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A Frictional Contact Problem with Wear for Two Electro-Viscoelastic Bodies

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Abstract: We consider a mathematical problem for the quasistatic contact between two electro-viscoelastic bodies. The contact is modelled with a version of normal compliance and the evolution of the wear function is described by Archard's law. We derive a variational formulation for the model and prove an existence and uniqueness result of the weak solution. The proof is based on the arguments of evolutionary variational inequalities, a classical existence and uniqueness result for parabolic inequalities and the Banach fixed point theorem.

Keywords: *electro-viscoelastic; fixed point; friction contact; piezoelectric; wear.*

Mathematics Subject Classification (2010): 74H20, 74H25, 49J40, 74M15.

1 Introduction

A considerable progress has been achieved recently in applied mathematics and mechanics for dynamic and quasistatic problems, the recent advances in the formulation of these problems are articulated around two main components, one devoted to the laws of behavior and the other devoted to the boundary conditions imposed on the body. The boundary conditions reflect the binding of the body with the outside world. The laws of behavior are stipulated by the nature of the materials under study, The authors utilize composite laws of behavior that combine materials with varying thermal and mechanical characteristics. These materials are referred to as thermo-mechanical materials. Alternatively, the authors also consider materials with combined mechanical and electrical behavior, which are known as piezoelectric materials. For the boundary conditions, the authors investigate the real processes such as adhesion, friction and wear to describe new

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problems, these processes can be described by several types of the models with normal compliance or a normal damped response version.

The piezoelectric effect is characterized by the coupling between the mechanical and electrical behavior of the materials.

Materials undergoing piezoelectric effects are called piezoelectric materials; their study requires techniques and results from electromagnetic theory and continuum mechanics. However, there are very few mathematical results concerning contact problems involving piezoelectric materials and therefore, there is a need to extend the results to the models for contact with deformable bodies which include coupling between mechanical and electrical properties. General models for elastic materials with piezoelectric effects can be found in [4, 6, 12, 13, 15, 16]. A static frictional contact problem for electric-elastic materials was considered in [4, 15]. A frictional contact problem for electro-viscoelastic materials was studied in [13]. Contact problems with friction and adhesion for electro elastic-viscoplastic materials were studied recently in [1].

Wear is one of the processes which reduce the lifetime of modern machine elements. It represents the unwanted removal of materials from the surfaces of contacting bodies occurring in relative motion. Wear arises when a hard rough surface slides against a softer surface, digs into it, and its asperities plough a series of grooves. When two surfaces come into contact, rearrangement of the surface asperities takes place. When they are in relative motion, some of the peaks break, and therefore, the harder surface removes the softer material. This phenomenon involves the wear of the contacting surfaces. The material loss by the wearing solids, the generation and circulation of free wear debris are the main effects of the wear process. The loose particles form a thin layer on the body surface. Tribological experiments show that this layer has a great influence on contact phenomena and the wear particles between sliding surfaces affect the frictional behavior. Realistically, wear cannot be totally eliminated.

Wear is a major problem for materials when two bodies come into contact with friction and sliding, the contact surfaces are found worn-out, the more rigid one wears out the other. The particles lost by contact surfaces form a thin layer between the two bodies, this layer can improve the sliding, it may get one body enters in the other.

Generally, a mathematical theory of friction and wear should be a generalization of experimental facts and it must be in agreement with the laws of thermodynamics of irreversible processes. The first attempts of a thermodynamical description of the friction and wear processes were provided in [3]. A bilateral frictional problem with wear for multidisciplinary bodies and foundation was studied in [6, 8, 9]. General models of quasi-static frictional contact with wear between deformable bodies were derived in [18] from thermodynamic considerations.

The goal of this paper is to analyse the coupling of two electro-viscoelastic materials and a frictional contact problem with wear. We study a quasistatic problem of frictional contact with wear. We model the materials behavior by an electro-viscoelastic constitutive law and the contact is frictional.

The paper is organized as follows. In Section 2, we introduce the notation and give some preliminaries. In Section 3, we describe the mathematical models for the frictional contact problem between two electro-viscoelastic bodies. The contact is modelled with normal compliance and wear, we introduce the list the assumptions on the problem's data and the variational formulation of the model. In Section 4, we state our main existence and uniqueness result, Theorem 4.1. The proof of the theorem is based on the arguments of evolutionary variational inequalities, a classical existence and uniqueness result on parabolic inequalities, differential equations and the Banach fixed point theorem.

2 Notation and Preliminaries

In this short section, we present the notation we shall use and some preliminary material. For more details, we refer the reader to [5, 10, 17]. We denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while $\|\cdot\|$ represents the Euclidean norm if it is applied to a vector on $\mathbb{S}^d$ and $\mathbb{R}^d$, respectively. Let $\Omega^k \subset \mathbb{R}^d$ be a bounded domain with the outer Lipschitz boundary Γ and let ν denote the unit outer normal on $\partial\Omega^k = \Gamma^k$. We shall use the following notation.

We recall that the inner products and the corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\mathbf{u}^k.\mathbf{v}^k = u_i^k.v_i^k, \quad \|\mathbf{v}^k\| = (\mathbf{v}^k.\mathbf{v}^k)^{\frac{1}{2}}, \quad \forall \mathbf{u}^k, \mathbf{v}^k \in \mathbb{R}^d,$$
$$\boldsymbol{\sigma}^k.\boldsymbol{\tau}^k = \boldsymbol{\sigma}_{ij}^k.\tau_{ij}^k, \quad \|\boldsymbol{\tau}^k\| = (\boldsymbol{\tau}^k \cdot \boldsymbol{\tau}^k)^{\frac{1}{2}}, \quad \forall \boldsymbol{\sigma}^k, \boldsymbol{\tau}^k \in \mathbb{S}^d.$$

Here and below, the indices i and j run between 1 and d and the summation convention over repeated indices is adopted. Now, to proceed with the variational formulation, we need the following function spaces:

$$H^k = \{\mathbf{v}^k = (v_i^k); v_i^k \in L^2(\Omega^k)\}, \quad H_1^k = \{\mathbf{v}^k = (v_i^k); v_i^k \in H^1(\Omega^k)\},$$
$$Q^k = \{\boldsymbol{\tau}^k = (\tau_{ij}^k); \tau_{ij}^k = \tau_{ji}^k \in L^2(\Omega^k)\}, \quad Q_1^k = \{\boldsymbol{\tau}^k = (\tau_{ij}^k) \in Q^k; div\ \boldsymbol{\tau}^k \in H^k\}.$$

The spaces H^k, H_1^k, Q^k and Q_1^k are the real Hilbert spaces endowed with the canonical inner products given by

$$(\mathbf{u}^k, \mathbf{v}^k)_{H^k} = \int_{\Omega^k} \mathbf{u}^k.\mathbf{v}^k dx, \quad (\mathbf{u}^k, \mathbf{v}^k)_{H_1^k} = \int_{\Omega^k} \mathbf{u}^k.\mathbf{v}^k dx + \int_{\Omega^k} \nabla\mathbf{u}^k.\nabla\mathbf{v}^k dx,$$
$$(\boldsymbol{\sigma}^k, \boldsymbol{\tau}^k)_{\mathbb{Q}^k} = \int_{\Omega^k} \boldsymbol{\sigma}^k.\boldsymbol{\tau}^k dx, \quad (\boldsymbol{\sigma}^k, \boldsymbol{\tau}^k)_{\mathbb{Q}_1^k} = \int_{\Omega^k} \boldsymbol{\sigma}^k.\boldsymbol{\tau}^k dx + \int_{\Omega^k} div\ \boldsymbol{\sigma}^k.Div\ \boldsymbol{\tau}^k dx$$

and the associated norms $\|\cdot\|_{H^k}$, $\|\cdot\|_{H_1^k}$, $\|\cdot\|_{Q^k}$, and $\|\cdot\|_{Q_1^k}$, respectively. Here and below we use the notation

$$\nabla\mathbf{u}^k = (u_{i,j}^k), \quad \varepsilon(\mathbf{u}^k) = (\varepsilon_{ij}(\mathbf{u}^k)), \quad \varepsilon_{ij}(\mathbf{u}^k) = \frac{1}{2}(u_{i,j}^k + u_{j,i}^k), \quad \forall u^k \in H_1^k,$$
$$Div\ \boldsymbol{\sigma}^k = (\boldsymbol{\sigma}_{ij,j}^k), \quad \forall \boldsymbol{\sigma}^k \in Q_1^k.$$

