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The Regularization Method For Solving Sub-Riemannian Geodesic Problems

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Abstract: In this paper, we used the regularization method to prove some properties of the sub-Riemannian geodesics in infinite dimension for a Hilbertian manifold. More precisely, we generalize the result obtained by S.Nikitin [14], so we prove that the sub-Riemannnian distance for the Hilbert-Schmidt distribution can be approximated by the smooth sub-Riemannian geodesics.

Keywords: *regularization method; geodesics; sub-Riemannian geometry; control problem; Hamilton's equation.*

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1 Introduction

In finite-dimension context, a sub-Riemannian distance between two fixed points is defined by the infimum length of curves connecting them and whose velocity is constrained to be tangent to sub-vector space (distribution) of the tangent space T_xM of a Riemannian manifold M, where $x \in M$. Such curves are called horizontal. The distance is finite if every pair of points can be connected by at least one horizontal curve and is achieved on the curves of minimal length. Finding a length minimizer is an optimal control problem, the extremals of this problem are called the sub-Riemannian geodesics. According to the Pontryagin maximum principle [6, 10, 15, 16], the optimal curves are of two types: abnormal curves and normal geodesics which are the projections of the Hammiltonian trajectories. In [14], in finite dimension, S.Nikitin presented conditions under which the sub-Riemannian distance can be measured by an infinitely smooth sub-Riemannian

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geodesics. This result arises from the fact that sometimes the sub-Riemannian distance is measured by abnormal extremals [10, 12, 13]. Our objective is to give a generalization of this result for an infinite dimensional manifold.

The first problem when we consider a control problem is that of controllability [1,9,17], which presents the first difference between the finite dimension and infinite-dimensional cases, so that the infimum could be not reached even for the Riemannian-Hilbertian manifold. The same is true in the general sub-Riemannian manifold. So the second difference is that the Pontryagin maximum principle is not available any longer. However, we still have the strong Chow-Rashevski theorem developed for the manifold modeled on Hilbert spaces and the maximum principle for certain special cases. Using them, we give analogue properties for the sub-Riemannian structure generated by a bilinear distribution of Hilbert-Schmidt.

In this work, we show that the problem of the length minimization is a control problem and we give a characterization of smooth geodesics where we use a variant of Pontryagin's maximum principle [3] and we also prove that in infinite dimension and under some conditions, we can approximate a sub-Riemannian distance by a normal sub-Riemannian geodesics. The structure of the paper is as follows. In Section 2, we introduce notations and briefly review some natural objects associated to a sub-Riemannian structure in an infinite dimensional manifold modeled on the Hilbert space. The results on the bilinear Hilbert-Schmidt distribution are given in Section 3. We characterize the sub-Riemannian geodesics in Section 4. To accomplish our objective, we replace the sub-Riemannian problem by the regularized one and we present certain conditions under which we prove, at the first step, the existence of the sub-Riemannian geodesics and, at the second step, we measure the sub-Riemannian distance by a normal geodesics. For the proof of all these results, we use some classical techniques of the functional analysis.

2 Preliminary Results

In this section, we will recall some basic notions of sub-Riemannian geometry in infinite dimension, for more details, we refer the reader to [1, 7].

2.1 Sub-Riemannian structure in infinite dimension

Let M be a connected manifold modelled on a Hilbert space E, TM be the tangent bundle of M, then according to [7], we have the following definition.

Definition 2.1 A sub-Riemannian structure is a triple $(M, \mathcal{F}, \mathbf{h})$, where

- M is a Hilbert connected manifold;
- $\mathcal{F}$ is a sub-bundle of TM.
- $\mathbf{h}$ is a Riemannian metric on $\mathcal{F}$.

Remark 2.1
- Given a Riemannian metric $\overline{h}$ on M, we get a Riemannian metric on $\mathcal{F}$ by restriction. On the other hand, there always exists a complementary $\mathcal{V}$ of $\mathcal{F}$ and so we can extend $\mathbf{h}$ into the Riemannian metric $\overline{h}$ on M, which means that $TM = \mathcal{V} + \mathcal{F}$.
- The requirement of the splitting is non-trivial if M is not modeled on a Hilbert space, see [7], this splitting implies that there exists a smooth projection from TM to $\mathcal{F}$.

