N-2)^2} \prod_{i=1}^{n} \mathtt{Q}_i^* \int_{\beta_1}^{1-\beta_1} \aleph(s,s) s^{\frac{2(N-1)}{2-N}} ds = 0.03584271890,
$$

$$
\mathfrak{N} = \max\left\{ \left[\beta_1 \frac{r_0^2}{(N-2)^2} \prod_{i=1}^{n} \mathtt{Q}_i^* \int_{\beta_1}^{1-\beta_1} \aleph(s,s) s^{\frac{2(N-1)}{2-N}} ds\right]^{-1}, \ 1 \right\} \approx 27.89966918.
$$

Let $\mathtt{q} = 2, \mathtt{p}_1 = \mathtt{p}_2 = 1/4$, then

$$\mathtt{M}_1 < \left[\frac{r_0^2}{(N-2)^2} \|\aleph\|_{\mathtt{q}} \prod_{i=1}^{n} \|\mathtt{Q}_i\|_{p_i} \right]^{-1} \approx 5.95134 \times 10^{-10}.$$

So, let $\mathtt{M}_1 = 5.5 \times 10^{-10}$. In addition, if we take

$$\mathtt{R}_k = 10^{-4k},\ \mathtt{S}_k = 10^{-(4k+2)},$$

then

$$\mathtt{R}_{k+1} = 10^{-(4k+4)} < \frac{1}{5} \times 10^{-(4k+2)} < \beta\mathtt{S}_k < \mathtt{S}_k = 10^{-(4k+2)} < \mathtt{R}_k = 10^{-4k},$$

and $\mathtt{g}_1, \mathtt{g}_2$ satisfy the following growth conditions:

$$\mathtt{g}_{\mathtt{j}}(\mathtt{u}) \leq \mathtt{M}_1\mathtt{R}_k = 5.5 \times 10^{-4k-10}, \quad \mathtt{u} \in \left[0, 10^{-4k}\right],$$
$$\mathtt{g}_{\mathtt{j}}(\mathtt{u}) \geq \mathfrak{N}\mathtt{S}_k = 27.89966918 \times 10^{-(4k+2)}, \quad \mathtt{u} \in \left[\frac{1}{5} \times 10^{-(4k+2)}, 10^{-(4k+2)}\right].$$

Then all the conditions of Theorem 3.3 are satisfied. Therefore, by Theorem 3.3, the boundary value problem (8) has denumerably many positive solutions $\{(\mathtt{u}_1^{[k]}, \mathtt{u}_2^{[k]})\}_{k=1}^{\infty}$ such that $10^{-(4k+2)} \leq \|\mathtt{u}_{\mathtt{j}}^{[k]}\| \leq 10^{-4k}$ for each $k = 1, 2, 3, \cdots$, and $\mathtt{j} = 1, 2$.

5 Conclusion

This paper focuses on establishing the existence of denumerably many positive radial solutions to the iterative system of nonlinear elliptic equations through the application of one of the most important fixed point theorems known as "Krasnoselskii's fixed point theorem". These ease the proof of the existence of the positive solution attached to the system under study.

In the future, we aim to expand this study by adapting some techniques used to other ideas and extracting new results that show the effectiveness of this study and its effect in the midst of scientific research. The closest result we would like to prove is the establishment of the multiple and sign-changing solutions for the iterative system of nonlinear elliptic equations with critical potential and critical parameters.

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Existence of Weak Solutions for a Class of $(p(b(u)), q(b(u)))$-Laplacian Problems

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Abstract: In this paper, we consider the existence of weak solutions for some parabolic $(p(b(u)), q(b(u)))$-Laplacian problem when (p, q) is a nonlocal quantity. The novelty of this work is the study of some problems involving the (p, q)-Laplacian operator in the nonlocal case. The motivation to study these nonlocal problems relies in the fact that in reality, the measurements of some physical quantities are not made pointwise but through some local averages.

Keywords: *$(p(b(u)), q(b(u)))$-Laplacian; weak solutions; parabolic problem, generalised Sobolev spaces.*

Mathematics Subject Classification (2010): 35J60, 35J05, 35-XX, 35Kxx.

1 Introduction

The study of partial differential equations involving the (p, q)-Laplacian generalized several types of problems not only in physics, but also in biophysics, plasma physics, and in the study of chemical reactions. These problems appear, for example, in a general reaction–diffusion system

$$u_t = -\operatorname{div}\left[\left(a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}\right)\nabla u\right] + f(x, u),$$

where $a_p, b_q \in \mathbb{R}^+$ are some positive constants, the function u generally describes the concentration, the term $\operatorname{div}\left[\left(a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}\right)\nabla u\right]$ corresponds to the diffusion with coefficient $D(u) = a_p|\nabla u|^{p-2} + b_q|\nabla u|^{q-2}$, and $f(x, u)$ is the reaction term related to the source and loss processes. In general, the reaction term $f(x, u)$ has a polynomial form with respect to the concentration u.

