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Chaos Anti-Synchronization between Fractional-Order Lesser Date Moth Chaotic System and Integer-Order Chaotic System by Nonlinear Control

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Abstract: This paper investigates the phenomenon of chaos anti-synchronization between the fractional-order lesser date moth and the integer-order chaotic systems based on the Lyapunov stability theory and numerical differentiation. The nonlinear feedback control is the method used to achieve the anti-synchronization of chaotic systems addressed in this paper. Numerical examples are implemented to illustrate and validate the results.

Keywords: *chaos; anti-synchronization; nonlinear control; fractional-order chaotic system; integer-order chaotic system.*

Mathematics Subject Classification (2010): 34H10, 37N35, 93C10, 93C15, 93C95.

1 Introduction

Chaos is a fascinating nonlinear phenomenon that has received a lot of attention in recent years. During the previous two decades, the chaos theory proved to be effective in a wide range of areas such as data encryption [20], financial systems [18, 19], biology [23] and biomedical engineering [2], etc. Fractional-order chaotic dynamical systems have begun to attract a lot of attention in recent years and can be seen as a generalization of chaotic dynamic integer-order systems. The synchronization between a fractional-order chaotic system and an integer-order chaotic system is thoroughly a new domain which has begun

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to attract much attention in recent years [9,21] because of its potential applications in secure communication and cryptography [11,12]. Obviously, the synchronization between a fractional-order chaotic system and an integer-order chaotic system is more difficult than the synchronization between a fractional-order chaotic system or an integer-order chaotic system for the different order of their error dynamical system. The synchronization between a fractional-order system and an integer-order system was first studied by Zhou et al. [21]. As a special case of generalized synchronization, anti-synchronization is achieved when the sum of the states of master and slave systems converge to zero asymptotically with time. In this research work, we apply nonlinear control theory to anti-synchronize two chaotic systems when a fractional-order system is chosen as the drive system and an integer-order system serves as the response system. The anti-synchronization capability of the approach is demonstrated using a fractional-order lesser date moth chaotic system and an integer-order chaotic system [15]. The paper is arranged in the following manner. In Section 2, we describe the problem formulation for a fractional-order and an integer-order chaotic system. In Section 3, we discuss the anti-synchronisation between a fractional-order lesser date moth chaotic system and an integer-order chaotic system using the nonlinear control. Section 4 gives a brief conclusion.

2 Problem Formulation for Fractional-Order and Integer-Order Chaotic System

Consider the following fractional-order chaotic system as a drive (master) system:

$$D^{\alpha} x_1 = A x_1 + g(x_1), \tag{1}$$

where $x_1 \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n \times n}$ is the linear part, $g(x_1)$ is a continuous nonlinear function, and D^{α} is the Caputo fractional derivative. Also, the response system (slave) can be described as

$$\dot{x}_2 = A x_2 + g(x_2) + u, \tag{2}$$

where $x_2 \in \mathbb{R}^n$ is the state vector, $A \in \mathbb{R}^{n \times n}$ is the linear part, $g(x_2)$ is a continuous nonlinear function and $u \in \mathbb{R}^n$ is the control.

Define the anti-synchronization errors as $e = x_2 + x_1$. The anti-synchronisation error system between the driving system (1) and the response system (2) can be expressed as

$$\dot{e} = \dot{x}_2 + \dot{x}_1,$$

where $\dot{x}_2$ is obtained from the response system (2), while no exact expressions of $\dot{x}_1$ can be obtained from the driving system (1). Therefore, a numerical differentiation method is used to obtain $\dot{x}_1$.

According to the definition of the derivative, the derivative is approximately expressed using the difference quotient as

$$g'(a) \approx \frac{g(a+h) - g(a)}{h}, \tag{3}$$

$$g'(a) \approx \frac{g(a) - g(a-h)}{h}, \tag{4}$$

where $(h > 0)$ is a small increment. Formulae (3) and (4) are called the pre-difference formula and the post-difference formula, respectively. The post-difference formula is used in this paper.

The global anti-synchronization problem is essentially to find a feedback controller u so as to stabilize the error dynamics for all initial conditions $e(0) \in \mathbb{R}^n$ (i.e., $\lim_{t\to\infty} \|e(t)\| = 0$).

