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Nonlinear Damped Oscillator with Varying Coefficients and Periodic External Forces

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Abstract: A modified harmonic balance method (MHBM) has been exhibited for operating the damped Duffing oscillator with varying coefficients and periodic external forces. The mentioned technique is able to convert a set of nonlinear algebraic equations into a set of linear algebraic equations using only a nonlinear algebraic equation and it makes the simplest form of the system and requires less computational effort than the classic harmonic balance method (HBM). On the contrary, a set of nonlinear algebraic equations is required to solve by the numerical technique in classic HBM. As a result, it needs a heavy computational attempt. The obtained results have been compared with the numerical solutions attained by the fourth order Runge-Kutta method in the Figures and Table. It is mentioned that the obtained results display a strong similarity with the corresponding numerical results.

Keywords: *harmonic balance method; nonlinear oscillators; varying coefficients and periodic forcing term.*

Mathematics Subject Classification (2010): 34E05, 34E10, 34M10.

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1 Introduction

Differential equations are a very important branch of science and engineering. They are linear or nonlinear differential equations. Actually, a greater portion of the real life physical and engineering problems are related to nonlinear differential equations. Solutions of differential equations provide a detailed information regarding the behavior of the systems. In this regard, nonlinear oscillators are very important in all areas of science and engineering. The appropriate solutions of these nonlinear oscillators are rarely obtained. Therefore, many researchers and scientists have focused their attention on developing numerical techniques as well as analytical methods. Numerical techniques are procedures for determining the true values for a set of discrete points. The true values are attained by the process of incremental steps. The proper initial guess values are required to perform the numerical techniques. Commonly, these techniques are comparatively simple but sometimes they need massive computational attempts and appropriate primary approximate values to achieve the desired solutions. Also, the numerical techniques are unable to provide overall feature of the nonlinear dynamical systems. It is also not possible to know the amplitude and phase by the numerical techniques. In contrast, analytical approximation methods have become more interesting to the scientists, physicists, engineers and applied mathematicians because of their analytical expression and suitability for parametric study. Many analytical approximation methods have been investigated for handling nonlinear dynamical systems, for example, the perturbation method [1- 10], homotopy analysis technique [11,12], homotopy perturbation technique [13-16], variational iteration technique [17,18], harmonic balance method (HBM) [19-28], modified multi-level residue harmonic balance method [25- 27], modified harmonic balance method (HBM) [28-33], etc. The perturbation methods [1- 10] are broadly used techniques for dealing with weakly nonlinear dynamical systems. Jones [8] investigated a technique to improve the scope of precision of the classical perturbation technique for large as well as small parameters. Cheung et al.[9] modified the Lindstedt–Poincare technique based on the idea of Jones [8]. Alam et al.[10] developed a modified Lindstedt-Poincare method to control oscillators with strong nonlinearities. The HBM and MHBM are also impressive methods for obtaining periodic solutions of nonlinear oscillators. In this method, the truncated Fourier series is selected as the trial solution of the nonlinear oscillators. According to the classical HBM, a set of nonlinear algebraic equations is handled by a numerical technique to find the values of the unknown coefficients. This method has been revised by some authors [18-28]. Rahman et al.[20] applied the HBM to study the Van der Pol equation. Wagner and Lentz [21] investigated the HBM for detecting the solutions to nonlinear oscillators. Wu [22] presented the HBM for the Yao-Cheng oscillator. Yeasmin et al. [24] presented an analytical technique to solve the free vibration problems with quadratic nonlinearity based on the HBM. Rahman and Lee [25] and Rahman et al. [26] exhibited a modified multi-level residue HBM. Hasan et al.[27] developed a multi-level residue harmonic balance solution for the nonlinear natural frequency of axially loaded beams with an internal hinge. Lee [28] presented an analytical solution for nonlinear multimode beam vibration using a modified harmonic balance approach and Vieta's substitution. Ullah et al. [29] exhibited MHBM to solve forced vibration problems with strong cubic and quadratic nonlinearities. Ullah et al.[30] extended this method to forced vibration problems with generalized nonlinearities. Further, Ullah et al.[31] exhibited the MHBM for the forced Van der Pol vibration equation. Recently, Ullah et al. [32] have exhibited the MHBM for free vibration analysis of nonlinear axially loaded beams. Ullah et al.[33] have handled a modified forced Van der Pol vibration equation using a modified harmonic balance method. Kandil et al.[35] have exhibited a HBM to obtain the steady-state solutions of the nonlinear problems. Alam et al. [36] have solved some strongly nonlinear oscillators with a combination of the modified Lindstedt-Poincare and the homotopy perturbation methods. Uddin and Sattar [34] developed an analytical procedure to solve the damped Duffing equation with varying coefficients without periodic external force combining KBM and homotopy perturbation methods. It is observed that MHBM has been remaining untouched for the controlling forced Duffing equation with varying coefficients and damping with strong nonlinearity. To fulfill this gap, a MHBM has been proposed to control the damped Duffing oscillator with varying coefficients and periodic external forces. The proposed technique reduces the heavy computational effort that cannot be avoided in the classical HBM.