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Because of the importance of this kind of problems, many authors have investigated the existence and uniqueness of different types of their solutions [5, 7, 14].

Our main interest in this work is to extend these results to the case when p, q may depend on the space variable x and the unknown solution u. We consider the case where the dependency of p, q on u is a nonlocal quantity. Namely, we study the following parabolic problem:

$$\begin{cases} u_t - \operatorname{div}(|\nabla u|^{p(b(u))-2}\nabla u) - \operatorname{div}(|\nabla u|^{q(b(u))-2}\nabla u) = f \ \text{ in } \Omega_T = \Omega \times (0,T), \\ u = 0 \quad \text{on } \Gamma = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) \quad \text{in } \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \geq 2$, $T > 0$, f, u_0 are given data, $p : \mathbb{R} \to [1, +\infty)$ and $q : \mathbb{R} \to [1, +\infty)$ are real functions such that

$$p, q \ \text{ are continuous } \quad \text{and} \quad 1 < \alpha \leq q < p \leq \beta < \infty \tag{2}$$

for some constants α, β. We denote by b a mapping from $W_0^{1,\alpha}(\Omega)$ into $\mathbb{R}$ such that

$$b \ \text{ is continuous and bounded,} \tag{3}$$

i.e., b sends the bounded sets of $W_0^{1,\alpha}(\Omega)$ into the bounded sets of $\mathbb{R}$. In this case, suitable examples for the mapping b in (3) can be chosen as

$$b(u) = \|\nabla u\|_{L^\alpha(\Omega)}.$$

This kind of problems was first introduced by Chipot and de Oliveira in [9]. The elliptic version of the problem (1) with local quantities p, q was studied by L. Yanru in [15], he obtained the existence of weak solutions by means of a singular perturbation technique and the Schauder fixed point theorem. We were inspired by the work of C. Zhang and X. Zhang (see [16]), where the authors proved the existence of weak solutions to the following parabolic $p(u)$-Laplacian problem:

$$\begin{cases} u_t - \operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) = f & \text{in } \Omega_T = \Omega \times (0,T), \\ u = 0 & \text{on } \Gamma = \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

The fact that in reality, the physical measurements of certain quantities are not made in a pointwise way but through local averages, is always the motivation to study non-local problems. The main difficulty in the analysis of these $p(u)$-problems relies in the fact that their weak formulations cannot be written as equalities in terms of duality in fixed Banach spaces. For more interesting features and results, we refer to [6, 9, 13, 15] and references therein.

This paper is organized as follows. In Section 2, we introduce the basic assumptions and we recall some definitions, basic properties of generalised Sobolev spaces that we will use later. Section 3 is devoted to showing the existence of weak solutions to the problem (1).

2 Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^N, N \geq 2$, with the Lipschitz-continuous boundary $\partial\Omega$. Given a measurable function $h : \Omega \to [1, \infty)$, we introduce the variable exponent Lebesgue space by

$$L^{h(\cdot)}(\Omega) = \{ u : \Omega \to \mathbb{R} \ / \ \rho_{h(\cdot)}(u) \ := \int_{\Omega} |u(x)|^{h(x)} dx < \infty\}. \tag{4}$$

Equipped with the Luxembourg norm

$$\|u\|_{h(\cdot)} \ := \inf \left\{ \lambda > 0 \ : \ \rho_{h(\cdot)} \left(\frac{u}{\lambda}\right) \leq 1 \right\}, \tag{5}$$

$L^{h(\cdot)}(\Omega)$ becomes a Banach space. If

$$1 < h_- \leq h_+ < \infty, \tag{6}$$

then $L^{h(\cdot)}(\Omega)$ is separable and reflexive. The dual space of $L^{h(\cdot)}(\Omega)$ is $L^{h'(\cdot)}(\Omega)$, where $h'(x)$ is the generalised Hölder conjugate of $h(x)$,

$$\frac{1}{h(x)} + \frac{1}{h'(x)} = 1.$$

From the definitions of the modular $\rho_{h(\cdot)}(u)$ and the norm (5), it can be proved that if (6) holds, then

$$\min \left\{ \|u\|_{h(\cdot)}^{h_-}, \ \|u\|_{h(\cdot)}^{h_+} \right\} \leq \rho_{h(\cdot)}(u) \leq \max \left\{ \|u\|_{h(\cdot)}^{h_-}, \ \|u\|_{h(\cdot)}^{h_+} \right\}. \tag{7}$$

One very useful consequence of (7) is

$$\|u\|_{h(\cdot)}^{h_-} - 1 \leq \rho_{h(\cdot)}(u) \leq \|u\|_{h(\cdot)}^{h_+} + 1. \tag{8}$$

