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Analysis of Solutions to Equations with a Generalized Derivative and Delay

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Abstract: This paper is concerned with the set-valued differential equations with a generalized derivative and constant delay. We introduce the notion of the initial problem solutions and establish conditions for their existence and uniqueness, also we provide a result on the continuous dependence of the solution of this problem on the initial function. It is found that the solutions of such equations can expand and contract, depending on the initial conditions. Also, in this paper we develop a numerical algorithm to calculate solutions to such problem approximately. By means of examples, we demonstrate how this algorithm works when solving different non-linear differential equations with generalized derivative with constant delay under different initial conditions.

Keywords: *set-valued differential equations; generalized derivative; delay; existence and uniqueness of solution; numerical algorithm.*

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1 Introduction

The study of the properties of a trajectory set and the construction of a reachability set for control systems plays an important role in the investigation of optimal control problems. Let the equation of motion of the control object have the form

$$\dot{x} = f(t, x, u), \quad u \in U, \quad x(t_0) = x_0, \tag{1}$$

where $x \in R^n$ is a phase vector, $t > t_0$, $u(t) \in U \in comp(R^k)$ is a control vector. Problem (1) can be replaced by the following problem [23]:

$$\dot{x} \in F(t, x), \quad x(t_0) = x_0, \tag{2}$$

where $F(t, x) = \{z \in R^n | z = f(t, x, u),\ u \in U\}$ is a multivalued mapping. It is also mentioned in [23] that the sets of solutions of equation (1) and inclusion (2) coincide. In the same book, it is mentioned that the solution of the corresponding equation with the Hukuhara derivative, in which the right-hand side contains the same multi-valued mapping from (2), bounds the solution of the differential inclusion (2).

Thus, differential equations with a set-valued right-hand side can be used to study solutions to the optimal control problem.

The first analysis of differential equations with a multivalued right-hand side was conducted by S. Zaremba [28] and A. Marchaud [12], [13], [14]. The main results were also presented in the works of T. Wazewski [26], [27], V.A. Plotnikov [22], [23], J.-P.Aubin [1], K. Deimling [7], M. Kisielewiez [9], [10] and others. The development of the theory of multivalued mappings has led to the clarification of the question of what is meant by a derivative of multivalued mappings. This is stated in the works of M. Hukuhara [8], T.F. Bridgland [4], H.T. Banks, M.Q. Jacobs [2], A.V. Plotnikov, N.V. Skripnik [19], [21], B. Bede, S.G. Gal [3], O. Carja, T. Donchev and A. I. Lazu [6].

Differential equations with set-valued right-hand side and generalized derivative appeared first in the works of A.V. Plotnikov, N.V. Skripnik [19], [21]. The existence and uniqueness of solutions to the Cauchy problems with such equations were studied there.

Let us note that the notion of generalized derivatives for multivalued maps was first introduced in [17], where the corresponding Cauchy problem was stated and the notion of solutions to such problems was provided. The initial condition in this Cauchy problem was given at a time point and the right-hand side of the equation depends on a time point rather than on a time interval. In contrary to [17], in the current paper, we consider the equations and initial states which depend on prehistory, that is, are defined on a time interval. Hence we consider equations with time delay which make the problems considered here essentially different from [17]. Hence our work extends the results of [17] to the case of equations with time delays. The presence of a time delay leads to essential changes in the approach of [17] and to other properties of solutions.

In this paper, the differential equation with a generalized derivative with a constant delay is considered, the theorem on the existence and uniqueness of solution of such equations is formulated and proved, the numerical algorithm for construction of these solutions is developed, and examples of the application of the numerical algorithm for construction of solutions of differential equations with a constant delay are given.

2 Main Results

2.1 Concept of solution

Consider a nonlinear differential equation with a generalized derivative with a constant delay:

$$DX = F\left(t, X\left(t\right), X\left(t-\Delta\right)\right), \quad X\left(s\right) = \rho\left(s\right), \quad s \in \left[-\Delta, t_0\right], \tag{3}$$

where $t \in I = \left[t_0, T\right], F : I \times conv\left(\mathbb{R}^n\right) \times conv\left(\mathbb{R}^n\right) \to conv\left(\mathbb{R}^n\right)$ is a multivalued mapping, $t_0 = 0, \Delta > 0$ is a constant delay, $\rho\left(\cdot\right) : \left[-\Delta, t_0\right] \to conv\left(\mathbb{R}^n\right)$.