For every element $\mathbf{v}^k \in H_1^k$, we also use the notation $\mathbf{v}^k$ for the trace of $\mathbf{v}^k$ on Γ^k and we denote by v_ν^k and $\mathbf{v}_\tau^k$ the *normal* and the *tangential* components of $\mathbf{v}^k$ on the boundary Γ^k given by

$$v_\nu^k = \mathbf{v}^k.\nu^k, \quad \mathbf{v}_\tau^k = \mathbf{v}^k - v_\nu^k \boldsymbol{\nu}^k.$$

Let H'_{Γ^k} be the dual of $H_{\Gamma^k} = H^{\frac{1}{2}}(\Gamma^k)^d$ and let $(\cdot,\cdot)_{-\frac{1}{2},\frac{1}{2},\Gamma^k}$ denote the duality pairing between H'_{Γ^k} and H_{Γ^k}. For every element $\boldsymbol{\sigma}^k \in Q_1^k$, let $\boldsymbol{\sigma}^k\boldsymbol{\nu}^k$ be the element of H'_{Γ^k} given by

$$(\boldsymbol{\sigma}^k\boldsymbol{\nu}^k, \mathbf{v}^k)_{-\frac{1}{2},\frac{1}{2},\Gamma^k} = (\boldsymbol{\sigma}^k, \varepsilon(\mathbf{v}^k))_{Q^k} + (Div\ \boldsymbol{\sigma}^k, \mathbf{v}^k)_{H^k}, \quad \forall \mathbf{v}^k \in H_1^k.$$

Denote by $\sigma_\nu^k \sigma_\nu^k$ and $\boldsymbol{\sigma}_\tau^k$ the *normal* and the *tangential* traces of $\boldsymbol{\sigma}^k \in Q_1^k$, respectively. If $\boldsymbol{\sigma}^k$ is continuously differentiable on $\Omega^k \cup \Gamma^k$, then

$$\sigma_\nu^k = (\boldsymbol{\sigma}^k \boldsymbol{\nu}^k) \cdot \boldsymbol{\nu}^k, \quad \boldsymbol{\sigma}_\tau^k = \boldsymbol{\sigma}^k \boldsymbol{\nu}^k - \sigma_\nu^k \boldsymbol{\nu}^k,$$
$$(\boldsymbol{\sigma}^k \boldsymbol{\nu}^k, \mathbf{v}^k)_{-\frac{1}{2},\frac{1}{2},\Gamma^k} = \int_{\Gamma^k} \boldsymbol{\sigma}^k \boldsymbol{\nu}^k \cdot \mathbf{v}^k da$$

for all $\mathbf{v}^k \in H_1^k$, where da is the surface measure element.

For the displacement field, we need the closed subspace of H_1^k defined by

$$V^k = \{\mathbf{v}^k \in H_1^k; \mathbf{v}^k = 0 \text{ on } \Gamma_1^k\}.$$

Since *meas* $\Gamma_1^k > 0$, the following Korn's inequality holds:

$$\|\varepsilon(\mathbf{v}^k)\|_{Q^k} \geqslant c_K \|\mathbf{v}^k\|_{H_1^k}, \quad \forall \mathbf{v}^k \in V^k, \tag{1}$$

where the constant c_K denotes a positive constant which may depend only on Ω^k, Γ_1^k (see [17]).

Over the space V^k, we consider the inner product given by

$$(\mathbf{u}^k, \mathbf{v}^k)_{V^k} = (\varepsilon(\mathbf{u}^k), \varepsilon(\mathbf{v}^k))_{Q^k}, \quad \forall \mathbf{u}^k, \mathbf{v}^k \in V^k, \tag{2}$$

and let $\|\cdot\|_{V^k}$ be the associated norm. It follows from Korn's inequality (1) that the norms $\|\cdot\|_{H_1^k}$ and $\|\cdot\|_{V^k}$ are equivalent on V^k. Then $(V^k, \|\cdot\|_{V^k})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem and (2), there exists a constant $c_0 > 0$ depending only on Ω^k, Γ_1^k and Γ_3 such that

$$\|\mathbf{v}^k\|_{L^2(\Gamma_3)^d} \leqslant c_0 \|\mathbf{v}^k\|_{V^k}, \quad \forall \mathbf{v}^k \in V^k. \tag{3}$$

We also introduce the spaces

$$W^k = \{\psi^k \in E_1^k; \psi^k = 0 \text{ on } \Gamma_a^k\},$$
$$W_1^k = \{\boldsymbol{D}^k = (D_i^k); D_i^k \in L^2(\Omega^k), div\ \boldsymbol{D}^k \in L^2(\Omega^k)\}.$$

Since *meas* $\Gamma_a^k > 0$, the following Friedrichs-Poincaré inequality holds:

$$\|\nabla \psi^k\|_{W^k} \geqslant c_F \|\psi^k\|_{H^1(\Omega^k)}, \quad \forall \psi^k \in W^k, \tag{4}$$

where $c_F > 0$ is a constant which depends only on Ω^k, Γ_a^k. In the space W^k, we consider the inner product

$$(\varphi^k, \psi^k)_{W^k} = \int_{\Omega^k} \nabla \varphi^k \cdot \nabla \psi^k dx, \tag{5}$$

and let $\|\cdot\|_{W^k}$ be the associated norm. It follows from (4) that $\|\cdot\|_{H^1(\Omega^k)}$ and $\|\cdot\|_{W^k}$ are equivalent norms on W^k and therefore $(W^k, \|\cdot\|_{W^k})$ is a real Hilbert space. Moreover, by the Sobolev trace theorem, there exists a constant $\mathbf{c}_0$ depending only on Ω^k, Γ_a^k and Γ_3 such that

$$\|\zeta^k\|_{L^2(\Gamma_3)} \leqslant c_0 \|\zeta^k\|_{W^k}, \quad \forall \zeta^k \in W^k. \tag{6}$$

The space W_1^k is a real Hilbert space with the inner product

$$(\boldsymbol{D}^k, \boldsymbol{\Phi}^k)_{W_1^k} = \int_{\Omega^k} \boldsymbol{D}^k \cdot \boldsymbol{\Phi}^k dx + \int_{\Omega^k} div\ \boldsymbol{D}^k \cdot div\ \boldsymbol{\Phi}^k dx,$$

where $div\ \boldsymbol{D}^k = (\boldsymbol{D}^k_{i,i})$, and the associated norm $\|\cdot\|_{W_1^k}$.

To simplify the notation, we define the product spaces

$$\mathbb{V} = V^1 \times V^2, \mathbb{H} = H^1 \times H^2, \quad \mathbb{H}_1 = H_1^1 \times H_1^2,$$
$$\mathbb{Q} = Q^1 \times Q^2, \quad \mathbb{Q}_1 = Q_1^1 \times Q_1^2, \quad \mathbb{W} = W^1 \times W^2, \mathbb{W}_1 = W_1^1 \times W_1^2.$$

The spaces $\mathbb{V}$, $\mathbb{W}$ and $\mathbb{W}_1$ are the real Hilbert spaces endowed with the canonical inner products denoted by $(\cdot,\cdot)_{\mathbb{V}}$, $(\cdot,\cdot)_{\mathbb{W}}$ and $(\cdot,\cdot)_{\mathbb{W}_1}$. The associate norms will be denoted by $\|\cdot\|_{\mathbb{V}}$, $\|\cdot\|_{\mathbb{W}}$ and $\|\cdot\|_{\mathbb{W}_1}$, respectively.

