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Inverse Problem of a Semilinear Parabolic Equation with an Integral Overdetermination Condition

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Abstract: The solvability of the semilinear parabolic problem with integral overdetermination condition for an inverse problem is investigated in this work. Accordingly, we solve the generated direct problem by using the so-called "energy inequality" method and then the inverse problem is handled with the use of the fixed point technique.

Keywords: *inverse problem; nonlocal integral condition; fixed point theorem.*

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1 Introduction

The goal of this research was to investigate the solvability of a pair of functions $\{y, f\}$ that satisfy the following semilinear parabolic problem:

$$y_t - a\frac{\partial^2 y}{dx^2} + by + cy^3 = f(t)h(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{1}$$

with the initial condition

$$y(x,0) = \varphi(x), \quad x \in \Omega, \tag{2}$$

the boundary condition

$$y(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3}$$

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and the nonlocal overdetermination condition

$$\int_{\Omega} y(x,t)v(x)dx = E(t), \quad t \in (0,T), \tag{4}$$

where Ω is a bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$, the functions g, φ, and E are well-known and β is a positive constant. In this case, supplementary or additional information about the solution of the main problem comes in the form of integral condition (4).

Inverse boundary value problems exist in a variety of domains, including seismology, biology and physics [1,2]. Inverse problems for parabolic equations satisfying the nonlocal overdetermination condition were first investigated in [3–5], whereas the references [6–8] discussed this subject for equations with time-independent coefficient under first and third-order boundary conditions. Several solvability investigations of the inverse problem and others were carried out in [9–12]. The theory of the existence and uniqueness of the inverse problem has been examined by many authors, see [13–17] and also [18–20]. In the present work, a new study for the inverse problem of a semilinear parabolic equation is presented. The existence and uniqueness of the classical solution to problem (1)-(4) are analysed by a fixed point technique.

2 Preliminaries

Let us now give certain notations and rules that we will use:

$$g^*(t) = \int_{\Omega} v(x)h(x,t)dx, \quad Q = \Omega \times (0,T).$$

We use also the well-known inequality (Cauchy's ε-inequality)

$$2|ab| \leq \varepsilon a^2 + \frac{1}{\varepsilon}b^2, \quad a,b \in \mathbb{R}.$$

Lemma 2.1 *(Gronwall's Lemma) Let $f \in L^{\infty}(0,T)$, $g \in L^1(0,T)$ and $f(t) \geq 0, g(t) \geq 0$. If we have*

$$f(t) \leq c + \int_0^{\tau} f(s)g(s)ds,$$

then

$$f(t) \leq cexp(\int_0^{\tau} g(s)ds).$$

Lemma 2.2 *(Poincare Inequality) If Ω is bounded in at least one direction, then there exists a constant $c = c_{\Omega,p} > 0$ such that*

$$\int_{\Omega} |u|^p dx \leq c(\sum_{i=1}^{n} \int_{\Omega} |\frac{\partial u}{\partial x_i}|^p dx),$$

or, what is equivalent,

$$\|u\|_{L^p(\Omega)} \leq c' \|\nabla u\|_{(L^p(\Omega))^n}, \forall u \in W_0^{1,p}(\Omega),$$

where c' is a constant dependant on c given by

$$c' = c^{\frac{1}{p}}.$$

3 Existence and Uniqueness of the Solution to the Direct Problem

3.1 Setting of the problem

In the rectangle $Q = (0,1) \times (0,T) = \Omega \times (0,T)$, with $T < \infty$, we consider the semilinear parabolic problem

$$(P) \qquad \begin{cases} y_t - a\dfrac{\partial^2 y}{dx^2} + by + cy^3 = f(x,t), & (x,t) \in \Omega \times (0,T), \\ y(x,0) = \varphi(x), \qquad x \in \Omega, \\ y(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \end{cases}$$

$$\mathcal{L}y = y_t - a\frac{\partial^2 y}{dx^2} + by + cy^3 = f(x,t), \tag{5}$$