Definition 2.2 A horizontal curve is a smooth curve $\gamma : [a,b] \to M$ such that

$$\dot{\gamma} \in \mathcal{F}_{\gamma(t)} \quad \textit{for every}\, t \in [a,b]. \tag{1}$$

According to the previous definitions, we can also define the following.

Definition 2.3 1. The length of a horizontal curve is given by

$$l(\gamma) = \int_a^b \sqrt{\mathbf{h}(\gamma(\dot{t}), \gamma(\dot{t}))}dt. \tag{2}$$

2. The sub-Riemannian energy functional (Action) is given by

$$e(\gamma) = \int_a^b \mathbf{h}(\gamma(\dot{t}), \gamma(\dot{t}))dt. \tag{3}$$

The sub-Riemannian distance between the distinct points is defined by

$$d_{\mathcal{F}}(x_0, x_1) = \inf \left\{ l(\gamma); \dot{\gamma} \in \mathcal{F}_{\gamma(t)}, \gamma(a) = x_0, \gamma(b) = x_1 \right\}. \tag{4}$$

The problem of the length minimization is equivalent to the problem of energy minimization, which allows us to formulate the first-order condition for length minimizer.

Definition 2.4 (see [7]) A horizontal curve γ is called a ***sub-Riemannian geodesics*** if

$$\partial_s e(\gamma^s)|_{s=0} = 0, \qquad \textit{for any} \quad \gamma^s \in \mathcal{J}_{\mathcal{F}}(\gamma), \tag{5}$$

where $\mathcal{J}_{\mathcal{F}}(\gamma)$ is the collection of all $\mathcal{F}$- horizontal variations of γ.

Our aim is to characterize the normal sub-Riemannian geodesics for the bilinear Hilbert-Schmidt distribution on the Hilbertian manifold. We consider a manifold modeled on the Hilbert space with a strong Riemannian metric.

3 Bilinear Distribution

From [3–5], we recall all definitions, properties and results we shall use in this work. Let E and F be two Hilbert spaces, and let $A \in \mathcal{L}(F;E)$ $(u \to A_u)$, $B \in \mathcal{L}(F;(E;E))$ $(u \to B_u)$ and $\tilde{B} \in \mathcal{L}(F \times E; E)$ be an operator associated to B and defined by

$$\forall\, u \in F,\ \forall\, x \in E\ \tilde{B}(x,u) = B_u x,$$

where $\mathcal{L}(F;E)$ is the space of linear bounded operators from F to E and $\mathcal{L}(F;(E;E))$ is the space of the bounded operators from F to $\mathcal{L}(E;E)$. Denote by $\{f_i; i \in \mathbb{N}\}$ a Hilbert basis for F and set

$$X_i(x) = A_{f_i} + B_{f_i}x.$$

We denote also by $\mathcal{F}$ the distribution spanned by $\{X_i, i \in \mathbb{N}\}$.

We consider the associated system defined by

$$\dot{x} = Au + \tilde{B}(u,x). \tag{6}$$

We will say that (6) is a bilinear system of E and $\mathcal{F}$ is a **bilinear distribution**. For a given bilinear distribution $\mathcal{F}$ on a Hilbertian manifold, for all horizontal curves $\gamma : [0,T] \to E$ that are tangent to $\mathcal{F}$, there exists a control $u : [0,T] \to F$ such that

$$\dot{\gamma} = Au + \tilde{B}(u, \gamma). \tag{7}$$

We can assume that all horizontal curves are defined on $[0,1]$, after changing the parametrization if necessary. Then for any horizontal curve, we can define its length $\mathcal{L}(\gamma)$ by

$$l(\gamma) = \int_0^1 \|u(t)\|_F dt, \tag{8}$$

and the energy of a horizontal curve is defined by

$$e(\gamma) = \frac{1}{2} \int_0^1 \|u(t)\|_F^2 dt, \tag{9}$$

where $u \in L^2([0,1], F)$ and $\|\|_F$ denotes the Hilbertian norm on F. In this case, all requirements of the previous definitions are satisfied.