3 Anti-Synchronisation of Fractional-Order Lesser Date Moth Chaotic System and Integer-Order Chaotic System by Nonlinear Control

3.1 Main results

In this section, to validate the nonlinear control method proposed in [5], we take the fractional-order lesser date moth chaotic system [15] as a drive system and the integer-order chaotic system as a response system.

Thus, the drive and response systems are as follows:

$$\begin{cases} D^\alpha x_1 = x_1(1-x_1) - \frac{x_1 y_1}{\beta + x_1}, \\ D^\alpha y_1 = -\delta y_1 + \frac{\gamma x_1 y_1}{\beta + x_1} - y_1 z_1, \\ D^\alpha z_1 = -\eta z_1 + \sigma y_1 z_1, \end{cases} \tag{5}$$

and

$$\begin{cases} \dot{x}_2 = x_2(1-x_2) - \frac{x_2 y_2}{\beta + x_2} + u_1, \\ \dot{y}_2 = -\delta y_2 + \frac{\gamma x_2 y_2}{\beta + x_2} - y_2 z_2 + u_2, \\ \dot{z}_2 = -\eta z_2 + \sigma y_2 z_2 + u_3, \end{cases} \tag{6}$$

where u_1, u_2, u_3 are the nonlinear controller. It is reported that the fractional-order lesser date moth system (5) with the fractional order of $\alpha = 0.95$ can behave chaotically [15]. The three-dimensional (3D) phase portraits of the lesser date moth chaotic system with fractional-order and integer-order, respectively, are shown in Figure 1 and Figure 2.

The anti-synchronization error e is defined by

$$\begin{cases} e_1 = x_1 + x_2, \\ e_2 = y_1 + y_2, \\ e_3 = z_1 + z_2. \end{cases} \tag{7}$$

The error dynamics is obtained as

$$\begin{cases} \dot{e_1} = \dot{x_1} + x_2(1-x_2) - \frac{x_2 y_2}{\beta + x_2} + u_1, \\ \dot{e_2} = \dot{y_1} - \delta y_2 + \frac{\gamma x_2 y_2}{\beta + x_2} - y_2 z_2 + u_2, \\ \dot{e_3} = \dot{z_1} - \eta z_2 + \sigma y_2 z_2 + u_3. \end{cases} \tag{8}$$

We consider the nonlinear controller defined by

$$\begin{cases} u_1 = -\dot{x_1} - x_2(1-x_2) + \frac{x_2 y_2}{\beta + x_2} - e_1, \\ u_2 = -\dot{y_1} - \delta y_1 - \frac{\gamma x_2 y_2}{\beta + x_2} + y_2 z_2, \\ u_3 = -\dot{z_1} - \eta z_1 - \sigma y_2 z_2. \end{cases} \tag{9}$$

Substituting (9) into (8), we obtain the linear system

$$\begin{cases} \dot{e_1} = -e_1, \\ \dot{e_2} = -\delta e_2, \\ \dot{e_3} = -\eta e_3. \end{cases} \tag{10}$$

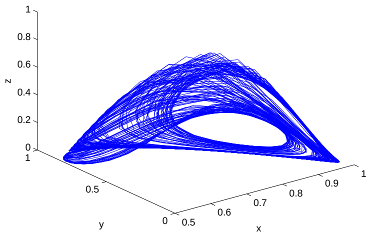

Figure 1: The 3D phase portrait of the fractional-order lesser date moth system.

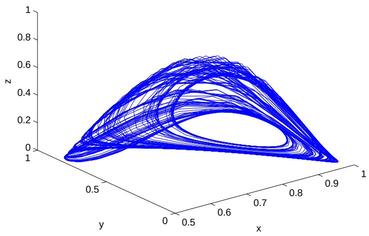

Figure 2: The 3D phase portrait of the integer-order lesser date moth system.

We consider the quadratic Lyapunov function defined by

$$V(e) = \frac{1}{2}e^T e = \frac{1}{2}(e_1^2 + e_2^2 + e_3^2), \tag{11}$$

which is a positive definite function on $\mathbb{R}^3$. A simple calculation gives

$$\dot{V(e)} = -e_1^2 - \delta e_2^2 - \eta e_3^2, \tag{12}$$

which is a negative definite function on $\mathbb{R}^3$.

Thus, by the Lyapunov stability theory [24], the error dynamics (10) is globally exponentially stable. Hence, we have proved the following result.

