Nonlinear Dynamics and Systems Theory, **23** (3) (2023) 261–272

Some Generalized Nonlinear Volterra-Fredholm Type Integral Inequalities with Delay of Several Variables and Applications

Ammar Boudeliou *

Department of Mathematics, University of Constantine 1, BP, 325, Ain El Bey Street, 25017, Algeria.

Received: March 4, 2023; Revised: July 21, 2023

Abstract: In the present paper, some new explicit bounds on solutions to a class of new nonlinear retarded integral inequalities of Volterra-Fredholm type for the functions of n-independent variables are established, which generalize some known integral inequalities. The derived results can be used as useful tools in the study of certain integral and differential equations of Volterra-Fredholm type. An application is given to illustrate the usefulness of our results.

Keywords: *delay integral inequality; Volterra-Fredholm type integral inequalities; explicit bounds; n-independent variables.*

Mathematics Subject Classification (2010): 26D15, 45B05, 45D05, 70K20.

1 Introduction

It is well known that the Gronwall-Bellman integral inequality [3,8] and its various generalizations which provide explicit bounds on unknown functions have played an important role in the study of existence, uniqueness, boundedness, and other qualitative properties of solutions of differential equations, integral equations and have been applied in the stability analysis of solutions to dynamic equations on time scale [1, 12]. Recently, many authors have further improved more general forms of this inequality [2,4,6]. In the past few decades, many such new interesting retarded integral inequalities of Volterra-Fredholm type were established [10, 15].

* Corresponding author: mailto:ammar_boudeliou@umc.edu.dz

In [13] and [14], respectively, Pachpatte has established the following useful linear Volterra-Fredholm type integral inequalities with delay:

$$u(t) \leq k + \int_{\alpha(t_0)}^{\alpha(t)} a(t,s) \left[f(s)u(s) + \int_{\alpha(t_0)}^{s} c(s,\tau)u(\tau)d\tau \right] ds + \int_{\alpha(t_0)}^{\alpha(T)} b(t,s)u(s)ds, \quad (1)$$

$$u(x,y) \leq c + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} a(x,y,s,t)u(s,t)dtds + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(y_0)}^{\beta(N)} b(x,y,s,t)u(s,t)dtds. \quad (2)$$

In [11], Ma and Pečarić discussed the following nonlinear retarded Volterra-Fredholm integral inequality:

$$\begin{aligned} u(x,y) \quad \leq \quad & k + \int \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(x_0)}^{\beta(x)} \sigma_1(s,t) \Big[f(s,t)\omega\left(u(s,t)\right) \\ & + \int_{\alpha(x_0)}^{s} \int_{\beta(x_0)}^{t} \sigma_2(\tau,\xi)\omega\left(u(\tau,\xi)\right) d\xi d\tau \Big] dtds \\ & + \int_{\alpha(x_0)}^{\alpha(M)} \int_{\beta(x_0)}^{\beta(M)} \sigma_1(s,t) \Big[f(s,t)\omega\left(u(s,t)\right) \\ & + \int_{\alpha(x_0)}^{s} \int_{\beta(x_0)}^{t} \sigma_2(\tau,\xi)\omega\left(u(\tau,\xi)\right) d\xi d\tau \Big] dtds. \quad (3) \end{aligned}$$

El-Deeb and Ahmed [5] have established the following useful Volterra-Fredholm type integral inequality with delay which generalizes some results obtained in [9]:

$$\omega^p(t) \leq c(t) + \int_a^{\alpha(t)} g(s)\omega(s)ds + \int_a^b f(s)\omega^p(s)ds. \quad (4)$$

However, in certain situations such as some classes of delay differential or integral equations of Volterra-Fredholm type, it is desirable to find some new delay inequalities in order to achieve a diversity of desired goals. In this paper, we discuss a class of retarded integral inequalities of Volterra-Fredholm type. We use some analysis techniques to get the explicit estimations of the unknown function in the inequality. Finally, we give an application to illustrate the usefulness of our results.

2 Main Results

Throughout this paper, we use the following notations: $I = [x^0, T] = I_1 \times ... \times I_n$, where $I_i = [x_i^0, T_i]$, $i = 1, ..., n$, and $x^0 = \left(x_1^0, ..., x_n^0\right)$, $T = (T_1, ..., T_n) \in \mathbb{R}^n$, $\Delta = \left\{(x,s) \in I^2 : x^0 \leq s \leq x \leq T\right\}$. If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ belong to $\mathbb{R}^n$, we write $x \leq y$ $(x < y)$ if and only if $x_i \leq y_i$ $(x_i < y_i)$, $i = 1, ..., n$. We also adopt the notation $x = (x_1, x_2, ..., x_n) = (x_1, x^1)$, where $x^1 = (x_2, ..., x_n)$, $(x^0)^1 = \left(x_2^0, ..., x_n^0\right)$, and

- $D_i = \frac{\partial}{\partial x_i}, i = 1, ..., n,$
- $dx^1 = dx_n ... dx_2,$
- $\int_{x^0}^{x} ...ds = \int_{x_1^0}^{x_1} ... \int_{x_n^0}^{x_n} ....ds_n ... ds_1 = \int_{x_1^0}^{x_1} \int_{(x^0)^1}^{x^1} ....ds^1 ds_1,$

- $\int_{\alpha(x^0)}^{\alpha(x)} ...ds = \int_{\alpha(x_1^0)}^{\alpha_1(x_1)} ... \int_{\alpha(x_n^0)}^{\alpha_n(x_n)} ...ds_n...ds_1$.

In the following, we establish some new generalized Volterra-Fredholm type integral inequalities in n-independent variables.