When A and B are the Hilbert-Schmidt operators, the associated distribution $\mathcal{F}$ is called a **Hilbert-Schmidt distribution.** Then we have the following result.

Lemma 3.1 *[3] If $\mathcal{F}$ is a bilinear Hilbert-Schmidt distribution, then to each horizontal curve we can associate a control u and conversely.*

Example 3.1 (see [3]) Let K, H be two separable Hilbert spaces, if we denote by $\{k_i; i \in \mathbb{N}\}$ a Hilbert basis for K and by $\{h_\alpha; \alpha \in \mathbb{N}\}$ a Hilbert basis for H, then the set $\{h_\alpha \otimes k_i\}$ is a Hilbert basis for the space of Hilbert-Schmidt operators $\mathcal{L}_{HS}(K;H)$. On $\mathcal{G} = \mathcal{L}_{HS}(K;H) \oplus K \oplus H$, we define a generalized Heisenberg-Lie algebra structure by setting $F = \mathcal{L}_{HS}(K;H) \oplus K$ and $Z = H$ with Lie brackets defined, with respect to the basis $Y_{\alpha i} = (h_\alpha \otimes k_i, 0)$ and $X_i = (0, k_i)$ of F, by

$$[Y_{\alpha i}, X_i] = C_{\alpha i} Z_\alpha,$$

where Z_α is the basis of Z and $C_{\alpha i}$ are constants, the other Lie brackets are zero.
Let G be a Lie group with Lie algebra $\mathcal{G}$. F induces on G a left invariant distribution $\mathcal{F}$, this distribution is a Hilbert-Schmidt distribution if $\sum_{\alpha i} C_{\alpha i} < \infty$.

4 Optimal Control Viewpoint

Looking for the sub-Riemannian geodesics between two points means solving the smooth infinite minimization problem

$$e(\gamma) = e(u) = \frac{1}{2} \int_0^1 \|u(t)\|_F^2 dt \to \inf, \tag{10}$$

where $u : [0,T] \to F$ with the following constraint:

$$\begin{cases} \dot{x}(t) = Au(t) + \tilde{B}(u(t), x(t)), \\ x(0) = x_0, x(1) = x_1, \end{cases} \tag{11}$$

where $A \in \mathcal{L}(F,E)$, $B \in \mathcal{L}(F,\mathcal{L}(E,E))$ and x_0, x_1 are two given points of E. So we have a bilinear control problem where the spaces of a control and state have an infinite dimension.

In finite dimension, we always use the maximum principle [2,15] to calculate the optimal trajectories. Unfortunately, in infinite dimension, we loose Pontryagine's Maximum Principle. However, in this context, we can apply a variant of the maximum principle [3,5] which gives us a characterization of the optimal curve for a bilinear distribution on the Hilbertian manifold. In the case when the set of control is contained in a closed bounded convex subset and the operators A, B are compact, this characterization is similar to the finite dimensional case, see [6].

Theorem 4.1 *(see [4]) Let $u \in L^2([0,T];K)$, where K is a closed bounded convex subset, B and A are compact for all $t \in [0,T]$, there exists a control $\bar{u}$ which minimizes the functional e and, moreover, $\bar{u}$ satisfies the following relation for almost all $t \in [0,T]$:*

$$\left\langle A\bar{u} + \tilde{B}(\bar{u},\bar{x}), \bar{p} \right\rangle + p_0\|\bar{u}\|^2 = \min_{v\in K} \left\langle Av + \tilde{B}(v,x), p \right\rangle + p_0\|v\|^2, \tag{12}$$

where $\bar{x}$ is the trajectory associated to $\bar{u}$ and where $\bar{p}$ is a mild solution of the adjoint system

$$\frac{d}{dt}\bar{p} = -B^*_{\bar{u}}p. \tag{13}$$

$B^*_{\bar{u}}$ is the adjoint of $B_{\bar{u}}$.

Under these assumptions, and following the terminology introduced in [1], we can distinguish two types of the extremal.

