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Contact Problem for Thermo-Elasto-Viscoplastic Material with Friction

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Abstract: We consider a quasistatic contact problem for thermo-elasto-viscoplastic material with thermal effects. The contact is modeled with the normal damped response condition, associated to Coulomb's law of dry friction. A variational formulation of the model is derived, and the existence of a unique weak solution is proved. The proofs are based on the arguments of evolutionary quasivariational inequality, the classical result of nonlinear first order evolution inequalities, and the fixed point arguments. We also study the dependence of the solution and prove a convergence result.

Keywords: *thermo-elasto-viscoplastic material; friction contact; normal damped response condition; Coulomb's friction; evolution equation; weak solution; fixed point.*

Mathematics Subject Classification (2010): 74F05, 74H25, 74M10, 74M25, 70K70, 70K75.

1 Introduction

Scientific research and recent papers in mechanics are articulated around two main components, one devoted to the laws of behavior and other devoted to the boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world. The frictional contact between deformable bodies can be frequently found in industry and everyday life. Because of the importance in metal forming and automotive industry, a considerable effort has been made towards the modeling and numerical simulations of contact problems and the engineering literature concerning this topic is rather extensive. An excellent reference in the field of contact problems with or without friction is [8]. The constitutive law with internal variables has been used in various publications in order to model the effect of internal variables on the behavior

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of real bodies such as metal and rocks polymers. Some of the internal state variables considered by many authors are the spatial display of dislocation, the work-hardening of materials, the absolute temperature and the damage fields. The cases of hardening, temperature and other internal state variables were considered in [2, 5, 17, 18], general models for contact processes with thermal effects can be found in [4, 10, 19]. Elastic or viscoelastic frictional contact problems, with thermal considerations, can be found in [1, 3, 14] and the references therein. The purpose of this paper is to make the coupling of an elasto-viscoplastic material with thermal effects and friction. We study a quasistatic problem of frictional contact with the normal damped response condition and the associated version of Coulomb's law of dry friction. We derive a variational formulation of the problem and prove that the proposed model has a unique weak solution by using the evolutionary quasivariational inequality. Also, we study the continuous dependence of the weak solution of the problem and prove a convergence result.

The paper is structured as follows. In Section 2, we present notation and some preliminaries. The model is described in Section 3, where the variational formulation is given. In Section 4, we present our existence and uniqueness result and the proof is based on the arguments for functional analysis concerning the evolutionary quasivariational inequality, the classical result for nonlinear first order evolution inequalities and the fixed point arguments. In Section 5, we study the dependence of the solution and prove a convergence result.

2 Notation and Preliminaries

In this section, we list the assumptions on the data, derive a variational formulation for the contact problem (9)–(18) and state our main existence and uniqueness result, Theorem 4.2. To this end, we need to introduce some notation and preliminary material.

We recall that the inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are given by

$$\begin{aligned} u.v &= u_i v_i, \qquad \|v\| = (v.v)^{\frac{1}{2}} \qquad \forall u, v \in \mathbb{R}^d, \\ \sigma.\tau &= \sigma_{ij}\tau_{ij}, \qquad \|\tau\| = (\tau.\tau)^{\frac{1}{2}} \qquad \forall \sigma, \tau \in \mathbb{S}^d. \end{aligned}$$

Here and everywhere in this paper, i, j run from 1 to d, the summation over repeated indices is used and the index which follows the comma represents the partial derivative. We use the classical notation for L^p and Sobolev spaces associated to Ω and Γ. Moreover, we use the notation H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ for the following spaces:

$$\begin{aligned} H &= L^2(\Omega)^d = \left\{v = (v_i) \ / \ v_i \in L^2(\Omega)\right\}, \\ \mathcal{H} &= \left\{\sigma = (\sigma_{ij}) \ / \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\right\}, \\ H_1 &= \{u = (u_i) \ / \ \varepsilon(u) \in \mathcal{H}\}, \\ \mathcal{H}_1 &= \{\sigma \in \mathcal{H} \ / \ \operatorname{Div}\sigma \in H\}. \end{aligned}$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are the real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned} (u,v)_H &= \int_\Omega u.v dx, \quad (\sigma,\tau)_{\mathcal{H}} = \int_\Omega \sigma.\tau dx, \\ (u,v)_{H_1} &= (u,v)_H + (\varepsilon(u),\varepsilon(v))_{\mathcal{H}}, \quad (\sigma,\tau)_{\mathcal{H}_1} = (\sigma,\tau)_{\mathcal{H}} + (\operatorname{Div}\sigma, \operatorname{Div}\tau)_H, \end{aligned}$$

and the associated norms $\|.\|_H$, $\|.\|_{\mathcal{H}}$, $\|.\|_{H_1}$ and $\|.\|_{\mathcal{H}_1}$, respectively. Here and below, we use the notation

$$\begin{aligned} \varepsilon(v) &= (\varepsilon_{ij}(v)), \quad \varepsilon_{ij}(v) = \frac{1}{2}(v_{i,j} + v_{j,i}) \quad \forall v \in H^1(\Omega)^d, \\ \operatorname{Div} \tau &= (\tau_{ij,j}) \qquad \forall \tau \in \mathcal{H}_1. \end{aligned}$$

For every element $v \in H_1$, we also write v for the trace of v on Γ and we denote by v_ν and v_τ the normal and tangential components of v on Γ given by $v_\nu = v.\nu$, $v_\tau = v - v_\nu \nu$. We also denote by σ_ν and σ_τ the normal and the tangential traces of a function $\sigma \in \mathcal{H}_1$, and we recall that when σ is a regular function, then $\sigma_\nu = (\sigma\nu).\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$, and the following Green's formula holds:

$$(\sigma, \varepsilon(v))_{\mathcal{H}} + (\operatorname{Div} \sigma, v)_H = \int_\Gamma \sigma\nu.v da \qquad \forall v \in H_1.$$

Let $T > 0$. For every real Banach space X, we use the notation $C(0,T;X)$ and $C^1(0,T;X)$ for the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively; $C(0,T;X)$ is a real Banach space with the norm $\|f\|_{C(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X$, while $C^1(0,T;X)$ is a real Banach space with the norm $\|f\|_{C^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \left\|\dot{f}(t)\right\|_X$. Finally, for $k \in \mathbb{N}$ and $p \in [1,\infty]$, we use the standard notation for the Lebesgue spaces $L^P(0,T;X)$ and for the Sobolev spaces $W^{k,p}(0,T;X)$. Moreover, for a real number r, we use r_+ to represent its positive part, that is, $r_+ = \max\{0, r\}$. Moreover, if X_1 and X_2 are real Hilbert spaces, then $X_1 \times X_2$ denotes the product Hilbert space endowed with the canonical inner product $(.,.)_{X_1 \times X_2}$.

Let X be a real Hilbert space with the inner product $(.,.)_X$ and the associated norm $\|.\|$, and consider the problem of finding $u : [0,T] \to X$ such that

$$\left\{ \begin{array}{l} (A\dot{u}(t), v - \dot{u}(t))_V + (Bu(t), v - \dot{u}(t))_V + j(\dot{u}(t), v) \\ -j\left(\dot{u}(t), \dot{u}(t)\right) \geqslant \left(f(t), v - \dot{u}(t)\right)_V \qquad \forall v \in X, t \in [0,T]. \\ u(0) = u_0. \end{array} \right. \tag{1}$$

To study problem (1), we need the following assumptions.