Theorem 2.1 *Let $u(x) \in C(I, \mathbb{R}_+)$, $f(x,s), \gamma_1(x,s), \gamma_2(x,s) \in C(\Delta, \mathbb{R}_+)$ and f, γ_1, γ_2 be nondecreasing in x for each $s \in I, \alpha(x) = (\alpha_1(x_1), ..., \alpha_n(x_n)) \in C^1(I, I)$, where $\alpha_i(x_i) \in C^1(I_i, I_i)$ are nondecreasing functions on I_i with $\alpha_i(x_i) \leq x_i$, $i = 1, ..., n$. Let $\psi, \omega, \omega_1 \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\{\psi, \omega, \omega_1\}(u) > 0$ for $u > 0$, and $\lim_{u \to +\infty} \psi(u) = +\infty$ and $F_1(v) = \int_{v_0}^{v} \frac{ds}{\omega(\psi^{-1}(s))\omega_1(\psi^{-1}(s))}$, $v \geq v_0 > 0$, $F_1(+\infty) = +\infty$. If $u(x)$ satisfies*

$$\begin{aligned}
\psi(u(x)) \leq\; & u_0 + \int_{\alpha(x^0)}^{\alpha(x)} \gamma_1(x,s)\omega(u(s)) \left[f(x,s)\omega_1(u(s)) + \int_{\alpha(x^0)}^{s} \gamma_2(s,\tau)\omega_1(u(\tau))d\tau \right] ds \\
& + \int_{\alpha(x^0)}^{\alpha(T)} \gamma_1(x,s)\omega(u(s)) \Big[f(x,s)\omega_1(u(s)) \\
& + \int_{\alpha(x^0)}^{s} \gamma_2(s,\tau)\omega_1(u(\tau))d\tau \Big] ds
\end{aligned} \tag{5}$$

for $x \in I$, where $u_0 \geq 0$ is a constant and

$$H_1(t) = F_1(2t - u_0) - F_1(t) \tag{6}$$

is increasing for $t \geq u_0$, then

$$\begin{aligned}
u(x) \leq\; & \psi^{-1} \Bigg\{ F_1^{-1} \Bigg(F_1 \Bigg[H_1^{-1} \Bigg(\int_{\alpha(x^0)}^{\alpha(T)} \gamma_1(x,s) \left[f(x,s) + \int_{\alpha(x^0)}^{s} \gamma_2(s,\tau)d\tau \right] ds \Bigg) \Bigg] \\
& + \int_{\alpha(x^0)}^{\alpha(x)} \gamma_1(x,s) \left[f(x,s) + \int_{\alpha(x^0)}^{s} \gamma_2(s,\tau)d\tau \right] ds \Bigg) \Bigg\}
\end{aligned} \tag{7}$$

for $x \in I$, F_1^{-1} and H_1^{-1} are the inverse functions of F_1 and H_1, respectively.

Proof. Let $u_0 > 0$ and fix any arbitrary $X = (X_1, ..., X_n) \in I$, then for $x^0 \leq x \leq X \leq T$, we define a positive and nondecreasing function $z(x)$ on I by the right-hand side of (5) for $x \in I$, so we have

$$u(x) \leq \psi^{-1}(z(x)), \tag{8}$$

and

$$\begin{aligned}
D_1...D_n z(x) \leq\; & \gamma_1(X, \alpha(x))\omega(\psi^{-1}(z(\alpha(x)))) \Big[f(X, \alpha(x))\omega_1(\psi^{-1}(z(\alpha(x)))) \\
& + \int_{\alpha(x^0)}^{\alpha(x)} \gamma_2(\alpha(x), \tau)\omega_1(\psi^{-1}(z(\tau)))d\tau \Big] \alpha'(x) \\
\leq\; & \gamma_1(X, \alpha_1(x_1), ..., \alpha_n(x_n))(\omega\omega_1)(\psi^{-1}(z(\alpha_1(x_1), ..., \alpha_n(x_n)))) \\
& \Big[f(X, \alpha_1(x_1), ..., \alpha_n(x_n)) + \int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)} ... \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} \gamma_2(\alpha(x), \tau_1, ..., \tau_n) \\
& d\tau_n...d\tau_1 \Big] \alpha_1'(x_1)...\alpha_n'(x_n).
\end{aligned}$$

So

$$\begin{aligned}\frac{D_1...D_n z(x)}{(\omega\omega_1)(\psi^{-1}(z(x)))} \quad &\leq \quad \gamma_1(X, \alpha_1(x_1), ..., \alpha_n(x_n))\Big[f(X, \alpha_1(x_1), ..., \alpha_n(x_n)) + \\ &\int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)} ... \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} \gamma_2(\alpha(x), ..., \alpha_n(x_n), \tau_1, ..., \tau_n) d\tau_n ... d\tau_1\Big] \\ &\alpha_1'(x_1) \times ... \times \alpha_n'(x_n),\end{aligned}$$

then

$$\begin{aligned}D_n\left(\frac{D_1...D_{n-1} z(x)}{(\omega\omega_1)(\psi^{-1}(z(x)))}\right) \quad &\leq \quad \gamma_1(X, \alpha_1(x_1), ..., \alpha_n(x_n))\Big[f(X, \alpha_1(x_1), ..., \alpha_n(x_n)) + \\ &\int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)} ... \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} \gamma_2(\alpha_1(x_1), ..., \alpha_n(x_n), \tau_1, ..., \tau_n) \\ &d\tau_n ... d\tau_1\Big] \alpha_1'(x_1) \times ... \times \alpha_n'(x_n). \qquad (9)\end{aligned}$$

Keeping $x_1, ..., x_{n-1}$ fixed in (9), setting $x_n = s_n$ and integrating with respect to s_n from x_n^0 to x_n, we get

$$\begin{aligned}\frac{D_1...D_{n-1} z(x)}{(\omega\omega_1)(\psi^{-1}(z(x)))} \quad &\leq \quad \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} \gamma_1(X, \alpha_1(x_1), ..., \alpha_{n-1}(x_{n-1}), s_n) \\ &\Big[f(X, \alpha_1(x_1), ..., \alpha_{n-1}(x_{n-1}), s_n) + \\ &\int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)} ... \int_{\alpha_n(x_n^0)}^{s_n} \gamma_2(\alpha_1(x_1), ..., \alpha_{n-1}(x_{n-1}), s_n, \tau_1, ..., \tau_n) \\ &d\tau_n ... d\tau_1\Big] \alpha_1'(x_1) \times ... \times \alpha_{n-1}'(x_{n-1}) ds_n.\end{aligned}$$