Definition 2.1 A multivalued mapping $X\left(\cdot\right) : \left[t_0, T\right] \to conv\left(\mathbb{R}^n\right)$ is called a solution of differential equation (3) if it is absolutely continuous and satisfies (3) almost everywhere on $[t_0, T]$.

But similarly to a differential equation with a generalized derivative without delay, in this case, it is impossible to ensure the unity of the solution [21]. Next, consider the differential equation of the form

$$DX \overset{h}{-} \Phi\left(-\varphi\left(t\right)\right) F_1\left(t, X\left(t\right), X\left(t-\Delta\right)\right) = \Phi\left(\varphi\left(t\right)\right) F_2\left(t, X\left(t\right), X\left(t-\Delta\right)\right),$$
$$X\left(s\right) = \rho\left(s\right), s \in \left[-\Delta, t_0\right], \tag{4}$$

where $t \in \left[t_0, T\right], X\left(\cdot\right) : \left[t_0, T\right] \to conv\left(\mathbb{R}^n\right), t_0 = 0, \Delta > 0$ is a constant delay, $\rho\left(\cdot\right) : \left[-\Delta, t_0\right] \to conv\left(\mathbb{R}^n\right), F_1, F_2\left(\cdot,\cdot,\cdot\right)\left[t_0, T\right] \times conv\left(\mathbb{R}^n\right) \times conv\left(\mathbb{R}^n\right) \to conv\left(\mathbb{R}^n\right)$ is a multivalued mappings, $\varphi\left(\cdot\right) : \left[t_0, T\right] \to \mathbb{R}$ is a continuous function.

$$\Phi\left(\varphi\right) = \begin{cases} 1, & \varphi > 0, \\ 0, & \varphi \leq 0. \end{cases}$$

Definition 2.2 A multivalued mapping $X\left(\cdot\right) : \left[t_0, T\right] \to conv\left(\mathbb{R}^n\right)$ is called a solution of differential equation (4) if it is absolutely continuous and on any segment $\left[\tau_i, \tau_{i+1}\right] \subset \left[t_0, T\right]$, where the function $\varphi\left(\cdot\right)$ on the interval $\left(t_i, t_{i+1}\right)$ has a constant sign, satisfies the integral equation

$$X\left(t\right) + \int\limits_{\tau_i}^{t} \Phi\left(-\varphi\left(s\right)\right) F_1\left(s, X\left(s\right), X\left(s-\Delta\right)\right) \mathrm{d}s =$$
$$= X\left(\tau_i\right) + \int\limits_{\tau_i}^{t} \Phi\left(\varphi\left(s\right)\right) F_2\left(s, X\left(s\right), X\left(s-\Delta\right)\right) \mathrm{d}s. \tag{5}$$

If on the interval $\left(\tau_i, \tau_{i+1}\right)$, a function $\varphi\left(t\right) > 0$, then $X\left(\cdot\right)$ satisfies the integral equation

$$X\left(t\right) = X\left(\tau_i\right) + \int\limits_{\tau_i}^{t} F_2\left(s, X\left(s\right), X\left(s-\Delta\right)\right) \mathrm{d}s$$

for $t \in \left[\tau_i, \tau_{i+1}\right]$ and $diam\left(X\left(t\right)\right)$ is a growing function.
If on the interval $\left(\tau_i, \tau_{i+1}\right)$, a function $\varphi\left(t\right) < 0$, then $X\left(\cdot\right)$ satisfies the integral equation

$$X\left(\tau_i\right) = X\left(t\right) + \int\limits_{\tau_i}^{t} F_1\left(s, X\left(s\right), X\left(s-\Delta\right)\right) \mathrm{d}s,$$

that is,

$$X(t) = X(\tau_i) \overset{h}{-} \int_{\tau_i}^{t} F_1(s, X(s), X(s-\Delta))\, \mathrm{d}s$$

for $t \in [\tau_i, \tau_{i+1}]$ and $diam\,(X(t))$ is a decreasing function.
If on the interval (τ_i, τ_{i+1}), a function $\varphi(t) = 0$, then $X(t) = X(\tau_i)$ for $t \in [\tau_i, \tau_{i+1}]$ and $diam\,(X(t))$ is a constant function.