For any functions $u \in L^{h(\cdot)}(\Omega)$ and $v \in L^{h'(\cdot)}(\Omega)$, the generalized Hölder inequality holds:

$$\int_{\Omega} uv dx \leq (\frac{1}{h_-} + \frac{1}{h'_-}) \|u\|_{h(\cdot)} \|v\|_{h'(\cdot)} \leq 2 \|u\|_{h(\cdot)} \|v\|_{h'(\cdot)}. \tag{9}$$

We define also the generalised Sobolev space by

$$W^{1,h(\cdot)}(\Omega) := \{u \in L^{h(\cdot)}(\Omega) : \nabla u \in L^{h(\cdot)}(\Omega)\},$$

which is a Banach space for the norm

$$\|u\|_{1,h(\cdot)} \ := \|u\|_{h(\cdot)} + \|\nabla u\|_{h(\cdot)}. \tag{10}$$

Now, we introduce the following function space:

$$W_0^{1,h(\cdot)}(\Omega) := \{u \in \mathrm{W}_0^{1,1}(\Omega) : \nabla u \in L^{h(\cdot)}(\Omega)\},$$

which we endow with the norm

$$\|u\|_{W_0^{1,h(\cdot)}(\Omega)} \ := \|u\|_1 + \|\nabla u\|_{h(\cdot)}. \tag{11}$$

If $h \in C(\overline{\Omega})$, then the norm in $W_0^{1,h(\cdot)}(\Omega)$ is equivalent to $\|\nabla u\|_{h(\cdot)}$. If h is log-Hölder continuous, then $C_0^\infty(\Omega)$ is dense in $W_0^{1,h(\cdot)}(\Omega)$. Let h be a measurable function in Ω satisfying $1 \leq h_- \leq h_+ < d$ and being log-Hölder continuous, then

$$\|u\|_{h^*(\cdot)} \leq C\|\nabla u\|_{h(\cdot)} \quad \forall u \in W_0^{1,h(\cdot)}(\Omega),$$

for some positive constant C, where

$$h^*(x) := \begin{cases} \frac{Nh(x)}{N-h(x)} & \text{if } h(x) < N, \\ \infty & \text{if } h(x) \geq N. \end{cases}$$

On the other hand, if $h_- > N$, then

$$\|u\|_\infty \leq C\|\nabla u\|_{h(\cdot)} \quad \forall u \in W_0^{1,h(\cdot)}(\Omega) \ ,$$

where C is another positive constant.

Lemma 2.1 *[9] Assume that*

$$1 < \alpha \leq q_n(x) \leq \beta < \infty \quad \forall n \in \mathbb{N},$$

$$\textit{for a.e. } x \in \Omega, \textit{ for some constants } \alpha \textit{ and } \beta, \tag{12}$$

$$q_n \to q \ \textit{ a.e. in } \Omega, \textit{ as } n \to \infty, \tag{13}$$

$$\nabla u_n \to \nabla u \textit{ in } L^1(\Omega)^d, \textit{ as } n \to \infty, \tag{14}$$

$$\||\nabla u_n|^{q_n(x)}\|_1 \leq C, \textit{ for some positive constant } C \textit{ not depending on } n. \tag{15}$$

Then $\nabla u \in L^{q(\cdot)}(\Omega)^d$ *and*

$$\liminf_{n \to \infty} \int_\Omega |\nabla u_n|^{q_n(x)} dx \geq \int_\Omega |\nabla u|^{q(x)} dx. \tag{16}$$

3 Main Results

In this section, we will give a reasonable definition for weak solutions and prove the existence of weak solutions to problem (1). We introduce the functional space

$$X(\Omega_T) := \left\{ u \in L^\infty\left(0,T; L^2(\Omega)\right) : |\nabla u| \in L^{p(b(u))}(\Omega_T), u(\cdot,t) \in V_t(\Omega) \text{ a.e. } t \in (0,T) \right\},$$

where

$$V_t(\Omega) := \left\{ u \in L^2(\Omega) \cap W_0^{1,\alpha}(\Omega) : |\nabla u| \in L^{p(b(u(\cdot,t)))}(\Omega) \right\}.$$

In the same way, we define $Y(\Omega_T)$ associated with the nonlinear exponent function $q(b(u))$. We denote their dual spaces by $X(\Omega_T)^*$ and $Y(\Omega_T)^*$, respectively.
Now, we give a definition of weak solutions for the parabolic problem (1).