Finally, for any real Hilbert space X, we use the classical notation for the spaces $L^p(0,T;X)$, $W^{k,p}(0,T;X)$, where $1 \le p \le \infty$, $k \ge 1$. We denote by $\mathcal{C}(0,T;X)$ and $\mathcal{C}^1(0,T;X)$ the space of continuous and continuously differentiable functions from $[0,T]$ to X, respectively, with the norms

$$\|f\|_{\mathcal{C}(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X,$$
$$\|f\|_{\mathcal{C}^1(0,T;X)} = \max_{t\in[0,T]} \|f(t)\|_X + \max_{t\in[0,T]} \|\dot{f}(t)\|_X.$$

3 The Model and Variational Problem

Let us consider two electro-viscoelastic bodies occupying two bounded domains Ω^1, Ω^2 of the space $\mathbb{R}^d (d = 2,3)$. For each domain Ω^k, the boundary Γ^k is assumed to be Lipschitz continuous, and is partitioned into three disjoint measurable parts Γ_1^k, Γ_2^k and Γ_3^k on one hand, and into two measurable parts Γ_a^k and Γ_b^k on the other hand, such that $meas\ \Gamma_1^k > 0$, $meas\ \Gamma_a^k > 0$. Let $T > 0$ and let $[0,T]$ be the time interval of interest. The body Ω^k is subjected to $\boldsymbol{f}_0^k$ forces and volume electric charges of density q_0^k. The bodies are assumed to be clamped on $\Gamma_1^k \times [0,T]$. The surface tractions $\boldsymbol{f}_2^k$ act on $\Gamma_2^k \times [0,T]$. We also assume that the electrical potential vanishes on $\Gamma_a^k \times [0,T]$ and a surface electric charge of density q_2^k is prescribed on $\Gamma_b^k \times [0,T]$. The two bodies can enter in contact along the common part $\Gamma_3^1 = \Gamma_3^2 = \Gamma_3$, the bodies are in contact with wear.

We denote by $\mathbf{u}^k$ the displacement field, by $\boldsymbol{\sigma}^k$ the stress tensor field and by $\boldsymbol{\varepsilon}(\mathbf{u}^k)$ the linearized strain tensor. We use an electro-viscoelastic constitutive law given by

$$\boldsymbol{\sigma}^k(t) = \mathcal{A}^k \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^k(t)) + \mathcal{G}^k \boldsymbol{\varepsilon}(\mathbf{u}^k(t)) + (\mathcal{E}^k)^* \nabla \varphi^k(t). \tag{7}$$

Here $\mathcal{A}^k$ is a given nonlinear operator, $\mathcal{G}^k$ represents the elasticity operator. $E(\varphi^k) = -\nabla\varphi^k$ is the electric field, $\mathcal{E}^k$ represents the third order piezoelectric tensor, $(\mathcal{E}^k)^*$ is its transposition. In (7) and everywhere in this paper, the dot above a variable represents the derivative with respect to the time variable t.

We now briefly describe the boundary conditions on the contact surface Γ_3 based on the model derived in [18]. We introduce the wear function $w : \Gamma_3 \times [0,T] \to \mathbb{R}^+$ which measures the wear of the surface.

The wear is identified as the normal depth of the material that is lost. Let g be the initial gap between the two bodies and let p_ν and p_τ denote the normal and tangential compliance functions. We denote by $\mathbf{v}^*$ and $\alpha^* = \|\mathbf{v}^*\|$ the tangential velocity and the tangential speed of the contact surface, respectively. We use the modified version of Archard's law $\dot{w} = -k_w \alpha^* \sigma_\nu$ to describe the evolution of wear, where $k_w > 0$ is a wear coefficient. We introduce the unitary vector $\delta : \Gamma_3 \to \mathbb{R}^d$ defined by $\delta = \mathbf{v}^* / \|\mathbf{v}^*\|$. In the reference configuration, there is a gap between Γ_3 of the two bodies, measured along the direction of ν, denoted by g. When the contact occurs, some material of the contact surface is worn out and immediately removed from the system. This process is measured by the wear function w.

Then, the classical formulation of the mechanical problem of a frictional contact with wear between two electro-viscoelastics bodies may be stated as follows.

Problem $\mathcal{P}$

For $k = 1, 2$, find a displacement field $\mathbf{u}^k : \Omega^k \times [0,T] \to \mathbb{R}^d$, a stress field $\boldsymbol{\sigma}^k : \Omega^k \times [0,T] \to \mathbb{S}^d$, an electric potential $\varphi^k : \Omega^k \times [0,T] \to \mathbb{R}$, a wear function $w : \Gamma_3 \times [0,T] \to \mathbb{R}_+$ and an electric displacement field $\boldsymbol{D}^k : \Omega^k \times [0,T] \to \mathbb{R}^d$ such that

$$\boldsymbol{\sigma}^k = \mathcal{A}^k \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^k) + \mathcal{G}^k \boldsymbol{\varepsilon}(\mathbf{u}^k) + (\mathcal{E}^k)^* \nabla \varphi^k, \text{in } \Omega^k \times [0,T], \tag{8}$$

$$\boldsymbol{D}^k = \mathcal{E}^k \varepsilon(\mathbf{u}^k) - \mathcal{B}^k \nabla \varphi^k \quad \text{in } \Omega^k \times [0,T], \tag{9}$$

$$Div\, \boldsymbol{\sigma}^k + \boldsymbol{f}_0^k = 0 \quad \text{in } \Omega^k \times [0,T], \tag{10}$$

$$div\, \boldsymbol{D}^k - q_0^k = 0 \quad \text{in } \Omega^k \times [0,T], \tag{11}$$

$$\mathbf{u}^k = 0 \quad \text{on } \Gamma_1^k \times [0,T], \tag{12}$$

$$\boldsymbol{\sigma}^k \boldsymbol{\nu}^k = \boldsymbol{f}_2^k \quad \text{on } \Gamma_2^k \times [0,T], \tag{13}$$

$$\left.\begin{aligned} \sigma_\nu^1 = \sigma_\nu^2 \equiv \sigma_\nu, \\ \sigma_\nu = p_\nu (u_\nu - w - g), \end{aligned}\right\} \quad \text{on } \Gamma_3 \times [0,T], \tag{14}$$

$$\left.\begin{aligned} \boldsymbol{\sigma}_\tau^1 = -\boldsymbol{\sigma}_\tau^2 \equiv \boldsymbol{\sigma}_\tau, \\ \boldsymbol{\sigma}_\tau = -p_\tau (u_\nu - w - g) \frac{\mathbf{v}^*}{\|\mathbf{v}^*\|}, \end{aligned}\right\} \quad \text{on } \Gamma_3 \times [0,T], \tag{15}$$

$$u_\nu^1 + u_\nu^2 = 0, \qquad \text{on } \Gamma_3 \times [0,T], \tag{16}$$

$$\dot{w} = -k_w \alpha^* \sigma_\nu = k_w \alpha^* p_\nu (u_\nu - w - g), \quad \text{on } \Gamma_3 \times [0,T], \tag{17}$$

$$\varphi^k = 0 \quad \text{on } \Gamma_a^k \times [0,T], \tag{18}$$

$$\boldsymbol{D}^k \cdot \boldsymbol{\nu}^k = q_2^k \quad \text{on } \Gamma_b^k \times [0,T], \tag{19}$$

$$\mathbf{u}^k(0) = \mathbf{u}_0^k, \qquad \text{in } \Omega^k, \tag{20}$$

$$w(0) = w_0 \quad \text{on } \Gamma_3. \tag{21}$$

First, equations (8) and (9) represent the electro-viscoelastic constitutive law. Equations (10) and (11) are the equilibrium equations for the stress and electric-displacement fields, respectively, in which "*Div*" and "*div*" denote the divergence operator for the tensor and vector valued functions, respectively. Next, the equations (12) and (13) represent the displacement and traction boundary condition, respectively. Conditions (14), (15) represent the frictional contact with the wear described above. Equation (16) means that the two bodies are inseparable.