The operator $A : X \to X$ is Lipschitz continuous and strongly monotone, i.e.,

$$\left\{ \begin{array}{l} a) \text{ There exists } L_{\mathcal{A}} > 0 \text{ such that} \\ \|\mathcal{A}u_1 - \mathcal{A}u_2\|_X \leqslant L_{\mathcal{A}} \|u_1 - u_2\|_X \quad \forall u_1, u_2 \in X, \\ c) \text{ There exists } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2)_X \geqslant m_{\mathcal{A}} \|u_1 - u_2\|_X^2 \qquad \forall u_1, u_2 \in X. \end{array} \right. \tag{2}$$

The nonlinear operator $B : X \to X$ is Lipschitz continuous, i.e.,

$$\left\{ \begin{array}{l} \text{There exists } L_B > 0 \text{ such that} \\ \|Bu_1 - Bu_2\|_X \leqslant L_B \|u_1 - u_2\|_X \quad \forall u_1, u_2 \in X. \end{array} \right. \tag{3}$$

The functional $j : X \times X \to R$ satisfies the following conditions:

$$\left\{ \begin{array}{l} a) \ j(u,.) \text{ is convex and i.s.c on X for all } u \in X. \\ b) \text{ There exists } \alpha > 0 \text{ such that} \\ j(u_1, v_2) + j(u_1, v_1) + j(u_2, v_1) + j(u_2, v_2) \\ \leqslant \alpha \|u_1 - u_2\|_X \|v_1 - v_2\|_X, \quad \forall u_1, u_2, v_1, v_2 \in X. \end{array} \right. \tag{4}$$

$$f \in C(0,T;X), \tag{5}$$

$$u_0 \in X, \tag{6}$$

$$m_{\mathcal{A}} > \alpha. \tag{7}$$

We have the following existence and uniqueness result which can be found in [16].

Theorem 2.1 *Assume that (2)-(7) hold. Then there exists a unique solution u to problem (1). Moreover, the solution satisfies $u \in C^1([0,T];X)$.*

3 Mechanical and Variational Formulations

We consider a thermo-elasto-viscoplastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^d\,(d = 2.3)$ with a Lipschitz continuous boundary Γ that is divided into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that meas $\Gamma_1 > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body is clamped on $\Gamma_1 \times (0,T)$, so the displacement field vanishes there. The surface tractions of density f_2 act on $\Gamma_2 \times (0,T)$, and the body force of density f_0 acts in $\Omega \times (0,T)$. The contact between the body and the foundation, over the contact surface Γ_3, is modeled with the normal damped response and the associated general version of Coulomb's law of dry friction. Moreover, the process is quasistatic, i.e., the intertial terms are neglected in the equation of motion. The material is assumed to behave according to the general elasto-viscoplastic constitutive law with thermal effects given by

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{F}\varepsilon\left(u\right) + \int_0^t \mathcal{G}\left(\sigma\left(s\right) - \mathcal{A}\varepsilon\left(\dot{u}\left(s\right)\right), \varepsilon\left(u\left(s\right)\right)\right) ds - \mathcal{M}\theta\left(t\right), \tag{8}$$

where σ denotes the stress tensor, u represents the displacement field, $\dot{u}$ is the velocity, $\varepsilon(u)$ is the small strain tensor, and θ is the temperature field. Here, $\mathcal{A}$ and $\mathcal{F}$ are nonlinear operators describing the purely viscous and the elastic properties of the material, respectively. $\mathcal{G}$ is a general nonlinear constitutive function describing the viscoplastic behavior of the material. $\mathcal{M} = (m_{ij})$ represents the thermal expansion tensor. We use dots for derivatives with respect to the time variable t. It follows from (8) that at each time moment, the stress tensor $\sigma(t)$ is split into two parts: $\sigma(t) = \sigma^V(t) + \sigma^R(t)$, where $\sigma^V(t) = \mathcal{A}\varepsilon\left(\dot{u}\right)$ represents the purely viscous part of the stress, whereas $\sigma^R(t)$ satisfies a rate-type thermo-elasto-viscoplastic relation

$$\sigma^R(t) = \mathcal{F}\varepsilon\left(u\right) + \int_0^t \mathcal{G}\left(\sigma^R\left(s\right), \varepsilon\left(u\left(s\right)\right)\right) ds - \mathcal{M}\theta\left(t\right).$$

The evolution of the temperature field θ is governed by the heat equation (*see* [1]), obtained from the conservation of energy, and defined by the following differential equation for the temperature:

$$\dot{\theta} - \operatorname{div}(k\nabla\theta) = q - \mathcal{M}\nabla\dot{u},$$

where $K = (k_{ij})$ represents the thermal conductivity tensor, $\operatorname{div}(k\nabla\theta) = (k_{ij}\theta_{,i})_{,i}$ and q represents the density of volume heat sources.

The associated temperature boundary condition on Γ_3 is described by

$$k_{ij}\theta_{,i}n_j = -k_e\left(\theta - \theta_R\right) + h_\tau\left(\left|\dot{u}_\tau\right|\right) \qquad \text{on } \Gamma_3 \times (0,T),$$

where θ_R is the temperature of the foundation, k_e is the heat exchange coefficient between the body and the obstacle and $h_\tau : \Gamma_3 \times R_+ \to R_+$ is a given tangential function.

Then, the classical formulation of the mechanical problem is as follows.

Problem P: Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times [0,T] \to S^d$ and a temperature $\theta : \Omega \times [0,T] \to \mathbb{R}$ such that

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{F}\varepsilon\left(u\right) + \int_0^t \mathcal{G}\left(\sigma\left(s\right) - \mathcal{A}\varepsilon\left(\dot{u}\left(s\right)\right), \varepsilon\left(u\left(s\right)\right)\right) ds \tag{9}$$

$$-\mathcal{M}\theta\left(t\right) \text{ in } \Omega \times (0,T),$$

$$\dot{\theta} - \operatorname{div}(k\nabla\theta) = q - \mathcal{M}\nabla\dot{u} \text{ in } \Omega \times (0,T), \tag{10}$$

$$\operatorname{Div}\sigma + f_0 = 0 \quad \text{in } \Omega \times (0,T), \tag{11}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{12}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{13}$$

$$-\sigma_\nu = p_\nu\left(\dot{u}_\nu\right) \text{ on } \Gamma_3 \times (0,T), \tag{14}$$

$$\begin{cases} \|\sigma_\tau\| \leqslant \mu p_\nu\left(\dot{u}_\nu\right) \\ \|\sigma_\tau\| < \mu p_\nu\left(\dot{u}_\nu\right) \quad \Rightarrow \dot{u}_\tau = 0 \\ \|\sigma_\tau\| = \mu p_\nu\left(\dot{u}_\nu\right) \quad \Rightarrow \exists\lambda \geqslant 0 \quad \sigma_\tau = -\lambda\dot{u}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{15}$$

$$-k_{ij}\frac{\partial\theta}{\partial\nu} = k_e\left(\theta - \theta_R\right) - h_\tau\left(\left|\dot{u}_\tau\right|\right) \quad \text{on } \Gamma_3 \times (0,T), \tag{16}$$

$$\theta = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{17}$$

$$u\left(0\right) = u_0, \theta\left(0\right) = \theta_0 \quad \text{in } \Omega. \tag{18}$$

We now provide some comments on the equations and conditions of problem (9)–(18).