Definition 3.1 A function $u \in X(\Omega_T) \cap Y(\Omega_T) \cap C\left([0,T]; L^2(\Omega)\right)$ is said to be a weak solution to problem (1) if for any $\varphi \in C^1\left(\overline{\Omega}_T\right)$ with $\varphi(\cdot,T) = 0$, we have

$$\begin{aligned} &-\int_\Omega u_0(x)\varphi(x,0)dx + \int_0^T \int_\Omega -u\varphi_t dxdt \\ &+ \int_0^T \int_\Omega \left(|\nabla u|^{p(b(u))-2}\nabla u + |\nabla u|^{q(b(u))-2}\nabla u\right) \cdot \nabla\varphi dxdt = \int_0^T \int_\Omega f\varphi dxdt. \end{aligned} \tag{17}$$

Theorem 3.1 *Assume that* (2) *and* (3) *hold together with* $\alpha > 2N/(N+2)$, $u_0 \in L^2(\Omega)$ *and* $f \in L^{\alpha'}(\Omega_T)$. *Then there exists at least one weak solution to problem* (1) *in the sense of Definition 3.1.*

Proof. We denote $h = T/N_0$, where N_0 is a positive integer. We consider the following time-discrete problem:

$$\begin{cases} \frac{u_k - u_{k-1}}{h} - \operatorname{div}\left(|\nabla u_k|^{p(b(u_k))-2}\,\nabla u_k\right) \\ \qquad\qquad\qquad - \operatorname{div}\left(|\nabla u_k|^{q(b(u_k))-2}\,\nabla u_k\right) = [f]_h\left((k-1)h\right), \quad x \in \Omega, \\ u_k|_{\partial\Omega} = 0, \quad k = 1, 2, \ldots, N_0, \end{cases} \tag{18}$$

where the Steklov average $[f]_h$ of f is defined as

$$[f]_h(x,t) = \frac{1}{h}\int_t^{t+h} f(x,\tau)d\tau \in L^{\alpha'}(\Omega).$$

For $k = 1$, we consider the problem

$$\begin{cases} \frac{u-u_0}{h} - \operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) - \operatorname{div}\left(|\nabla u|^{q(b(u))-2}\nabla u\right) = [f]_h(0), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0. \end{cases} \tag{19}$$

Set

$$W = W_0^{1,p(b(u))}(\Omega) \cap W_0^{1,q(b(u))}(\Omega) \cap L^2(\Omega).$$

First, we show that the problem (19) has a weak solution $u_1 \in W$.

Step 1: Approximation
For each $\varepsilon > 0$, we consider the auxiliary problem

$$\begin{cases} \frac{u-u_0}{h} - \operatorname{div}\left(|\nabla u|^{p(b(u))-2}\nabla u\right) - \operatorname{div}\left(|\nabla u|^{q(b(u))-2}\nabla u\right) \\ \qquad\qquad\qquad -\varepsilon \operatorname{div}\left(|\nabla u|^{\beta-2}\nabla u\right) = [f]_h(0), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{20}$$

where

$$\frac{2N}{N+2} < \alpha < q(b(u)) \leqslant p(b(u)) \leqslant \beta < \infty \quad \forall u \in \mathbb{R}.$$

Lemma 3.1 *For each* $\varepsilon > 0$, *there exists a weak solution* u_ε *to the problem* (20).

Proof. Let $\omega \in L^2(\Omega)$ be given. We have

$$\frac{2N}{N+2} < \alpha < q(b(w)) \leqslant p(b(w)) \leqslant \beta < \infty \quad \text{for a.e. } x \in \Omega.$$

Observing that $[f]_h(0) \in L^{\alpha'}(\Omega) \subset W^{-1,\alpha'}(\Omega) \subset W^{-1,\beta'}(\Omega)$, by the usual theory of monotone operators, there exists a unique solution $u_w \in W_0^{1,\beta}(\Omega)$ such that

$$\begin{aligned} &\int_\Omega \frac{u_w - u_0}{h} v dx + \int_\Omega |\nabla u_w|^{p(b(w))-2} \nabla u_w \cdot \nabla v dx \\ &+ \int_\Omega |\nabla u_w|^{q(b(w))-2} \nabla u_w \cdot \nabla v dx + \varepsilon \int_\Omega |\nabla u_w|^{\beta-2} \nabla u_w \cdot \nabla v dx = \int_\Omega [f]_h(0) v dx \end{aligned} \tag{21}$$

for all $v \in W_0^{1,\beta}(\Omega)$.

By taking $v = u_w$ in (21), we get

$$\begin{aligned} &\frac{1}{2h} \int_\Omega u_w^2 dx + \int_\Omega |\nabla u_w|^{p(b(w))} dx + \int_\Omega |\nabla u_w|^{q(b(w))} dx \\ &\qquad + \varepsilon \int_\Omega |\nabla u_w|^\beta dx \leqslant \frac{1}{2h} \int_\Omega u_0^2 dx + C \|\nabla u_w\|_{L^\beta(\Omega)} \end{aligned}$$

for some positive constant $C = C(\alpha, \beta, \Omega, [f]_h(0))$. Then, using Young's inequality, we obtain

$$\|u_w\|_{L^2(\Omega)} + \|\nabla u_w\|_{L^\beta(\Omega)} \leqslant C \tag{22}$$

for some positive constant $C = C(\alpha, \beta, \Omega, [f]_h(0), \varepsilon, h, N)$. Hence

$$\|u_w\|_{L^2(\Omega)} \leqslant C.$$

Let us now consider the mapping

$$T \ni w \to u_w \in T,$$

where $T := \{v \in L^2(\Omega) \ : \ \|v\|_2 \leqslant C\}$. Firstly, we prove that this mapping is continuous, then by Schauder's fixed point theorem, it will have a fixed point. We suppose that w_n is a sequence in $L^2(\Omega)$ such that

$$w_n \to w \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad n \to \infty. \tag{23}$$