Next, the equation (17) represents the ordinary differential equation which describes the evolution of the wear function. Equations (18) and (19) represent the electric boundary conditions. (20) represents the initial displacement field. Finally, (21) represents the initial condition in which w_0 is the given initial wear field.

We now list the assumptions on the problem's data.

The *viscosity function* $\mathcal{A}^k : \Omega^k \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following conditions:

$$\left\{\begin{array}{l} \text{(a) There exists } L_{\mathcal{A}^k} > 0 \text{ such that} \\ \quad \|\mathcal{A}^k(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}^k(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathcal{A}^k}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \text{ for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, \text{a.e. } \boldsymbol{x} \in \Omega^k. \\ \text{(b) There exists } m_{\mathcal{A}^k} > 0 \text{ such that} \\ \quad (\mathcal{A}^k(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{A}^k(\boldsymbol{x},\boldsymbol{\varepsilon}_2)) \cdot (\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2) \geqslant m_{\mathcal{A}^k}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|^2 \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d, a.e.\ \boldsymbol{x} \in \Omega^k. \\ \text{(c) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^k(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega^k \\ \qquad \text{for any } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(d) The mapping } \boldsymbol{x} \mapsto \mathcal{A}^k(\boldsymbol{x},\mathbf{0}) \text{ belongs to } \mathbb{Q}. \end{array}\right. \tag{22}$$

The *elasticity operator* $\mathcal{G}^k : \Omega^k \times \mathbb{S}^d \to \mathbb{S}^d$ satisfies the following conditions:

$$\left\{\begin{array}{l} \text{(a) There exists a constant } L_{\mathcal{G}^k} > 0 \text{ such that} \\ \quad \|\mathcal{G}^k(\boldsymbol{x},\boldsymbol{\varepsilon}_1) - \mathcal{G}^k(\boldsymbol{x},\boldsymbol{\varepsilon}_2)\| \leqslant L_{\mathcal{G}^k}\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\| \\ \quad \text{for all } \boldsymbol{\varepsilon}_1,\boldsymbol{\varepsilon}_2 \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega. \\ \text{(b) The mapping } \boldsymbol{x} \to \mathcal{G}^k(\boldsymbol{x},\boldsymbol{\varepsilon}) \text{ is Lebesgue measurable on } \Omega^k \\ \quad \text{for all } \boldsymbol{\varepsilon} \in \mathbb{S}^d. \\ \text{(c) The mapping } \boldsymbol{x} \to \mathcal{G}^k(\boldsymbol{x},0) \in \mathbb{Q}. \end{array}\right. \tag{23}$$

The *piezoelectric tensor* $\mathcal{E}^k : \Omega^k \times \mathbb{S}^d \to \mathbb{R}^d$ satisfies the following conditions:

$$\left\{\begin{array}{l} \text{(a) } \mathcal{E}^k(\boldsymbol{x},\tau) = (e^k_{ijk}(\boldsymbol{x})\tau_{jk}) \text{ for all } \tau = (\tau_{ij}) \in \mathbb{S}^d \text{ a.e. } \boldsymbol{x} \in \Omega^k. \\ \text{(b) } e^k_{ijk} = e^k_{ikj} \in L^\infty(\Omega^k), 1 \leqslant i,j,k \leqslant d. \end{array}\right. \tag{24}$$

Recall also that the transposed operator $(\mathcal{E}^k)^*$ is given by $(\mathcal{E}^k)^* = (e^{k,*}_{ijk})$, where $e^{k,*}_{ijk} = e^k_{kij}$ and the following equality holds:

$$\mathcal{E}^k\sigma.\mathbf{v} = \sigma \cdot (\mathcal{E}^k)^*\mathbf{v} \quad \forall\sigma \in \mathbb{S}^d,\ \forall\mathbf{v} \in \mathbb{R}^d.$$

The *electric permittivity operator* $\mathcal{B}^k = (b^k_{ij}) : \Omega^k \times \mathbb{R}^d \to \mathbb{R}^d$ satisfies the following conditions:

$$\left\{\begin{array}{l} \text{(a) } \mathcal{B}^k(\boldsymbol{x},\mathbf{E}) = (b^k_{ij}(\boldsymbol{x})E_j) \text{ for all } \mathbf{E} = (E_i) \in \mathbb{R}^d,\ a.e.\ \boldsymbol{x} \in \Omega^k. \\ \text{(b) } b^k_{ij} = b^k_{ji}, b^k_{ij} \in L^\infty(\Omega^k), 1 \leqslant i,j \leqslant d. \\ \text{(c) There exists } m_{\mathcal{B}^k} > 0, \text{ such that } \mathcal{B}^k\mathbf{E}\cdot\mathbf{E} \geqslant m_{\mathcal{B}^k}|\mathbf{E}|^2 \text{ for all } \mathbf{E} = (E_i) \in \mathbb{R}^d, \\ \qquad \text{a.e. } \boldsymbol{x} \in \Omega^k. \end{array}\right. \tag{25}$$

The *normal compliance function* $p_\nu : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies the following conditions:

$$\left\{\begin{array}{l} \text{(a) There exists } \mathcal{L}_\nu > 0 \text{ such that} \\ \quad |p_\nu(\boldsymbol{x},r_1) - p_\nu(\boldsymbol{x},r_2)| \leqslant \mathcal{L}_\nu|r_1 - r_2| \text{ for all } r_1,r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto p_\nu(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}. \\ \text{(c) } p_\nu(\boldsymbol{x},r) = 0 \text{ for all } r \leqslant 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{26}$$

The *tangential contact function* $p_\tau : \Gamma_3 \times \mathbb{R} \to \mathbb{R}_+$ satisfies the following conditions:

$$\left\{\begin{array}{l} \text{(a) There exists } \mathcal{L}_\tau > 0 \text{ such that} \\ \quad |p_\tau(\boldsymbol{x},r_1) - p_\tau(\boldsymbol{x},r_2)| \leqslant \mathcal{L}_\tau|r_1 - r_2| \text{ for all } r_1,r_2 \in \mathbb{R},\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \\ \text{(b) The mapping } \boldsymbol{x} \mapsto p_\tau(\boldsymbol{x},r) \text{ is measurable on } \Gamma_3 \text{ for all } r \in \mathbb{R}. \\ \text{(c) } p_\tau(\boldsymbol{x},r) = 0 \text{ for all } r \leqslant 0,\ \text{a.e. } \boldsymbol{x} \in \Gamma_3. \end{array}\right. \tag{27}$$

We also suppose the following regularities:

$$\begin{aligned} \mathbf{f}_0^k &\in C(0,T;L^2(\Omega^k)^d), \quad \mathbf{f}_2^k \in C(0,T;L^2(\Gamma_2^k)^d), \\ q_0^k &\in C(0,T;L^2(\Omega^k)), \quad q_2^k \in C(0,T;L^2(\Gamma_b^k)), \end{aligned} \tag{28}$$

$$\mathbf{u}_0^k \in \boldsymbol{V}^k, \tag{29}$$

$$w_0 \in L^2(\Gamma_3), \tag{30}$$

$$p_\nu\,(.,r) \in L^2(\Gamma_3), p_\tau\,(.,r) \in L^2(\Gamma_3), \forall r \in \mathbb{R} \tag{31}$$

$$g \in L^2(\Gamma_3), g \geqslant 0 \text{ a.e on } \Gamma_3. \tag{32}$$

Using the Riesz representation theorem, we define the linear mappings $\mathbf{f} = (\ \mathbf{f}^1,\ \mathbf{f}^2) : [0,T] \to \mathbb{V}$ and $q = (q^1, q^2) : [0,T] \to \mathbb{W}$ as follows:

$$(\mathbf{f}(t), \mathbf{v})_{\mathbb{V}} = \sum_{k=1}^{2} \int_{\Omega^k} \mathbf{f}_0^k(t) \cdot \mathbf{v}^k \, dx + \sum_{k=1}^{2} \int_{\Gamma_2^k} \mathbf{f}_2^k(t) \cdot \mathbf{v}^k \, da \quad \forall \mathbf{v} \in \mathbb{V}, \tag{33}$$

$$(q(t), \zeta)_{\mathbb{W}} = \sum_{k=1}^{2} \int_{\Omega^k} q_0^k(t) \zeta^k \, dx - \sum_{k=1}^{2} \int_{\Gamma_b^k} q_2^k(t) \zeta^k \, da \quad \forall \zeta \in \mathbb{W}. \tag{34}$$