Repeating this, we find (after $n-1$ steps)

$$\begin{aligned}\frac{D_1 z(x)}{\omega\omega_1(\psi^{-1}(z(x)))} \quad &\leq \quad \int_{\alpha_2(x_2^0)}^{\alpha_2(x_2)} ... \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} \gamma_1(X, \alpha_1(x_1), s_2, ..., s_n)\Big[f(X, \alpha_1(x_1), s_2, ..., s_n) \\ &+ \int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)} \int_{\alpha_2(x_2^0)}^{s_2} ... \int_{\alpha_n(x_n^0)}^{s_n} \gamma_2(\alpha_1(x_1), s_2, ..., s_n, \tau_1, ..., \tau_n) \\ &d\tau_n ... d\tau_1\Big] \alpha_1'(x_1) ds_n ... ds_2. \qquad (10)\end{aligned}$$

Keeping $x^1 = (x_2, ..., x_n)$ fixed in (10), replacing x_1 by s_1 and then integrating with respect to s_1 from x_1^0 to x_1, we obtain

$$z(x) \leq F_1^{-1}\left(F_1(z(x_1^0, x^1)) + \int_{\alpha(x^0)}^{\alpha(x)} \gamma_1(X, s)\left[f(X, s) + \int_{\alpha(x^0)}^{s} \gamma_2(s, \tau) d\tau\right] ds\right) \qquad (11)$$

for $x \in I$. From the equation

$$z(x_1^0, x^1) = u_0 + \int_{\alpha(x^0)}^{\alpha(T)} \gamma_1(X, s)\omega(u(s))\left[f(X, s)\omega_1(u(s)) + \int_{\alpha(x^0)}^{s} \gamma_2(s, \tau)\omega_1(u(\tau)) d\tau\right] ds,$$

we observe that

$$z(T) = 2z(x_1^0, x^1) - u_0 = u_0 + 2\int_{\alpha(x^0)}^{\alpha(T)} \gamma_1(X,s)\omega(u(s))\Big[f(X,s)\omega_1(u(s)) + \int_{\alpha(x^0)}^{s} \gamma_2(s,\tau)\omega_1(u(\tau))d\tau\Big] ds.$$

Using (11), we get

$$2z(x_1^0, x^1) - u_0 \le F_1^{-1}\left(F_1(z(x_1^0,x^1)) + \int_{\alpha(x^0)}^{\alpha(T)} \gamma_1(X,s)\left[f(X,s) + \int_{\alpha(x^0)}^{s}\gamma_2(s,\tau)d\tau\right]ds\right),$$

or

$$F_1\left(2z(x_1^0,x^1) - u_0\right) - F_1(z(x_1^0,x^1)) \le \int_{\alpha(x^0)}^{\alpha(T)} \gamma_1(X,s)\left[f(X,s) + \int_{\alpha(x^0)}^{s}\gamma_2(s,\tau)d\tau\right]ds, \tag{12}$$

then $H_1\left(z(x_1^0,x^1)\right) \le \int_{\alpha(x^0)}^{\alpha(T)} \gamma_1(X,s)\left[f(X,s) + \int_{\alpha(x^0)}^{s}\gamma_2(s,\tau)d\tau\right]ds$. Since H_1 is increasing, for $t \ge u_0$, we get

$$z(x_1^0,x^1) \le H_1^{-1}\left(\int_{\alpha(x^0)}^{\alpha(T)} \gamma_1(X,s)\left[f(X,s) + \int_{\alpha(x^0)}^{s}\gamma_2(s,\tau)d\tau\right]ds\right). \tag{13}$$

Since $X \in I$ is chosen arbitrary, now substituting (13) into (11) and from (8), we obtain the desired inequality (7). If $u_0 = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of u_0 and subsequently let $\varepsilon \to 0$. □

Remark 2.1 For $\gamma_1 = 1$, $\gamma_2 = 0$, $\psi(u) = \omega_1(u) = u$, $\omega(u) = 1$ and $x^2 = (x_3, ..., x_n)$ fixed, inequality (5) reduces to inequality (2).

Remark 2.2 For $\psi(u) = u$, $\gamma_1(x,s) = \gamma_1(s)$, $\gamma_2(s,\tau) = \gamma_2(\tau)$, $f(x,s) = f(s)$, $\omega(u) = 1$ and x^1 fixed, (5) reduces to (3). Further, for $\psi(u) = u$, $\gamma_1(x,s) = \gamma_1(s)$, $\gamma_2(s,\tau) = \gamma_2(\tau)$, $f(x,s) = f(s)$, $\omega(u) = 1$ and x^2 fixed, Theorem 2.1 reduces to Theorem 3.1 in [11].

