Nonlinear Dynamics and Systems Theory, **23** (2) (2023) 183–194

Global Optimization Method of Multivariate non-Lipschitz Functions Using Tangent Minorants

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Received: January 12, 2023; Revised: March 20, 2023

Abstract: This paper deals with the multidimensional global optimization problem where the objective function f is non-Lipschitz over a hyper-rectangle of $\mathbb{R}^n$. The generalization of Piyavskii's algorithm to the multivariate case requires finding the intersection of many non-linear hyper-surfaces. In this paper, we propose an algorithm which is composed of two steps. The first one is to transform the multivariate function f into a single variable function $\mathbf{f}(t)$ using the α-dense curves and the second one is to apply the extended version of Piyavskii's algorithm to $\mathbf{f}(t)$. For minimizing $\mathbf{f}(t)$, we construct a sequence of lower bounding piecewise tangent functions. A convergence result is proved and the numerical experiments on some test functions are given and compared with the existing methods.

Keywords: *global optimization; non-Lipschitz multivariate functions; lower bounding function; Piyavskii's algorithm.*

Mathematics Subject Classification (2010): 93-03, 93A30, 93B40, 93C35, 90C26.

1 Introduction

Let us consider the box constrained global optimization problem

$$\min_{x\in\mathbf{P}=\prod_{i=1}^{n}[a_i,b_i]} f(x), \tag{P}$$

where f is a real continuous multi-extremal function defined on the hyper-rectangle $\mathbf{P}$ and satisfies the following condition:

$$|f(x)-f(y)| \leq h\,\|x-y\|^{1/m}, \qquad \forall x,y\in\mathbf{P}, \tag{1}$$

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with two parameters $h > 0$ and $1/m$ $(m > 1)$, where $\|.\|$ stands for the Euclidean norm. The last condition is called the Hölder condition (it is clear that if $m = 1$, we have the Lipschitz case) [7]. Global optimization is of interest in many complex industrial applications. But it can also be applied to a variety of other multidimensional problems such as the resolution of systems of nonlinear functional equations [6] involving objective functions, which are only continuous and do not possess strong mathematical proprieties such as convexity or differentiability, and which should be optimized [1]. The kind of problem (P) arises in several applications, for instance, the simple plant location problem under a uniform delivered price policy, see Hanjoul et al. [10], infinite horizon optimization problems, see Kiatsupaibul et al. [12], etc. The local irregularity of the objective function, particularly when the value of m is large, is what causes the problem to be complex to solve in this case. When applied to higher dimensions, the traditional multidimensional global optimization methods present significant challenges. Some researchers have considered reducing the dimension of certain problems to convert them into others that are simpler [5], [17]. There are numerous methods for reducing a multidimensional global optimization problem to one or more optimization problems with a smaller dimension, especially with one dimension. Many authors have explored the strategy based on filling the feasible region with a curve, see, for example, Butz [4], Strongin [18], and Sergeyev et al. [17]. For this, they take into account the Peano-type curve approximations. These curves, known as space-filling curves, were first presented by Peano (1890), subsequently by Hilbert (1891), and have the property of passing through all points of a hyper-rectangle of $\mathbb{R}^n$. On the other hand, Cherruault [5], Guettal and Ziadi [9], [15], [16] and their collaborators have consistently improved the reducing transformation method in recent years, their method depends on reducing a multidimensional problem to a unidimensional one by using the space-filling curves like α-dense type curves to fill the feasible domain, and then, using a one-dimensional global optimization algorithm, to approximate the global minimizer. Gourdin et al. [8] have suggested solving this problem by the generalization of Piyavskii's algorithm to the multivariate situation [8]. Indeed, Piyavskii's approach cannot be directly generalized since finding the intersection of many parabolic hyper-surfaces is necessary to find the local minima of the sub-estimators of the objective function on $\mathbf{P}$. The authors in [8] proposed a procedure for partitioning and eliminating (Branch-and-Bound) hyper-rectangles of no interest by constructing piecewise constant sub-estimator functions. Here, we present a novel method for deterministic global optimization that relies on a methodology for reducing the dimension of the problem (P) and is referred to as the "method of the reducing transformation". Finding the global minima of multivariate functions with a lot of local minima has proven to be quite effective with the Alienor method coupled with some covering one-dimensional methods. The concept is to densify the hyper-rectangle $\mathbf{P}$ as accurately as we need, using pretty regular so-called "α-dense curves", and then approach the objective function f with n variables defined on the hyper-rectangle $\mathbf{P}$, by a function $\mathbf{f}$ with a single variable t on a real interval A of $\mathbb{R}$, which will be specified later. This allows the multidimensional optimization problem to be reduced to a one-dimensional optimization problem, which can then be solved using one-dimensional methods that are well-known for their effectiveness and performance. This coupling has proved to be efficient for solving diverse non-Lipschitz global optimization problems. For minimizing the function $\mathbf{f}$ on A, we construct a sequence of lower bounding piecewise tangent functions.