For every $n \in \mathbb{N}$, let u_n be the solution to the problem (20) associated to $w = w_n$. From (22), we have

$$\|\nabla u_n\|_\beta \leq C$$

for some positive constant C which does not depend on n. It follows that

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,\beta}(\Omega), \quad \text{as} \quad n \to \infty, \tag{24}$$

$$u_n \to u \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty. \tag{25}$$

From (21), one has

$$\begin{aligned} &\int_\Omega \frac{u_n - u_0}{h} v dx + \int_\Omega |\nabla u_n|^{p(b(w_n))-2} \nabla u_n \cdot \nabla v dx + \int_\Omega |\nabla u_n|^{q(b(w_n))-2} \nabla u_n \cdot \nabla v dx \\ &\qquad + \varepsilon \int_\Omega |\nabla u_n|^{\beta-2} \nabla u_n \cdot \nabla v dx = \int_\Omega [f]_h(0) v dx, \ \forall v \in W_0^{1,\beta}(\Omega). \end{aligned} \tag{26}$$

Using the monotonicity, one also has

$$\int_\Omega \frac{(u_n - v)^2}{h} dx + \int_\Omega \Big(|\nabla u_n|^{p(b(w_n))-2} \nabla u_n + |\nabla u_n|^{q(b(w_n))-2} \nabla u_n + \varepsilon |\nabla u_n|^{\beta-2} \nabla u_n \Big) \nabla (u_n - v) \, dx - \int_\Omega \Big(|\nabla v|^{p(b(w_n))-2} \nabla v + |\nabla v|^{q(b(w_n))-2} \nabla v + \varepsilon |\nabla v|^{\beta-2} \nabla v \Big) \nabla (u_n - v) \, dx \geqslant 0 \quad (27)$$

for all $v \in W_0^{1,\beta}(\Omega)$. Taking $u_n - v$ as a test function in (26) and using (27), we get

$$\int_\Omega \frac{u_0 - v}{h} (u_n - v) \, dx + \int_\Omega [f]_h(0) (u_n - v) \, dx - \int_\Omega \Big(|\nabla v|^{p(b(w_n))-2} \nabla v + |\nabla v|^{q(b(w_n))-2} \nabla v + \varepsilon |\nabla v|^{\beta-2} \nabla v \Big) \cdot \nabla (u_n - v) \, dx \geqslant 0 \quad (28)$$

for all $v \in W_0^{1,\beta}(\Omega)$. From (23), we may assume that for some subsequence

$$w_n \to w \quad a.e. \text{ in } \Omega, \quad \text{as} \quad n \to \infty.$$

According to the assumptions of p, q and Lebesgue's theorem, we know that for any $v \in W_0^{1,\beta}(\Omega)$,

$$|\nabla v|^{p(b(w_n))-2} \nabla v \to |\nabla v|^{p(b(w))-2} \nabla v \quad \text{strongly in} \quad L^{\beta'}(\Omega)^d, \quad \text{as} \quad n \to \infty,$$
$$|\nabla v|^{q(b(w_n))-2} \nabla v \to |\nabla v|^{q(b(w))-2} \nabla v \quad \text{strongly in} \quad L^{\beta'}(\Omega)^d, \quad \text{as} \quad n \to \infty. \quad (29)$$

Using (25), (28) and (29), we obtain

$$\int_\Omega \frac{u_0 - v}{h} (u - v) \, dx + \int_\Omega [f]_h(0) (u - v) \, dx - \int_\Omega \Big(|\nabla v|^{p(b(w))-2} \nabla v + |\nabla v|^{q(b(w))-2} \nabla v + \varepsilon |\nabla v|^{\beta-2} \nabla v \Big) \cdot \nabla (u - v) \, dx \geqslant 0$$

for all $v \in W_0^{1,\beta}(\Omega)$. We take $v = u \pm \theta z$ in (3), with $z \in W_0^{1,\beta}(\Omega)$ and $\theta > 0$, we get

$$\pm \Big[\int_\Omega \frac{u_0 - (u \mp \delta z)}{h} z dx + \int_\Omega [f]_h(0) z dx - \int_\Omega \Big(|\nabla (u \mp \delta z)|^{p(b(w))-2} \nabla (u \mp \delta z) + |\nabla (u \mp \delta z)|^{q(b(w))-2} \nabla (u \mp \delta z) + \varepsilon |\nabla (u \mp \delta z)|^{\beta-2} \nabla (u \mp \delta z) \Big) \cdot \nabla z dx \Big] \geqslant 0.$$