Theorem 2.2 *Let $u, f, g, h \in C(I, \mathbb{R}_+)$ and $\alpha(x) = (\alpha_1(x_1), ..., \alpha_n(x_n)) \in C^1(I, I)$, where $\alpha_i(x_i) \in C^1(I_i, I_i)$ are nondecreasing functions on I_i with $\alpha_i(x_i) \le x_i$, $i = 1, ..., n$. Let $\omega_1, \omega_2, \omega_3, \frac{\omega_3}{\omega_2} \in C(\mathbb{R}_+, \mathbb{R}_+)$ be nondecreasing with $\omega_i(u) > 0$ $(i = 1, 2, 3)$ for $u > 0$, and*

$$G_1(r) = \int_{r_0}^{r} \frac{ds}{\omega_1\omega_2(s)}, G_2(r) = \int_{r_0}^{r} \frac{\omega_2\left(G_1^{-1}(s)\right)ds}{\omega_3\left(G_1^{-1}(s)\right)}, r \ge r_0 > 0, G_1(\infty) = G_2(\infty) = \infty, \tag{14}$$

$$H_1(u) = G_2\left(G_1(2u - u_0)\right) - G_2\left(G_1(u) + \int_{\alpha(x^0)}^{\alpha(T)} f(s)g(s)ds\right) - \int_{\alpha(x^0)}^{\alpha(T)} f(s)\left(\int_{\alpha(x^0)}^{s} h(\tau)d\tau\right)ds \tag{15}$$

is increasing, and $H_1(u)=0$ has a solution c for $u \geq u_0$. If $u(x)$ satisfies

$$\begin{aligned} u(x) \quad &\leq \quad u_0+\int_{\alpha(x^0)}^{\alpha(x)} f(s)\omega_1(u(s))\left[g(s)\omega_2\left(u(s)\right)+\int_{\alpha(x^0)}^{s} h(\tau)\omega_3(u(\tau))d\tau\right]ds \\ &\quad +\int_{\alpha(x^0)}^{\alpha(T)} f(s)\omega_1(u(s))\left[g(s)\omega_2\left(u(s)\right)+\int_{\alpha(x^0)}^{s} h(\tau)\omega_3(u(\tau))d\tau\right]ds, \end{aligned} \tag{16}$$

then

$$u(x)\leq G_1^{-1}\left\{G_2^{-1}\left[G_2\left(G_1(c)+\int_{\alpha(x^0)}^{\alpha(x)} f(s)g(s)ds\right)+\int_{\alpha(x^0)}^{\alpha(x)} f(s)\left(\int_{\alpha(x^0)}^{s} h(\tau)d\tau\right)ds\right]\right\} \tag{17}$$

for $x\in I$, where G_1^{-1}, G_2^{-1} are the inverse functions of G_1, G_2, respectively.

Proof. Let $u_0>0$ and $z(x)$ denote the function on the right-hand side of (16), which is positive and nondecreasing function on I. Then we have

$$u(x)\leq z(x), \tag{18}$$

and

$$z\left(x_1^0,x^1\right)=u_0+\int_{\alpha(x^0)}^{\alpha(T)} f(s)\omega_1(u(s))\left[g(s)\omega_2\left(u(s)\right)+\int_{\alpha(x^0)}^{s} h(\tau)\omega_3(u(\tau))d\tau\right]ds.$$

Differentiating $z(x)$ with respect to x, using (18), we have

$$D_1...D_nz(x)\leq\alpha'(x)f\left(\alpha(x)\right)\omega_1(z(\alpha(x)))\left[g(\alpha(x))\omega_2\left(z(\alpha(x))\right)+\int_{\alpha(x^0)}^{\alpha(x)} h(\tau)\omega_3(z(\tau))d\tau\right]$$

by the monotonicity of ω_1, ω_2, and z and the property of α. From the above inequality, we have

$$\frac{D_1...D_nz(x)}{(\omega_1\omega_2)(z(x))}\leq\alpha'(x)f\left(\alpha(x)\right)\left[g(\alpha(x))+\int_{\alpha(x^0)}^{\alpha(x)} h(\tau)\frac{\omega_3(z(\tau))}{\omega_2\left(z(\tau)\right)}d\tau\right],$$

or

$$\begin{aligned} D_n\left(\frac{D_1...D_{n-1}z(x)}{\omega_1\omega_2(z(x))}\right) \quad &\leq \quad \alpha_1'(x_1)...\alpha_n'(x_n)f\left(\alpha_1(x_1),...,\alpha_n(x_n)\right)\Big[g(\alpha_1(x_1),...,\alpha_n(x_n)) \\ &\quad +\int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)}...\int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} h(\tau_1,...,\tau_n)\frac{\omega_3(z(\tau_1,...,\tau_n))}{\omega_2\left(z(\tau_1,...,\tau_n)\right)}d\tau_n...d\tau_1\Big]. \end{aligned}$$

Keeping $x_1,...,x_{n-1}$ fixed, integrating both sides of the above inequality from x_n^0 to x_n, we obtain

$$\begin{aligned} \frac{D_1..D_{n-1}z(x)}{\omega_1\omega_2(z(x))} \quad &\leq \quad \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} f\left(\alpha_1(x_1),...,\alpha_{n-1}(x_{n-1}),s_n\right)\Big[g(\alpha_1(x_1)...,\alpha_{n-1}(x_{n-1}),s_n) \\ &\quad +\int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)}...\int_{\alpha_{n-1}(x_{n-1}^0)}^{\alpha_{n-1}(x_{n-1})}\int_{\alpha_n(x_n^0)}^{s_n} h(\tau_1,...,\tau_n)\frac{\omega_3(z(\tau_1,...,\tau_n))}{\omega_2\left(z(\tau_1,...,\tau_n)\right)} \\ &\quad d\tau_n...d\tau_1\Big]\alpha_1'(x_1)\times...\times\alpha_{n-1}'(x_{n-1})ds_n. \end{aligned}$$

Continuing this process, we obtain (after $n-1$ steps)

$$\begin{aligned}\frac{D_1 z(x)}{\omega_1\omega_2(z(x))} &\le \int_{\alpha_2(x_2^0)}^{\alpha_2(x_2)} \cdots \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} f\left(\alpha_1(x_1), s_2, ..., s_n\right) \Big[g(\alpha_1(x_1), s_2, ..., s_n) + \\ & \int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)} \int_{\alpha_2(x_2^0)}^{s_2} \cdots \int_{\alpha_n(x_n^0)}^{s_n} h(\tau_1, ..., \tau_n) \frac{\omega_3(z(\tau_1, ..., \tau_n))}{\omega_2\left(z(\tau_1, ..., \tau_n)\right)} d\tau_n ... d\tau_1 \Big] \\ & \alpha_1'(x_1) ds^1.\end{aligned}$$