The use of (33) permits to verify that

$$\mathbf{f} \in \mathcal{C}(0,T;\mathbb{V}). \tag{35}$$

Next, we define the mappings $j : L^2(\Gamma_3) \times \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ by

$$\begin{aligned} j(w, \mathbf{u}, \mathbf{v}) = \int_{\Gamma_3} \left(p_\nu \left(u_\nu - w - g\right) v_\nu\right) \, da + &\int_{\Gamma_3} p_\tau \left(u_\nu - w - g\right) .\delta.\mathbf{v}_\tau da, \\ &\text{for all } \mathbf{u}, \mathbf{v} \in V, w \in L^2\left(\Gamma_3\right). \end{aligned} \tag{36}$$

Now, we give the following variational formulation of the mechanical problem (8)–(21).

Problem $\mathcal{PV}$

Find a displacement field $\mathbf{u} = (\mathbf{u}^1, \mathbf{u}^2) : [0,T] \to \mathbb{V}$, a stress field $\boldsymbol{\sigma} = (\boldsymbol{\sigma}^1, \boldsymbol{\sigma}^2) : [0,T] \to \mathbb{Q}$, an electric potential $\varphi = (\varphi^1, \varphi^2) : [0,T] \to \mathbb{W}$, a wear $w : [0,T] \to L^2(\Gamma_3)$ and an electric displacement field $\boldsymbol{D} = (\boldsymbol{D}^1, \boldsymbol{D}^2) : [0,T] \to \mathbb{W}_1$ such that

$$\boldsymbol{\sigma}^k = \mathcal{A}^k \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^k) + \mathcal{G}^k \boldsymbol{\varepsilon}(\mathbf{u}^k) + (\mathcal{E}^k)^* \nabla \varphi^k \ \text{ in } \ \Omega^k \times [0,T], \tag{37}$$

$$\boldsymbol{D}^k = \mathcal{E}^k \varepsilon(\mathbf{u}^k) - \mathcal{B}^k \nabla \varphi^k \quad \text{in } \Omega^k \times [0,T], \tag{38}$$

$$\begin{aligned} &\sum_{k=1}^{2} (\boldsymbol{\sigma}^k, \varepsilon(\mathbf{v}^k))_{Q^k} + j(w(t), \mathbf{u}(t), \mathbf{v}) = (\mathbf{f}(t), \mathbf{v})_{\mathbb{V}} \\ &\qquad\qquad \forall \mathbf{v} \in \mathbb{V}, \text{a.e. } t \in (0,T), \end{aligned} \tag{39}$$

$$\begin{aligned} &\sum_{k=1}^{2} (\mathcal{B}^k \nabla \varphi^k(t), \nabla \phi^k)_{H^k} - \sum_{k=1}^{2} (\mathcal{E}^k \varepsilon(\mathbf{u}^k(t)), \nabla \phi^k)_{H^k} = (q(t), \phi)_{\mathbb{W}}, \\ &\qquad\qquad \forall \phi \in \mathbb{W}, \ \text{a.e. } t \in (0,T), \end{aligned} \tag{40}$$

$$\dot{w} = k_w \alpha^* p_\nu \left(u_\nu - w - g\right), \quad \text{a.e. } (0,T), \tag{41}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad w(0) = w_0. \tag{42}$$

We notice that the variational Problem $\mathcal{PV}$ is formulated in terms of a displacement field, a stress field, an electrical potential, a wear and an electric displacement field. The existence of the unique solution of Problem $\mathcal{PV}$ is stated and proved in the next section.

4 Existence and Uniqueness of a Solution

Our main existence and uniqueness result is the following.

Theorem 4.1 *Assume that* (22)–(32) *hold and also assume the smallness assumption:*

$$(\mathcal{L}_\nu + \mathcal{L}_\tau) < \alpha_0, \tag{43}$$

where $\alpha_0 = \frac{m_{\mathcal{A}^k}}{c_0^2}$ *such that* $m_{\mathcal{A}^k}$ *is defined in* (22) *and* c_0 *is defined in* (3). *Then there exists a unique solution of Problem* $\mathcal{PV}$. *Moreover, the solution satisfies the following conditions*

$$\mathbf{u} \in \mathcal{C}^1(0,T;\mathbb{V}), \tag{44}$$

$$\boldsymbol{\sigma} \in \mathcal{C}(0,T;\mathbb{Q}_1), \tag{45}$$

$$w \in \mathcal{C}^1(0,T;L^2(\Gamma_3)), \tag{46}$$

$$\varphi \in \mathcal{C}(0,T;\mathbb{W}), \tag{47}$$

$$\boldsymbol{D} \in \mathcal{C}(0,T;\mathbb{W}_1). \tag{48}$$

Then $\{\mathbf{u},\ \boldsymbol{\sigma}, w, \varphi, \boldsymbol{D}\}$ which satisfy (37)-(42) are called a weak solution of the contact Problem $\mathcal{P}$. We conclude that, under the assumptions (22)–(32), the mechanical problem (8)–(21) has a unique weak solution satisfying (44)–(48).

The proof of Theorem (4.1) is carried out in several steps and is based on the following abstract result for evolutionary variational inequalities.

We turn now to the proof of Theorem (4.1) which will be carried out in several steps and is based on the arguments of nonlinear equations with monotone operators, a classical existence and uniqueness result on parabolic inequalities and fixed-point arguments. To this end, we assume in what follows that (22)–(32) hold, and we consider that C is a generic positive constant which depends on Ω^k, Γ_1^k, Γ_1^k, Γ_3, p_ν,p_τ, $\mathcal{A}^k$, $\mathcal{G}^k$, $\mathcal{E}^k$ but does not depend on t or the rest of input data, and whose value may change from place to place.

First step.
Let $\eta = (\eta^1, \eta^2) \in \mathcal{C}(0,T;\mathbb{V})$.
We consider the following variational problem.

Problem $\mathcal{PV}_\eta^u$.

Find a displacement field $\mathbf{u}_\eta = (\mathbf{u}_\eta^1, \mathbf{u}_\eta^2) : [0,T] \to \mathbb{V}$ such that

$$\sum_{k=1}^{2} (\mathcal{A}^k \boldsymbol{\varepsilon}(\dot{\mathbf{u}}^k), \varepsilon(\mathbf{v}^k))_{Q^k} + (\eta(t), \mathbf{v})_{\mathbb{V}} = (\mathbf{f}(t), \mathbf{v})_{\mathbb{V}}, \tag{49}$$

$$\mathbf{u}_\eta\,(0) = \mathbf{u}_0 \tag{50}$$

for all $\mathbf{v} \in \mathbb{V}$ a.e $t \in (0, T)$.

We have the following result for $\mathcal{PV}_\eta^u$.

Lemma 4.1 *There exists a unique solution* $\mathbf{u}_\eta = (\mathbf{u}_\eta^1, \mathbf{u}_\eta^2) \in \mathcal{C}^1(0, T; \mathbb{V})$ *to the problem* (49) *and* (50).