2 Method

We guess a damped nonlinear oscillator [29-33, 34] with varying coefficients and periodic external force

$$\ddot{x} + 2k\ \dot{x} + e^{-\tau}x + \epsilon f(x) = F\ \cos(\omega t), \tag{1}$$

where the dots above denote differentiation with respect to time t, $2k$ is the coefficient of viscous damping, $f(x)$ is a certain nonlinear function, ϵ is a positive parameter which is not necessarily small, $\tau = \epsilon t$ is the slow varying time, F and ω represent the amplitude and frequency of excitation, respectively. All of the parameters are positive. According to the proposed method, the approximate solution of Eq.(1) is assumed [29-33]to be of the following form:

$$x = c_1\ \cos(\omega t) + d_1\ \sin(\omega t) + c_3\ \cos(3\omega t) + d_3\ \sin(3\omega t) + \dots, \tag{2}$$

where c_1, d_1, c_3, d_3... are unknown coefficients in the Fourier series. Now, differentiating Eq.(2) twice with respect to t and then putting into Eq.(1) and expanding $f(x)$ as a truncated Fourier series expansion and then comparing similar harmonics, we accomplish the following set of algebraic equations

$$c_1(-\omega^2 + e^{-\tau}) + 2d_1k\omega + \epsilon A_1(c_1, d_1, c_3, d_3, ...) = F, \tag{3a}$$

$$d_1(-\omega^2 + e^{-\tau}) - 2c_1k\omega + \epsilon B_1(c_1, d_1, c_3, d_3, ...) = 0, \tag{3b}$$

$$c_3(-9\omega^2 + e^{-\tau}) + 6d_3k\omega + \epsilon A_3(c_1, d_1, c_3, d_3, ...) = 0, \tag{3c}$$

$$d_3(-9\omega^2 + e^{-\tau}) - 6c_3k\omega + \epsilon B_3(c_1, d_1, c_3, d_3, ...) = 0. \tag{3d}$$

Deducting ω^2 from the Eqs.(3b)-(3d), utilizing Eq.(3a), and removing the terms whose responses are small, we get Eqs.(3a)-(3d) in the form

$$\omega^2 = e^{-\tau} + 2d_1k\omega/c_1 + \epsilon A_1(c_1, d_1, c_3, d_3, ...) - F/c_1, \tag{4a}$$

$$-2c_1k\omega - 2d_1^2k\omega/c_1 - \epsilon A_1(c_1, d_1, c_3, d_3, ...) + \epsilon B_1(c_1, d_1, c_3, d_3, ...) + d_1F/c_1 = 0, \tag{4b}$$

$$-8c_3e^{-\tau} - 18c_3d_1k\omega/c_1 + 6d_3k\omega - \epsilon A_1(c_1, d_1, c_3, d_3, ...) + \epsilon A_3(c_1, d_1, c_3, d_3, ...) - 9c_3F/c_1 = 0, \tag{4c}$$

$$-8d_3e^{-\tau} - 18d_1d_3k\omega/c_1 + 6c_3k\omega - \epsilon A_1(c_1, d_1, c_3, d_3, ...) + \epsilon B_3(c_1, d_1, c_3, d_3, ...) + 9d_3F/c_1 = 0. \tag{4d}$$

Utilizing Eq.(4b), terminating ω from the Eqs.(4c)-(4d) and taking into account only the linear terms of c_3, d_3, a set of linear algebraic equations of c_3, d_3 is achieved. After simplifying, c_3, d_3 are acquired in terms of c_1,d_1. Finally, inserting c_3, d_3 in Eq.(4b), and expanding d_1 into a power series of $\lambda(k, \omega, F)$, we acquire

$$d_1 = l_0 + l_1\lambda + l_2\lambda^2 + l_3\lambda^3 + \dots, \tag{5}$$

where l_0, l_1, $l_2, \dots$ are the functions of c_1 and λ is a small parameter. After inserting c_3, d_3 and d_1 in Eq.(4a) and solving, c_1 is obtained. Systematically, d_1, c_3 and d_3 are obtained.