First, (9)-(10) represent the thermo-elasto-viscoplastic constitutive law and the evolution equation of the heat field, respectively. (11) is the equilibrium equation. (12) and (13) represent the displacement and traction boundary conditions, respectively. Conditions (16) and (17) represent the temperature boundary conditions, where (17) means that the temperature vanishes on $(\Gamma_1 \cup \Gamma_2) \times (0,T)$. Conditions (14) and (15) are Colomb's friction law, where $\mu \geqslant 0, \lambda \geqslant 0$, and they state a general normal damped response condition, where $\dot{u}_\nu$ represents the normal velocity, p_ν is a prescribed function, σ_ν is the normal stress, $\dot{u}_\tau$ denotes the tangential velocity and σ_τ represents the tangential force on the contact boundary. Denote by u_0 and θ_0 the initial displacement and the initial temperature, respectively. To simplify the notation, we do not indicate explicitly the dependence of various functions on the variables $x \in \Omega \cup \Gamma$ and $t \in [0,T]$. To obtain a variational formulation of the problem (9)–(18), we need additional notations. Let E denote the closed subspace of $H^1(\Omega)$ given by

$$E = \left\{\gamma \in H^1\left(\Omega\right) / \ \gamma = 0 \qquad on \ \ \Gamma_1 \cup \Gamma_2\right\}.$$

Let us now consider the closed subspace of H_1 defined by

$$V = \left\{v \in H_1 / \ v = 0 \ on \ \Gamma_1\right\}.$$

Since meas $(\Gamma_1) > 0$, the following Korn's inequality holds:

$$\|\varepsilon(v)\|_{\mathcal{H}} \geqslant c_k \|v\|_{H_1} \qquad \forall v \in V, \tag{19}$$

where $c_k > 0$ is a constant which depends only on Ω and Γ_1. On the space V, we consider the inner product and the associated norm given by

$$(u,v)_V = (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \qquad \|v\|_V = \|\varepsilon(v)\|_{\mathcal{H}} \qquad \forall u, v \in V. \tag{20}$$

It follows from Korn's inequality that $\|.\|_{H_1}$ and $\|.\|_V$ are equivalent norms on V. Therefore $(V, \|.\|_V)$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (20), there exists a constant $c_0 > 0$ depending only on the domain Ω , Γ_1 and Γ_3 such that

$$\|v\|_{L^2(\Gamma_3)^d} \leqslant c_0 \|v\|_V \qquad \forall v \in V. \tag{21}$$

In the study of the mechanical problem (9)–(18), we assume that the viscosity operator $\mathcal{A} : \Omega \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the conditions:

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{A}} > 0 \text{ such that} \\ \|\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)\| \leqslant L_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\| \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. x \in \Omega. \\ (b) \text{ There exists a constant } m_{\mathcal{A}} > 0 \text{ such that} \\ (\mathcal{A}(x,\varepsilon_1) - \mathcal{A}(x,\varepsilon_2)) . (\varepsilon_1 - \varepsilon_2) \geqslant m_{\mathcal{A}} \|\varepsilon_1 - \varepsilon_2\|^2, \ \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^d, a.e. x \in \Omega. \\ (c) \text{ The mapping } x \mapsto \mathcal{A}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \varepsilon \in \mathbb{S}^d. \\ (d) \text{ The mapping } x \mapsto \mathcal{A}(x,0) \in \mathcal{H}. \end{cases} \tag{22}$$

The elasticity operator $\mathcal{F} : \Omega \times S^d \times \mathbb{R} \to S^d$ satisfies the conditions:

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{F}} > 0 \text{ such that} \\ \|\mathcal{F}(x,\varepsilon_1) - \mathcal{F}(x,\varepsilon_2)\| \leqslant L_{\mathcal{F}} \|\varepsilon_1 - \varepsilon_2\| \quad \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } x \in \Omega. \\ \forall \varepsilon_1, \varepsilon_2 \in S^d, \text{ a.e. } x \in \Omega. \\ (b) \text{ The mapping } x \to \mathcal{F}(x,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \varepsilon \in S^d. \\ (c) \text{ The mapping } x \to \mathcal{F}(x,0) \in \mathcal{H}. \end{cases} \tag{23}$$

The visco-plasticity operator $\mathcal{G} : \Omega \times S^d \times S^d \to S^d$ satisfies the conditions:

$$\begin{cases} (a) \text{ There exists a constant } L_{\mathcal{G}} > 0 \text{ such that} \\ \|\mathcal{G}(x,\sigma_1,\varepsilon_1) - \mathcal{G}(x,\sigma_2,\varepsilon_2)\| \leqslant L_{\mathcal{G}}(\|\varepsilon_1 - \varepsilon_2\| + \|\sigma_1 - \sigma_2\|) \\ \forall \varepsilon_1, \varepsilon_2, \sigma_1, \sigma_2 \in S^d, \text{ a.e. } x \in \Omega. \\ (b) \text{ The mapping } x \to \mathcal{G}(x,\sigma,\varepsilon) \text{ is Lebesgue measurable on } \Omega, \text{ for any } \varepsilon, \sigma \in S^d. \\ (c) \text{ The mapping } x \to \mathcal{G}(x,0,0) \in \mathcal{H}. \end{cases} \tag{24}$$

The contact function $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}^+$ satisfies the conditions:

$$\begin{cases} (a) \text{ There exists a constant } L_\nu \succ 0 \text{ such that} \\ \|p_\nu(x,r_1) - p_\nu(x,r_2)\| \leq L_\nu \|r_1 - r_2\| \quad \forall r_1, r_2 \in \mathbb{R}, \ a.e. x \in \Gamma_3. \\ (d) \text{ The mapping } x \longmapsto p_\nu(x,r) \text{ is Lebesgue measurable on } \Gamma_3, \text{ for any } r \in \mathbb{R}. \\ (f) \text{ The mapping } x \longmapsto p_\nu(x,r) \text{ belongs to } L^2(\Gamma_3). \end{cases} \tag{25}$$

The tangential function $h_\tau : \Gamma_3 \times \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the conditions:

$$\begin{cases} (a) \text{ There exists a constant } L_h > 0 \text{ such that} \\ \|h_\tau(x,r_1) - h_\tau(x,r_2)\| \leq L_h \|r_1 - r_2\| \quad \forall r_1, r_2 \in \mathbb{R}^+, \ a.e. x \in \Gamma_3. \\ (b) \text{ The mapping } x \to h_\tau(x,r) \in L^2(\Gamma_3) \text{ is Lebesgue measurable on } \Gamma_3, \forall r \in \mathbb{R}^+. \end{cases} \tag{26}$$

The body forces and surface tractions have the regularity

$$f_0 \in C\left(0,T;H\right), \qquad f_2 \in C\left(0,T;L^2(\Gamma_2)^d\right). \tag{27}$$

The coefficient μ satisfies the following conditions:

$$\mu \in L^\infty(\Gamma_3) \quad \mu(x) \geqslant 0 \text{ a.e. on } \Gamma_3. \tag{28}$$

The thermal tensors and the heat source density satisfy the conditions:

$$\begin{cases} \mathcal{M} = (m_{ij}), & m_{ij} = m_{ji} \in L^\infty(\Omega). \\ K = (k_{ij}), & k_{ij} = k_{ji} \in L^\infty(\Omega), k_{ij}\zeta_i\zeta_i \geqslant c_k\zeta_i\zeta_i, \\ & \text{for some } c_k > 0, \text{ for all } (\zeta_i) \in R^d. \\ q \in L^2(0,T;L^2(\Omega)). & \end{cases} \tag{29}$$

Finally, the boundary and initial data verify that

$$u_0 \in V, \quad \theta_0 \in E, \quad \theta_R \in L^2(0,T;L^2(\Gamma_3)), \quad k_e \in L^\infty(\Omega, R^+). \tag{30}$$

We define the function $f : [0,T] \to V$ by

$$(f(t), v) = \int_\Omega f_0(t).vdx + \int_{\Gamma_2} f_2(t).vda. \quad \forall v \in V, \forall t \in [0,T]. \tag{31}$$

Next, we denote by $j : V \times V \to \mathbb{R}$ the functional defined by

$$j(u,v) = \int_{\Gamma_3} p_\nu(u).v_\nu da + \int_{\Gamma_3} \mu p_\nu(u).\left\|v_\tau\right\| da \qquad \forall u,v \in V. \tag{32}$$

We note that condition (27) implies

$$f \in C\left([0,T], V\right). \tag{33}$$

Using standard arguments, we obtain the variational formulation of the mechanical problem (9)-(18).