Integrating the above inequality from x_1^0 to x_1, using (14), we obtain

$$\begin{aligned} G_1\left(z(x)\right) &\le G_1\left(z(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(x)} f(s) \left[g(s) + \int_{\alpha(x^0)}^{s} h(\tau_1, ..., \tau_n) \frac{\omega_3(z(\tau))}{\omega_2\left(z(\tau)\right)} d\tau \right] ds \\ &\le G_1\left(z(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(X)} f(s) g(s) ds \\ &\quad + \int_{\alpha(x^0)}^{\alpha(x)} f(s) \left(\int_{\alpha(x^0)}^{s} h(\tau_1, ..., \tau_n) \frac{\omega_3(z(\tau))}{\omega_2\left(z(\tau)\right)} d\tau \right) ds \qquad (19)\end{aligned}$$

for all $x \in [x^0, X]$, $X \in I$, and X is chosen arbitrarily. Let $v(x)$ denote the function on the right-hand side of (19), which is positive and nondecreasing in each variable $x \in [x^0, X]$.

From (19), we have

$$z(x) \le G_1^{-1}\left(v(x)\right), \quad \forall x \in [x^0, X], \qquad (20)$$

$$v(x_1^0, x^1) = G_1\left(z(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(X)} f(s) g(s) ds.$$

Differentiating $v(x)$ with respect to x, by the monotonicity of v, G_1^{-1}, and $\frac{\omega_3}{\omega_2}$, the property of α, and (20), we have

$$D_1 ... D_n v(x) \le \alpha'(x) f\left(\alpha(x)\right) \frac{\omega_3(G_1^{-1}\left(v(x)\right))}{\omega_2\left(G_1^{-1}\left(v(x)\right)\right)} \int_{\alpha(x^0)}^{\alpha(x)} h(\tau_1, ..., \tau_n) d\tau$$

for all $x \in [x^0, X]$. Then we have

$$\frac{\omega_2\left(G_1^{-1}\left(v(x)\right)\right) D_1 ... D_n v(x)}{\omega_3(G_1^{-1}\left(v(x)\right))} \le \alpha'(x) f\left(\alpha(x)\right) \int_{\alpha(x^0)}^{\alpha(x)} h(\tau_1, ..., \tau_n) d\tau,$$

$$D_n \left(\frac{\omega_2\left(G_1^{-1}\left(v(x)\right)\right) D_1 ... D_{n-1} v(x)}{\omega_3(G_1^{-1}\left(v(x)\right))} \right) \le \alpha'(x) f\left(\alpha(x)\right) \int_{\alpha(x^0)}^{\alpha(x)} h(\tau_1, ..., \tau_n) d\tau.$$

Keeping x_1 fixed, integrating both sides of the above inequality with respect to $x_2, ..., x_n$, respectively, we obtain (after $n-1$ steps)

$$\begin{aligned}\frac{\omega_2\left(G_1^{-1}\left(v(x)\right)\right) D_1 v(x)}{\omega_3(G_1^{-1}\left(v(x)\right))} &\le \int_{\alpha_2(x_2^0)}^{\alpha_2(x_2)} \cdots \int_{\alpha_n(x_n^0)}^{\alpha_n(x_n)} f\left(\alpha_1(x_1), s_2, ..., s_n\right) \times \\ & \left(\int_{\alpha_1(x_1^0)}^{\alpha_1(x_1)} \int_{\alpha_2(x_2^0)}^{s_2} \cdots \int_{\alpha_n(x_n^0)}^{s_n} h(\tau_1, ..., \tau_n) d\tau_n ... d\tau_1 \right) ds^1 \alpha_1'(x).\end{aligned}$$

Integrating both sides of the above inquality from x_1^0 to x_1, using (14), we obtain

$$G_2\left(v(x)\right) \leq G_2\left(v(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(x)} f(s)\left(\int_{\alpha(x^0)}^{s} h(\tau)d\tau\right)ds,$$

or

$$v(x) \leq G_2^{-1}\left[G_2\left(v(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(x)} f(s)\left(\int_{\alpha(x^0)}^{s} h(\tau)d\tau\right)ds\right], \quad \forall x \in [x^0, X]. \tag{21}$$

From (20) and (21), we have

$$z(x) \leq G_1^{-1}\left(v(x)\right) \leq G_1^{-1}\left\{G_2^{-1}\left[G_2\left(v(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(x)} f(s)\left(\int_{\alpha(x^0)}^{s} h(\tau)d\tau\right)ds\right]\right\}.$$

Substituting $v(x_1^0, x^1)$ into the above inequality, and since X is chosen arbitrarily, we have

$$\begin{aligned} z(x) \quad \leq \quad & G_1^{-1}\left\{G_2^{-1}\left[G_2\left(G_1\left(z(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(x)} f(s)g(s)ds\right)\right.\right. \\ & \left.\left. + \int_{\alpha(x^0)}^{\alpha(x)} f(s)\left(\int_{\alpha(x^0)}^{s} h(\tau)d\tau\right)ds\right]\right\}. \end{aligned} \tag{22}$$

By the definition of z and the expression of $z\left(x_1^0, x^1\right)$, we have $2z\left(x_1^0, x^1\right) - u_0 = z(T)$.
From (22), we have

$$\begin{aligned} 2z\left(x_1^0, x^1\right) - u_0 \quad \leq \quad & G_1^{-1}\left\{G_2^{-1}\left[G_2\left(G_1\left(z(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(T)} f(s)g(s)ds\right)\right.\right. \\ & \left.\left. + \int_{\alpha(x^0)}^{\alpha(T)} f(s)\left(\int_{\alpha(x^0)}^{s} h(\tau)d\tau\right)ds\right]\right\}, or \end{aligned}$$

$$\begin{aligned} G_2\left(G_1\left(2z\left(x_1^0, x^1\right) - u_0\right)\right) \quad \leq \quad & G_2\left(G_1\left(z(x_1^0, x^1)\right) + \int_{\alpha(x^0)}^{\alpha(T)} f(s)g(s)ds\right) \\ & + \int_{\alpha(x^0)}^{\alpha(T)} f(s)\left(\int_{\alpha(x^0)}^{s} h(\tau)d\tau\right)ds. \end{aligned} \tag{23}$$