The remainder of the work is organized as follows. Section 2 contains the Alienor reducing transformation method. Section 3 presents some covering methods to find the global minima of univariate functions. Section 4 shows the modified mixed Alienor method with covering methods and their convergence. Section 5 gives some numerical experiments confirming theoretical results and showing a reliable performance of the proposed method and Section 6 concludes the paper.

2 A Multivariate non-Lipschitz Method

2.1 The Alienor reducing transformation method

Global optimization is essentially the purpose of the Alienor reducing transformation approach [5], [19], [20]. But it can also be applied to a variety of other multidimensional problems such as the resolution of systems of non-linear functional equations and the approximation of functions of many variables by functions of a single variable. The essential idea behind this approach is to perform a transformation that turns multidimensional optimization problems into single-variable ones before using an effective algorithm for one-dimensional optimization problems. The transformation is thus based on the creation of a specific α-dense parametrized curve $\zeta(t) = (\zeta_1(t), \zeta_2(t), ..., \zeta_n(t))$ in the feasible set $\mathbf{P}$.

Definition 2.1 Let A be an interval of $\mathbb{R}$. We say that a parametrized curve of $\mathbb{R}^n$ defined by $\zeta : A \to \mathbf{P}$ is α-dense in $\mathbf{P}$ if for all $x \in \mathbf{P}$, $\exists t \in A$ such that

$$d(x, \zeta(t)) \leq \alpha,$$

where d stands for the Euclidean distance in $\mathbb{R}^n$.

2.2 Building α-dense curves

In order to create α-dense curves in $\mathbf{P}$, let us assume that the function $\zeta(t)$ is defined on the closed and bounded interval $A = [0, T]$ of $\mathbb{R}$ with values in $\mathbf{P}$, where T is the upper bound of the domain of definition of ζ. The number α is supposed strictly positive and extremely small the dimension of the hyper-rectangle $\mathbf{P} = \prod_{i=1}^{n} [a_i, b_i]$. We define by a constructive way an α-dense curve in an arbitrary hyper-rectangle of $\mathbb{R}^n$ thanks to the following results.

Theorem 2.1 *Let $\zeta(t) = (\zeta_1(t), \zeta_2(t), ..., \zeta_n(t))$ be a function defined from $[0, T]$ into the hyper-rectangle $\mathbf{P}$, $\alpha > 0$, and μ be the Lebesgue measure such that*
(1) *$(\zeta_i)_{1\leq i\leq n}$ are continuous and surjective.*
(2) *$(\zeta_i)_{2\leq i\leq n}$ are periodic, respectively, of periods $(t_i)_{2\leq i\leq n}$.*
(3) *For any interval I of $[0, T]$ and for any $i \in \{2, ..., n\}$, we have*

$$\mu(I) \leq t_i \Rightarrow \mu(\zeta_{i-1}(I)) < \alpha.$$

Then for $t \in [0, T]$, the function $\zeta(t)$ represents a parametrized $\sqrt{n-1}\alpha$-dense curve in $\mathbf{P}$. (The proof can be found in [20]).

Corollary 2.1 [20] *Let $\zeta(t) = (\zeta_1(t), \zeta_2(t), ..., \zeta_n(t)) : \left[0, \frac{\pi}{\alpha_1}\right] \to \prod_{i=1}^{n} [a_i, b_i]$ be a function defined by*

$$\zeta_i(t) = \frac{a_i - b_i}{2} \cos(\alpha_i t) + \frac{a_i + b_i}{2}, \qquad i = 1, 2, ..., n,$$

where $\alpha_1, \alpha_2, ..., \alpha_n$ *are given strictly positive constants satisfying the relationships*

$$\alpha_i \geq \frac{\pi}{\alpha}(b_{i-1} - a_{i-1})\alpha_{i-1}, \qquad \forall i = 2, ..., n.$$

Then the curve defined by the parametric curve $\zeta(t)$, *is* $\sqrt{n-1}\alpha$*-dense in* $\mathbf{P}$.