Problem PV**.** Find a displacement field $u : [0,T] \to V$, a stress field $\sigma : [0,T] \to \mathcal{H}$ and a temperature field $\theta : [0,T] \to E$ such that for all $t \in [0,T]$,

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{F}\varepsilon\left(u\right) + \int_0^t \mathcal{G}\left(\sigma\left(s\right) - \mathcal{A}\varepsilon\left(\dot{u}\left(s\right)\right), \varepsilon\left(u\left(s\right)\right)\right) ds - \mathcal{M}\theta\left(t\right), \tag{34}$$

$$\left(\sigma\left(t\right), \varepsilon(v) - \varepsilon\left(\dot{u}\right)\right)_{\mathcal{H}} + j\left(\dot{u}(t), v\right) - j\left(\dot{u}(t), \dot{u}(t)\right) \geqslant \left(f(t), v - \dot{u}\right)_V. \tag{35}$$

$$\dot{\theta}(t) + K\theta(t) = R\dot{u}(t) + Q(t) \quad \text{ in } E', \tag{36}$$

$$u(0) = u_0, \quad \theta(0) = \theta_0, \tag{37}$$

where $K : E \to E'$, $R : V \to E'$ and $Q : [0,T] \to E'$ are given by

$$(K\tau, \omega)_{E' \times E} = \sum_{i,j=1}^{d} \int_\Omega k_{ij} \frac{\partial \tau}{\partial x_j} \frac{\partial \omega}{\partial x_i} dx + \int_{\Gamma_3} k_e \tau \omega da,$$

$$(Rv, \omega)_{E' \times E} = \int_{\Gamma_3} h_\tau(|v_\tau|)\omega da - \int_\Omega m_{ij} \frac{\partial v_i}{\partial x_j} \omega dx,$$

$$(Q(t), \omega)_{E' \times E} = \int_{\Gamma_3} k_e \theta_R(t) \omega da + \int_\Omega q(t) \omega dx$$

for all $v \in V$, $\tau, \omega \in E$.

4 Existence and Uniqueness Result

Now, we propose our existence and uniqueness result.

Theorem 4.1 *Assume that (22)-(30) hold. Then there exists $L_0 > 0$ depending only on $\Omega, \Gamma_1, \Gamma_3$ and $\mathcal{A}$ such that if $L_\nu(\|\mu\|_{L^\infty(\Gamma_3)} + 1) < L_0$, problem PV has a unique solution which satisfies the conditions:*

$$u \in C^1([0,T], V), \ \sigma \in C([0,T], \mathcal{H}_1), \tag{38}$$

$$\theta \in W^{1,2}(0,T;E') \cap L^2(0,T;E) \cap C(0,T;L^2(\Omega)). \tag{39}$$

The functions u, σ and θ which satisfy (34)-(37) are called a weak solution of the contact problem P. We conclude that, under the assumptions (22)–(30), the mechanical problem (9)-(18) has a unique weak solution satisfying (38)-(39).

The proof of Theorem 4.2 is carried out in several steps that we prove in what follows, everywhere in this section we suppose that the assumptions of Theorem 4.2 hold, and we consider that C is a generic positive constant which is independent of time and whose value may change from one occurrence to another.

Let $\eta \in C(0,T;\mathcal{H})$ be given; in the first step, we consider the following variational problem.

Problem PV_η : Find a displacement field $u_\eta : [0,T] \to V$ such that

$$\begin{aligned}\left(\mathcal{A}\varepsilon(\dot{u}_\eta), \varepsilon(v) - \varepsilon(\dot{u}_\eta)\right)_{\mathcal{H}} + \left(\mathcal{F}\varepsilon(u_\eta), \varepsilon(v) - \varepsilon(\dot{u}_\eta)\right)_{\mathcal{H}} + \left(\eta(t), \varepsilon(v) - \varepsilon(\dot{u}_\eta)\right)_{\mathcal{H}} \\ + j\left(\dot{u}_\eta(t), v\right) - j\left(\dot{u}_\eta(t), \dot{u}_\eta(t)\right) \geqslant \left(f(t), v - \dot{u}_\eta\right)_V.\end{aligned} \tag{40}$$

$$u_\eta(0) = u_0. \tag{41}$$

We have the following result for the problem.

Lemma 4.1 *There exists L_0 depending only on $\Omega, \Gamma_1, \Gamma_3$ and $\mathcal{A}$ such that if $L_\nu(\|\mu\|_{L^\infty(\Gamma_3)} + 1) < L_0$, the problem PV has a unique solution $u_\eta \in C^1([0,T], V)$.*

Proof. We define the operators $A : V \to V$, $F : V \to V$ and the function f_η :$[0,T] \to V$ by

$$(Au, v)_V = (\mathcal{A}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{42}$$

$$(Fu, v)_V = (\mathcal{F}\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \tag{43}$$

$$(f_\eta, v)_V = (f(t), v)_V - (\eta(t), \varepsilon(v))_{\mathcal{H}} \tag{44}$$

for all $u, v \in V$ and $t \in [0,T]$.

We use (42), (22)(b) and (22)(c) to find that

$$\|Au_1 - Au_2\| \leqslant L_{\mathcal{A}} \|u_1 - u_2\|_V. \tag{45}$$

$$(Au_1 - Au_2, u_1 - u_2)_V \geqslant m_{\mathcal{A}} \|u_1 - u_2\|_V^2. \tag{46}$$

From (23)(a) and (43), we have

$$\|Fu_1 - Fu_2\| \leqslant L_{\mathcal{F}} \|u_1 - u_2\|_V. \tag{47}$$

From (46) and (45), A is a strongly monotone Lipschitz continuous operator, then from (47), F is a Lipschitz continuous operator. We use (27), we find that the function f

defined by (31) satisfies $f \in C([0,T],V)$, and keeping in mind that $\eta \in C([0,T],\mathcal{H})$, we deduce by (44) that $f_\eta \in C([0,T],V)$ and $u_0 \in V$. We use (25), (28) and (21), we find that the function j given by (32) satisfies the condition (4)(a). Moreover,

$$\begin{aligned} &j(u_1,v_2) - j(u_1,v_1) + j(u_2,v_1) - j(u_2,v_2) \\ &\leqslant c_0^2 L_\nu(\|\mu\|_{L^\infty(\Gamma_3)} + 1)\,\|u_1 - u_2\|_V\,\|v_1 - v_2\|_V \end{aligned} \tag{48}$$

for all $u_1,u_2,v_1,v_2 \in V$, which implies that the function j satisfies the condition (4)(b) on $X = V$ with $\alpha = c_0^2 L_\nu(\|\mu\|_{L^\infty(\Gamma_3)} + 1)$. Let $L_0 = \frac{m_{\mathcal{A}}}{c_0^2}$ and note that L_0 depends only on $\Omega, \Gamma_1, \Gamma_3$ and $\mathcal{A}$. Then, if $L_\nu(\|\mu\|_{L^\infty(\Gamma_3)} + 1) < L_0$, we have

$$m_{\mathcal{A}} > \alpha, \tag{49}$$

and it follows from Theorem 4.1 that there exists a unique function $u_\eta \in C^1([0,T],V)$ such that

$$\begin{aligned} &\big(A\dot{u}_\eta(t), v - \dot{u}_\eta(t)\big)_V + \big(Fu_\eta(t), v - \dot{u}_\eta(t)\big)_V + j\left(\dot{u}_\eta(t), v\right) \\ &- j\left(\dot{u}_\eta(t), \dot{u}_\eta(t)\right) \geqslant \big(f_\eta(t), v - \dot{u}_\eta(t)\big)_V . \qquad \forall v \in V, t \in [0,T]. \end{aligned} \tag{50}$$

$$u_\eta(0) = u_0. \tag{51}$$

We use (42), (43), (50) and (51) to see that u_η is the unique solution to PV_η.