with the initial condition

$$ly = y(x,0) = \varphi(x), \qquad x \in \Omega, \tag{6}$$

and the Dirichlet boundary condition

$$y(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{7}$$

where the functions $f(x,t)$ and $y_0(x)$ are known functions and a, b, c are also given constants that verify the following hypothesis:

$$A1 : a \geq 0, b \geq 0, c \geq 0.$$

The operator L is defined from E to F, where E is the Banach space, which contains all functions $y(x,t)$ with finite norms

$$\|y\|_E^2 = \|y\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\frac{\partial y}{\partial x}\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} + \|y\|^4_{L^4(Q)}.$$

Besides, F represents the Hilbert space, which includes all elements $\mathcal{F} = (f, \varphi)$ for which the norm

$$\|\mathcal{F}\|_F^2 = \|f\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(\Omega)}$$

is finite.

3.1.1 A priori estimate

Theorem 3.1 *Let condition A1 be satisfied. Then for any function $y \in D(L)$, we have the inequality*

$$\|y\|_E \leq C\|Ly\|_F,$$

where C is a positive constant independent of y and $D(L)$ denotes the domain of definition of the operator L, which can be defined by

$$D(L) = \{y \setminus y, y_t, \frac{\partial y}{\partial x}, \frac{\partial^2 y}{\partial x^2} \in L^2(Q), y \in L^4(Q)\}.$$

Proof. Taking the scalar product in $L^2(Q)$ of (5) and $My = y$, we have

$$\begin{aligned} < \mathcal{L}y, My >_{L^2(Q)} &=< y_t, y >_{L^2(Q)} - a < \frac{\partial^2 y}{dx^2}, y >_{L^2(Q)} + b < y, y >_{L^2(Q)} + c < y^3, y >_{L^2(Q)} \\ &=< f, y >_{L^2(Q)} . \end{aligned} \tag{8}$$

Integrating (3.1.1) and applying Cauchy's ε-inequality yield

$$\begin{aligned} &\frac{1}{2}\|y(.,\tau)\|^2_{L^2(\Omega)} + a\|\frac{\partial y}{\partial x}\|^2_{L^2(Q)} + b\|y\|^2_{L^2(Q)} + c\|y\|^4_{L^4(Q)} \\ &\qquad \le \frac{1}{2\varepsilon}\|f\|^2_{L^2(Q)} + \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)} + \frac{\varepsilon}{2}\int_0^T \|y\|^2_{L^2(\Omega)} dt. \end{aligned}$$

Using Gronwall's lemma and the fact that the right-hand side is not related to τ, we substitute the left-hand side with its upper bound with respect to τ from 0 to T to obtain

$$\|y\|^2_{L^\infty(0,T,L^2(\Omega))} + \|\frac{\partial y}{\partial x}\|^2_{L^2(Q)} + \|y\|^2_{L^2(Q)} + \|y\|^4_{L^4(Q)} \le C(\|f\|^2_{L^2(Q)} + \|\varphi\|^2_{L^2(\Omega)}),$$

where

$$C = \frac{max(\frac{c'}{2}, \frac{c'}{2\varepsilon})}{min(\frac{1}{2}, a, b, c)} \quad \text{and} \quad c' = exp(\frac{\varepsilon T}{2}).$$

Consequently, we have

$$\|y\|_E \le C\|Ly\|_F. \tag{9}$$

Proposition 3.1 *The operator L from E to F has a closure.*

Proof. Let $(y_n)_{n\in\mathbb{N}} \subset D(L)$ be a sequence such that

$$y_n \longrightarrow 0 \qquad in \quad E$$

and

$$Ly_n \longrightarrow (f, \varphi) \qquad in \quad F. \tag{10}$$

Herein, we should prove that

$$f \equiv 0, \varphi \equiv 0 \quad in \quad F.$$

The convergence of y_n to 0 in E entails that

$$y_n \longrightarrow 0 \quad in \quad D'(Q). \tag{11}$$

According to the continuity of the derivation of $D'(Q)$ and the continuity distribution of the function y^2, relation (11) involves