3 Example

Consider a nonlinear damped oscillator having varying coefficients with periodic external force [29-33, 34] of the following form:

$$\ddot{x} + 2k\ \dot{x} + e^{-\tau}x + \epsilon x^3 = F\ \cos(\omega t). \tag{6}$$

The solution of Eq.(6) is supposed as [29-33]

$$x(t) = c_1\ \cos(\omega t) + d_1\ \sin(\omega t) + c_3\ \cos(3\omega t) + d_3\ \sin(3\omega t). \tag{7}$$

Eq.(7) is treated as the truncated Fourier series. The unknown constants c_1, d_1, c_3 and d_3 are to be found to get the desired results. Putting Eq.(7) in Eq.(6) and then comparing the coefficients of similar harmonics and removing the terms whose effects are negligible, we carry out

$$\begin{aligned} &c_1e^{-\tau} - c_1\omega^2 + 3\epsilon c_1^3/4 + 3\epsilon c_1^2c_3/4 + 3\epsilon c_1c_3^2/2 + 2k\omega d_1 + 3\epsilon c_1d_1^2/4 \\ &- 3\epsilon c_3d_1^2/4 + 3\epsilon c_1d_1d_3/2 + 3\epsilon c_1d_3^2/2 = F, \end{aligned} \tag{8a}$$

$$\begin{aligned} &- 2k\omega c_1 + d_1e^{-\tau} - d_1\omega^2 + 3\epsilon c_1^2d_1/4 - 3\epsilon c_1c_3d_1/2 + 3\epsilon c_3^2d_1/2 + 3\epsilon d_1^3/4 \\ &+ 3\epsilon c_1^2d_3/4 - 3\epsilon d_1^2d_3/4 + 3\epsilon d_1d_3^2/2 = 0, \end{aligned} \tag{8b}$$

$$\begin{aligned} &\epsilon c_1^3/4 + c_3e^{-\tau} - 9c_3\omega^2 + 3\epsilon c_1^2c_3/2 + 3\epsilon c_3^3/4 - 3\epsilon c_1d_1^2/4 + 3\epsilon c_3d_1^2/2 \\ &+ 6k\omega d_3 + 3\epsilon c_3d_3^2/4 = 0, \end{aligned} \tag{8c}$$

$$\begin{aligned} &- 6k\omega c_3 + 3\epsilon c_1^2d_1/4 - \epsilon d_1^3/4 + d_3e^{-\tau} - 9d_3\omega^2 + 3\epsilon c_1^2d_3/2 + 3\epsilon c_3^2d_3/4 \\ &+ 3\epsilon d_1^2d_3/2 + 3\epsilon d_3^3/4 = 0. \end{aligned} \tag{8d}$$

Deducting ω^2 from the Eqs.(8b)-(8d), utilizing Eq.(8a), and removing the terms whose responses are small, we receive

$$- 8k\omega c_1^2 + 4Fd_1 - 9\epsilon c_1^2c_3d_1 - 8k\omega d_1^2 + 3\epsilon c_3d_1^3 + 3\epsilon c_1^3d_3 - 9\epsilon c_1d_1^2d_3 = 0, \tag{9a}$$

$$\begin{aligned} &\epsilon c_1^4 + 36Fc_3 - 32c_1c_3e^{-\tau} - 21\epsilon c_1^3c_3 - 72k\omega c_3d_1 - 3\epsilon c_1^2d_1^2 - 21\epsilon c_1c_3d_1^2 \\ &+ 24k\omega c_1d_3 = 0, \end{aligned} \tag{9b}$$

$$\begin{aligned} &- 24k\omega c_1c_3 + 3\epsilon c_1^3d_1 - \epsilon c_1d_1^3 + 36Fd_3 - 32c_1d_3e^{-\tau} - 21\epsilon c_1^3d_3 - 72k\omega d_1d_3 \\ &- 21\epsilon c_1d_1^2d_3 = 0. \end{aligned} \tag{9c}$$