Proof. Let $A : \mathbb{V} \to \mathbb{V}$ be a semi-continuous and monotone operator which satisfies the condition

$$(A\mathbf{u}, \mathbf{v})_{\mathbb{V}\times\mathbb{V}} = \sum_{k=1}^{2} (\mathcal{A}^k \boldsymbol{\varepsilon}(\mathbf{u}^k), \boldsymbol{\varepsilon}(\mathbf{v}^k))_{Q^k}. \tag{51}$$

It follows from hypothesis (22) that

$$\|A\mathbf{u} - A\mathbf{v}\|_{\mathbb{V}} \leqslant L_{\mathcal{A}^K} \|\mathbf{u} - \mathbf{v}\|_{\mathbb{V}} \quad \forall \mathbf{u}, \mathbf{v} \in \mathbb{V}.$$

This proves that A is bounded and semi-continuous on $\mathbb{V}$.

On the other hand, by (22) and Korn's inequality, we find, for every $\mathbf{v} \in \mathbb{V}$,

$$\frac{(A\mathbf{v}, \mathbf{v})_{\mathbb{V}\times\mathbb{V}}}{\|\mathbf{v}\|_{\mathbb{V}}} \geqslant c_0^2 m_{\mathcal{A}^k} \|\mathbf{v}\|_{\mathbb{V}} \ .$$

The passage to the limit in this inequality when $\|\mathbf{v}\|_{\mathbb{V}} \to +\infty$ implies that A is coercive in $\mathbb{V}$

Next, by the definition of A, the use of (22) and Korn's inequality permits also to obtain

$$(A\mathbf{u} - A\mathbf{v}, \mathbf{u} - \mathbf{v})_{\mathbb{V}\times\mathbb{V}} > c_0^2 m_{\mathcal{A}^K} \|\mathbf{u} - \mathbf{v}\|_{\mathbb{V}} \quad \text{if } \mathbf{u} \neq \mathbf{v}.$$

Then A is strict monotone. Therefore, we put

$$\mathbf{f}_\eta(t) = \mathbf{f}(t) - \eta(t) \in \mathcal{C}(0, T; \mathbb{V}).$$

From (33) and the condition $\eta \in \mathcal{C}(0, T; \mathbb{V})$, we have $\mathbf{f}_\eta \in \mathcal{C}(0, T; \mathbb{V})$. Then, from the Cauchy-Lipschitz theorem, there exists a unique function $\mathbf{v}_\eta$ satisfying the relations

$$A\mathbf{v}_\eta(t) = \mathbf{f}_\eta(t) \text{ a.e } t \in (0, T)\,,$$

$$\mathbf{u}_\eta = \int_0^t \mathbf{v}_\eta\,(s)\,ds + \mathbf{u}_0, \ \forall t \in [0, T]\,.$$

We recall that by (35), we have $\mathbf{F}_\eta \in \mathcal{C}(0, T; \mathbb{V})$. Keeping in mind that the operator A is strict monotone, semi-continuous, bounded and coercive, and by using the classical arguments of functional analysis concerning parabolic equations [5, 14], we can easily prove the existence and uniqueness of $\mathbf{u}_\eta$ satisfying (49)–(50) and the regularity (44).

Second step.

In the second step, we consider the following variational problem.

4.1 Problem $\mathcal{PV}_\eta^w$

Find the wear function $w_\eta : [0,T] \to L^2(\Gamma_3)$ such that

$$\dot{w}_\eta(t) = k_w \alpha^* p_\nu (u_\nu - w - g), \quad (52)$$

$$w_\eta(0) = w_0 \quad \text{in } \Gamma_3. \quad (53)$$

We have the following result for $\mathcal{PV}_\eta^w$.

Lemma 4.2 *There exists a unique solution $w_\eta \in \mathcal{C}^1\left(0,T; L^2(\Gamma_3)\right)$ to the problem $\mathcal{PV}_\eta^w$.*

Proof. We use a version of the classical Cauchy–Lipschitz theorem when considering the mapping $\mathcal{F}_\eta : [0,T] \times L^2(\Gamma_3) \to L^2(\Gamma_3)$ defined by

$$\mathcal{F}_\eta(t, w_\eta) = k_w \alpha^* p_\nu (u_\nu - w_\eta - g) \ , \forall w_\eta \in L^2(\Gamma_3), t \in [0,T].$$

It is easy to see that $\mathcal{F}_\eta$ is Lipschitz continuous with respect to the second variable, uniformly in time. Thus, by the Cauchy–Lipschitz theorem, there exists a unique solution w_η which satisfies (52)–(53).

Third step.

In the third step, we consider the following variational problem.

4.2 Problem $\mathcal{PV}_\eta^\varphi$

Find the electric potential $\varphi_\eta : [0,T] \to \mathbb{W}$ such that

$$\sum_{k=1}^{2}(\mathcal{B}^k \nabla\varphi_\eta^k(t), \nabla\phi^k)_{H^k} - \sum_{k=1}^{2}(\mathcal{E}^k \varepsilon(\mathbf{u}_\eta^k(t)), \nabla\phi^k)_{H^k} = (q(t), \phi)_{\mathbb{W}} \quad (54)$$

for all $\phi \in \mathbb{W}$, a.e. $t \in (0,T)$. We have the following result.

Lemma 4.3 *There exists a unique solution $\varphi_\eta \in C(0,T;W)$ to the problem $\mathcal{PV}_\eta^\varphi$.*

Proof. We define a bilinear form $b(\cdot,\cdot) : \mathbb{W} \times \mathbb{W} \to \mathbb{R}$ such that

$$b(\varphi, \phi) = \sum_{k=1}^{2}(\mathcal{B}^k \nabla\varphi^k, \nabla\phi^k)_{H^k} \quad \forall \varphi, \phi \in \mathbb{W}. \quad (55)$$

We use (4), (5), (25) and (55) to show that the bilinear form $b(\cdot,\cdot)$ is continuous, symmetric and coercive on $\mathbb{W}$, moreover, using (34) and the Riesz representation theorem, we may define an element $q_\eta : [0,T] \to \mathbb{W}$ such that

$$(q_\eta(t), \phi)_{\mathbb{W}} = (q(t), \phi)_{\mathbb{W}} + \sum_{k=1}^{2}(\mathcal{E}^k \varepsilon(\mathbf{u}_\eta^k(t)), \nabla\phi^k)_{H^k} \quad \forall \phi \in \mathbb{W}, t \in [0,T].$$

We apply the Lax-Milgram theorem to deduce that there exists a unique element $\varphi_\eta(t) \in \mathbb{W}$ such that

$$b(\varphi_\eta(t), \phi) = (q_\eta(t), \phi)_{\mathbb{W}} \quad \forall \phi \in \mathbb{W}. \quad (56)$$

We conclude that φ_η is a solution of Problem $\mathcal{PV}_\eta^\varphi$. Let $t_1, t_2 \in [0,T]$, it follows from (54) that

$$\|\varphi_\eta(t_1) - \varphi_\eta(t_2)\|_{\mathbb{W}} \leqslant C\big(\|\mathbf{u}_\eta(t_1) - \mathbf{u}_\eta(t_2)\|_{\mathbb{V}} + \|q(t_1) - q(t_2)\|_{\mathbb{W}}\big). \tag{57}$$

We also note that assumptions (28), $\mathbf{u}_\eta \in \mathcal{C}^1(0,T;\mathbb{V})$ and inequality (57) imply that $\varphi_\eta \in C(0,T;\mathbb{W})$.