Let $u_\eta : [0,T] \to V$ be the function defined by

$$u = \int_0^t v_\eta(s)ds + u_0, \quad \forall t \in [0,T]. \tag{52}$$

In the second step, let $\eta \in C([0,T],\mathcal{H})$, we use the displacement field u_η obtained in Lemma 4.1 and we consider the following variational problem.

Problem QV_η. Find the temperature field $\theta_\eta : [0,T] \to E$ such that

$$\dot{\theta}_\eta(t) + K\theta_\eta(t) = R\dot{u}_\eta(t) + Q(t), \tag{53}$$

$$\theta_\eta(0) = \theta_0. \tag{54}$$

We have the following result.

Lemma 4.2 *Problem QV_η has a unique solution θ_η which satisfies the regularity (39), then we have for all $t \in [0,T]$,*

$$\|\theta_{\eta_1}(t) - \theta_{\eta_2}(t)\|^2_{L^2(\Omega)} \leqslant C \int_0^t \left\|\dot{u}_{\eta_1}(s) - \dot{u}_{\eta_2}(s)\right\|_V^2 ds. \tag{55}$$

Proof. We use a classical result for the first order evolution equation given in [15]. We have the Gelfand triple

$$E \subset L^2(\Omega) \equiv (L^2(\Omega))' \subset E'.$$

The operator K is linear and coercive. By Korn's inequality

$$(K\tau,\tau)_{E'\times E} \geqslant C\,|\tau|_E^2, \qquad C > 0.$$

Now, for $\theta_{\eta_i} \in E, i = 1,2$, let $t \in [0,T]$.

We have

$$\begin{aligned}\left(\dot{\theta}_{\eta_1}(t)\ -\ \dot{\theta}_{\eta_2}(t),\theta_{\eta_1}(t)-\theta_{\eta_2}(t)\right)_{E'\times E}&+\left(K\theta_{\eta_1}(t)-K\theta_{\eta_2}(t),\theta_{\eta_1}(t)-\theta_{\eta_2}(t)\right)_{E'\times E}\\ =\left(R\dot{u}_{\eta_1}(t)-R\dot{u}_{\eta_2}(t),\theta_{\eta_1}(t)-\theta_{\eta_2}(t)\right)_{E'\times E},&\end{aligned}\tag{56}$$

we integrate (56) over $(0,t)$ and we use the coercivity of K and the Lipschitz continuity of $R:V\to E'$ to deduce that (55) is satisfied for all $t\in[0,T]$.

In the third step, we use the displacement field u_η obtained in Lemma 4.1 and the temperature field θ_η obtained in Lemma 4.2 to construct the following Cauchy problem for the stress field.

Problem $PV\sigma_\eta$. Find the stress field $\sigma_\eta:[0,T]\to\mathcal{H}$ such that

$$\sigma_\eta(t)=\mathcal{F}\varepsilon\left(u_\eta(t)\right)+\int_0^t\mathcal{G}\left(\sigma_\eta\left(s\right),\varepsilon\left(u_\eta\left(s\right)\right)\right)ds-\mathcal{M}\theta_\eta\left(t\right)\quad\forall t\in[0,T].\tag{57}$$

In the study of problem $PV\sigma_\eta$, we have the following result.

Lemma 4.3 *There exists a unique solution of problem $PV\sigma_\eta$ and it satisfies $\sigma_\eta\in C^1([0,T],\mathcal{H})$. Moreover, if u_i, σ_i and θ_i represent the solutions of the problems PV_{η_i}, $PV\sigma_{\eta_i}$ and QV_{η_i}, respectively, for $\eta_i\in C(0,T;\mathcal{H})$, $i=1,2$, then there exists $C>0$ such that*

$$\begin{aligned}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}}^2\leqslant C(&\|u_1(t)-u_2(t)\|_V^2+\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2\\&+\textstyle\int_0^t\|u_1(s)-u_2(s)\|_V^2\,ds),\qquad\forall t\in[0,T].\end{aligned}\tag{58}$$

Proof. Let $\Lambda_\eta:C(0,T;\mathcal{H})\to C(0,T;\mathcal{H})$ be the operator given by

$$\Lambda_\eta\sigma(t)=\mathcal{F}\varepsilon\left(u_\eta(t)\right)+\int_0^t\mathcal{G}\left(\sigma\left(s\right),\varepsilon\left(u_\eta\left(s\right)\right)\right)ds-\mathcal{M}\theta_\eta\left(t\right)\tag{59}$$

for $\sigma\in C(0,T;\mathcal{H})$ and $t\in[0,T]$. For $\sigma_1,\sigma_2\in C(0,T;\mathcal{H})$, we obtain for all $t\in[0,T]$,

$$\|\Lambda_\eta\sigma_1-\Lambda_\eta\sigma_2\|_{\mathcal{H}}\leqslant L_{\mathcal{G}}\int_0^t\|\sigma_1(s)-\sigma_2(s)\|\,ds.$$

It follows from this inequality that for p large enough, the operator Λ_η^p is a contraction on the Banach space $C(0,T;\mathcal{H})$ and, therefore, there exists a unique element $\sigma_\eta\in C(0,T;\mathcal{H})$ such that $\Lambda_\eta\sigma=\sigma_\eta$. Moreover, σ_η is the unique solution of problem $PV\sigma_\eta$ and, when using (57), the regularity of u_η, the regularity of θ_η and the properties of the operators $\mathcal{F}$ and $\mathcal{G}$, it follows that $\sigma_\eta\in C^1(0,T;\mathcal{H})$.

Consider now $\eta_1,\eta_2\in C(0,T;\mathcal{H})$ and for $i=1,2$, denote $u_{\eta_i}=u_i,\sigma_{\eta_i}=\sigma_i$ and $\theta_{\eta_i}=\theta_i$. We have

$$\sigma_i(t)=\mathcal{F}\varepsilon\left(u_i(t)\right)+\int_0^t\mathcal{G}\left(\sigma_i\left(s\right),\varepsilon\left(u_i\left(s\right)\right)\right)ds-\mathcal{M}\theta_i\left(t\right),\quad\forall t\in[0,T],$$

and, using the properties (23) and (24) of $\mathcal{F}$ and $\mathcal{G}$, we find

$$\begin{aligned}\|\sigma_1(t)-\sigma_2(t)\|_{\mathcal{H}}^2\leqslant C(&\|u_1(t)-u_2(t)\|_V^2+\textstyle\int_0^t\|\sigma_1(s)-\sigma_2(s)\|_{\mathcal{H}}^2\,ds\\&+\textstyle\int_0^t\|u_1(s)-u_2(s)\|_V^2\,ds+\|\theta_1(t)-\theta_2(t)\|_{L^2(\Omega)}^2)\qquad\forall t\in[0,T].\end{aligned}$$

We use the Gronwall argument in the obtained inequality to deduce the estimate (58).

Finally, we consider the operator $\Lambda : C(0,T;\mathcal{H}) \to C(0,T;\mathcal{H})$ defined by

$$\Lambda\eta = \int_0^t \mathcal{G}\left(\sigma_\eta(s), \varepsilon\left(u_\eta(s)\right)\right) ds - \mathcal{M}\theta_\eta. \tag{60}$$

Here, for every $\eta \in C(0,T;\mathcal{H})$, u_η, θ_η and σ_η represent the displacement field, the temperature field and the stress field which are obtained in Lemma 4.1, Lemma 4.2 and Lemma 4.3, respectively. We have the following result.