Definition 4.1 An extremal of minimization problem (10)-(11), i.e., a couple $(\bar{x}, \bar{p},)$ meeting the condition of Theorem 4.1 is called the *normal bi-extremal* if $p_0 \neq 0$ (which can be normalized to 1), and the *abnormal bi-extremal* if $p_0 = 0$.

In the sequel of this work, we assume that $u(t) \in K$, where K is a closed bounded convex subset, B and A are compact.

4.1 Characterization of normal geodesics

The following proposition gives the link between the normal extremal of Theorem 4.1 and the normal geodesics.

Proposition 4.1 *Let γ be a horizontal curve, then the following assertions are equivalent:*

1. *γ is a critical point of the energy function with a fixed end point;*
2. *there exists a covector p such that the couple (γ, p) is a normal bi-extremal of maximum principle.*

Proof. The proof of this result is an adaptation, step by step, of the proof of the corresponding result of Proposition 2 in [1].

Remark 4.1 By the previous proposition, we deduce that the normal geodesics is a solution to the Hamiltonian system

$$\begin{cases} \dot{x} & = \frac{\partial}{\partial p} H(x,p), \\ \dot{p} & = -\frac{\partial}{\partial x} H(x,p), \\ H(x,p) & = \frac{1}{2} \left\| (A+B(x))^{*} p \right\|^{2}. \end{cases} \tag{14}$$

We recall that our objective is to generalize the result obtained by S.Nikitin in [14] to infinite dimension. We present new conditions under which the sub-Riemannian distance can be approximated by a normal sub-Riemannian geodesics. To attain this goal, we use the regularization method.

5 Regularization Procedure

We use the regularization method to replace a minimization problem with constraint by another one without constraint.

5.1 Regularized problem

At first, we need the following hypotheses

Let $L : E \times TE \times F \to E, \eta = (x,u) \to L(x,u) = \dot{x} - (A + B(x))u$, and $G : E \times F \to F, \eta = (x,u) \to u$,

where $A \in \mathcal{L}(F,E)$, $B \in \mathcal{L}(F, \mathcal{L}(E,E))$ and TE is the tangent space of E.

We assume that L and G satisfy the following assumptions.

Assumption 5.1 The set

$$U_L = \left\{ \begin{array}{c} (x,u) \in E \times F\colon \|L(x,u)\|_{L^2} = \|\dot{x} - (A+B(x))\,u\|_{L^2} = \\ \mu = \inf\limits_{(x,u)\in D} \|\dot{x} - (A+B(x))\,u\|_{L^2}\,, \end{array} \right\}$$

is not empty, where $D = E \times K$. We define also

$$\hat{U} = \left\{ (x,u) \in E \times F\colon \|G(x,u)\| = \|u(t)\| = \nu_F = \inf_{(x,u)\in U_L} \|u\,(t)\|_{\ell^2} \right\},$$

where $L^2([0,1],F)$ is identified to the space $L^2([0,1],l^2(\mathbb{N}))$ via the Hilbertian basis $\{f_i; i \in \mathbb{N}\}$ of F (see [3]).

Assumption 5.2 There exists $c > 0$ such that

$$W_C = \{(x,u) \in E \times F\colon \|L(x,u)\| \le c, \|G(x,u)\| \le c\}$$

is not empty and bounded.

The regularized problem is

$$J_\alpha(\gamma) = \frac{1}{2}\,\|L(x,u)\|_{\mathbb{E}} + \frac{\alpha}{2}\,\|G(x,u)\|_{\mathbb{F}} \to \inf, \tag{P_α}$$

where $\alpha > 0$ denotes the regularization parameter.