By the definition of H_1, the assumption of Theorem 2.2, and (23), we observe that

$$H_1\left(z(x_1^0, x^1)\right) \leq 0 = H_1\left(c\right). \tag{24}$$

Since H_1 is increasing, from (18), (22), and (24), we have the desired estimation (17). If $u_0 = 0$, we carry out the above procedure with $\varepsilon > 0$ instead of u_0 and subsequently let $\varepsilon \to 0$. □

Remark 2.3 If $\omega_2 = \omega_3$, and for $x^2 = (x_3, ..., x_n)$ fixed, $G_2(u) = u - u_0$, and $G_2^{-1}(u) = u + u_0$, (17) is equivalent to

$$u(x) \quad \leq \quad G_1^{-1}\left\{G_1(c) + \int_{\alpha(x^0)}^{\alpha(x)} f(s)\left[g(s) + \int_{\alpha(x^0)}^{s} h(\tau)d\tau\right]ds\right\}.$$

Theorem 2.2 reduces to Theorem 3.1 in [11].

Theorem 2.3 *Let u, f, α, u_0 be as in Theorem 2.1, $a(x,s)$, $b(x,s)$, $c(x,s)$, $g(x,s)$, $d(x,s)$ be the functions of $C(\Delta, \mathbb{R}_+)$ nondecreasing in x for each $s \in I$, and $0 < p < 1$ be a constant. If $u(x)$ satisfies*

$$\begin{aligned} u(x) \quad &\leq \quad u_0 + \int_{\alpha(x^0)}^{\alpha(x)} a(x,s)\left[f(x,s)u(s) + \int_{\alpha(x^0)}^{s} b(s,\tau)u(\tau)d\tau\right] ds \\ &\quad + \int_{\alpha(x^0)}^{\alpha(T)} c(x,s)\left[g(x,s)u^p(s) + \int_{\alpha(x^0)}^{s} d(s,\tau)u^p(\tau)d\tau\right] ds \end{aligned} \tag{25}$$

for $x \in I$, and

$$\exp\left(\int_{\alpha(x^0)}^{\alpha(T)} \gamma_1^*(x,s)\left[f^*(x,s) + \int_{\alpha(x^0)}^{s} \gamma_2^*(s,\tau)d\tau\right] ds\right) < 2, \tag{26}$$

then

$$\begin{aligned} u(x) \quad &\leq \quad \left\{\left(1 + (c)^{1-p}\right)\exp\left((1-p)\int_{\alpha(x^0)}^{\alpha(x)} \gamma_1^*(x,s)\left[f^*(x,s) + \int_{\alpha(x^0)}^{s} \gamma_2^*(s,\tau)d\tau\right] ds\right)\right. \\ &\quad \left. -1\right\}^{\frac{1}{1-p}} \end{aligned} \tag{27}$$

for $x \in I$, where c is the solution of the equation

$$H_2(t) = \frac{1}{1-p}\ln\frac{1 + (2t - u_0)^{1-p}}{1 + t^{1-p}} - \int_{\alpha(x^0)}^{\alpha(T)} \gamma_1^*(x,s)\left[f^*(x,s) + \int_{\alpha(x^0)}^{s} \gamma_2^*(s,\tau)d\tau\right] ds = 0 \tag{28}$$

for $t \geq u_0$, where $\gamma_1^*(x,s) = \max\{a(x,s), c(x,s)\}$, $f^*(x,s) = \max\{f(x,s), g(x,s)\}$, and $\gamma_2^*(x,s)) = \max\{b(x,s), d(x,s)\}$.

Proof. Let $W \in C(\mathbb{R}_+, \mathbb{R}_+)$ so that $W(u) = u + u^p$ is nondecreasing, so it is obvious that $u, u^p \leq W(u)$. From (25) and the assumptions, we get

$$\begin{aligned} u(x) \quad &\leq \quad u_0 + \int_{\alpha(x^0)}^{\alpha(x)} \gamma_1^*(x,s)\left[f^*(x,s)W(u(s)) + \int_{\alpha(x^0)}^{s} \gamma_2^*(s,\tau)W(u(\tau))\, d\tau\right] ds \\ &\quad + \int_{\alpha(x^0)}^{\alpha(T)} \gamma_1^*(x,s)\left[f^*(x,s)W(u(s)) + \int_{\alpha(x^0)}^{s} \gamma_2^*(s,\tau)W(u(\tau))\, d\tau\right] ds. \end{aligned}$$

Fix any arbitrary $X = (X_1, ..., X_n) \in I$, then for $x^0 \leq x \leq X \leq T$, define a positive and nondecreasing function $z(x)$ on I by

$$\begin{aligned} z(x) \quad &= \quad u_0 + \int_{\alpha(x^0)}^{\alpha(x)} \gamma_1^*(X,s)\left[f^*(X,s)W(u(s)) + \int_{\alpha(x^0)}^{s} \gamma_2^*(s,\tau)W(u(\tau))\, d\tau\right] ds \\ &\quad + \int_{\alpha(x^0)}^{\alpha(T)} \gamma_1^*(X,s)\left[f^*(X,s)W(u(s)) + \int_{\alpha(x^0)}^{s} \gamma_2^*(s,\tau)W(u(\tau))\, d\tau\right] ds, \end{aligned}$$

so we have $u(x) \leq z(x)$, by the same steps as in the proof of Theorem 2.1, we obtain