We will also introduce another equivalent definition of the solution of the equation (4).

Definition 2.3 A multivalued mapping $X(\cdot) : [t_0, T] \to conv\,(\mathbb{R}^n)$ is called a solution of differential equation (4) if it is absolutely continuous, satisfying (4) almost everywhere on $[t_0, T]$, and

$$diam\,(X(t)) = \begin{cases} \textit{increase}, & \varphi(t) > 0, \\ \textit{constant}, & \varphi(t) = 0, \\ \textit{decrease}, & \varphi(t) < 0. \end{cases}$$

In the equation (4), the multivalued mappings F_1 and F_2 determine the rate of change (”compression” and ”expansion”) of the multivalued mapping $X(t)$ and how it changes in the space $conv\,(\mathbb{R}^n)$, and a function $\varphi(t)$ determines when a diameter $X(t)$ increases, decreases or is constant. These mappings are considered different because the laws of ”compression” and ”expansion” may be different.

2.2 A condition for the existence of a unique solution.

Based on [24], we can formulate and prove the following theorems.

Theorem 2.1 *Let F_1 and F_2 be continuous mappings and, in some neighborhood, points $(t_0, \rho(t_0), \rho(t_0 - \Delta))$ satisfy the Lipschitz condition with respect to the 2nd and 3rd variables with a constant λ. Let the initial function $\rho(s)$ be continuous and the delay Δ be non-negative. Then there is a unique solution $X(t)$ of equation (4) for $t_0 \le t \le t_0 + \sigma$, where σ is arbitrarily small.*

Proof. Consider the function $\varphi(t)$ on the segment $t \in [t_0; t_0 + \sigma]$. As mentioned above, it can take a negative, positive and zero value.

1. $\varphi(t) = 0$. Then
$$X(t) = \rho(t). \tag{6}$$

2. $\varphi(t) > 0$. Then we obtain a differential equation with the Hukuhara derivative with a constant delay, which has a unique solution [24].

3. $\varphi(t) < 0$. Then we transform the system (4) into the integral equation

$$X(t) = \rho(t) \overset{h}{-} \int_{t_0}^{t} F_1(s, X(s), X(s-\Delta))\, ds \tag{7}$$

and prove that it has a unique solution on the segment $[t_0; t_0 + d]$.

Suppose the opposite. Let the equation (7) have at least two solutions $X(t)$ and $Y(t)$ such that

$$\overline{\omega} = \max_{t \in [t_0; t_0 + d]} h(X(t), Y(t)) > 0,$$

where $[t_0; t_0 + d]$ is the total period of existence of solutions $X(t)$ and $Y(t)$. We have

$$X(t) \equiv \rho(t) \overset{h}{-} \int_{t_0}^{t} F_1(s, X(s), X(s - \Delta))\, ds,$$

$$Y(t) \equiv \rho(t) \overset{h}{-} \int_{t_0}^{t} F_1(s, Y(s), Y(s - \Delta))\, ds,$$

whence, using the Lipschitz condition and the Hausdorff distance properties, we obtain

$$h(X(t), Y(t)) =$$

$$= h\left(\rho(t) \overset{h}{-} \int_{t_0}^{t} F_1(s, X(s), X(s - \Delta))\, ds, \rho(t) \overset{h}{-} \int_{t_0}^{t} F_1(s, Y(s), Y(s - \Delta))\, ds\right) =$$

$$= h\left(\int_{t_0}^{t} F_1(s, X(s), X(s - \Delta))\, ds, \int_{t_0}^{t} F_1(s, Y(s), Y(s - \Delta))\, ds\right) \leq$$

$$\leq \int_{t_0}^{t} h(F_1(s, X(s), X(s - \Delta)), F_1(s, Y(s), Y(s - \Delta)))\, ds \leq$$

$$\leq \lambda \left(\int_{t_0}^{t} h(X(s), Y(s))\, ds + \int_{t_0}^{t} h(X(s - \Delta), Y(s - \Delta))\, ds\right).$$