Theorem 5.1 *Under Assumptions* 5.1 *and* 5.2*, the problem* (P_α) *has a solution.*

Proof. Let $\{\gamma_\alpha^n\}$ be a minimizing sequence

$$\gamma_\alpha^n = \{(x_\alpha^n(t), u_\alpha^n(t))\,; t \in [0,1]\} \subset D \qquad n = 1, 2, ..., \tag{15}$$

such that $m_\alpha \leq J_\alpha(\gamma^n) \leq m_\alpha + \frac{1}{n}$, $n = 1, 2, ...$, then we have

$$\|L(x_\alpha^n, u_\alpha^n)\|_{\mathbb{E}} = \|\dot{x}_\alpha^n - (A + B(x_\alpha^n)\, u_\alpha^n\| \leq (m_\alpha + 1)^{\frac{1}{2}}\,,$$

$$\|G(x_\alpha^n, u_\alpha^n)\|_{\mathbb{F}} = \|u_\alpha^n\| \leq \left(\frac{m_\alpha + 1}{\alpha}\right)^{\frac{1}{2}}.$$

We take

$$c = max\left\{\left(\frac{m_\alpha + 1}{\alpha}\right)^{\frac{1}{2}}, (m_\alpha + 1)^{\frac{1}{2}}\right\}.$$

It is clear that $\{\gamma_\alpha^n\} \subset W_c$, as we have already noticed that the set W_c is weakly compact, then the sequence (γ_α^n) is weakly convergent, i.e.,

$$((x_\alpha^n(t), u_\alpha^n(t))) \overset{\text{weakly}}{\longrightarrow} (x_0(t), u_0(t)) \qquad\qquad (\text{in } H = E \times F)$$

$$\dot{x}_\alpha^n(t) - (A + B(x_\alpha^n)\, u_\alpha^n(t) \overset{\text{weakly}}{\longrightarrow} r$$
$$u_\alpha^n(t) \overset{\text{weakly}}{\longrightarrow} u_0(t).$$

As the operators L, G are jointly weakly closed on D, then we have

$$\dot{x}_0(t) - (A + B(x_0)\, u_0(t) = r.$$

It remains to prove that $(x_0(t), u_0(t))$ is a solution of the problem (P_α).

We use the lower semi-continuity of the norm in a Hilbert space, we find that

$$m_\alpha \leq J_\alpha(\gamma^0) \leq \lim_{n\to\infty} \inf J_\alpha(\gamma^n) \leq \lim_{n\to\infty} \sup J_\alpha(\gamma^n) \leq m_\alpha,$$

then $J_\alpha(\gamma^0) = m_\alpha$.

Now we define a new hypothesis to show that under these conditions and Assumptions 5.1-5.2, the sub-Riemannian distance can be measured by normal minimizers.

Assumption 5.3 The distribution $\mathcal{F}$ satisfies the strong Chow-Rashevsky property [1], then there exists a control $v(t)$ which steers the system

$$\dot{x} = (A + B(x))v$$

from the state x_0 to the state x_1.

Assumption 5.4 The system (6) satisfies the following condition (at points x_0 and x_1): if there exist real numbers $\delta > 0$, $P > 0$ and $Q > 0$ such that

$$\forall 0 < \alpha \leq \delta \qquad \left\|\sqrt{\alpha} p_\alpha(0)\right\|_E \leq P \;\Rightarrow\; \|p_\alpha(0)\|_E \leq Q, \tag{16}$$

$p(0)$ should be chosen so that $x\ (1) = x_1$, where x is the solution to the following Hamiltonian system:

$$\begin{cases} \dot{x} &= \frac{\partial}{\partial p} H_\alpha(x,p), \\ \dot{p} &= -\frac{\partial}{\partial x} H_\alpha(x,p), \\ x(0) &= x_0, \end{cases} \tag{17}$$

$$H_\alpha(x,p) = \frac{1}{2}\left\|(A + B(x))^* p\right\|^2 + \frac{\alpha}{2}\|p\|^2 .$$

Our principal result is the following theorem.

Theorem 5.2 *Consider the sub-Riemannian problem*

$$\int_0^1 \|u(t)\|^2 \, dt \to \inf,$$

where u is the unique solution of the system

$$\dot{x} = (A + B(x))u. \tag{$\sum$}$$

Suppose that Assumptions 5.1,5.2,5.3 *and* 5.4 *are satisfied. Then, for all given x_0, $x_1 \in E$, the regularized solutions converge to the normal geodesics solution, i.e.,*

$$\|x_\alpha - \tilde{x}(t)\| \to 0$$

and

$$H_\alpha \to H,$$

where H_α, H are given in Assumption 5.4 *and* (14).