Lemma 4.4 *The operator Λ has a unique fixed point $\eta^* \in C(0,T;\mathcal{H})$ such that $\Lambda\eta^* = \eta^*$.*

Proof. Let now $\eta_1, \eta_2 \in C(0,T;\mathcal{H})$. We use the notation $u_{\eta_i} = u_i$, $\dot{u}_{\eta_i} = v_{\eta_i} = v_i, \dot{\theta}_{\eta_1} = \dot{\theta}_{\eta_1}$ and $\sigma_{\eta_i} = \sigma_i$ for $i = 1, 2$. Using (24), (20), (29) and (60), we deduce that

$$\begin{aligned} \|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{\mathcal{H}}^2 \quad &\leqslant C\big(\textstyle\int_0^t \|\sigma_1(s) - \sigma_2(s)\|_{\mathcal{H}}^2 ds + \int_0^t \|u_1(s) - u_2(s)\|_V^2 ds \\ &\quad + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2\big). \end{aligned} \tag{61}$$

We use the estimate (58) to obtain

$$\begin{aligned} \|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{\mathcal{H}}^2 \quad &\leqslant C\big(\textstyle\int_0^t \|u_1(s) - u_2(s)\|_V^2 ds + \|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \\ &\quad + \textstyle\int_0^t \|\theta_1(s) - \theta_2(s)\|_{L^2(\Omega)}^2 ds\big). \end{aligned}$$

Moreover, from (40), we obtain

$$\begin{aligned} &\left(\mathcal{A}\varepsilon(v_1) - \mathcal{A}\varepsilon(v_2), \varepsilon(v_1) - \varepsilon(v_2)\right)_{\mathcal{H}} - \left(\mathcal{F}\varepsilon(u_1) - \mathcal{F}\varepsilon(u_2), \varepsilon(v_2) - \varepsilon(v_1)\right)_{\mathcal{H}} \\ &- \left(\eta_1(t) - \eta_2, \varepsilon(v_2) - \varepsilon(v_1)\right)_{\mathcal{H}} \leqslant j(v_1, v_2) - j(v_1, v_1) + j(v_2, v_1) - j(v_2, v_2). \end{aligned}$$

We use the assumptions (22), (23) and the estimation (48) to find that

$$\begin{aligned} m_{\mathcal{A}} \|v_1 - v_2\|_V^2 \quad &\leqslant L_{\mathcal{F}} \|u_1 - u_2\|_V \|v_1 - v_2\|_V + \|\eta_1 - \eta_2\|_{\mathcal{H}} \|v_1 - v_2\|_V \\ &\quad + c_0^2 L_\nu (\|\mu\|_{L^\infty(\Gamma_3)} + 1) \|v_1 - v_2\|_V^2. \end{aligned}$$

Then, by (49), we have

$$\|v_1 - v_2\|_V \leqslant C(\|u_1 - u_2\|_V + \|\eta_1 - \eta_2\|_{\mathcal{H}}). \tag{62}$$

Since

$$u_i(t) = \int_0^t v_i(s) ds + u_0 \qquad \forall t \in [0,T],$$

we have

$$\|u_1(t) - u_2(t)\|_V \leqslant C \int_0^t \|v_1(s) - v_2(s)\|_V ds. \tag{63}$$

Next, we use (62), (63) and we apply Gronwall's inequality to deduce

$$\|v_1(t) - v_2(t)\|_V^2 \leqslant C \|\eta_1(t) - \eta_2(t)\|_{\mathcal{H}}^2, \tag{64}$$

and from (56) and (64), we obtain

$$\|\theta_1(t) - \theta_2(t)\|_{L^2(\Omega)}^2 \leqslant C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2 ds. \tag{65}$$

We substitute (63),(64) and (65) to obtain

$$\|\Lambda\eta_1 - \Lambda\eta_2\|_{\mathcal{H}}^2 \leqslant C \int_0^t \|\eta_1(s) - \eta_2(s)\|_{\mathcal{H}}^2 \, ds.$$

Reiterating this inequality m times leads to

$$\|\Lambda^m\eta_1 - \Lambda^m\eta_2\|_{C(0,T;\mathcal{H})}^2 \leqslant \frac{C^m T^m}{m!} \|\eta_1 - \eta_2\|_{C(0,T;\mathcal{H})}^2 .$$

For m sufficiently large, Λ^m is a contraction on the Banach space $C(0,T;\mathcal{H})$, and so Λ has a unique fixed point.

Now, we have all the ingredients needed to prove Theorem 4.1.

Proof. **Existence.** Let $\eta^* \in C(0,T;\mathcal{H})$ be the fixed point of Λ defined by (60), and let $u_{\eta^*}, \sigma_{\eta^*}$ and $\dot{\theta}_{\eta^*}$ be the solutions of the problems PV_η , $PV\sigma\eta$ and QV_η, respectively, for $\eta = \eta^*$, and denote

$$u = u_{\eta^*}, \quad \dot{u} = \dot{u}_{\eta^*}, \quad \theta = \theta_{\eta^*}, \tag{66}$$

$$\sigma = \mathcal{A}\varepsilon\left(\dot{u}\right) + \mathcal{F}\varepsilon\left(u\right) + \sigma_{\eta^*}. \tag{67}$$

We prove that (u, σ, θ) satisfies (34)-(37) and (38)-(39). Indeed, we write (57) for $\eta = \eta^*$ and use (66)-(67) to obtain (34). We consider (40) for $\eta = \eta^*$ and use the equality $\Lambda\eta^* = \eta$ combined with (60) and (66)-(67) to conclude that (35) is satisfied. We write (53) for $\eta = \eta^*$ and use (66) to find that (36) is also satisfied. Next, (37) and the regularities (38)-(39) follow from Lemmas 4.1 and 4.2. The regularity of σ is a consequence of Lemmas 4.1, 4.2, 4.3, the relations (66)-(67) and the assumptions on $\mathcal{A}$ and $\mathcal{F}$.

Uniqueness. The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (60) and the unique solvability of the problems PV_η, QV_η and $PV\sigma_\eta$.

5 Convergence Results

In this section, we study the dependence of the solution to problem PV when we introduce the perturbation of certain data. We suppose that the assumptions (22)-(30) are satisfied. Moreover, we assume that $L_\nu(\|\mu\|_{L^\infty(\Gamma_3)} + 1) < L_0$, where $L_0 = \frac{m_{\mathcal{A}}}{c_0^2}$. Let (u, σ, θ) be the solution of PV which is obtained by Theorem 4.1 for every $\rho > 0$, let $\mathcal{F}_\rho$, p_ν^ρ and L_ν^ρ be the perturbations of $\mathcal{F}$, p_ν and L_ν, respectively, which satisfy the conditions (23) and (25).

We define the function $j_\rho : V \times V \to \mathbb{R}$ by

$$j_\rho(u, v) = \int_0^t p_\nu^\rho(u_\nu).v_\nu da + \int_0^t \mu p_\nu^\rho(u_\nu).\|v_\tau\| \, da \qquad \forall u, v \in V. \tag{68}$$

Under these assumptions, we consider the following variational problem.