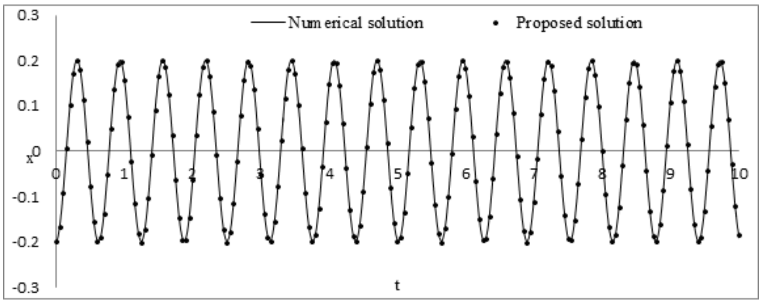

Figure 1: Comparison between the outcomes attained by the mentioned technique and the numerical technique of Eq.(6) for $\omega = 10, \epsilon = 0.5, k = 0.5, F = 20$.

Utilizing Eq.(9a), terminating ω from the Eqs.(9b) and (9c) and taking into account only the linear expressions of c_3, d_3 and omitting the expressions whose effects are insignificant, we obtain

$$\begin{aligned} &8\epsilon kc_1^6 + 288kFc_1^2c_3 - 256kc_1^5c_3e^{-\tau} - 168\epsilon kC_1^3c_3 - 16\epsilon kc_1^4d_1^2 - 256kc_1c_3d_1^2e^{-\tau} \\ &- 16\epsilon kc_1^4d_1^2 - 256kc_1c_3d_1^2e^{-\tau} - 336\epsilon kc_1^3c_3d_1^2 - 24\epsilon kc_1^2d_4 - 168\epsilon kc_1c_3d_1^4 = 0, \end{aligned} \tag{10a}$$

$$\begin{aligned} &24\epsilon kc_1^5d_1 + 16\epsilon kc_1^3d_1^3 - 8\epsilon kc_1d_1^5 + 288kFc_1^2d_3 - 256kc_1^3d_3e^{-\tau} - 168\epsilon kc_1^5d_3 \\ &- 256kc_1d_1^2d_3e^{-\tau} - 336\epsilon kc_1^3d_1^2d_3 - 168\epsilon kc_1d_1^4d_3 = 0. \end{aligned} \tag{10b}$$

By simplifying Eqs.(10a) and (10b), c_3 and d_3 are obtained as follows:

$$\begin{aligned} &c_3 = \epsilon c_1(c_1^4 - 2c_1^2d_1^2 - 3d_1^4)e^{-\tau} \\ &/(-36Fc_1e^{\tau} + 32c_1^2 + 21\epsilon c_1^4e^{\tau} + 32d_1^2 + 42\epsilon c_1^2d_1^2e^{\tau} + 21\epsilon d_1^4e^{\tau}), \end{aligned} \tag{11a}$$

$$\begin{aligned} &d_3 = \epsilon d_1(-c_1^4 - 2c_1^2d_1^2 + d_1^4)e^{\tau} \\ &/(36Fc_1e^{\tau} - 32c_1^2 - 21\epsilon c_1^4e^{\tau} - 32d_1^2 - 42\epsilon c_1^2d_1^2e^{\tau} - 21\epsilon d_1^4e^{\tau}). \end{aligned} \tag{11b}$$

Inserting c_3 and d_3 in Eq.(9a) and expanding d_1 into a power series of λ, we acquire

$$d_1 = l_0 + l_1\lambda + l_2\lambda^2 + l_3\lambda^3 + \ldots, \tag{12}$$

where $\lambda = 2k\omega/E$, $l_0 = 2c_1^2k\omega/F$, $l_1 = 16c_1^4k^2\omega^2/F^2$, $l_2 = 16c_1^6k^3\omega^3/F^3$, $l_3 = 80c_1^8k^4\omega^4/F^4$. Finally, after inserting c_3, d_3 and d_1 into Eq.(8a) and solving, c_1 is obtained. Systematically, d_1, c_3 and d_3 are obtained.