Proof. According to [19], the solution $(x_\alpha(t), u_\alpha(t))$ (normal) of regularized problem (P_α) necessarily satisfies the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}} L_\alpha(\dot{x}, x, u)\right) - \frac{\partial}{\partial x} L_\alpha(\dot{x}, x, u) = 0, \frac{\partial}{\partial u} L_\alpha(\dot{x}, x, u) = 0,$$

where L_α is the Lagrangian which is given by

$$L_\alpha(\dot{x}, x, u) = \frac{1}{2}\|L(x,u)\|_{\mathbb{E}} + \frac{\alpha}{2}\|G(x,u)\|_{\mathbb{F}} = \frac{1}{2}\|\dot{x} - (A + B(x))\, u\|^2_{\mathbb{E}} + \frac{\alpha}{2}\|u\|^2_{\mathbb{F}} .$$

As the Lagrangian L_α is hyper regular, then according to [8], we can define p as

$$p = \frac{1}{\alpha}\frac{\partial}{\partial \dot{x}} L_\alpha(\dot{x}, x, u). \tag{18}$$

Using the Euler-Lagrange equations, and according to [18], we can easily write

$$\begin{cases} \dot{x}_\alpha &= \frac{\partial}{\partial p} H_\alpha(x_\alpha, p_\alpha), \\ \dot{p}_\alpha &= -\frac{\partial}{\partial x} H_\alpha(x_\alpha, p_\alpha), \\ u_\alpha &= (A + B(x_\alpha))^* p_\alpha, \end{cases}$$

where

$$H_\alpha(x_\alpha, p_\alpha) = \frac{1}{2}\left\|(A + B(x_\alpha))^t p_\alpha\right\|^2 + \frac{\alpha}{2}\|p_\alpha\|^2 .$$

By Assumption 5.3, there exists a control $v(t)$ which steers the system

$$\dot{x} = (A + B(x))\, v$$

from x_0 to x_1, for which we have

$$\int_0^1 \frac{1}{2} \|\dot{x}_\alpha(t) - (A + B(x_\alpha(t))u_\alpha(t))\|_E^2\, dt + \frac{\alpha}{2} \|u_\alpha(t)\|_F^2\, dt \le \frac{\alpha}{2} \int_0^1 \|v(t)\|_F^2\, dt.$$

Set

$$k = \frac{1}{2} \int_0^1 \|v(t)\|_F^2\, dt.$$

As

$$H_\alpha = \frac{1}{\alpha} L_\alpha,$$

then

$$H_\alpha(x_\alpha, p_\alpha) \le k \quad \forall \alpha > 0 \quad \forall t \in [0,1],$$

it implies

$$\|\alpha p_\alpha\| \to 0 \quad \text{while} \qquad \alpha \to 0.$$

According to Assumption 5.1, there exists a positive constant δ such that

$$\|\alpha p_\alpha\| \le \delta,$$

the function $x_\alpha(t)$ is bounded on $[0,1]$, i.e.

$$\|x_\alpha(t)\| \le \beta,$$

and as

$$\left\| \frac{\partial}{\partial p} H_\alpha(x_\alpha, p_\alpha) \right\|$$

is bounded, we have the same for $\dot{x}_\alpha(t)$, i.e.,

$$\left\| \dot{x_\alpha}(t) \right\| \le G_1,$$

where G_1 is a positive constant which does not depend on α; on the other hand,

$$\frac{1}{2} \|L(x_\alpha, u_\alpha)\|_{\mathbb{E}} + \frac{\alpha}{2} \|G(x_\alpha, u_\alpha)\|_{\mathbb{F}} \le \frac{1}{2} \|L(x, u)\|_{\mathbb{E}} + \frac{\alpha}{2} \|G(x, u)\|_{\mathbb{F}} \tag{19}$$

for all $u \in U_L$, then we have

$$\|G(x_\alpha, u_\alpha)\|_{\mathbb{F}} \le \|G(x, u)\|_{\mathbb{F}}, \|L(x_\alpha, u_\alpha)\|_{\mathbb{E}} \le \mu_L + \alpha \nu_F. \tag{20}$$