So we get

$$h(X(t), Y(t)) \leq \lambda \int_{t_0}^{t} \overline{\omega} ds = \lambda \overline{\omega} (t - t_0) \leq \lambda \overline{\omega} d,$$

$$h(X(t), Y(t)) \leq \lambda \int_{t_0}^{t} \overline{\omega}(s - t_0)\, ds = \frac{\lambda^2 \overline{\omega} (t - t_0)^2}{2} \leq \frac{\lambda^2 \overline{\omega} d^2}{2} \ldots$$

Using the method of complete mathematical induction, we have that for any natural m on the segment $[t_0; t_0 + d]$, there is an inequality

$$h(X(t), Y(t)) \leq \frac{\lambda^m \overline{\omega} d^m}{m!}.$$

Then

$$\overline{\omega} = \max_{t \in [t_0; t_0 + d]} h(X(t), Y(t)) \leq \frac{\lambda^m \overline{\omega} d^m}{m!},$$

from here, by virtue of the positivity $\overline{\omega}$, we have that for any natural m,

$$1 \le \frac{(\lambda d)^m}{m!}. \tag{8}$$

In view of the sign of the d'Alembert series, $\sum\limits_{m=1}^{\infty} \frac{(\lambda d)^m}{m!}$ converges and from here, the necessity of the condition $\lim\limits_{m\to\infty} \frac{(\lambda d)^m}{m!} = 0$. This means that for $\varepsilon = \frac{1}{2}$, there exists $m \in \mathbb{N}$ such that $\frac{(\lambda d)^m}{m!} < \frac{1}{2}$. Then, by virtue of (8), we get that $1 < \frac{1}{2}$. We have obtained a contradiction, and so we have that the equation (7) and the equivalent equation (4) have a unique solution.

4. In the case when the function $\varphi(t)$ changes the sign on the segment $[t_0; t_0 + d]$, the existence of a unique solution is proved by the combination of cases $1) - 3)$.

The theorem is proved.

Theorem 2.2 *Let all conditions of Theorem 2.1 be satisfied. Then the solution of the equation* (4) *continuously in the space* $comp(\mathbb{R}^n)$ *depends on the initial function, and at* $h(\rho_1(s), \rho_2(s)) \le \delta$, $\delta > 0$, $s \in [-\Delta; t_0]$, *we have*

$$h(X_1(t), X_2(t)) \le \delta e^{2\lambda(t-t_0)}, t \ge t_0. \tag{9}$$

Proof. Similarly to the previous theorem, consider 3 cases for the function $\varphi(t)$.

1. $\varphi(t) = 0$. We have
$$h(\rho_1(s), \rho_2(s)) < \delta \le \delta e^{2\lambda(t-t_0)},$$
which implies (9).

2. $\varphi(t) > 0$. We have
$$h\left(\int\limits_{t_0}^{t} F_2(s, X_1(s), X_1(s-\Delta))\,ds, \int\limits_{t_0}^{t} F_2(s, X_2(s), X_2(s-\Delta))\,ds\right) \le$$
$$\le \int\limits_{t_0}^{t} h(F_2(s, X_1(s), X_1(s-\Delta)), F_2(s, X_2(s), X_2(s-\Delta)))\,ds \le$$
$$\le \lambda \int\limits_{t_0}^{t} [h(X_1(s), X_2(s)) + h(X_1(s-\Delta), X_2(s-\Delta))]\,ds. \tag{10}$$
Let
$$z(t) = \max\left\{\delta, \max_{t_0 \le s \le t} h(X_1(s), X_2(s))\right\}.$$
From (10), we get
$$z(t) \le \delta + 2\lambda \int\limits_{t_0}^{t} z(s)\,ds. \tag{11}$$
From (11), by the Gronwall-Bellman lemma, we get (9).