$$\mathcal{L}y_n \longrightarrow 0 \qquad in \quad D'(Q). \tag{12}$$

Also, the convergence of Ly_n to f in $L^2(Q)$ gives

$$\mathcal{L}y_n \longrightarrow f \qquad in \quad D'(Q). \tag{13}$$

By means of the uniqueness of the limit in $D'(Q)$, we can deduce from (12) and (13) that $f \equiv 0$. Therefore, it can be generated from (10) that

$$ly_n \longrightarrow \varphi \qquad in \quad L^2(\Omega).$$

On the other hand, we have

$$\|y_n\|_E \geq \|y_n\|^2_{L^\infty(0,T,L^2(\Omega))},$$

i.e.,

$$\|y_n\|_E \geq \|\varphi\|^2_{L^2(\Omega)}.$$

Immediately, we have

$$y_n \longrightarrow 0 \qquad in \quad E,$$

which implies

$$\|y_n\|^2_E \longrightarrow 0 \qquad in \quad \mathbb{R}.$$

So, we get $\varphi \equiv 0$, and as a result, the operator L is closable.

Definition 3.1 Let $\overline{L}$ be the closure of L and $D(\overline{L})$ be the definition domain of $\overline{L}$. The solution of the equation

$$\overline{L}y = F$$

is called a strong solution to problem (5)-(7). Then a priori estimate (9) can be extended to the strong solution, i.e., we have the following inequality:

$$\|y\|_E \leq C\|\overline{L}y\|_F, \forall y \in D(\overline{L}). \tag{14}$$

Corollary 3.1 *The range $R(\overline{L})$ of the operator $\overline{L}$ is closed in F and equal to the closure $\overline{R(L)}$ of $R(L)$.*

Proof. First, we prove the uniqueness of the solution if it exists. Let y_1 and y_2 be two different solutions. If we put $\eta = y_1 - y_2$, then η satisfies

$$(P') \qquad \begin{cases} \eta_t - a\dfrac{\partial^2\eta}{\partial x^2} + c(y_1^3 - y_2^3) + b\eta = 0, & (x,t) \in Q, \\ \eta(x,0) = 0, \qquad x \in \Omega, \\ \eta(x,t) = 0, \qquad (x,t) \in \partial\Omega \times (0,T) \end{cases} \tag{15}$$

$$\eta_t - a\frac{\partial^2\eta}{\partial x^2} + c(y_1^3 - y_2^3) + b\eta = 0, \qquad (x,t) \in Q. \tag{16}$$

By multiplying (16) by η and integrating the result over Ω, we get

$$\int_\Omega \eta_t(x,t).\eta(x,t)dx - a\int_\Omega \frac{\partial^2\eta}{\partial x^2}.\eta(x,t)dx + c\int_\Omega (y_1^3 - y_2^3)(y_1 - y_2)dx + b\int_\Omega \eta^2(x,t)dx = 0.$$

Consequently, we can get

$$\frac{1}{2}\frac{d}{dt}\|\eta\|^2_{L^2(\Omega)} + a\|\frac{\partial\eta}{\partial x}\|^2_{L^2(\Omega)} + b\|\eta\|^2_{L^2(\Omega)} + c\int_\Omega (y_1^3 - y_2^3)(y_1 - y_2)dx = 0. \tag{17}$$

As the function λ^3 is a monotone function over Ω, we can conclude that the last term of the left-hand side of (17) is positive, so it follows that

$$\frac{d}{dt}\|\eta\|^2_{L^2(\Omega)} \leq 0,$$

which implies that for all $t \in (0,T)$, we have $y_1(t) = y_2(t)$ in E. Now, we will return to the proof of Corollary 3.1. To this end, we let $z \in R(\overline{L})$. Then there exists a Cauchy sequence $(z_n)_{n\in\mathbb{N}}$ in $R(L)$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