From the previous inequality, the families $\{(x_\alpha, u_\alpha)\}$ $\{G(x_\alpha, u_\alpha)\}$ $\{L(x_\alpha, u_\alpha)\}$ are weakly compact, there exist weakly convergent sub-families, i.e.,

$$(x_{\alpha_j}, u_{\alpha_j}) \to (\tilde{x}, \tilde{u}),$$

$$L(x_{\alpha_j}, u_{\alpha_j}) \to r.$$

So

$$L(\tilde{x}, \tilde{u}) = r,$$

we replace $(\tilde{x}, \tilde{u})$ in (19) and (20), we find

$$\lim_{\alpha_j \to 0} \left\| L(x_{\alpha_j}, u_{\alpha_j}) - L(\tilde{x}, \tilde{u}) \right\|_{\mathbb{E}} = 0,$$

$$\lim_{\alpha_j \to 0} \left\| G(x_{\alpha_j}, u_{\alpha_j}) - G(\tilde{x}, \tilde{u}) \right\|_{\mathbb{F}} = 0.$$

Using the previous results and the Gronwall inequality, we prove that the solution $x_{\alpha_j}(t)$ will converge strongly to $\tilde{x}$,

$$\alpha_j \to 0 \quad \text{while} \quad j \to \infty$$

and

$$\left\| x_{\alpha_j}(t) - \tilde{x}(t) \right\| \to 0 \qquad \text{while} \quad j \to \infty.$$

On the other hand, for the control $v(t)$ which steers the system

$$\dot{x}(t) = (A + B(x(t))v(t)$$

from x_0 to x_1 and for any $\alpha > 0$,

$$\int_0^1 \|u_\alpha(t)\|^2 \, dt \leq \int_0^1 \|v(t)\|^2 \, dt.$$

This proves that $\tilde{x}(t)$ is a minimizing curve which measures the sub-Riemannian distance between x_0 and x_1.

The functions $\dot{p}_\alpha$ and p_α are bounded,

$$\dot{p}_\alpha = -p_\alpha^t \left(A + B(x)\right) B^t p_\alpha,$$

but

$$\left(A + B(x)\right)^t p_\alpha = u_\alpha,$$

by substituting u_α in the above expression, we obtain

$$\begin{array}{rcl} \|p_\alpha(t)\| & = & \left\| p_\alpha(0) + \int_0^t -u_\alpha(s) \, B^t p_\alpha(s) ds \right\| \\ & \leq & \|p_\alpha(0)\| + \left\| \int_0^t -u_\alpha(s) \, B^t p_\alpha(s) ds \right\|, \end{array}$$

then

$$\sqrt{\alpha} \, \|p_\alpha(t)\| \leq \sqrt{2k} \quad \forall t \in [0,1].$$

From Assumption 5.4, we have

$$\|p_\alpha(0)\| \leq \delta,$$

then

$$\|p_\alpha(t)\| \leq c \int_0^t \|p_\alpha(s)\| \, ds.$$

According to the Gronwall inequality, we get

$$\|p_\alpha(t)\| \leq G.$$

We also have

$$\left\| \frac{\partial}{\partial x} H_\alpha(x_\alpha, u_\alpha) \right\| \leq l.$$

It follows that $\dot{p}_\alpha$ is also bounded. The proof of the convergence of p_α to a continuous function p is similar to x_α; by passing to the limit in H_{α_j}, we obtain

$$H_{\alpha_j} \to H.$$

6 Conclusion

In this paper, we studied some properties of bilinear extremals in infinite dimension, these properties have a direct application in sub-Riemannian geometry, especially in the case of a sub-Riemannian structure generated by a bilinear Hilbert-Schmidt distribution. We prove also that, under some conditions, a sub-Riemannian distance can be approximated by a normal geodesics. These results remain valid for a manifold modeled on a Hilbert space, we can also generalize these results for the Banach manifold.

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