4 Results and Discussion

The proposed method is easy and straightforward. We have successfully applied this technique to solve the strongly nonlinear forced dynamical damped problems with varying coefficients and cubic nonlinearity. The solutions have been assimilated with the corresponding numerical outcomes to rationalize the precision and the correctness of the mentioned scheme. Comparisons between the solutions acquired by the mentioned scheme and the numerical technique have been displayed in Figs. 1–5 for nonlinear forced

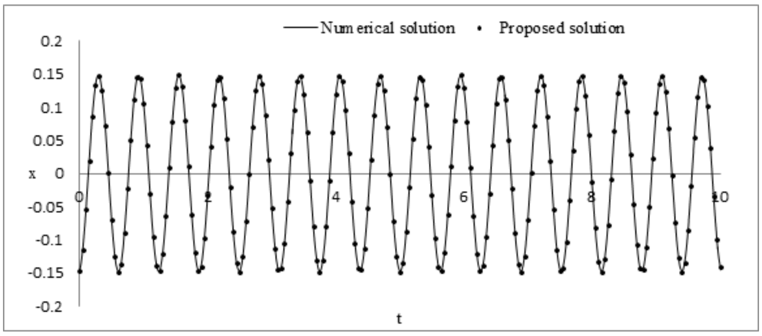

Figure 2: Comparison between the outcomes attained by the mentioned technique and the numerical technique of Eq.(6) for $\omega = 10, \epsilon = 1.0, k = 1.0, F = 15$.

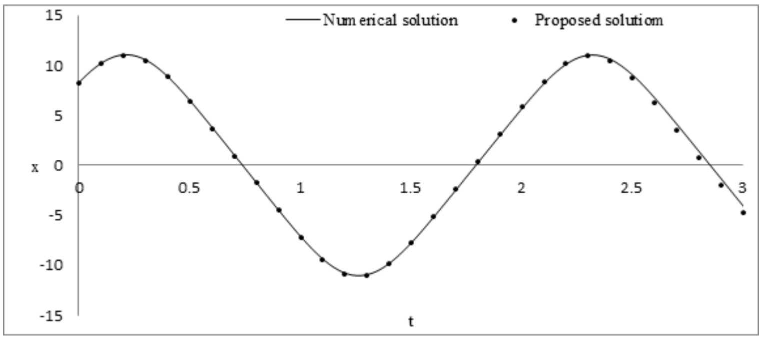

Figure 3: Comparison between the outcomes attained by the mentioned technique and the numerical technique of Eq.(6) for $\omega = 3, \epsilon = 0.1, k = 0.1, F = 10$.

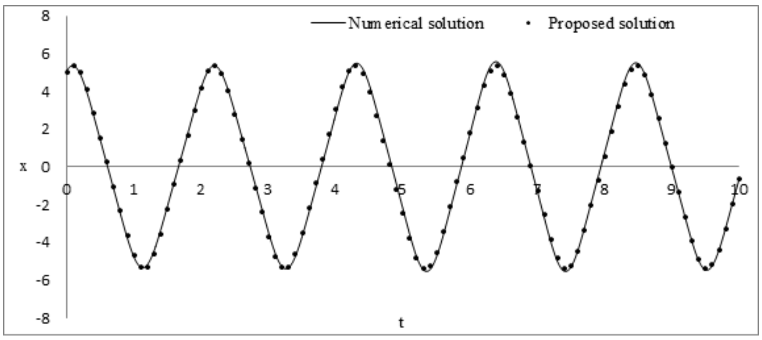

Figure 4: Comparison between the outcomes attained by the mentioned technique and the numerical technique of Eq.(6) for $\omega = 3, \epsilon = 0.5, k = 0.2, F = 10$.

vibration problems with varying coefficients for various damping. Moreover, the phase planes have been traced for different values in Figs. 6 and 7.

Geometrical representation is very important to visualize the behavior of the physical

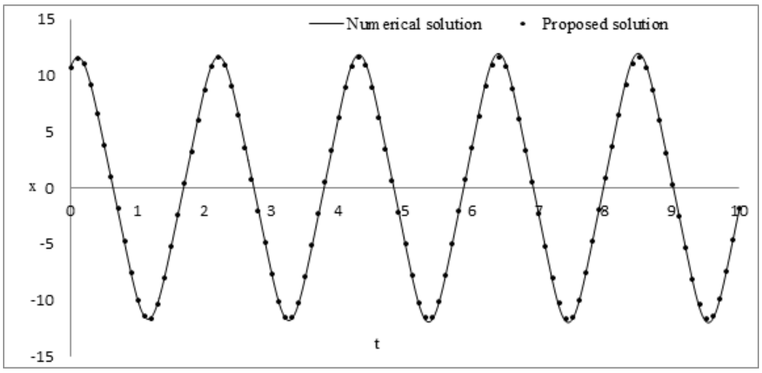

Figure 5: Comparison between the outcomes attained by the mentioned technique and the numerical technique of Eq.(6) for $\omega = 3, \epsilon = 0.1, k = 0.1, F = 20$.