Finally, as a consequence of these results, and using the properties of the operator $\mathcal{E}^k$ and the functional j , for $t \in [0,T]$, we consider the element

$$\Lambda : \mathcal{C}(0,T;\mathbb{V}) \to \mathcal{C}(0,T;\mathbb{V}) \tag{58}$$

defined by the equations

$$\begin{aligned}(\Lambda\eta(t), \mathbf{v})_{\mathbb{V}} = &\sum_{k=1}^{2} (\mathcal{G}^k \boldsymbol{\varepsilon}(\mathbf{u}_\eta^k(t)), \mathbf{v})_{\mathbb{V}} + j(w_\eta(t), \mathbf{u}_\eta(t), \mathbf{v}) \\ &+ \sum_{k=1}^{2} \big((\mathcal{E}^k)^* \nabla\varphi_\eta^k(t), \varepsilon(\mathbf{v}^k)\big)_{Q^k}, \forall \mathbf{v} \in \mathbb{V}.\end{aligned} \tag{59}$$

Here, for every $\eta \in \mathcal{C}(0,T;\mathbb{V})$, $\mathbf{u}_\eta$, w_η and φ_η represent the displacement field, wear field and the potential electric field obtained in Lemmas 4.1, 4.2 and 4.3, respectively, and $\boldsymbol{\sigma}_\eta^k$ is denoted by

$$\boldsymbol{\sigma}_\eta^k(t) = \mathcal{A}^k \boldsymbol{\varepsilon}(\dot{\mathbf{u}}_\eta^k(t)) + \mathcal{G}^k \boldsymbol{\varepsilon}(\mathbf{u}_\eta^k(t)) + (\mathcal{E}^k)^* \nabla\varphi_\eta^k(t) \text{ in } \Omega^k \times [0,T]. \tag{60}$$

We have the following result.

Lemma 4.4 *There exists a unique $\eta^* \in \mathcal{C}(0,T;\mathbb{V})$ such that $\Lambda\eta^* = \eta^*$.*

Proof. Let η_1, $\eta_2 \in \mathcal{C}(0,T;\mathbb{V})$ and denote by $\mathbf{u}_i$, w_i, φ_i and $\boldsymbol{\sigma}_i$ the functions obtained in Lemmas 4.1, 4.2, 4.3 and the relation (60), respectively, for $\eta = \eta_i$, $i = 1,2$. Let $t \in [0,T]$, we have

$$\begin{aligned}\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{\mathbb{V}} \leqslant &\sum_{k=1}^{2} \|\mathcal{G}^k \boldsymbol{\varepsilon}(\mathbf{u}_1^k(t)) - \mathcal{G}^k \boldsymbol{\varepsilon}(\mathbf{u}_2^k(t))\|_{Q^k} \\ &+ |j(w_1(t), \mathbf{u}_1(t), \mathbf{v}) - j(w_2(t), \mathbf{u}_2(t), \mathbf{v})| \\ &+ \sum_{k=1}^{2} \|(\mathcal{E}^k)^* \nabla\varphi_1^k(t) - (\mathcal{E}^k)^* \nabla\varphi_2^k(t)\|_{Q^k}.\end{aligned}$$

We use (23) and (24), we have

$$\begin{aligned}\|\Lambda\eta_1(t) - \Lambda\eta_2(t)\|_{\mathbb{V}} \leqslant C\Big(&\|\mathbf{u}_1(t) - \mathbf{u}_2(t))\|_{\mathbb{V}} + \|\varphi_1(t) - \varphi_2(t)\|_{\mathbb{W}} \\ &+ |j(w_1(t), \mathbf{u}_1(t), \mathbf{v}) - j(w_2(t), \mathbf{u}_2(t), \mathbf{v})|\Big).\end{aligned} \tag{61}$$

From (3),(26),(36) and (27), we get

$$\begin{aligned}&\|j(w_1(t), \mathbf{u}_1(t), \mathbf{v}) - j(w_2(t), \mathbf{u}_2(t), \mathbf{v})\|_{L^2(\Gamma_3)} \\ &\qquad \leqslant c_0\left(\mathcal{L}_\nu + \mathcal{L}_\tau\right)\Big(c_0\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbb{V}} + \|w_1(t) - w_2(t)\|_{L^2(\Gamma_3)}\Big)\|\mathbf{v}\|_{\mathbb{V}}. \\ &\qquad \forall \mathbf{u}_1, \mathbf{u}_2, \mathbf{v} \in \mathbb{V}, w_1, w_2 \in L^2(\Gamma_3).\end{aligned}$$

Recall that $u^k_{\eta\nu}$ and $\mathbf{u}^k_{\eta\tau}$ denote the normal and the tangential component of the function $\mathbf{u}^k_\eta$, respectively.

Also, since

$$\mathbf{u}^k_i(t) = \int_0^t \dot{\mathbf{u}}^k_i(s)ds + \mathbf{u}^k_0(t), \quad t \in [0,T],\ k = 1,2,$$

we have

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|_{\mathbb{V}} \leqslant \int_0^t \|\dot{\mathbf{u}}_1(s) - \dot{\mathbf{u}}_2(s))\|_{\mathbb{V}}\, ds. \tag{62}$$

Using now (22),(26),(27), (59) and (60), we get

$$(m_{\mathcal{A}^k} - (\mathcal{L}_\nu + \mathcal{L}_\tau))\, \|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t))\|_{\mathbb{V}} \leqslant \|\eta_1(s) - \eta_2(s))\|_{\mathbb{V}}.$$

It follows from (49) that

$$\|\dot{\mathbf{u}}_1(t) - \dot{\mathbf{u}}_2(t))\|^2_{\mathbb{V}} \leqslant C\|\eta_1(s) - \eta_2(s))\|^2_{\mathbb{V}},$$

and using this inequality in (62) yields

$$\|\mathbf{u}_1(t) - \mathbf{u}_2(t)\|^2_{\mathbb{V}} \leqslant C \int_0^t \|\eta_1(s) - \eta_2(s))\|^2_{\mathbb{V}}\, ds. \tag{63}$$

On the other hand, from the Cauchy problem (52)–(53), we can write

$$w_i(t) = w_0 - \int_0^t k_w \alpha^* p_\nu \left(u_\nu\left(s\right) - w_i\left(s\right) - g\left(s\right)\right) ds,$$

and then

$$\begin{aligned}\|w_1(t) - w_2(t)\|_{L^2(\Gamma_3)} \leqslant C\Big(&\int_0^t \|k_w \alpha^* p_\nu \left(u_\nu\left(s\right) - w_1\left(s\right) - g\left(s\right)\right)\|_{L^2(\Gamma_3)} ds \\ &+ \int_0^t \|k_w \alpha^* p_\nu \left(u_\nu\left(s\right) - w_2\left(s\right) - g\left(s\right)\right)\|_{L^2(\Gamma_3)} ds\Big).\end{aligned}$$

Using (26),(27), and writing $w_1 = w_1 - w_2 + w_2$, we obtain

$$\begin{aligned}\big\|w_1(t) - w_2(t)\big\|_{L^2(\Gamma_3)} \leqslant C\Big(&\int_0^t \|w_1(s) - w_2(s)\|_{L^2(\Gamma_3)} ds \\ &+ \int_0^t \big\|\mathbf{u}_1(s) - \mathbf{u}_2(s)\big\|_{\mathbb{V}} ds\Big).\end{aligned}$$

Next, we apply Gronwall's inequality to deduce

$$\|w_1(t) - w_2(t)\|_{L^2(\Gamma_3)} \leqslant C \int_0^T \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|_{\mathbb{V}} ds,$$

and from the relation (3), we obtain

$$\|w_1(t) - w_2(t)\|^2_{L^2(\Gamma_3)} \leqslant C \int_0^T \|\mathbf{u}_1(s) - \mathbf{u}_2(s)\|^2_{\mathbb{V}} ds. \tag{64}$$

We use now (4), (24),(25) and (54) to find

$$\|\varphi_1(t)-\varphi_2(t)\|_W^2 \leqslant C\|\mathbf{u}_1(t)-\mathbf{u}_2(t)\|_{\mathbb{V}}^2. \tag{65}$$

We substitute (63), (64) and (65) in (61) to obtain

$$\|\Lambda\eta_1(t)-\Lambda\eta_2(t)\|_{\mathbb{V}}^2 \leqslant C\int_0^T \|\eta_1(s)-\eta_2(s)\|_{\mathbb{V}}^2\,ds.$$

Reiterating this inequality m times, we obtain

$$\|\Lambda^m\eta_1-\Lambda^m\eta_2\|_{\mathcal{C}(0,T;\mathbb{V})}^2 \leqslant \frac{C^mT^m}{m!}\|\eta_1-\eta_2\|_{\mathcal{C}(0,T;\mathbb{V})}^2.$$

Thus, for m sufficiently large, the operator Λ^m is a contraction on the Banach space $\mathcal{C}(0,T;\mathbb{V})$, and so Λ has a unique fixed point.