3. $\varphi(t) < 0$. Similar to the previous case, we have

$$h\left(\int_{t_0}^{t} F_2(s, X_1(s), X_1(s-\Delta))\,ds, \int_{t_0}^{t} F_2(s, X_2(s), X_2(s-\Delta))\,ds\right) \le$$

$$\le \lambda \int_{t_0}^{t} \left[h(X_1(s), X_2(s)) + h(X_1(s-\Delta), X_2(s-\Delta))\right] ds.$$

Next, from (11) and by the Gronwall-Bellman lemma, we get (9).

The theorem is proved.

2.3 A numerical algorithm for construction of solutions of differential equations with a generalized derivative with a constant delay

Based on Definitions 2.2 and 2.3, Theorems 2.1, and 2.2 and [21], we can formulate a numerical algorithm for constructing a solution of a differential equation with a generalized derivative with delay.

Consider the equation (4)

$$DX \overset{h}{-} \Phi(-\varphi(t)) F_1(t, X(t), X(t-\Delta)) = \Phi(\varphi(t)) F_2(t, X(t), X(t-\Delta)),$$
$$X(s) = \rho(s),\, s \in [-\Delta, t_0],$$

where $t \in [t_0, T], X(\cdot) : [t_0, T] \to conv(\mathbb{R}^n), t_0 = 0, \Delta > 0$ is a constant delay, $\rho(\cdot) : [-\Delta, t_0] \to conv(\mathbb{R}^n), F_1, F_2(\cdot,\cdot,\cdot)\, [t_0, T] \times conv(\mathbb{R}^n) \times conv(\mathbb{R}^n) \to conv(\mathbb{R}^n)$ are multivalued mappings, $\varphi(\cdot) : [t_0, T] \to \mathbb{R}$ is a continuous function.

$$\Phi(\varphi) = \begin{cases} 1, & \varphi > 0, \\ 0, & \varphi \le 0. \end{cases}$$

Let the dimension of the space $n = 2$. Next, we write the formula for a counterpart of Euler's method in the case of differential equation (4)

$$X_m(t) = \begin{cases} X_m(t_k) + (t - t_k) F_2(t_k, X(t_k), X(t_k - \Delta)), & \varphi(t) > 0, \\ X_m(t_k) \overset{h}{-} (t - t_k) F_1(t_k, X(t_k), X(t_k - \Delta)), & \varphi(t) < 0, \\ X_m(t_k), & \varphi(t) = 0. \end{cases}$$

$$t \in [t_k, t_{k+1}],\, k = \overline{0, m-1}, X_m(s) = \rho(s),\, s \in [-\Delta, t_0].$$

Using the apparatus of support functions, we obtain

$$C(X_m(t), \psi) = \begin{cases} C(X_m(t_k) + (t - t_k) F_2(t_k, X(t_k), X(t_k - \Delta)), \psi), \\ C\left(X_m(t_k) \overset{h}{-} (t - t_k) F_1(t_k, X(t_k), X(t_k - \Delta)), \psi\right), \\ C(X_m(t_k), \psi). \end{cases} =$$

$$= \begin{cases} C(X_m(t_k), \psi) + (t - t_k) C(F_2(t_k, X(t_k), X(t_k - \Delta)), \psi), \\ C(X_m(t_k), \psi) - (t - t_k) C(F_1(t_k, X(t_k), X(t_k - \Delta)), \psi), \\ C(X_m(t_k), \psi), \end{cases}$$

where ψ is a unit vector.

For $t = t_{k+1}$ we have formulas:

$$C\left(X_m\left(t_{k+1}\right), \psi\right) = \begin{cases} C\left(X_m\left(t_k\right), \psi\right) + \delta C\left(F_2\left(t_k, X\left(t_k\right), X\left(t_k - \Delta\right)\right), \psi\right), \\ C\left(X_m\left(t_k\right), \psi\right) - \delta C\left(F_1\left(t_k, X\left(t_k\right), X\left(t_k - \Delta\right)\right), \psi\right), \\ C\left(X_m\left(t_k\right), \psi\right). \end{cases} \tag{12}$$

To construct an external approximation of the set $X_m\left(t_{k+1}\right)$, we find $C\left(X_m\left(t_{k+1}\right), \psi_i\right)$, where $\psi_i = \begin{pmatrix} cos\gamma_i \\ sin\gamma_i \end{pmatrix}$, $\gamma_i = \frac{2\pi i}{p}$, $i = \overline{0, p-1}$.