By letting $\theta \to 0$, we obtain that

$$\int_\Omega \frac{u - u_0}{h} z dx + \int_\Omega |\nabla u|^{p(b(w))-2} \nabla u \cdot \nabla z dx + \int_\Omega |\nabla u|^{q(b(w))-2} \nabla u \cdot \nabla z dx + \varepsilon \int_\Omega |\nabla u|^{\beta-2} \nabla u \cdot \nabla z dx = \int_\Omega [f]_h(0) z dx, \quad \forall z \in W_0^{1,\beta}(\Omega).$$

Hence $u = u_w$. Since the limit is uniquely determined and by (25), we get

$$u_n \to u_w \quad \text{strongly in} \quad L^2(\Omega), \quad \text{as} \quad n \to \infty,$$

which proves the continuity of the mapping. By Schauder's fixed point theorem, this mapping has a fixed point, and thus concludes the proof of Lemma 3.1.

Step 2: Passage to the limit as $\varepsilon \to 0$
From Lemma 3.1, it can be obtained that for each $\varepsilon > 0$, there exists $u_\varepsilon \in W_0^{1,\beta}(\Omega)$ such that

$$\int_\Omega \frac{u_\varepsilon - u_0}{h} v dx + \int_\Omega |\nabla u_\varepsilon|^{p(b(u_\varepsilon))-2} \nabla u_\varepsilon \cdot \nabla v dx + \int_\Omega |\nabla u_\varepsilon|^{q(b(u_\varepsilon))-2} \nabla u_\varepsilon \cdot \nabla v dx$$
$$+\varepsilon \int_\Omega |\nabla u_\varepsilon|^{\beta-2} \nabla u_\varepsilon \cdot \nabla v dx = \int_\Omega [f]_h(0) v dx \tag{30}$$

for all $v \in W_0^{1,\beta}(\Omega)$.

By taking $v = u_\varepsilon$ in (30), we get

$$\frac{1}{2h} \int_\Omega u_\varepsilon^2 dx + \int_\Omega |\nabla u_\varepsilon|^{p(b(u_\varepsilon))} dx + \int_\Omega |\nabla u_\varepsilon|^{q(b(u_\varepsilon))} dx + \varepsilon \int_\Omega |\nabla u_\varepsilon|^\beta dx$$
$$\leqslant \frac{1}{2h} \int_\Omega u_0^2 dx + \int_\Omega [f]_h(0) u_\varepsilon dx.$$

Then we conclude that

$$\int_\Omega |\nabla u_\varepsilon|^{p(b(u_\varepsilon))} dx + \int_\Omega |\nabla u_\varepsilon|^{q(b(u_\varepsilon))} dx + \frac{\varepsilon}{2} \|\nabla u_\varepsilon\|_{L^\beta(\Omega)}^\beta \leqslant C,$$

and

$$\|\nabla u_\varepsilon\|_{L^\alpha(\Omega)} \leqslant C,$$

where C is a positive constant which does not depend on ε.

Using the compact embedding $W_0^{1,\alpha}(\Omega) \hookrightarrow L^2(\Omega)$ due to the fact that $\alpha > 2N/(N+2)$, we have

$$\begin{aligned} &u_\varepsilon \rightharpoonup u \quad \text{in } W_0^{1,\alpha}(\Omega), \\ &\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } \left(L^\alpha(\Omega)\right)^N, \\ &u_\varepsilon \to u \quad \text{strongly in } L^2(\Omega), \\ &u_\varepsilon \to u \quad \text{a.e. in } \Omega, \end{aligned}$$
$$p\left(b\left(u_\varepsilon\right)\right) \to p(b(u)) \quad \text{a.e. in } \Omega, \qquad q\left(b\left(u_\varepsilon\right)\right) \to q(b(u)) \quad \text{a.e. in } \Omega.$$

By the application of Lemma 2.1, we get

$$u \in W_0^{1,p(b(u))}(\Omega) \text{ and } u \in W_0^{1,q(b(u))}(\Omega).$$

Therefore,

$$u \in W_0^{1,p(b(u))}(\Omega) \cap W_0^{1,q(b(u))}(\Omega).$$

By using the monotonicity trick in [9], we can establish that (19) has a weak solution $u_1(x)$ in W.