When using the reducing transformation approach, we first explicitly provide a parametric representation $x_i = \zeta_i(t)$, where $i = 1, ..., n$, of the α-dense curve in the hyper-rectangle $\mathbf{P}$, for $t \in \left[0, \frac{\pi}{\alpha_1}\right]$. Let us specify the following function:

$$\zeta(t) = (\zeta_1(t), \zeta_2(t), ..., \zeta_n(t)) : \left[0, \frac{\pi}{\alpha_1}\right] \to \mathbf{P}$$

with

$$\zeta_i(t) = \frac{a_i - b_i}{2}\cos(\alpha_i t) + \frac{a_i + b_i}{2}, \qquad i = 1, ..., n,$$

where α and $(\alpha_i)_{1 \leq i \leq n}$ are provided by

$$\alpha = \left(\frac{\varepsilon}{2h}\right)^m \frac{1}{\sqrt{n-1}}, \ \alpha_1 = 1 \text{ and } \alpha_i = \frac{\pi}{\alpha}(b_{i-1} - a_{i-1})\alpha_{i-1}, \qquad i = 2, ..., n.$$

According to Corollary 2.1, the parametrized curve $\zeta(t)$ is α-dense in the hyper-rectangle $\mathbf{P}$. Moreover, the function ζ is Lipschitzian on $\left[0, \frac{\pi}{\alpha_1}\right]$ with the constant

$$L = \frac{1}{2}\left(\sum_{i=1}^{n}(b_i - a_i)^2\alpha_i^2\right)^{\frac{1}{2}}.$$

Then the objective function f is approximated by the function of a single variable defined by $\mathbf{f}(t) = f(\zeta(t))$. The minimization problem (P) is then approximated by the one-dimensional minimization problem

$$\min_{t \in \left[0, \frac{\pi}{\alpha_1}\right]} \mathbf{f}(t).$$

Theorem 2.2 *The function* $\mathbf{f}(t) = f(\zeta(t))$ *for* $t \in \left[0, \frac{\pi}{\alpha_1}\right]$ *satisfies the condition* (1) *with the constant* $\mathbf{h}$ *and exponent* $1/m$, *where* $\mathbf{h}$ *is given by* $\mathbf{h} = hL^{1/m}$.

Proof. For t_1 and t_2 in $\left[0, \frac{\pi}{\alpha_1}\right]$, we have

$$|\mathbf{f}(t_1) - \mathbf{f}(t_2)| = |f(\zeta(t_1)) - f(\zeta(t_2))| \leq h \left\|\zeta(t_1) - \zeta(t_2)\right\|^{1/m}.$$

As the function ζ is Lipschitzian on $\left[0, \frac{\pi}{\alpha_1}\right]$ with the constants L, we have

$$\left\|\zeta(t_1) - \zeta(t_2)\right\| \leq L\left|t_1 - t_2\right|,$$

then

$$|\mathbf{f}(t_1) - \mathbf{f}(t_2)| \leq h\left(L\left|t_1 - t_2\right|\right)^{1/m},$$

whence

$$|\mathbf{f}(t_1) - \mathbf{f}(t_2)| \leq hL^{1/m}\left|t_1 - t_2\right|^{1/m}.$$

This permits us to use one of the unidimensional algorithms to solve the multidimensional problem (P) shown in Section 3.

3 A Single Variable non-Lipschitz Method

The following unidimensional optimization problem will be defined by

$$\min_{t\in\left[0,\frac{\pi}{\alpha_1}\right]} \mathbf{f}(t), \tag{P$'$}$$

where $\mathbf{f}$ is defined on the interval $\left[0, \frac{\pi}{\alpha_1}\right]$ and satisfies the condition (1) with the constant $\mathbf{h}$ and exponent $1/m$, $(m > 1)$. When minimizing a non-convex function $\mathbf{f}$, the general principle behind most deterministic global optimization methods is to relax the original non-convex problem in order to make the relaxed problem convex by utilizing an under-estimator of the objective function [11], [14].

Definition 3.1 A function $\mathcal{F}$ is said to be an under-estimator of a function $\mathbf{f}$ on a set X if

$$\mathcal{F}(t) \leq \mathbf{f}(t), \qquad \forall t \in X,$$

with the possibility that $\mathcal{F}$ may not reach $\mathbf{f}$ at any point in X.