It follows from (12) that

$$C\left(X_m\left(t_{k+1}\right), \psi\right) = \begin{cases} C\left(X_m\left(t_k\right) + \delta F_2\left(t_k, X\left(t_k\right), X\left(t_k - \Delta\right)\right), \psi_i\right), \\ C\left(X_m\left(t_k\right) \overset{h}{-} \delta F_1\left(t_k, X\left(t_k\right), X\left(t_k - \Delta\right)\right), \psi_i\right), \\ C\left(X_m\left(t_k\right), \psi_i\right). \end{cases} =$$

$$= \begin{cases} C\left(X_m\left(t_k\right), \psi\right) + \delta C\left(F_2\left(t_k, X\left(t_k\right), X\left(t_k - \Delta\right)\right), \psi_i\right), \\ C\left(X_m\left(t_k\right), \psi\right) - \delta C\left(F_1\left(t_k, X\left(t_k\right), X\left(t_k - \Delta\right)\right), \psi_i\right), \\ C\left(X_m\left(t_k\right), \psi_i\right). \end{cases}$$

Thus, we can get the values of the support functions $C\left(X_m\left(t_k\right), \psi_i\right)$, $k = \overline{0, m}$, $i = \overline{0, p-1}$.

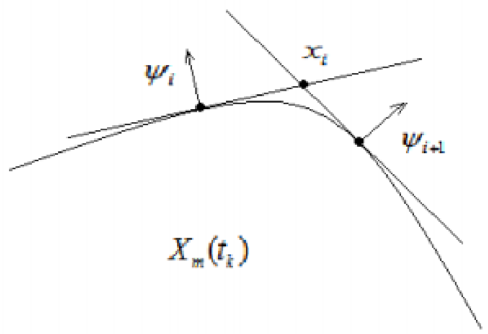

Figure 1: Construction of boundary points of the numerical approximation of a convex set.

To construct the approximation (Fig. 1), find the points of intersection of the support hyperplanes to the set $X_m\left(t_k\right)$ in the directions of the vectors ψ_i and ψ_{i+1}, $i = \overline{0, p-1}$, $\psi_p = \psi_0$:

$$\begin{cases} \left(x, \psi_i\right) = C\left(X_m\left(t_k\right), \psi_i\right), \\ \left(x, \psi_{i+1}\right) = C\left(X_m\left(t_k\right), \psi_{i+1}\right). \end{cases}$$

This is a linear system relatively unknown vector $x \in \mathbb{R}^2$ with determinant

$$\Delta = \begin{pmatrix} cos\gamma_i & sin\gamma_i \\ cos\gamma_{i+1} & sin\gamma_{i+1} \end{pmatrix} = sin(\gamma_{i+1} - \gamma_i) = sin\frac{2\pi}{p} \neq 0.$$

Let us denote the solution of the system by x_i, $i = \overline{0, p-1}$. Construct a polygon with vertices at points x_0, x_1, ..., x_{p-1}, which we denote Q_k^p. The criterion for account termination is

$$\left|\text{square } Q_{k+1}^{p+1} - \text{square } Q_k^p\right| < \varepsilon,$$

where ε is a predefined number.

2.4 Construction of solutions of differential equation with a generalized derivative with a constant delay

Using the Octave package, we constructed a solution of differential equations with a generalized derivative with a delay with different initial sets X_0, a partition m, the Euler number of "broken lines" p and a constant delay Δ on the time interval $t \in [0; T]$. It should be noted that the delay Δ must be a multiple of the time step $h = \frac{T - t_0}{m}$. The following examples show how this program works.