So, there exists a corresponding sequence $(y_n)_{n\in\mathbb{N}}$ in $D(L)$ such that $Ly_n = z_n$. Now, let $\varepsilon, n \geq n_0$ and $m, m' \in \mathbb{N}$ such that $m \geq m'$ and $y_m, y_{m'}$ are two solutions, i.e.,

$$Ly_m = f \quad and \quad Ly_{m'} = f.$$

We put $\phi = y_m - y_{m'}$ and we apply to ϕ the same procedure that we used to demonstrate the uniqueness of the solution in the previous step. This yields $\phi = 0$. It means that for all $t \in (0,T)$, we have

$$0 \leq \|y_m(t) - y_{m'}(t)\|_E \leq 0 \tag{18}$$

$$\leftrightarrow \forall\varepsilon \geq 0, \quad \exists n_0 \in \mathbb{N}\backslash\forall m, m' \geq n_0 : \|y_m(t) - y_{m'}(t)\|_E \leq \varepsilon.$$

As a result, $(y_n)_n$ is a Cauchy sequence in the Banach space E. So, there is $y \in E$ such that

$$\lim_{n\longrightarrow+\infty} y_n = y.$$

By virtue of the definition of $\overline{L}$ (i.e., $\lim_{n\longrightarrow+\infty} y_n = y$ if $\lim_{n\longrightarrow+\infty} Ly_n = \lim_{n\longrightarrow+\infty} z_n = z$, and so $\lim_{n\longrightarrow+\infty} \overline{L}y_n = z$ as $\overline{L}$ is closed, which implies that $\overline{L}y = z$), the function y verifies

$$y \in D(\overline{L}),\ \overline{L}y = z.$$

Thus $z \in R(\overline{L})$, and so $\overline{R(L)} \subset R(\overline{L})$. In the same regard, we can also deduce that $R(\overline{L})$ is closed because it is a Banach space. It remains to prove the reverse inclusion. For this purpose, we observe that $z \in R(\overline{L})$, and then there exists a sequence of $(z_n)_n$ in F consisting of the elements of the set $R(\overline{L})$ such that

$$\lim_{n\longrightarrow+\infty} z_n = z.$$

As a result, there exists a corresponding sequence $(v_n)_n \subset D(\overline{L})$ such that

$$\lim_{n\longrightarrow+\infty} \overline{L}v_n = z_n.$$

On the other hand, we have $(v_n)_n$ is a Cauchy sequence in F. So, there is $v \in E$ such that

$$\lim_{n\longrightarrow+\infty} v_n = v, \qquad v \in E.$$

This implies

$$\lim_{n\longrightarrow+\infty} \overline{L}v_n = z.$$

Consequently, $z \in \overline{R(L)}$, and hence we conclude that $R(\overline{L}) \subset \overline{R(L)}$.

3.1.2 Solvability of the direct problem

To prove the existence of the solution, we must prove that $R(L)$ is dense in F for all $y \in E$ and for arbitrary $\mathcal{F} = (f, \varphi) \in F$.

Theorem 3.2 *Suppose that A1 is satisfied. Then for each $\mathcal{F} = (f, \varphi) \in F$, there is a unique strong solution $y = L^{-1}\mathcal{F} = \overline{L^{-1}}\mathcal{F}$ to problem (P).*

Proof. First, we prove that $R(L)$ is dense in F for all $y \in D(L)$ for the exceptional case when $D(L)$ is reduced to $D_0(L)$, where

$$D_0(L) = \{y, y \in D(L) : ly = 0\}.$$

Proposition 3.2 *Let the conditions of Theorem 3.2 be satisfied. If for $w \in L^2(Q)$ and for each $y \in D_0(L)$, we have*

$$\int_Q \mathcal{L}y.wdxdt = 0, \tag{19}$$

then w vanishes almost everywhere in Q.