$$z(x) \leq F_2^{-1}\left(F_2(z(x_1^0, x^1)) + \int_{\alpha(x^0)}^{\alpha(x)} \gamma_1^*(x,s)\left[f^*(x,s) + \int_{\alpha(x^0)}^{s} \gamma_2^*(s,\tau)d\tau\right] ds\right), \tag{29}$$

where

$$F_2(v)=\int_{v_0}^{v}\frac{ds}{W(s)}=\int_{v_0}^{v}\frac{ds}{s+s^p}=\frac{1}{1-p}\ln\frac{1+v^{1-p}}{1+v_0^{1-p}},v\geq v_0>0, \tag{30}$$

then

$$F_2^{-1}(v)=\left[\left(1+v_0^{1-p}\right)\exp\left((1-p)\,v\right)-1\right]^{\frac{1}{1-p}}. \tag{31}$$

We have

$$H_2(t)=F_2\left(2t-u_0\right)-F_2(t)-\int_{\alpha(x^0)}^{\alpha(T)}\gamma_1^*(x,s)\left[f^*(x,s)+\int_{\alpha(x^0)}^{s}\gamma_2^*(s,\tau)d\tau\right]ds,$$

so

$$H_2(t)=\frac{1}{1-p}\ln\frac{1+(2t-u_0)^{1-p}}{1+t^{1-p}}-\int_{\alpha(x^0)}^{\alpha(T)}\gamma_1^*(x,s)\left[f^*(x,s)+\int_{\alpha(x^0)}^{s}\gamma_2^*(s,\tau)d\tau\right]ds,$$

so we have

$$H_2{}'(t)=\frac{u_0+2t^p-(2t-u_0)^p}{\left[2t-u_0+(2t-u_0)^p\right](t+t^p)}>0 \tag{32}$$

for $t\geq u_0$ and

$$H_2(u_0)=-\int_{\alpha(x^0)}^{\alpha(T)}\gamma_1^*(x,s)\left[f^*(x,s)+\int_{\alpha(x^0)}^{s}\gamma_2^*(s,\tau)d\tau\right]ds<0, \tag{33}$$

and from (26), we get

$$\lim_{t\to+\infty}H_2(t)=\ln 2-\int_{\alpha(x^0)}^{\alpha(T)}\gamma_1^*(x,s)\left[f^*(x,s)+\int_{\alpha(x^0)}^{s}\gamma_2^*(s,\tau)d\tau\right]ds>0. \tag{34}$$

By (32)-(34), we obtain that (28) has a unique solution $c>u_0$. Now by (29), (30) and (31), we get (27). □

3 Application

In this section, we apply our results to obtain the estimate of the solution of the retarded Volterra-Fredholm integral equation with delay in n-independent variables.

Example 3.1. Consider the following differential boundary value problem system in n-independent variables

$$\begin{cases} D_1...D_nz(x)=D_1...D_nf(x)+A\left(x,s,z\left(s-\beta(s)\right)\right)+B\left(x,s,z\left(s-\beta(s)\right)\right), \\ z(x_1,...,x_{n-1},x_n^0)=f(x_1,...,x_{n-1},x_n^0),...,z(x_1^0,...,x_n)=f(x_1^0,...,x_n), \end{cases} \tag{35}$$

where $z,f\in C^1\left(I,\mathbb{R}\right)$, $A,B\in C(\Delta\times\mathbb{R},\mathbb{R})$, $I=\left[x^0,T\right]\subset\mathbb{R}^n$,
$\Delta=\left\{(x,s)\in I^2:x^0\leq s\leq x\leq T\right\}\subset\mathbb{R}^n$ and $\beta\in C^1(I,I)$ is nonincreasing on I such that $\beta(x)=(\beta_1(x_1),...,\beta_n(x_n))$, $x_i-\beta_i(x_i)\geq 0$, $\beta_i'(x_i)<1$, and $\beta_i(x_i^0)=0$ for $i=1,...,n$, $x=(x_1,...,x_n)$, $x^0=\left(x_1^0,...,x_n^0\right)\in\mathbb{R}^n$. Integrating both sides of (35) with respect to $x_1,...,x_n$, respectively, we obtain (after n steps)

$$z(x)=f(x)+\int_{x^0}^{x}A\left(x,s,z\left(s-\beta(s)\right)\right)ds+\int_{x^0}^{T}B\left(x,s,z\left(s-\beta(s)\right)\right)ds. \tag{36}$$

Theorem 3.1 *Assume that the functions f, A, B in (36) satisfy the conditions*

$$|f(x)| \leq u_0, \tag{37}$$

$$|A(x,s,z)| \leq a(x,s)\,|z|\,, \tag{38}$$

$$|B(x,s,z)| \leq b(x,s)\,|z|^p\,, \tag{39}$$

where u_0, $a(x,s)$, $b(x,s)$ are as in Theorem 2.3, $0 < p < 1$ *is a constant. Let*

$$M_i = \max_{x_i \in I_i} \frac{1}{1-\beta_i'(x_i)}, i = 1, ..., n, \tag{40}$$

and

$$\exp\left(M \int_{\alpha(x^0)}^{\alpha(T)} \gamma(x,s)ds \right) < 2, \tag{41}$$

where $M = M_1 \times ... \times M_n$, $\alpha(x) = x - \beta(x) \in C^1(I, I)$ is increasing on I,
$\gamma(x,s) = \max\left\{a(x,\alpha^{-1}(s)), b(x,\alpha^{-1}(s))\right\}$*. If $z(x)$ is a solution of (35) on I, then*

$$|z(x)| \leq \left\{ \left(1 + (c_3)^{1-p}\right) \exp\left((1-p)M \int_{\alpha(x^0)}^{\alpha(x)} \gamma(x,s)ds \right) - 1 \right\}^{\frac{1}{1-p}} \tag{42}$$

for $x \in I$, where c_3 is the solution of the equation

$$\bar{H}_3(t) = \frac{1}{1-p} \ln \frac{1 + (2t-u_0)^{1-p}}{1 + t^{1-p}} - M \int_{\alpha(x^0)}^{\alpha(T)} \gamma(x,s)ds = 0, t \geq u_0.$$