3.1 Constructing a sequence of under-estimators

The idea is to build an increasing sequence of piecewise functions that minorize the objective function $\mathbf{f}$ and are constructed in such a way that their global minima converge to the desired global minimum. From the condition (1), if a point $t' \in \left[0, \frac{\pi}{\alpha_1}\right]$ is fixed, then we have

$$\mathcal{F}(t) = \mathbf{f}(t') - \mathbf{h}\,|t - t'|^{1/m} \leq \mathbf{f}(t), \qquad \forall t \in \left[0, \frac{\pi}{\alpha_1}\right],$$

i.e., $\mathcal{F}$ is an under-estimator of $\mathbf{f}$ on $\left[0, \frac{\pi}{\alpha_1}\right]$. Let us define the first under-estimator by

$$\mathcal{F}_1(t) = \mathbf{f}(t_1) - \mathbf{h}\,|t - t_1|^{1/m} \leq \mathbf{f}(t), \qquad \forall t \in \left[0, \frac{\pi}{\alpha_1}\right],$$

where t_1 is chosen arbitrarily, we then determine a point $t_2 = \arg\min_{t\in[0,\frac{\pi}{\alpha_1}]} \mathcal{F}_1(t)$, we thus obtain a new under-estimator of $\mathbf{f}$,

$$\mathcal{F}_2(t) = \max_{1\leq i\leq 2} \left\{ \mathbf{f}(t_i) - \mathbf{h}\,|t - t_i|^{1/m} \right\}.$$

At step k, the function

$$\mathcal{F}_k(t) = \max_{1\leq i\leq k} \left\{ \mathbf{f}(t_k) - \mathbf{h}\,|t - t_k|^{1/m} \right\}.$$

In the search interval $\left[0, \frac{\pi}{\alpha_1}\right]$, the restriction of $\mathcal{F}_k$ on each sub-interval $[t_{i-1}, t_i]$, $i = 2, \ldots, k$, can be expressed as

$$\mathcal{F}_i(t) = \max_i \left\{ \underbrace{\mathbf{f}(t_{i-1}) - \mathbf{h}\,(t - t_{i-1})^{1/m}}_{\Phi_{i-1}(t)},\ \underbrace{\mathbf{f}(t_i) - \mathbf{h}\,(t_i - t)^{1/m}}_{\Phi_i(t)} \right\}.$$

The function $F_i(t)$ is convex and non-differentiable in $[t_{i-1}, t_i]$ and its global minimum value can be computed by locating the point where the two parabolic curves intersect, i.e., it necessitates solving a non-linear algebraic equation on $[0, \frac{\pi}{\alpha_1}]$,

$$\mathbf{f}(t_{i-1}) - \mathbf{h}\,(t - t_{i-1})^{1/m} = \mathbf{f}(t_i) - \mathbf{h}\,(t_i - t)^{1/m}. \tag{2}$$

Determining the unique point of intersection of two parabolic curves is generally easy only for certain cases of m. Gourdin et al. [8] give the analytical expression for the intersection point when m is the integers $2, 3, 4$ and $\mathbf{h}$ is known. Lera and Sergeyev proposed the secant method (SM) [13] when they utilized a different concept based on changing the intersection point of the parabolic curves at each sub-interval $[t_{i-1}, t_i]$ to the intersection point $\overline{\mathbf{t}}_i$ of two linked linear interpolations l_{i-1} (resp. l_i) of the parabolas Φ_{i-1} (resp. Φ_i). Then the constant lower bound of the objective function on $[t_{i-1}, t_i]$ is defined by

$$\mathbf{w}_i = \min\{\Phi_{i-1}(\overline{\mathbf{t}}_i), \Phi_i(\overline{\mathbf{t}}_i)\}.$$

Here we suggest another technique noted TM, when changing the solution of the equation (2) by an intersection point ω_i of two tangents T_{i-1} (resp. T_i) at the same middle point of the interval $[t_{i-1}, t_i]$, related to these two parabolas Φ_{i-1} (resp. Φ_i) and defined by

$$\begin{cases} T_{i-1}(t) = -(\mathbf{h}/m)e_i^{(1/m)-1}t + \mathbf{h}e_i^{1/m}(\frac{v_i}{me_i} - 1) + \mathbf{f}(t_{i-1}), \\ T_i(t) = (\mathbf{h}/m)e_i^{(1/m)-1}t - \mathbf{h}e_i^{1/m}(\frac{v_i}{me_i} + 1) + \mathbf{f}(t_i) \end{cases} \tag{3}$$

such as $v_i = \frac{t_i + t_{i-1}}{2}$ and $e_i = \frac{t_i - t_{i-1}}{2}$.