Proof. The scalar product of F is defined as follows:

$$(Ly, W)_F = \int_Q \mathcal{L}y.wdxdt, W = (w, 0) \in D(L). \tag{20}$$

If we put $y = w$, the equality (19) can be written as follows:

$$\int_Q y_t(t,x).y(t,x)dxdt - a\int_Q \frac{\partial^2 y}{\partial x^2}.y(t,x)dxdt + b\int_Q y^2(t,x)dxdt + c\int_Q y^4(t,x)dxdt = 0. \tag{21}$$

Integrating (21) by parts yields

$$a\|\frac{\partial y}{\partial x}\|^2_{L^2(Q)} + b\|y\|^2_{L^2(Q)} + c\|y\|^4_{L^4(Q)} = \frac{-1}{2}\|y\|^2_{L^2(\Omega)}.$$

So, we can deduce that $\|y\|^2_{L^2(Q)} \leq 0$, i.e., $y \equiv 0$ in Q, and hence $w \equiv 0$. Now, we return to the proof of Theorem 3.2. To this end, we suppose that $W = (w, w_1) \in R^{\perp}(L)$. This implies

$$(Ly, W)_F = \int_Q \mathcal{L}y.wdxdt + \int_\Omega ly.w_1dx = 0, \forall y \in D(L). \tag{22}$$

By means of the last proposition and by putting $y \in D_0(L)$, we obtain $w \equiv 0$. Thus, (22) becomes

$$\int_\Omega ly.w_1dx = 0,\ \forall y \in D(L). \tag{23}$$

The range of the trace operator l is dense in the Hilbert space F, then the equality (23) implies that $w_1 = 0$. As a result, we can conclude that $W = 0$, and this completes the proof of Theorem 3.2.

4 Existence and Uniqueness of Solution of the Inverse Problem

In this section, we will suppose that the functions appearing in the problem data are measurable and satisfy the following conditions:

$$(H) \qquad \begin{cases} h \in C(0,T,L^2(\Omega)), v \in V = \{v, \dfrac{\partial v}{\partial x} \in L^2(\Omega), v \in L^4(\Omega)\}, \quad E \in W_2^2(0,T), \\ \|h(x,t)\| \leq m; |g^*(t)| \geq r > 0, \quad for \quad r \in \mathbb{R}, \quad (x,t) \in Q, \\ \varphi(x) \in W_2^1(\Omega). \end{cases} .$$

The relation between f and y is given by the following linear operator:

$$A : L^2(0,T) \longrightarrow L^2(0,T), \tag{24}$$

with the value

$$Af(t) = \frac{1}{g^*}\{a \int_\Omega \frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dx + c\int_\Omega y^3(t,x)v(x)dx\}. \tag{25}$$

As a result, the preceding relationship between f and y may be expressed as a second-order linear equation for the function f over $L^2(0,T)$ such that

$$f = Af + \mu, \tag{26}$$

where

$$\mu = \frac{E' + bE}{g^*}. \tag{27}$$

Theorem 4.1 *Assume that the input of data of the inverse problem (1)-(4) verifies condition (H). Then the following statements are equivalent:*

- *If the inverse problem (1)-(4) is solved, then so is equation (26).*
- *If equation (26) has a solution and the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$ is true, then the inverse problem (1)-(4) has also a solution.*

Proof.

- Assume that the inverse problem (1)-(4) is solved. We denote its solution by $\{y, f\}$. Now, multiplying (1) by v and then integrating the result over Ω yield

$$\frac{d}{dt}\int_\Omega y(t,x)v(x)dx + a\int_\Omega \frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dx + b\int_\Omega y(x,t)v(x)dx + c\int_\Omega y^3(t,x)v(x)dx = f(t)g^*(t). \tag{28}$$

Using (4) and (24) implies

$$\frac{E' + bE}{g^*} + Af = f.$$

This gives that f solves equation (26).