In the same way, we show that (18) has weak solutions u_k for $k = 2, \ldots, N_0$. It means that, for every $\varphi \in W$, we have

$$\int_\Omega \frac{u_k - u_{k-1}}{h} \varphi dx + \int_\Omega |\nabla u_k|^{p(b(u_k))-2} \nabla u_k \cdot \nabla \varphi dx + \int_\Omega |\nabla u_k|^{q(b(u_k))-2} \nabla u_k \cdot \nabla \varphi dx$$
$$= \int_\Omega [f]_h\left((k-1)h\right) \varphi dx. \tag{31}$$

For any $h = T/N_0$, we define $u_h(x,t)$ by

$$u_h(x,t) = \begin{cases} u_0(x), & t = 0, \\ u_1(x), & 0 < t \leqslant h, \\ \vdots & \vdots \\ u_j(x), & (j-1)h < t \leqslant jh, \\ \vdots & \vdots \\ u_{N_0}(x), & (N_0 - 1)\, h < t \leqslant N_0 h = T. \end{cases}$$

By taking $\varphi = u_k$ in (31), we obtain

$$\begin{aligned} \frac{1}{2} \int_\Omega u_k^2 dx + h \int_\Omega |\nabla u_k|^{p(b(u_k))}\, dx + h \int_\Omega |\nabla u_k|^{q(b(u_k))}\, dx \\ \leqslant \frac{1}{2} \int_\Omega u_{k-1}^2 dx + h \, \|[f]_h((k-1)h)\|_{L^{\alpha'}(\Omega)} \cdot \|u_k\|_{L^\alpha(\Omega)} \\ \leqslant \frac{1}{2} \int_\Omega u_{k-1}^2 dx + Ch \, \|[f]_h((k-1)h)\|_{L^{\alpha'}(\Omega)} \cdot \|\nabla u_k\|_{L^\alpha(\Omega)} . \end{aligned} \tag{32}$$

By the Hölder inequality, one has

$$\begin{aligned} \int_\Omega |\nabla u_k|^\alpha \, \mathrm{d}x &\leqslant C \, \| |\nabla u_k|^\alpha \|_{L^{(p(b(u_k))/\alpha}(\Omega)} \\ &\leqslant C \left(\int_\Omega |\nabla u_k|^{p(b(u_k))}\, dx + 1 \right). \end{aligned}$$

By using Young's inequality, one deduces that

$$\|[f]_h((k-1)h)\|_{L^{\alpha'}(\Omega)} \cdot \|\nabla u_k\|_{L^\alpha(\Omega)} \leqslant \varepsilon \int_\Omega |\nabla u_k|^{p(b(u_k))}\, dx + C.$$

From (32), we get

$$\int_\Omega u_k^2 dx + h \int_\Omega |\nabla u_k|^{p(b(u_k))}\, dx + h \int_\Omega |\nabla u_k|^{q(b(u_k))}\, dx \leqslant \int_\Omega u_{k-1}^2 dx + Ch. \tag{33}$$

By summing up inequalities in (33), we deduce that

$$\int_\Omega u_h^2(x,t) dx + \int_0^T \int_\Omega \left(|\nabla u_h(x,t)|^{p(b(u_h))} + |\nabla u_h(x,t)|^{q(b(u_h))} \right) dxdt \leqslant \int_\Omega u_0^2 dx + CT.$$

Hence

$$\begin{aligned} \|u_h\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla u_h\|_{L^{p(b(u_h))}(\Omega_T)} + \|u_h\|_{L^\alpha\left(0,T;W_0^{1,p(b(u_h))}(\Omega)\right)} \\ + \|\nabla u_h\|_{L^{q(b(u_h))}(\Omega_T)} + \|u_h\|_{L^\alpha\left(0,T;W_0^{1,q(b(u_h))}(\Omega)\right)} \leqslant C. \end{aligned}$$

Thus we have for some subsequence still labeled with h and some u,

$$\begin{aligned}
&u_h \longrightarrow u \quad \text{weakly-* in } L^\infty\left(0,T;L^2(\Omega)\right),\\
&u_h \longrightarrow u \quad \text{in } L^\alpha\left(0,T;W_0^{1,\alpha}(\Omega)\right),\\
&|\nabla u_h|^{p(b(u_h))-2}\nabla u_h \longrightarrow \xi \quad \text{in } \left(L^{\alpha'}(\Omega_T)\right)^N,\\
&|\nabla u_h|^{q(b(u_h))-2}\nabla u_h \longrightarrow \chi \quad \text{in } \left(L^{\alpha'}(\Omega_T)\right)^N.
\end{aligned}$$

Lemma 3.2 *u is a weak solution to problem* (1).

Proof. For every $k \in \{1,2,\ldots,N_0\}$, we take $\varphi(x,kh)$ as a test function in (31), where $\varphi \in C^1\left(\overline{\Omega}_T\right)$, $\varphi(\cdot,T)=0$ and $\varphi(x,t)|_\Gamma = 0$, we obtain

$$\begin{aligned}
&\frac{1}{h}\int_\Omega u_k(x)\varphi(x,kh)dx - \frac{1}{h}\int_\Omega u_{k-1}(x)\varphi(x,kh)dx\\
&-\int_\Omega \left(|\nabla u_k|^{p(b(u_k))-2}\nabla u_k\right)(x)\cdot\nabla\varphi(x,kh)dx - \int_\Omega \left(|\nabla u_k|^{q(b(u_k))-2}\nabla u_k\right)(x)\cdot\nabla\varphi(x,kh)dx\\
&\qquad = \int_\Omega [f]_h((k-1)h)\varphi(x,kh)dx.
\end{aligned}$$