Proof. Using the conditions (37)-(39) for (36), we have

$$\begin{aligned} |z(x)| &\leq u_0 + \int_{x^0}^{x} a(x,s)\,|z(s-\beta(s))|\,ds + \int_{x^0}^{T} b(x,s)\,|z(s-\beta(s))|^p\,ds \\ &\leq u_0 + \int_{x^0}^{x} a(x,s)\,|z(\alpha(s))|\,ds + \int_{x^0}^{T} b(x,s)\,|z(\alpha(s))|^p\,ds, \end{aligned}$$

with a suitable change of variables and using (40), we get

$$\begin{aligned} |z(x)| &\leq u_0 + M \int_{\alpha(x^0)}^{\alpha(x)} a(x,\alpha^{-1}(s))\,|z(s)|\,ds + M \int_{x^0}^{T} b(x,\alpha^{-1}(s))\,|z(s)|^p\,ds \\ &\leq u_0 + M \int_{\alpha(x^0)}^{\alpha(x)} \gamma(x,s)\,|z(s)|\,ds + M \int_{x^0}^{T} \gamma(x,s)\,|z(s)|^p\,ds \end{aligned} \tag{43}$$

for $x \in I$. The application of Theorem 2.3, with $f = g = 1$, $b = d = 0$, to (43) yields (42). □

Remark 3.1 In (35), if we replace $\mathbb{R}$ by any time scale $\mathbf{T}$, we obtain a dynamic boundary value problem system as follows:

$$\begin{cases} z^{\Delta_1 \cdots \Delta_n}(x) = f^{\Delta_1 \cdots \Delta_n}(x) + A\left(x,s,z(s)\right) + B\left(x,s,z(s)\right), \\ z(x_1,...,x_{n-1},x_n^0) = f(x_1,...,x_{n-1},x_n^0), ..., z(x_1^0,...,x_n) = f(x_1^0,...,x_n), \end{cases} \tag{44}$$

(44) can be restated as follows:

$$z(x) = f(x) + \int_{x^0}^{x} A\left(x,s,z\left(s\right)\right)\Delta s + \int_{x^0}^{T} B\left(x,s,z\left(s\right)\right)\Delta s,$$

which can be applied in the dynamic analysis of stability of solutions to dynamic Volterra-Fredholm integral equations on time scales.

4 Conclusion

Some new generalized Gronwall-Bellman-Volterra-Fredholm type nonlinear integral inequalities with delay have been established in this paper, which extend some known results obtained in [11, 14]. In the last section, to illustrate the usefulness of our results, we give an application to the research of boundedness of solutions of certain Volterra-Fredholm integral equations in n-independent variables.

Acknowledgment

The author is very grateful to the editor and the anonymous referees for their helpful comments and valuable suggestions.

References

[1] S. V. Babenko and A. A. Martynyuk. Nonlinear dynamic inequalities and stability quasilinear systems on time scales. *IFAC PapersOnLine* **50** (1) (2017) 14879–14883.

[2] D. Bainov, and P. Simeonov. *Integral inequalities and applications.* Kluwer Academic Publishers, Dordrecht, 1992.

[3] R. Bellman. The stability of solutions of linear differential equations. *Duke Math. J.* **10** (1943) 643–647.

[4] A. Boudeliou. On certain new nonlinear retarded integral inequalities in two independent variables and applications. *Appl. Math. Comput.* **335** (2018) 103–111.

[5] A. A. El-Deeb and R. G. Ahmed. On some generalizations of certain nonlinear retarded integral inequalities for Volterra–Fredholm integral equations and their applications in delay differential equations. *J. Egypt. Math. Soci.* **25** (2017) 279–285.

[6] S. Dhar and J. T. Neugebauer. Lyapunov-Type Inequalities for a Fractional Boundary Value Problem with a Fractional Boundary Condition. *Nonlinear Dynamics and Systems Theory* **22** (2) (2022) 133–143.

[7] Q. Feng, F. Meng and B. Fu. Some new generalized Volterra–Fredholm type finite difference inequalities involving four iterated sums. *Appl. Math. Comput.* **219** (2013) 8247–8258.

[8] T. H. Gronwall. Note on the derivatives with respect to a parameter of the solutions of a system of differential equations. *Ann Math.* **20** (1919) 292–296.

[9] S. D. Kendre, S. G. Latpate and S. S. Ranmal. Some nonlinear integral inequalities for Volterra-Fredholm Integral equations. *Adv. Inequal. Appl.* **2014** (2014), Article ID 21.

[10] H. Liu. On some nonlinear retarded Volterra-Fredholm type integral inequalities on time scales and their applications. *J. Inequal. Appl.* (2018) 2018:211.

[11] Q. H. Ma and J. Pecaric. Estimates on solutions of some new nonlinear retarded Volterra-Fredholm type integral inequalities. *Nonlinear Analysis* **69** (2008) 393–407.

[12] A. A. Martynyuk. *Stability Theory for Dynamic Equations on Time Scales.* Birkhauser, 2016.

[13] B. G. Pachpatte. Explicit bound on a retarded integral inequality. *Math. Inequal. Appl.* **7** (2004) 7–11.

[14] B. G. Pachpatte. On a certain retarded integral inequality and its applications. *JIPAM. J. Inequal. Pure Appl. Math.* **5** (2004) 1–22.

[15] O. M. Stanzhytskyi, S. G. Karakenova, and R. E. Uteshova. Averaging method and boundary value problems for systems of Fredholm integro-differential equations. *Nonlinear Dynamics and Systems Theory* **21** (1) (2021) 100–113.