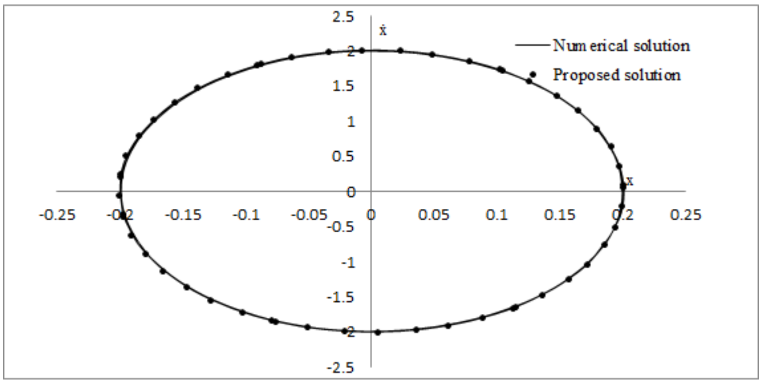

Figure 6: Comparison between the outcomes attained by the mentioned technique and the numerical technique of Eq.(6) in the phase plane when $\omega = 10, \epsilon = 0.5, k = 0.5, F = 20$.

systems since it provides an overall view of the behavior of the nonlinear dynamical systems. The approximate methods have become more interesting to the scientists, physicists, engineers and applied mathematicians because of their analytical expression and suitability for parametric study. From the figures presented, it is noticed that the obtained results have agreed nicely with the numerical results determined by the fourth order Runge-Kutta method. In Table 1, a comparison between the results obtained by the proposed method and the numerical method is given. From the figures and table, it is observed that the acquired outcomes comply almost accurately with the numerical outcomes acquired by the fourth order Runge-Kutta technique.

5 Conclusion

A MHBM is exhibited for managing nonlinear forced dynamical equations with varying coefficients and damping. The convenience of the mentioned scheme is that only one nonlinear equation is requisite to handle instead of a set of nonlinear algebraic equations.

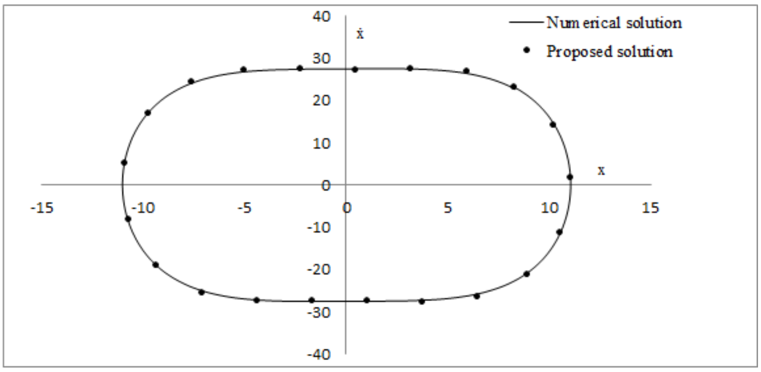

Figure 7: Comparison between the outcomes attained by the mentioned technique and the numerical technique of Eq.(6) in the phase plane when $\omega = 3, \epsilon = 0.1, k = 0.1, F = 10$.

Table 1: Comparison between the outcomes achieved by the mentioned and the numerical techniques.

	$E = 15, \omega = 10,$ $\epsilon = 0.1, k = 0.1$		$E = 20, \omega = 10,$ $\epsilon = 0.5, k = 0.5$	
Time, t	Analytical Solution, x_{app}	Numerical Solution, x_{nu}	Analytical Solution, x_{app}	Numerical Solution, x_{nu}
0	-0.151	-0.151	-0.2	-0.2
0.5	-0.046	-0.046	-0.076	-0.076
1	0.125	0.125	0.157	0.157
1.5	0.117	0.117	0.165	0.165
2	-0.059	-0.059	-0.063	-0.061
2.5	-0.151	-0.15	-0.201	-0.89
3	-0.026	-0.026	-0.051	-0.051
3.5	0.136	0.134	0.172	0.17
4	0.103	0.103	0.148	0.148
4.5	-0.077	-0.077	-0.088	-0.085
5	-0.147	-0.146	-0.88	-0.86

It requires less computational effort than the harmonic balance method. The outcomes acquired by the mentioned technique show a nice similarity with the numerical outcomes in the figures and table. The mentioned scheme may play an important role for tackling the forced dynamical systems with varying coefficients and damping.

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