Consider the equation of the form

$$DX \overset{h}{-} \Phi(t-a) \frac{1}{2} X(t-\Delta) = \Phi(a-t) X(t), X(s) = X_0(s), s \in [-\Delta, 0]. \quad (13)$$

1. Let $X_0 = S_{100} \begin{pmatrix} 0 \\ t \end{pmatrix}$, then $c(X_0, \psi) = t\psi_2 + 100 \|\psi\|$.

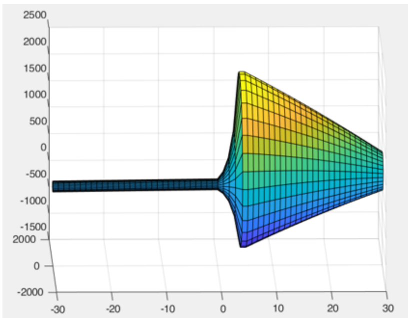

Figure 2: Equation (13) sol. graph for $m = 30$, $p = 30$, $\Delta = 30$, $a = 5$, $T = 30$, init. conditions (1).

2. Let $X_0 = K_{10000} \begin{pmatrix} t \\ 2t^2 \end{pmatrix}$, then $c(X_0, \psi) = t\psi_1 + 2t^2\psi_2 + 10000|\psi_1| + 10000|\psi_2|$.

Consider the equation with a generalized derivative with a delay of the form

$$DX \overset{h}{-} \Phi(diam(X(t)) - diam(X_0) - 100) \frac{1}{2} X(t-\Delta) =$$

$$= \Phi(diam(X_0) + 100 - diam(X(t))) X(t), \quad (14)$$

$$X(s) = X_0(s), s \in [-\Delta, 0].$$

There are no proved theorems on the existence and uniqueness of the solution for this equation, so we will consider it as an experiment.

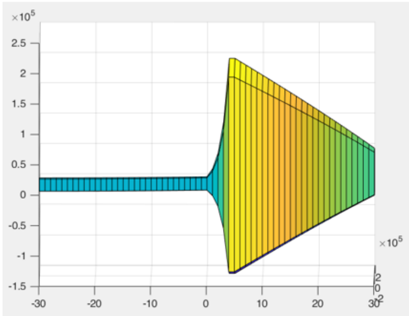

Figure 3: Equation (13) sol. graph for $m = 30$, $p = 30$, $\Delta = 30$, $a = 5$, $T = 30$, init. conditions (2).

1. Let $X_0 = \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix} S_1 \begin{pmatrix} sint \\ 2cost \end{pmatrix}$, then $c(X_0, \psi) = sint\psi_1 + 2cost\psi_2 + $ $+ \left\| \begin{pmatrix} 10 & 0 \\ 0 & 20 \end{pmatrix} \psi \right\|$.

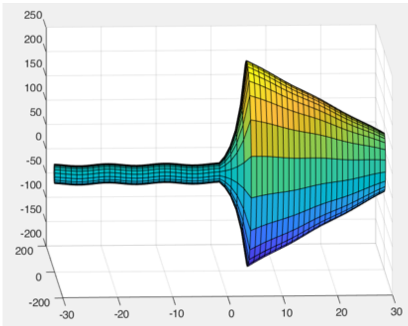

Figure 4: Equation (14) sol. graph for $m = 30$, $p = 30$, $\Delta = 30$, $T = 30$, init. conditions (1).

2. Let $X_0 = \begin{pmatrix} \text{rectangle with half sides 100 and 300} \\ \text{and the center at the point } \begin{pmatrix} sint \\ 2cost \end{pmatrix} \end{pmatrix}$, then $c(X_0, \psi) = sint\psi_1 + 2cost\psi_2 + 100|\psi_1| + 300|\psi_2|$.

3 Conclusion

The theorem on the existence of a unique solution of a differential equation with a generalized derivative with a delay and the theorem on the continuous dependence of this solution on the initial function are formulated and proved. A numerical algorithm for solving such equations is developed. The paper presents examples of this algorithm for different types of equations, different initial conditions, partitions, and delays.

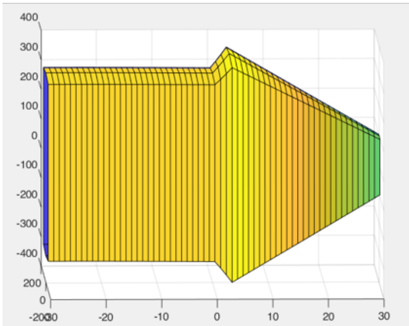

Figure 5: Equation (14) sol. graph for $m = 30$, $p = 30$, $\Delta = 30$, $T = 30$, init. conditions (2).

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