In this case, the point ω_i can be calculated even if m is large enough or not integer, by

$$\omega_i = v_i + \frac{m(\mathbf{f}(t_{i-1}) - \mathbf{f}(t_i))}{2\mathbf{h}e_i^{(1/m)-1}}. \tag{4}$$

Proposition 3.1 *Let $\mathbf{f}$ be a real univariate function satisfying the condition* (1) *with the constant $\mathbf{h} > 0$ and exponent $1/m$ defined on the interval $\left[0, \frac{\pi}{\alpha_1}\right]$. Let the value $\mathbf{T}_i = \min\{\Phi_{i-1}(\omega_i), \Phi_i(\omega_i)\}$ as a constant lower bound of $\mathbf{f}$ on $[t_{i-1}, t_i] \subset \left[0, \frac{\pi}{\alpha_1}\right]$, then we have*

$$\mathbf{T}_i = \min\left\{\mathbf{f}(t_{i-1}) - \mathbf{h}\left(e_i + \frac{m(\mathbf{f}(t_{i-1}) - \mathbf{f}(t_i))}{2\mathbf{h}e_i^{(1/m)-1}}\right)^{1/m}, \mathbf{f}(t_i) - \mathbf{h}\left(e_i + \frac{m(\mathbf{f}(t_i) - \mathbf{f}(t_{i-1}))}{2\mathbf{h}e_i^{(1/m)-1}}\right)^{1/m}\right\}$$

and

$$\mathbf{T}_i < \mathbf{f}(t), \qquad \forall t \in [t_{i-1}, t_i]. \tag{5}$$

Proof. The value $\mathbf{T}_i$ is given by replacing the variable t in the two functions $\Phi_{i-1}(t)$ and $\Phi_i(t)$ by the expression (4). Since $F_i(t) < \mathbf{f}(t)$, $\forall t \in \,]t_{i-1}, t_i[$, where $F_i(t) = \max\{\Phi_{i-1}(t), \Phi_i(t)\}$, we have

$$\min\{\Phi_{i-1}(t), \Phi_i(t)\} \leq \min_{[t_{i-1}, t_i]} F_i(t) \leq \mathbf{f}(t), \qquad \forall t \in [t_{i-1}, t_i].$$

In particular, for $t = \omega_i$, it then follows

$$\mathbf{T}_i = \min\{\Phi_{i-1}(\omega_i), \Phi_i(\omega_i)\} < \mathbf{f}(t), \qquad \forall t \in \,]t_{i-1}, t_i[.$$

4 The Modified Mixed Alienor-TM Method

In order to determine the global minimum of $f(x)$, the modified mixed Alienor-TM Method consists of two steps: the reducing transformation step and the application of the TM algorithm to the function $\mathbf{f}(t) = f(\zeta(t))$, which satisfies the condition (1) with the constant $\mathbf{h} = hL^{1/m}$.

Algorithm 4.1 (Alienor-TM)

Input: $\mathbf{P} = \prod_{i=1}^{n} [a_i, b_i]$ is the search domain, f is the objective function (multivariate non-Lipschitz function). The parameters h, m, ε and the dimension n.

Output: ***Part* 1 :** $\zeta(t)$ is the parametric curve,
$\mathbf{f}$ is the univariate non-Lipschitz function.
***Part* 2 :** $\mathbf{f}_{opt}$ is the best global minimum of $\mathbf{f}$.

***Part* 1 :**

$\alpha = (\frac{\varepsilon}{2h})^m$, $\alpha_1 = 1$.
for $i = 2$ to n **do**
$\alpha_i = \frac{\pi}{\alpha}(b_i - a_i)\alpha_{i-1}$.
end for
for $i = 1$ to n **do**
$\zeta_i(t) = \frac{a_i - b_i}{2}\cos(\alpha_i t) + \frac{a_i + b_i}{2}$.
end for
$\zeta(t) = (\zeta_1(t), \zeta_2(t), ..., \zeta_n(t))$ and $\mathbf{f}(t) = f(\zeta(t))$.