Problem PV_ρ**.** Find a displacement field $u_\rho : [0,T] \to V$, a stress field $\sigma_\rho : [0,T] \to \mathcal{H}$ and a temperature field $\theta_\rho : [0,T] \to E$ such that for all $t \in [0,T]$,

$$\begin{aligned}\sigma_\rho(t) = \mathcal{A}\varepsilon\left(\dot{u_\rho}(t)\right) + \mathcal{F}\varepsilon\left(u_\rho(t)\right) + \textstyle\int_0^t \mathcal{G}\left(\sigma_\rho(s) - \mathcal{A}\varepsilon\left(\dot{u}_\rho(s)\right), \varepsilon\left(u_\rho(s)\right)\right) ds \\ - \mathcal{M}\theta_\rho(t).\end{aligned} \tag{69}$$

$$\begin{aligned}\left(\sigma_\rho\left(t\right),\varepsilon(v)-\varepsilon\left(\dot u_\rho\right)\right)_{\mathcal{H}}+j_\rho\left(\dot u_\rho(t),v\right)-j_\rho\left(\dot u_\rho(t),\dot u_\rho(t)\right)\\ \geqslant\left(f(t),v-\dot u_\rho(t)\right)_V.\end{aligned}\tag{70}$$

$$\dot\theta_\rho(t)+K\theta_\rho(t)=R\dot u_\rho(t)+Q(t)\ \text{ in } E',\tag{71}$$

$$u_\rho(0)=u_0,\quad \theta_\rho(0)=\theta_0.\tag{72}$$

Assume that

$$L_\nu^\rho(\|\mu\|_{L^\infty(\Gamma_3)}+1)<L_0\qquad \forall\rho>0.$$

We deduce from Theorem 4.1 that for each $\rho>0$, the problem PV_ρ has a unique solution $(u_\rho,\sigma_\rho,\theta_\rho)$ satisfying $u_\rho\in C^1\left([0,T],V\right)$, $\sigma_\rho\in C\left([0,T],\mathcal{H}_1\right)$ and $\theta_\rho\in W^{1,2}(0,T;E')\cap L^2(0,T;E)\cap C(0,T;L^2(\Omega))$.

Let us suppose $\mathcal{F}_\rho$, $\mathcal{F}$, p_ν^ρ and p_ν satisfy the following assumptions:

$$\begin{cases}\text{There exists } B:\mathbb{R}_+\to\mathbb{R}_+ \text{ such that}\\ a)\ \|\mathcal{F}_\rho(x,\varepsilon)-\mathcal{F}(x,\varepsilon)\|\leqslant B(\rho)\\ \forall\varepsilon\in S^d,\ a.e.x\in\Omega,\ \text{ for each } \rho>0.\\ b)\ \lim\limits_{\rho\to0}B(\rho)=0.\end{cases}\tag{73}$$

$$\begin{cases}\text{There exists } G_\nu:\mathbb{R}_+\to\mathbb{R}_+ \text{ such that}\\ a)\ |p_\nu^\rho(x,r)-p_\nu(x,r)|\ \leqslant G_\nu(\rho)\\ \forall r\in\mathbb{R},\ a.e.x\in\Gamma_3,\ \text{ for each } \rho>0.\\ b)\ \lim\limits_{\rho\to0}G_\nu(\rho)=0.\end{cases}\tag{74}$$

We have the following convergence result.

Theorem 5.1 *Assume that (73)-(74) hold, the solution $(u_\rho,\sigma_\rho,\theta_\rho)$ of the problem PV_ρ converges to the solution (u,σ,θ) of problem PV_η,*

$$u_\rho\to u \ \text{ in } C^1(0,T;V\ )\ \text{ as } \rho\to0;\tag{75}$$

$$\sigma_\rho\to\sigma\ \text{ in } C(0,T;\mathcal{H}_1)\ \text{ as } \rho\to0;\tag{76}$$

$$\theta_\rho\to\theta\ \text{ in } C(0,T;L^2(\Omega))\ \text{ as } \rho\to0.\tag{77}$$

In addition to the mathematical interest of convergence result (75)-(77), it is important in mechanical applications because it indicates that small perturbations of the contact conditions and of the elasticity operator lead to small perturbations of the weak solution of the problem P.

Proof. Let $\rho>0$ and $t\in[0,T]$, we use $v=\dot u(t)$ in (70) and $v=\dot u_\rho(t)$ in (35), then in addition to the two inequalities, we get

$$\begin{aligned}\left(\sigma_\rho(t)-\sigma(t),\varepsilon(\dot u_\rho(t))-\varepsilon\left(\dot u(t)\right)\right)_{\mathcal{H}}\leqslant j_\rho\left(\dot u_\rho(t),\dot u(t)\right)-j_\rho\left(\dot u_\rho(t),\dot u_\rho(t)\right)\\ +j\left(\dot u(t),\dot u_\rho(t)\right)-j\left(\dot u(t),\dot u(t)\right).\end{aligned}\tag{78}$$

We have

$$\sigma_\rho^R(t)=\sigma_\rho(t)-\mathcal{A}\varepsilon\left(\dot{u_\rho}(t)\right),\qquad \sigma^R(t)=\sigma(t)-\mathcal{A}\varepsilon\left(\dot u(t)\right),\tag{79}$$

where

$$\sigma_\rho^R(t)=\mathcal{F}_\rho\varepsilon\left(u_\rho(t)\right)+\int_0^t\mathcal{G}\left(\sigma_\rho^R\left(s\right),\varepsilon\left(u_\rho\left(s\right)\right)\right)ds-\mathcal{M}\theta_\rho\left(t\right),\tag{80}$$

$$\sigma^R(t) = \mathcal{F}\varepsilon\left(u(t)\right) + \int_0^t \mathcal{G}\left(\sigma^R\left(s\right), \varepsilon\left(u\left(s\right)\right)\right) ds - \mathcal{M}\theta\left(t\right). \tag{81}$$

We combine (78) and (79) to obtain

$$\begin{aligned} &\left(\mathcal{A}\varepsilon\left(\dot{u_\rho}(t)\right) - \mathcal{A}\varepsilon\left(\dot{u}(t)\right), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} + \left(\sigma_\rho^R(t) - \sigma^R(t), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \\ &\quad \leqslant j_\rho\left(\dot{u}_\rho(t), \dot{u}(t)\right) - j_\rho\left(\dot{u}_\rho(t), \dot{u}_\rho(t)\right) + j\left(\dot{u}(t), \dot{u}_\rho(t)\right) - j\left(\dot{u}(t), \dot{u}(t)\right). \end{aligned} \tag{82}$$

Moreover, from (22), it follows that for a.e. $t \in [0,T]$,

$$\left(\mathcal{A}\varepsilon\left(\dot{u_\rho}(t)\right) - \mathcal{A}\varepsilon\left(\dot{u}(t)\right), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \geqslant m_{\mathcal{A}} \left\|\dot{u}_\rho(t) - \dot{u}(t)\right\|_V^2. \tag{83}$$

Using (80) and (81), we get

$$\begin{aligned} &\sigma_\rho^R(t) - \sigma^R(t) = \mathcal{F}_\rho\varepsilon\left(u_\rho(t)\right) - \mathcal{F}\varepsilon\left(u(t)\right) + \textstyle\int_0^t \mathcal{G}\left(\sigma_\rho^R\left(s\right), \varepsilon\left(u_\rho\left(s\right)\right)\right) ds \\ &- \textstyle\int_0^t \mathcal{G}\left(\sigma^R\left(s\right), \varepsilon\left(u\left(s\right)\right)\right) ds + \mathcal{M}\theta\left(t\right) - \mathcal{M}\theta_\rho\left(t\right). \end{aligned}$$

We now use (20), (23), (24), (29) and (73) to obtain

$$\begin{aligned} \left\|\sigma_\rho^R(t) - \sigma^R(t)\right\|_{\mathcal{H}} &\leqslant B(\rho) + L_{\mathcal{F}}\left\|u_\rho(t) - u(t)\right\|_V + L_{\mathcal{G}} \textstyle\int_0^t \left\|\sigma_\rho^R(s) - \sigma^R(s)\right\|_{\mathcal{H}} ds \\ &\quad + L_{\mathcal{G}} \textstyle\int_0^t \left\|u_\rho(s) - u(s)\right\|_V ds + \|\mathcal{M}\| \left\|\theta_\rho(t) - \theta(t)\right\|_{L^2(\Omega)}. \end{aligned}$$

By the Gronwall inequality, we find

$$\begin{aligned} &\left\|\sigma_\rho^R(t) - \sigma^R(t)\right\|_{\mathcal{H}} \leqslant B(\rho) + L_{\mathcal{F}}\left\|u_\rho(t) - u(t)\right\|_V \\ &+ L_{\mathcal{G}} \textstyle\int_0^t \left\|u_\rho(s) - u(s)\right\|_V ds + \|\mathcal{M}\| \left\|\theta_\rho(t) - \theta(t)\right\|_{L^2(\Omega)}. \end{aligned} \tag{84}$$