Theorem 1 *The fractional-order lesser date moth chaotic system and the integer-order chaotic systems (5) and (6) are exponentially and globally anti-synchronized for any initial conditions with the nonlinear controller u defined by (9).*

3.2 Numerical results

For the numerical simulations, we use some documented data for some parameters like $\gamma = 3$, $\delta = \eta = 1$, $\sigma = 3, \beta = 1.15$, $h = 0.85$, $\alpha = 0.95$, then we have $(x_1, y_1, z_1) = (0.7, 0.3, 0.8)$ and $(x_2, y_2, z_2) = (-0.68, -0.91, -0.65)$. The simulation results are illustrated in Figure 3.

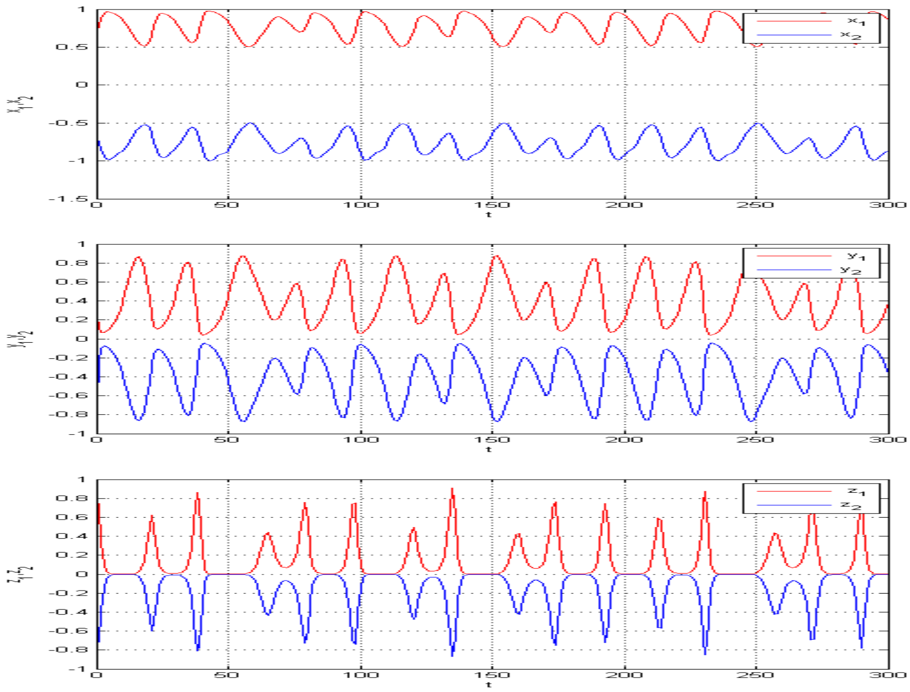

Figure 3: Anti-synchronization between response system (6) and drive system (5).

4 Conclusion

Anti-synchronizing different chaotic systems have important applications in many physical and biological systems, as well as in secure communication using chaotic signals, where one cannot assume that the equations and parameters of the drive and response systems are identical. Furthermore, in the literature, few studies apply nonlinear control theory to anti-synchronize two chaotic systems when a fractional-order system is chosen as the drive system and an integer-order system is the response system. And there is no study regarding the anti-synchronization capability of the approach demonstrated using a fractional-order lesser date moth chaotic system and an integer-order chaotic system. Our goal in this paper was to study the phenomenon of chaos anti-synchronization between a fractional-order lesser date moth chaotic system and an integer-order chaotic system. Our findings show that chaos anti-synchronization can be performed between fractional-order chaotic systems and integer-order chaotic systems using nonlinear control techniques. The numerical outcomes are consistent with the theoretical analyses.

References

[1] A. E. Matouk. Chaos, feedback control and synchronization of a fractional-order modified autonomous Van der Pol-Duffing circuit. *Commun Nonlinear Sci. Numer. Simulat.* **16** (2011) 975–986.

[2] B. Zsolt. Chaos theory and power spectrum analysis in computerized cardiotocography. *Eur. J. Obstet. Gynecol. Reprod. Biol.* **71** (2) (1997) 163–168.

[3] D. Matignon. Stability result on fractional differential equations with applications to control processing. *Computational Engineering in Systems and Application multi-conference, IMACS. In: IEEE-SMC Proceedings.* Lille, France **2** (1996) 963–968.

[4] D. Pazo, M. A. Zaks and J. Kurths. Role of unstable periodic orbit in phase and lag synchronization between coupled chaotic oscillators. *Chaos.* **13** (2003) 309–318.