***Part* 2 :**

Initialization: $k \leftarrow 2$, $\mu \leftarrow 2$, $t_1 \leftarrow 0$, $t_2 \leftarrow \frac{\pi}{\alpha_1}$.
Step k**:** $t_1, t_2, \ldots, t_k$ are ordered such that $0 = t_1 < t_2 < \cdots < t_k = \frac{\pi}{\alpha_1}$.
for $i = 2$ to k **do**
$\omega_i = v_i + \frac{m(\mathbf{f}(t_{i-1}) - \mathbf{f}(t_i))}{2\mathbf{h}e_i^{(1/m)-1}}$,
$\mathbf{T}_i = \min\left\{\mathbf{f}(t_{i-1}) - \mathbf{h}(\omega_i - t_{i-1})^{1/m}, \mathbf{f}(t_i) - \mathbf{h}(t_i - \omega_i)^{1/m}\right\}$.
end for

$$\mathbf{T}_\mu \leftarrow \min\left\{\mathbf{T}_i, 2 \le i \le k\right\}, \tag{6}$$

$t_\mu \leftarrow \omega_\mu$.
if $|t_\mu - t_{\mu-1}| > \epsilon = \left(\frac{\varepsilon}{2\mathbf{h}}\right)^m$, **then**

$$t_{k+1} \leftarrow \omega_\mu \tag{7}$$

$k \leftarrow k + 1$
Go to Step k
else
$\mathbf{f}_{opt} = \min\left\{\mathbf{f}(t_i) : 1 \le i \le k\right\}$ and Stop.
end if
return $\mathbf{f}_{opt}$

5 Convergence Results of TM and Alienor-TM Algorithms

Theorem 5.1 *Let* $\mathbf{f}(t)$ *be a real non-Lipschitz function defined on a closed interval* $[0, \frac{\pi}{\alpha 1}]$, *with* $\mathbf{h} > 0$ *and* $1/m$, $(m > 1)$. *Let* t^* *be a global minimizer of* $\mathbf{f}(t)$. *Then the sequence* $(t_k)_{k\geq 1}$ *generated by the* TM *algorithm converges to* t^*, *i.e.,*

$$\lim_{k\to+\infty} \mathbf{f}(t_k) = \mathbf{f}(t^*).$$

Proof. Let $t_1, t_2, t_3, ...$ be the sampling sequence satisfying (4), (6), (7). Let us consider that $t_s \neq t_{s'}$ for all $s \neq s'$, the set of the elements of the sequence $(t_k)_{k\geq 1}$ is then infinite and therefore has at least one limit point in $[0, \frac{\pi}{\alpha_1}]$. Let $\mathbf{z}$ be any limit point of $(t_k)_{k\geq 1}$ such that $\mathbf{z} \neq 0$, $\mathbf{z} \neq \frac{\pi}{\alpha_1}$, then the convergence to $\mathbf{z}$ is bilateral (one can see [13]). Consider now an interval $[t_{\rho(k)-1}, t_{\rho(k)}]$ which contains $\mathbf{z}$, using (4), (6) and (7), we obtain

$$\lim_{k\to+\infty} (t_{\rho(k)-1} - t_{\rho(k)}) = 0. \tag{8}$$

In addition, the value $\mathbf{T}_{\rho(k)}$ that corresponds to $[t_{\rho(k)-1}, t_{\rho(k)}]$, is given by

$$\mathbf{T}_{\rho(k)} = \min\left\{\mathbf{f}(t_{\rho(k)-1}) - \mathbf{h}(\omega_\rho - t_{\rho(k)-1})^{1/m}, \mathbf{f}(t_{\rho(k)}) - \mathbf{h}(t_{\rho(k)} - \omega_\rho)^{1/m}\right\}, \tag{9}$$

where z_ρ is obtained by replacing i by ρ in (4). As $\mathbf{z} \in [t_{\rho(k)-1}, t_{\rho(k)}]$ and from (8), we have

$$\lim_{k\to+\infty} \mathbf{T}_{\rho(k)} = \mathbf{f}(\mathbf{z}). \tag{10}$$

On the other hand, according to (5),

$$\mathbf{T}_{j(k)} \leq \mathbf{f}(t), \qquad \forall t \in [t_{j(k)-1}, t_{j(k)}]. \tag{11}$$

From (6), $\mathbf{T}_{\rho(k)} = \min\{\mathbf{T}_j, j = 2, ..., k\}$, then

$$\mathbf{T}_{\rho(k)} \leq \mathbf{T}_{j(k)}, \qquad \forall t \in [t_{j(k)-1}, t_{j(k)}],$$

and since $[0, \frac{\pi}{\alpha_1}] = \bigcup_{j=2}^{k} [t_{j(k)-1}, t_{j(k)}]$, we have

$$\lim_{k\to+\infty} \mathbf{T}_{\rho(k)} \leq \mathbf{T}_{j(k)}, \qquad \forall t \in [0, \frac{\pi}{\alpha_1}], \tag{12}$$

and from (11), (12) we get

$$\lim_{k\to+\infty} \mathbf{T}_{\rho(k)} \leq \mathbf{f}(t), \qquad \forall t \in [0, \frac{\pi}{\alpha_1}].$$