Using the definition of $u_h(x,t)$ and the fact that $\varphi(\cdot,N_0h)=0$, we get

$$\begin{aligned}
&h\sum_{k=1}^{N_0-1}\int_\Omega u_h(x,kh)\frac{\varphi(x,kh)-\varphi(x,(k+1)h)}{h}dx - \int_\Omega u_0(x)\varphi(x,h)dx\\
&-h\sum_{k=1}^{N_0}\int_\Omega \left(|\nabla u_h|^{p(b(u_h))-2}\nabla u_h + |\nabla u_h|^{q(b(u_h))-2}\nabla u_h\right)(x,kh)\cdot\nabla\varphi(x,kh)dx\\
&\qquad = h\sum_{k=1}^{N_0}\int_\Omega [f]_h((k-1)h)\varphi(x,kh)dx. \qquad (34)
\end{aligned}$$

Since $C^1\left(\overline{\Omega}_T\right)$, one has

$$\begin{aligned}
&h\sum_{k=1}^{N_0}\int_\Omega \left(|\nabla u_h|^{p(b(u_h))-2}\nabla u_h + |\nabla u_h|^{q(b(u_h))-2}\nabla u_h\right)(x,kh)\cdot\nabla\varphi(x,kh)dx\\
&= \int_0^T\int_\Omega \left(|\nabla u_h|^{p(b(u_h))-2}\nabla u_h + |\nabla u_h|^{q(b(u_h))-2}\nabla u_h\right)(x,\tau)\cdot\nabla\varphi(x,\tau)dxd\tau\\
&+\sum_{k=1}^{N_0}\int_{(k-1)h}^{kh}\int_\Omega \left(|\nabla u_h|^{p(b(u_h))-2}\nabla u_h\right)(x,\tau)\cdot(\nabla\varphi(x,kh)-\nabla\varphi(x,\tau))dxd\tau\\
&+\sum_{k=1}^{N_0}\int_{(k-1)h}^{kh}\int_\Omega \left(|\nabla u_h|^{q(b(u_h))-2}\nabla u_h\right)(x,\tau)\cdot(\nabla\varphi(x,kh)-\nabla\varphi(x,\tau))dxd\tau\\
&\longrightarrow \int_0^T\int_\Omega \xi\cdot\nabla\varphi(x,\tau)dxd\tau + \int_0^T\int_\Omega \chi\cdot\nabla\varphi(x,\tau)dxd\tau, \quad \text{as } h\to 0.
\end{aligned}$$

From (34), we deduce that

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}dx\ \mathrm{d}\tau-\int_\Omega u_0(x)\varphi(x,0)dx-\int_0^T\int_\Omega\xi\cdot\nabla\varphi dxd\tau$$
$$-\int_0^T\int_\Omega\chi\cdot\nabla\varphi dxd\tau=\int_0^T\int_\Omega f\varphi dxd\tau.$$

By using the monotonicity method as in [9, 15], we show that $\xi=|\nabla u|^{p(b(u))-2}\nabla u$ a.e. in Ω_T and $\chi=|\nabla u|^{q(b(u))-2}\nabla u$ a.e. in Ω_T. By applying Lemma 2.1, we can show that $\nabla u\in\left(L^{p(b(u))}\left(\Omega_T\right)\right)^N$ and $\nabla u\in\left(L^{q(b(u))}\left(\Omega_T\right)\right)^N$.
Choosing $\varphi\in C_0^\infty\left(\Omega_T\right)$, we get

$$-\int_0^T\int_\Omega u\frac{\partial\varphi}{\partial t}dxd\tau=\int_0^T\int_\Omega\xi\cdot\nabla\varphi dxd\tau+\int_0^T\int_\Omega\chi\cdot\nabla\varphi dxd\tau+\int_0^T\int_\Omega f\varphi dxd\tau,$$

therefore $u_t\in X\left(\Omega_T\right)^*$ and $u_t\in Y\left(\Omega_T\right)^*$. Since $u\in X\left(\Omega_T\right)\cap Y\left(\Omega_T\right)$, we can deduce that $u\in C\left([0,T];L^2(\Omega)\right)$ (see [10, 12]). Then u is a weak solution to problem (1) in the sense of Definition 3.1.

4 Conclusion

In this paper, we proved the existence of weak solutions to some parabolic $(p(b(u)),q(b(u)))$-Laplacian problems stated in (1). By using a singular perturbation technique, we proved the existence of weak solutions for the discretized problem associated with problem (1). We finished this paper by proving the existence of a solution for problem (1) as a limit of the solutions of the approximated problem (18). This work provides a qualitative addition to the study of problems involving the (p,q)-Laplacian operators, especially the general reaction–diffusion system [5, 7, 14].

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