[5] E. W. Bai and K. E. Lonngren. Synchronization of two Lorenz systems using active control. *Chaos, Solitons and Fractals.* **9** (1998) 1555-1561.

[6] G. Alvarez, S. Li, F. Montoya, G. Pastor and M. Romera. Breaking projective chaos synchronization secure communication using filtering and grneralized synchronization. *Chaos, Solutons and Fractals.* **24** (2005) 775–783.

[7] G.M. Mahmoud, T. Bountis, G.M. AbdEl-Latif and Emad E. Mahmoud. Chaos synchronization of two different chaotic complex Chen and Lü systems. *Nonlinear Dynamics* **55** (2009) 43–53.

[8] G. P. Jiang, K. S. Tang and G. Chen. A simple global synchronization criterion for coupled chaotic systems. *Chaos, Soliton and Fractals.* **15** (2003) 925–935.

[9] G. Q. Si, Z. Y. Sun, and Y. B. Zhang. A general method for synchronizing an integer-order chaotic system and a fractionalorder chaotic system. *Chinese Physics B.* **20** (8) (2011), 080505.

[10] H. Targhvafard and G. H. Enjace. Phase and anti-phase synchronization of fractional-order chaotic systems via active control. *Commun Nonlinear Sci. Numer. Simul.* **16** (2011) 4079–4408.

[11] K. Murali and M. Lakshmanan. Secure communication using a compound signal from generalized synchronizable chaotic system. *Phys. Letters A* **241** (1998) 303–310.

[12] L. Kocarev and U. Parlitz. General approach for chaotic synchronization with applications to communication. *Phys. Rev. Lett.* **74** (1995) 5028–5030.

[13] L. X. Jia, H. Dai and M. Hui. Nonlinear feedback synchronisation control between fractional-order and integer-order chaotic systems. *Chinese Physics B.* **19** (11) (2010), Article ID 110509.

[14] M. C. Ho, and Y. C. Hung. Synchronization of two different chaotic systems by using generalized active control. *Physics Letters A* **301** (2002) 424–428.

[15] M. El-Shahed, Juan J. Nieto, A. M. Ahmed and I. M. E. Abdelstar. Fractional-order model for biocontrol of the lesser date moth in palm trees and its discretization. *Advance in Difference Equations* (2017) 2017–2295.

[16] M. Labid and N. Hamri. Chaos Synchronization and Anti-Synchronization of two Fractional-Order Systems via Global Synchronization and Active Control. *Nonlinear Dynamics and Systems Theory* **19** (3) (2019) 416–426.

[17] M. Labid and N. Hamri. Chaos Synchronization between Fractional-Order Lesser Date Moth Chaotic System and Integer-Order Chaotic System via Active Control. *Nonlinear Dynamics and Systems Theory* **22** (4) (2022) 407–413.

[18] M. S. Abd-Elouhab, N. Hamri and J. Wang. Chaos Control of a Fractional-Order Financial System. *Mathamatical Problems in Engineering* (2010), Article ID270646, 18 pages, doi: 10.1155/2010/270646.

[19] N. Laskin. Fractional market dynamics. *Physics A* **287** (2000) 482–492.

[20] N. Zhou, Y. Wang, L. Gong, H. He and J. Wu. Novel single-channel color image encryption algorithm based on chaos and fractional Fourier transform. *Optics Communications* **284** (2011) 2789–2796.

[21] P. Zhou, Y. M. Chen and F. Kuang. Synchronization between fractional-order chaotic systems and integer orders chaotic systems (fractional-order chaotic systems). *Chinese Physics B* **19** (9) (2010), 090503.

[22] R. Mainieri and J. Rehacek. Projective synchronization in three dimensional chaotic systems. *Phys. Rev. Lett.* **82** (1999) 3042–3045.

[23] S. Vaidyanathan. Lotka-Volterra two-species mutualistic biology models and their ecological monitoring. *Pharm Tech Research.* **8** (2015) 199–212.

[24] W. Hahn. *The Stability of Motion.* Springer-Verlag, Berlin, 1967.

[25] W. H. Deng and C. P. Li. Chaos synchronization of the fractional Lü system, *Physica A.* **353** (2005) 61–72.

[26] Y. Zhang and J. Sun. Chaotic synchronization and anti-synchronization based on suitable separation. *Physics Letters A* **330** (2004) 442–447.