Since t^* is the global minimizer of $\mathbf{f}$ over $[0, \frac{\pi}{\alpha_1}]$,

$$\lim_{k\to+\infty} \mathbf{T}_{\rho(k)} \leq \mathbf{f}(t^*) \leq \mathbf{f}(\mathbf{z}),$$

from (10), we have

$$0 \leq \mathbf{f}(\mathbf{z}) - \mathbf{f}(t^*) \leq \mathbf{f}(\mathbf{z}) - \lim_{k\to+\infty} \mathbf{T}_{\rho(k)} = 0,$$

then

$$\mathbf{f}(\mathbf{z}) = \mathbf{f}(t^*).$$

The function $\mathbf{f}$ is non-Lipschitz on $[0, \frac{\pi}{\alpha_1}]$, so $\mathbf{f}$ must be continuous so that

$$\mathbf{f}(\mathbf{z}) = \mathbf{f}\left(\lim_{k\to+\infty} t_k\right) = \lim_{k\to+\infty} \mathbf{f}(t_k) = \mathbf{f}(t^*).$$

Theorem 5.2 *Let f be a non-Lipschitz function satisfying the condition* (1) *over* $\mathbf{P}$ *and M be the global minimum of f on* $\mathbf{P}$. *Then the mixed Alienor-TM algorithm converges to the global minimum with an accuracy at least equal to ε.*

Proof. Denote by M^* the global minimum of $\mathbf{f}$ on $[0, \frac{\pi}{\alpha_1}]$, where $\mathbf{f}(t) = f(\zeta(t))$. On the other hand, let us designate by $\mathbf{f}_\varepsilon$ the global minimum of the problem (P$'$) obtained by the Alienor-TM method.
Let us show that

$$\mathbf{f}_\varepsilon - M \leq \varepsilon.$$

a) As f is continuous on $\mathbf{P}$, there exists a point $\mathbf{x} \in \mathbf{P}$ such that $M = f(\mathbf{x})$. Moreover, there exists $t_0 \in [0, \frac{\pi}{\alpha_1}]$ such that $\|\mathbf{x} - \zeta(t_0)\| \leq \left(\frac{\varepsilon}{2h}\right)^m$ so that $\|f(\mathbf{x}) - f(\zeta(t_0))\| \leq \frac{\varepsilon}{2}$. And therefore

$$f(\zeta(t_0)) - M \leq \frac{\varepsilon}{2}.$$

But from $M \leq M^* \leq f(\zeta(t_0))$, we deduce that

$$M^* - M \leq \frac{\varepsilon}{2}. \tag{13}$$

b) As $\mathbf{f}$ is continuous on $[0, \frac{\pi}{\alpha_1}]$, there exists a point $t^* \in [0, \frac{\pi}{\alpha_1}]$ such that $M^* = \mathbf{f}(t^*)$, involving t^* as a global minimizer of $\mathbf{f}$. Then t^* is a limit point of the sequence $(t_k)_{k\geq 1}$ obtained by the mixed algorithm.
Hence $t^* \in [t_{\rho(k)-1}, t_{\rho(k)}]$ and $\lim\limits_{k\to+\infty}(t_{\rho(k)} - t_{\rho(k)-1}) = 0$, i.e.,

$$\exists t_\varepsilon \in [t_{s-1}, t_s] : |t_s - t_{s-1}| \leq \left(\frac{\varepsilon}{2\mathbf{h}}\right)^m \text{ and } \mathbf{f}_\varepsilon = \mathbf{f}(t_\varepsilon)$$

so that

$$\begin{cases} \mathbf{T}_s = \min\left\{\mathbf{f}(t_{s-1}) - \mathbf{h}\,|t - t_{s-1}|^{1/m}, \mathbf{f}(t_s) - \mathbf{h}\,|t - t_s|^{1/m}\right\}, \\ \mathbf{T}_s \leq \mathbf{f}(t^*) \leq \mathbf{f}(t_\varepsilon) \text{ and } t^* \in [t_{s-1}, t_s]. \end{cases}$$

Consequently,

$$\mathbf{f}_\varepsilon - M^* = \mathbf{f}(t_\varepsilon) - \mathbf{f}(t^*) \leq \mathbf{h}\,|t_\varepsilon - t^*|^{1/m} \leq \frac{\varepsilon}{2}. \tag{14}$$

Finally, from (13) and (14), the result of Theorem 5.2 is proved.