- According to the assumption, the equation (25) has a solution, say f. By substituting f into equation (1), the resulting relationships (1)-(3) can be then treated as a direct problem with a unique solution. It is yet up to us to show that y verifies the integral overdetermination (4). By the equation (28), the function y is subject to the following relation:

$$E' + bE + a\int_{\Omega} \frac{\partial y}{\partial x}\frac{\partial v}{\partial x}dx + c\int_{\Omega} y^3(t,x)v(x)dx = f(t)g^*(t). \tag{29}$$

  Subtracting equation (28) from (29) yields

$$\frac{d}{dt}\int_{\Omega} y(t,x)v(x)dx + b\int_{\Omega} y(x,t)v(x)dx = E' + bE. \tag{30}$$

  Now, integrating the above differential equation and using the compatibility condition $E(0) = \int_{\Omega}\varphi(x)v(x)dx$ lead us to the conclusion that y satisfies the integral condition (4). As a result, we can conclude that $\{y, f\}$ is the solution of the inverse problem (1)-(4).

In what follows, we aim to introduce some properties connected to the operator A.

Lemma 4.1 *If (H1) holds, then there exists a positive δ for which the operator A is a contracting operator in $L^2(0,T)$.*

Proof. We obtain from (25) the following estimate:

$$|Af(t)|^2 \leq \frac{2}{r^2}[a^2\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)}\|\frac{\partial v}{\partial x}\|^2_{L^2(\Omega)} + \gamma\|v\|^2_{L^4(\Omega)}\|y\|^4_{L^4(\Omega)}],$$

where $\gamma = \|y\|^2_{L^\infty(0,T,L^4(\Omega))} \geq 0$. Now, integrating the above equality over $(0,T)$ yields

$$\int_0^T |Af(t)|^2 \leq \frac{2}{r^2}max(a^2\|\frac{\partial v}{\partial x}\|^2_{L^2(\Omega)}, \gamma\|v\|^2_{L^4(\Omega)})\int_0^T (\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)} + \|y\|^4_{L^4(\Omega)})dt. \tag{31}$$

So, we get

$$\|Af\|_{L^2(0,T)} \leq K(\int_0^T (\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)} + \|y\|^4_{L^4(\Omega)})dt)^{\frac{1}{2}},$$

where

$$K = \frac{1}{r}\sqrt{2max(a^2\|\frac{\partial v}{\partial x}\|^2_{L^2(\Omega)}, \gamma\|v\|^2_{L^4(\Omega)})}.$$

By multiplying both sides of (1) by y in $L^2(Q)$ and then by integrating the resulting expression by parts with the use of Cauchy's ε-inequality and the Poincare inequality, we get

$$\frac{1}{2}\|y\|^2_{L^2(\Omega)} + (a - \frac{c''\varepsilon}{2})\|\frac{\partial y}{\partial x}\|^2_{L^2(Q)} + b\|y\|^2_{L^2(Q)} + c\|y\|^4_{L^4(Q)} \leq \frac{m^2}{2\varepsilon}\|f\|^2_{L^2(0,T)} + \frac{1}{2}\|\varphi\|^2_{L^2(\Omega)}, \tag{32}$$

with $a - \dfrac{c''\varepsilon}{2} > 0$. With the help of passing to the maximum and omitting some terms, we get

$$\int_0^T (\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)} + \|y\|^4_{L^4(\Omega)})dt \leq M'\|f\|^2_{L^2(0,T)}, \tag{33}$$

where

$$M' = \frac{\frac{m^2}{2\varepsilon}}{min(a - \frac{c''\varepsilon}{2}, c)}.$$

It means that

$$\left(\int_0^T (\|\frac{\partial y}{\partial x}\|^2_{L^2(\Omega)} + \|y\|^4_{L^4(\Omega)})dt\right)^{\frac{1}{2}} \leq M''\|f\|_{L^2(0,T)}, \tag{34}$$

where $M'' = \sqrt{M'}$. Consequently, we get

$$\|Af\|_{L^2(0,T)} \leq \delta\|f\|_{L^2(0,T)}, \tag{35}$$

with $\delta = KM''$. It is obvious from the above assertion that there exists a positive δ such that $\delta \leq 1$. Thus, inequality (35) demonstrates that the operator A is a contracting mapping in $L^2(0,T)$.