Now, we have all the ingredients to prove Theorem 4.1.

Proof. [Proof of Existence] Let $\eta^*\in\mathcal{C}(0,T;\mathbb{V})$ be the fixed point of Λ defined by (59), and if $\{\mathbf{u}_*,w_*,\varphi_*\}$ are the solutions of $\mathcal{PV}_\eta^u,\mathcal{PV}_\eta^w$ and $\mathcal{PV}_\eta^\varphi$, for $\eta=\eta^*$, we use the following notations:

$$\mathbf{u}_*=\mathbf{u}_{\eta^*},\quad \varphi_*=\varphi_{\eta^*},\quad w_*=w_{\eta^*}. \tag{66}$$

Let $\boldsymbol{\sigma}$ and $\boldsymbol{D}$ be the functions defined by

$$\begin{aligned}\boldsymbol{\sigma}_*^k &= \mathcal{A}^k\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^k)+\mathcal{G}^k\boldsymbol{\varepsilon}(\mathbf{u}_*^k)+(\mathcal{E}^k)^*\nabla\varphi_*^k, &(67)\\ \boldsymbol{D}_*^k &= \mathcal{E}^k\varepsilon(\mathbf{u}_*^k)-\mathcal{B}^k\nabla\varphi_*^k. &(68)\end{aligned}$$

We prove that $\{\mathbf{u}_*,\boldsymbol{\sigma}_*,w_*,\varphi_*,\boldsymbol{D}_*\}$ satisfies (37)–(42) and the regularities (44)–(48).

Clearly, (37), (41) and (42) are satisfied. We use now the equality $\Lambda\eta^*=\eta^*$, it follows that

$$(\Lambda\eta^*(t),\mathbf{v})_{\mathbb{V}}=(\eta^*(t),\mathbf{v})_{\mathbb{V}}. \tag{69}$$

From the problem $\mathcal{PV}_\eta^u$, we get

$$(\eta^*(t),\mathbf{v})_{\mathbb{V}}=-\sum_{k=1}^2(\mathcal{A}^k\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^k(t)),\varepsilon(\mathbf{v}^k(t)))_{Q^k}+(\mathbf{f}(t),\mathbf{v})_{\mathbb{V}},\forall\mathbf{v}\in\mathbb{V},\ \text{a.e. } t\in(0,T). \tag{70}$$

From the definition of Λ, we have

$$\begin{aligned}(\Lambda\eta^*(t),\mathbf{v})_{\mathbb{V}}=&\sum_{k=1}^2(\mathcal{G}^k\boldsymbol{\varepsilon}(\mathbf{u}_*^k(t)),\varepsilon(\mathbf{v}^k(t)))_{\mathbb{V}}+j(w_*(t),\mathbf{u}_*(t),\mathbf{v})\\ &+\sum_{k=1}^2\big((\mathcal{E}^k)^*\nabla\varphi_*^k(t),\varepsilon(\mathbf{v}^k(t))\big)_{Q^k},\\ &\forall\mathbf{v}\in\mathbb{V},\ \text{a.e. } t\in(0,T),\ k=1,2.\end{aligned} \tag{71}$$

From (69), (70) and (71), we deduce that

$$\begin{aligned}(\mathbf{f}(t),\mathbf{v})_{\mathbb{V}}=&\sum_{k=1}^2(\mathcal{A}^k\boldsymbol{\varepsilon}(\dot{\mathbf{u}}_*^k(t)),\varepsilon(\mathbf{v}^k(t)))_{Q^k}+\sum_{k=1}^2(\mathcal{G}^k\boldsymbol{\varepsilon}(\mathbf{u}_*^k(t)),\varepsilon(\mathbf{v}^k))_{\mathbb{V}}\\ &+j(w_*(t),\mathbf{u}_*(t),\mathbf{v})+\sum_{k=1}^2\big((\mathcal{E}^k)^*\nabla\varphi_*^k(t),\varepsilon(\mathbf{v}^k)\big)_{Q^k},\\ &\forall\mathbf{v}\in\mathbb{V},\ \text{a.e. } t\in(0,T),\ k=1,2.\end{aligned} \tag{72}$$

We use (67) and (72), we get

$$(\mathbf{f}(t),\mathbf{v})_{\mathbb{V}}=\sum_{k=1}^{2}(\boldsymbol{\sigma}_*^k(t),\varepsilon(\mathbf{v}^k))_{Q^k}+j(w_*(t),\mathbf{u}_*(t),\mathbf{v}). \tag{73}$$

We deduce that (39) is satisfied. Additionally, we use $\mathbf{u}_{\eta^*}$ in (52) and (66) to find

$$\dot{w}_*(t)=k_w\alpha^* p_{*\nu}\left(u_{*\nu}-w_*-g\right), \text{a.e.} t\in(0,T). \tag{74}$$

Now, relations (66), (67), (68), (73) and (74) allow us to conclude that $\{\mathbf{u}_*,\boldsymbol{\sigma}_*,w_*,\varphi_*,\boldsymbol{D}_*\}$ satisfies (37)–(42).

Next, (42) and the regularities (44), (46)–(47) follow from Lemmas 4.1, 4.2 and 4.3.

Since $\mathbf{u}_*, w_*$ and φ_* satisfy (44), (46) and (47), respectively, it follows from (66) and (67) that

$$\boldsymbol{\sigma}_*\in\mathcal{C}(0,T;\mathbb{Q}). \tag{75}$$

For $k=1,2$, we choose $\mathbf{v}=\dot{\mathbf{u}}\pm\phi$ in (73), with $\phi=(\phi^1,\phi^2)$, $\phi^k\in D(\Omega^k)^d$ and $\phi^{3-k}=0$ in (54), to obtain

$$Div\ \boldsymbol{\sigma}_*^k(t)=-\boldsymbol{f}_0^k(t)\quad\forall t\in[0,T],\quad k=1,2, \tag{76}$$

where $D(\Omega^k)$ is the space of infinitely differentiable real functions with a compact support in Ω^k. The regularity (45) follows from (28), (75) and (76). Let now $t_1,t_2\in[0,T]$, by (4),(24), (25) and (68), we deduce that

$$\|\boldsymbol{D}_*(t_1)-\boldsymbol{D}_*(t_2)\|_{\mathbb{H}}\leqslant C\left(\|\varphi_*(t_1)-\varphi_*(t_2)\|_{\mathbb{W}}+\|\mathbf{u}_*(t_1)-\mathbf{u}_*(t_2)\|_{\mathbb{V}}\right).$$

The regularity of $\mathbf{u}_*$ and φ_* given by (44) and (47) implies

$$\boldsymbol{D}_*\in C(0,T;\mathbb{H}). \tag{77}$$

For $k=1,2$, we choose $\phi=(\phi^1,\phi^2)$ with $\phi^k\in D(\Omega^k)^d$ and $\phi^{3-k}=0$ in (54) and using (34), we find

$$div\ \boldsymbol{D}_*^k(t)=q_0^k(t)\quad\forall t\in[0,T],\quad k=1,2. \tag{78}$$

Property (48) follows from (28), (77) and (78).

Finally, we conclude that the weak solution $\{\mathbf{u}_*,\boldsymbol{\sigma}_*,w_*,\varphi_*,\boldsymbol{D}_*\}$ of the Problem $\mathcal{PV}$ has the regularities (44)–(48), which concludes the existence part of Theorem 4.1.

Proof. [Proof of Uniqueness] The uniqueness of the solution is a consequence of the uniqueness of the fixed point of the operator Λ defined by (59) and the unique solvability of the Problems $\mathcal{PV}_\eta^u, \mathcal{PV}_\eta^w$ and $\mathcal{PV}_\eta^\varphi$.

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