From (71) and (36), we obtain

$$\left\|\theta_\rho(t) - \theta(t)\right\|_{L^2(\Omega)}^2 \leqslant C \int_0^t \left\|\dot{u}_\rho(s) - \dot{u}(s)\right\|_V^2 ds. \tag{85}$$

The estimation (84) becomes

$$\left\|\sigma_\rho^R(t) - \sigma^R(t)\right\|_{\mathcal{H}} \leqslant B(\rho) + C\left(\int_0^t \left\|\dot{u}_\rho(s) - \dot{u}(s)\right\|_V ds + \left\|\theta_\rho(t) - \theta(t)\right\|_{L^2(\Omega)}\right). \tag{86}$$

We use (85), the inequality (86) shows that

$$\begin{aligned} &-\left(\sigma_\rho^R(t) - \sigma^R(t), \varepsilon(\dot{u}_\rho(t)) - \varepsilon\left(\dot{u}(t)\right)\right)_{\mathcal{H}} \\ &\leqslant \left(B(\rho) + C \textstyle\int_0^t \left\|\dot{u}_\rho(s) - \dot{u}(s)\right\|_V ds\right) \left\|\dot{u}_\rho(t) - \dot{u}(t)\right\|_V \qquad a.e \;\; t \in [0,T]. \end{aligned} \tag{87}$$

We use the definition of j and j_p, $(73)(a)$ and $(24)(b)$, we find

$$\begin{aligned} &j_\rho\left(\dot{u}_\rho(t), \dot{u}(t)\right) - j_\rho\left(\dot{u}_\rho(t), \dot{u}_\rho(t)\right) + j\left(\dot{u}(t), \dot{u}_\rho(t)\right) - j\left(\dot{u}(t), \dot{u}(t)\right) \\ &\leqslant \textstyle\int_{\Gamma_3} \left(p_\nu^\rho(\dot{u}_{\rho\nu}) - p_\nu(\dot{u}_\nu)\right)\left(\dot{u}_\nu - \dot{u}_{\rho\nu}\right) da + \int_{\Gamma_3} (\mu p_\nu^\rho(\dot{u}_{\rho\nu}) - \mu p_\nu(\dot{u}_\nu))(\|\dot{u}_\tau\| - \|\dot{u}_{\rho\tau}\|) da \\ &\leqslant \textstyle\int_{\Gamma_3} \left|p_\nu^\rho(\dot{u}_{\rho\nu}) - p_\nu(\dot{u}_\nu)\right| \left|\dot{u}_\nu - \dot{u}_{\rho\nu}\right| da + \int_{\Gamma_3} \left|\mu p_\nu^\rho(\dot{u}_{\rho\nu}) - \mu p_\nu(\dot{u}_\nu)\right| \left|\|\dot{u}_\tau\| - \|\dot{u}_{\rho\tau}\|\right| da. \end{aligned}$$

Then we use (74) and after some calculations, we get

$$\begin{aligned} &j_\rho\left(\dot{u}_\rho(t), \dot{u}(t)\right) - j_\rho\left(\dot{u}_\rho(t), \dot{u}_\rho(t)\right) + j\left(\dot{u}(t), \dot{u}_\rho(t)\right) - j\left(\dot{u}(t), \dot{u}(t)\right) \\ &\quad \leqslant \operatorname{meas}(\Gamma_3)^{\frac{1}{2}} c_0 (1 + \|\mu\|_{L^\infty(\Gamma_3)}) G_\nu(\rho) \left\|\dot{u}_\rho(t) - \dot{u}(t)\right\|_V \\ &\qquad + c_0^2 (1 + \|\mu\|_{L^\infty(\Gamma_3)}) L_\nu \left\|\dot{u}_\rho(t) - \dot{u}(t)\right\|_V^2. \end{aligned} \tag{88}$$

We use (82), (83), (87) and (88) to obtain

$$\|\dot{u}_\rho(t) - \dot{u}(t)\|_V \leqslant \tfrac{1}{m_{\mathcal{A}} - c_0^2(1+\|\mu\|_{L^\infty(\Gamma_3)})L_\nu} B(\rho)$$
$$+ \tfrac{C}{m_{\mathcal{A}} - c_0^2(1+\|\mu\|_{L^\infty(\Gamma_3)})L_\nu} \int_0^t \left\|\dot{u}_\rho(s) - \dot{u}(s)\right\|_V ds + \tfrac{\text{meas}(\Gamma_3)^{\frac{1}{2}} c_0 (1+\|\mu\|_{L^\infty(\Gamma_3)})}{m_{\mathcal{A}} - c_0^2(1+\|\mu\|_{L^\infty(\Gamma_3)})L_\nu} G_\nu(\rho),$$

this inequality implies that

$$\|\dot{u}_\rho(t) - \dot{u}(t)\|_V \leq \delta \left(B(\rho) + G_\nu(\rho)\right) + \frac{C}{m_{\mathcal{A}} - c_0^2(1+\|\mu\|_{L^\infty(\Gamma_3)})L_\nu} \int_0^t \left\|\dot{u}_\rho(s) - \dot{u}(s)\right\|_V ds,$$

where $\delta = \max\left\{\tfrac{1}{m_{\mathcal{A}} - c_0^2(1+\|\mu\|_{L^\infty(\Gamma_3)})L_\nu}, \tfrac{\text{meas}(\Gamma_3)^{\frac{1}{2}} c_0 (1+\|\mu\|_{L^\infty(\Gamma_3)})}{m_{\mathcal{A}} - c_0^2(1+\|\mu\|_{L^\infty(\Gamma_3)})L_\nu}\right\}$.

Using the Gronwall inequality, we find

$$\|\dot{u}_\rho(t) - \dot{u}(t)\|_V \leq c \left(B(\rho) + G_\nu(\rho)\right). \tag{89}$$

We integrate (89) over $(0, t)$, using (52), (37) and (72), we get

$$\|u_\rho - u\|_V \leqslant c \int_0^t \|\dot{u}_\rho(t) - \dot{u}(t)\|_V ds \leqslant c \left(B(\rho) + G_\nu(\rho)\right). \tag{90}$$

It results from (90), (73)(b) and (74)(b) that (75) is satisfied.

It follows from (79) that

$$\sigma_\rho(t) - \sigma = \sigma_\rho^R(t) - \sigma^R(t) + \mathcal{A}\varepsilon\left(\dot{u}_\rho(t)\right) - \mathcal{A}\varepsilon\left(\dot{u}(t)\right), \qquad a.e \ \ t \in [0, T].$$

We use this inequality, the properties (22) of the operator $\mathcal{A}$, (87), (73) and (75), we see that (76) is satisfied. We conclude that (77) is a consequence of (85), (90), (73)(b) and (74)(b).

6 Conclution

Contact problems involving bodies arise in many industrial processes as well as in everyday life. For this reason, they have been widely studied in the recent years, with various constitutive laws and boundary conditions, including the normal compliance condition associated to a version of Coulomb's friction law. The studies concern the mechanical, mathematical and numerical modeling of the corresponding boundary value problems. In this paper, we consider a mathematical model which describes a quasistatic frictional contact between a body and a foundation. We study an elasto-viscoplastic material with thermal effects. The frictional contact is modeled with a normal damped response condition associated to a version of Coulomb's law of dry friction. These non standard contact conditions could model the contact with the deformable foundation covered by a lubricant, say oil, as already mentioned. We derive a variational formulation of the problem and prove that the proposed model has a unique weak solution by using evolutionary quasivariational inequality. Also, we study the dependence of the solution on the data and prove a convergence result.

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