6 Computational Experiments

In this section, we present a series of numerical results concerning two mixed Alienor-SM and Alienor-TM algorithms, applied to a set of non-Lipschitz test functions given in the literature. The analytical expressions of the objective functions are reported in Table 1 below including their sources.

Problem No.	Non-Lipschitz test functions.	Domain	h	m	Ref.
1	$\max\left\{\sqrt{\lvert x\rvert}, \sqrt{\lvert y\rvert}\right\}$	$[-1,1]^2$	1	2	[2]
2	$\sqrt{\lvert x\rvert + \lvert y\rvert}$	$[-1,1]^2$	$(\sqrt{2})^{\frac{1}{2}}$	2	[2]
3	$\sqrt{\lvert x\rvert} + \sqrt{\lvert y\rvert}$	$[-1,1]^2$	2	2	[2]
4	$\lvert x+y-0.25\rvert^{2/3} - 3\cos(\frac{x}{2})$	$[\frac{-1}{2}, \frac{1}{2}]^2$	2.42	$\frac{3}{2}$	[15]
5	$\sum_{k=1}^{3} \frac{1}{2k} \left\lvert \cos\left(\left(\frac{3}{2k}+1\right)x + \frac{1}{2k}\right)\right\rvert \lvert x-y\rvert^3$	$[0,3]^2$	15.8	3	[15]
6	$-\cos(x)\cos(y)\exp\left(1 - \frac{\sqrt{x^2+y^2}}{\pi}\right)$	$[-6,6]^2$	45.265	2	[3]
7	$-10\exp\left(-\sqrt{0.5\left(\lvert x\rvert + \lvert y\rvert\right)}\right)$	$[-2,12]^2$	$\frac{10}{\sqrt{2}}$	2	[3]

Table 1: The non-Lipschitz test functions.

The experiments have been carried out on PC with Intel(R) Core(TM)i5-7200U CPU 2.50 GHz and 8.00 RAM. The codes are implemented in MATLAB R2017a, with the parameter $\alpha = 0.1$. We give, in Table 2, the numerical results obtained by each method to solve the problem (P) and the comparison is made with respect to the number of evaluations Ev and the calculation time CPU. In Table 2, the bold form indicates the best results in terms of CPU and Ev.

Problem No.	Alienor-SM		Alienor-TM	
	Ev	CPU	Ev	CPU
1	**207**	**0.0655**	212	0.1506
2	**192**	**0.0901**	196	0.1731
3	283	0.1738	**248**	**0.0783**
4	214	0.1063	**206**	**0.0899**
5	4905	1.6163	**4865**	**1.6039**
6	65549	308.5771	**65546**	**307.6463**
7	**4792**	13.5927	4862	**12.9657**

Table 2: The numerical results.

According to Table 1, all the test functions satisfy the condition (1) with $m > 1$ and even for non-integer m. The results given in Table 2 show that the Alienor-TM mixed method gives relatively satisfactory results, either in terms of the calculation time CPU or the number of evaluations Ev. The dimensionality reduction Alienor method is rather effective for dealing with difficult problems and its numerical implementation is very simple. The number of evaluations Ev of $\mathbf{f}(t)$ depends on the length of the α-dense curve. This raises a particular interest when choosing other curves. In general, for a fixed value of α, the shorter the curve, the shorter the calculation time. It is therefore natural to look for other α-dense curves having a shorter length.

7 Conclusion

In this paper, we report a method for solving a multidimensional global optimization problem, where the objective function is non-Lipschitz over a hyper-rectangle of $\mathbb{R}^n$. The concept relies on using the α-dense curve for reducing the size of the space $\mathbb{R}^n$ to 1, then we apply the one-dimensional version of Piyavskii's algorithm based on constructing tangent minorant functions. This method is simple and easy to implement on any multivariate non-Lipschitz function even if m is not an integer. We suggested a series of numerical applications, followed by a comparative study of two mixed algorithms applied to the proposed problem. We see that the mixed Alienor-TM and Alienor-SM methods offer interesting prospects for reducing the computation time and the number of evaluations. Finally, we want to elaborate on these investigations in cases where the constant h is a priori unknown.

Acknowledgment

This work is funded by the General Directorate of Scientific Research and Technological Development (DGRSDT) of Algeria.

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