Theorem 4.2 *Let the compatibility condition $E(0) = \int_\Omega \varphi(x)v(x)dx$ and the condition (H) hold. Then the following statements are correct:*

- *With any initial iteration $f_0 \in L^2(0,T)$, the following approximations are correct:*

$$f_{n+1} = \mathcal{A}f_n, \tag{36}$$

which converge to f in the $L^2(0,T,L^2(0,T))$-norm.

- *The inverse problem (1)-(4) has a unique solution $\{y, f\}$.*

Proof.

- We have the following operator $\mathcal{A} : L^2(0,T) \longrightarrow L^2(0,T,L^2(0,T))$, which is defined by

$$\mathcal{A}f = Af + \frac{E' + bE}{g^*}, \tag{37}$$

where the operator A and the function g^* come from (25). As a result of (36), relation (26) can be expressed as

$$f = \mathcal{A}f. \tag{38}$$

As a result, it is sufficient to show that the operator $\mathcal{A}$ has a fixed point in the space $L^2(0,T,L^2(0,T))$. Accordingly, we can have

$$\mathcal{A}f_1 - \mathcal{A}f_2 = Af_1 - Af_2 = A(f_1 - f_2).$$

From estimate (35), we can deduce that

$$\|\mathcal{A}f_1 - \mathcal{A}f_2\|_{L^2(0,T)} \leq \delta\|f_1 - f_2\|_{L^2(0,T,L^2(0,T))}. \tag{39}$$

Based on (38), $\mathcal{A}$ is a contracting mapping on $L^2(0,T,L^2(0,T))$. As a result, $\mathcal{A}$ has a unique fixed point f in $L^2(0,T,L^2(0,T))$ and the successive approximations (36) converge to f in $L^2(0,T,L^2(0,T))$-norm, which is independent of the initial iteration $f_0 \in L^2(0,T,L^2(0,T))$.

- This demonstrates that equations (38) and (26) have a unique solution f in $L^2(0,T,L^2(0,T))$. The existence of a solution to the main problem is proved by Theorem 4.1, but it has to be proven that this solution is unique. Using the demonstration by contradiction, we assume that there are two distinct solutions $\{y_1, f_1\}$ and $\{y_2, f_2\}$ to problem (1)-(4). First, we claim that $f_1 \neq f_2$ almost everywhere on $(0,T)$. If $f_1 = f_2$, then by applying the uniqueness theorem to the related direct problem (5)-(7), we find $y_1 = y_2$ almost everywhere in Q. Given that both pairs have verified (28), we infer that the functions f_1 and f_2 are two distinct solutions to equation (38), which contradicts the uniqueness of the functions.

Corollary 4.1 *If the conditions of Theorem 4.2 are satisfied, then the solution f varies continuously with respect to the data μ of the equation (26).*

Proof. Let μ and ϑ be two sets of data that satisfy the assumptions of Theorem 4.2 and let f and g be two solutions of the equation (26), which correspond to μ and ϑ, respectively. As a result of (26), we have

$$f = Af + \mu, \quad g = Ag + \vartheta.$$

By calculating the difference $f - g$ and by using (35), we can have:

$$\|f - g\|_{L^2(0,T,L^2(0,T))} \leq \frac{1}{1-\delta}\|\mu - \vartheta\|_{L^2(0,T)}.$$

Therefore, the proof of this corollary is completed.

5 Conclusion

The novel contribution of this manuscript has been successfully made by investigating the solvability of the semilinear parabolic problem with the integral overdetermination condition for an inverse problem. In addition, we have solved the direct problem by using the "energy inequality" method and accordingly, we have dealt with the inverse problem by using the fixed point technique.

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