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Numerical Solution of the Black-Scholes Partial Differential Equation for the Option Pricing Model Using the ADM-Kamal Method

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Abstract: Option contracts are financial derivatives developed as investment alternatives which are useful for minimizing the risk of loss. One of the most well-known models for calculating option prices is the Black-Scholes equation. This equation is a partial differential equation (PDE) of the order of natural and fractional numbers. In this paper, we have proposed a combined method of the Adomian Decomposition Method (ADM) and the Kamal Integral Transform (KIT) to solve the Black-Scholes Fractional Partial Differential Equation (FPDE) for the Option Pricing Model (OPM). The Black-Scholes FPDE approach solution can be used to build a buy and sell option pricing model. Numerical simulation results show that this method has an accurate performance in determining option pricing.

Keywords: *price of buy and sell options; fractional partial differential equation; Black-Scholes; Kamal integral transform; Adomian decomposition method.*

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1 Introduction

Since its applications in numerous industries began to gain traction several decades ago, fractional-order calculus has made significant stride in many areas such as high-tech industry [1], spherical tank system for level process [2], heat and mass transfer for the elliptic inclined plate [3], web transport systems in process industries [4], image encryption process [5,6], robotic manipulators [7], photovoltaic solar energy [8], manufacturing industrial natural gas consumption [9], Field-Programmable Gate Array [10], lesser date moth system [11], magnetic levitation system [12] and spiral-plate heat exchanger [13].

Numerous economic models employ fractional order models of real dynamical objects and processes. For instance, a business cycle model includes an investment function and a general liquidity preference function [14], an IS-LM macroeconomic system [15], a financial risk chaotic system [16,17], an economic growth model [18,19] and Ivancevic option pricing model [20].

The Black-Scholes equation (BSE) is among the most important mathematical models for option pricing. Black and Scholes [21] first introduced the Black-Scholes PDE employed for calculating the price of European type call and put options, in which the underlying financial asset is the stock price without dividend payments. The symbol $C = C(S.t)$ denotes the price of the European call option at time t and the asset price S. Let E be the exercise price, σ be the price volatility of the asset, T be the maturity date or time, and r be the rate of interest at which there is no risk of loss. The BSE and the boundary conditions for pricing European type call options are as follows [22]:

$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC = 0, \tag{1}$$

where C is option, σ is the volatility of the underlying asset, r is the risk-free interest rate, $C(0,t) = 0$, $C(S,t) \sim S$ for $S \to \infty$, and $C(S,T) = \max\{S - E, 0\}$. It follows that the diffusion equation is similar to equation (1) but with more parameters. In order to simplify equation (1), make the following conversion:

$$S = Ee^x, t = T - \frac{2\tau}{\sigma^2}, \text{ and } C(S,t) = Ev(x,\tau), \tag{2}$$

which reduces to the following PDE:

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv, \tag{3}$$

where $k = \frac{2r}{\sigma^2}$ and the main criteria becomes $v(x,0) = \max\{e^x - 1,\ 0\}$.

Many Black-Scholes PDE for the option pricing model have been studied and solved. The semidiscretization technique was employed by Company *et al.* [23] to evaluate the computational efficiency of the Black-Scholes option pricing PDEs. They found that when incorporating transaction costs into a model of option pricing, the semidiscretization approach provides a highly accurate approximation. Song and Wang [24] explored the Black-Scholes time-fractional equation-based option pricing problems, where the fractional derivative is referred to as a modified Riemann-Liouville fractional derivative. The successful use of the finite difference method demonstrates the efficiency of this approach and the reduction in computational effort needed to solve fractional PDE. Wang [25] investigated the degenerate Black-Scholes equation, which governs option pricing by using a novel numerical strategy. The author has employed implicit temporal stepping and fitted finite volume spatial discretization. Edeki *et al.* [26] extended the idea from the classical Differential Transformation Method (DTM) for the Black–Scholes equation to define the Projected DTM Method (PDTM) for European Option Valuation. Due to the fact that the PDTM requires less computing work than the traditional DTM and other semi-analytical approaches, it is demonstrated that it is more effective, reliable, and superior. He and Lin [27], using a new two-step solution approach, investigated the prices of European option using the stochastic volatility finite moment log-stable model. Additionally, numerical examples are provided to illustrate the efficiency and accuracy of the newly developed formula. The generalized Black-Scholes PDE, which appears in European option pricing, can be solved numerically using a method proposed by Mohammadi [28]. The author demonstrates how the numerical outcomes demonstrate the method's effectiveness and validate the predicted behavior of the rates of convergence. Based on the idea of homotopy perturbation, the Sumudu transform, and He's polynomials, Elbeleze *et al.* [29] investigated the fractional Black-Scholes equation and presented an interesting result. They demonstrate how effective and powerful the new approach is at locating both approximate and numerical solutions. However, to the best of our knowledge, the Black-Scholes PDE for the option pricing model using the ADM-Kamal method has not been studied in the above literature.

The key innovation and contribution of this study is the investigation of a combined approach for solving the Black-Scholes Fractional Partial Differential Equation (FPDE) for the Option Pricing Model using the Adomian Decomposition Method (ADM) and the Kamal Integral Transform (KIT).

The rest of the study is as follows. In Section 2, we briefly introduce the basic theories and theorems related to the modification and development of the ADM merging theorem with the Kamal Integral Transform. In Section 3, the combined theorem of the ADM and KIT to find a solution to the Black-Scholes FPDE for the option pricing model is discussed. Section 4 and 5 present a detailed description of the numerical experiments and concluding remarks, respectively.

2 Preliminaries

2.1 Kamal Integral Transform (KIT)

Definition 2.1 [30] Based on the set of functions

$$S = \left\{ f(x) : \exists M, k_1, k_2 > 0, |f(x)| < M e^{\frac{|x|}{k_j}}, x \in (-1)^j \times [0, \infty) \right\},$$

the Kamal transformation of f to x is given as

$$\mathcal{G}[f(x)] = G(v) = \int_0^\infty f(x) e^{-\frac{x}{v}} dx = \lim_{b \to \infty} \int_0^b f(x) e^{-\frac{x}{v}} dx, \ x \geq 0, k_1 \leq v \leq k_2,$$

where either the integral is unreasonably convergent or the limit value exists and is finite. The inverse transformation is given as

$$\mathcal{G}^{-1}[G(v)] = f(x).x \geq 0.$$

According to Definition 2.1, for $f(x) = x^n$ with n being non negative integers and $x \geq 0$, the Kamal transformation of f is

$$\mathcal{G}[x^n] = n! v^{n+1}. \tag{4}$$

If $\alpha \in \mathbb{R}$, then equation (4) is rewritten as

$$\mathcal{G}\left[x^{\alpha}\right] = \Gamma\left(\alpha+1\right) v^{\alpha+1}, \tag{5}$$

where $\Gamma(x)$ denotes the gamma function. In addition, according to Definition 2.1, the Kamal transformation of the derivative of order n is re-written as

$$\mathcal{G}\left[f^{(n)}\left(x\right)\right] = \frac{G\left(v\right)}{v^{n}} - \sum_{k=0}^{n-1} \frac{f^{(k)}\left(0\right)}{v^{n-k-1}}.$$

Definition 2.2 [31] The fundamental Mittag-Leffler function is represented by $E_{\alpha}\left(z\right)$ for $\alpha \in \mathbb{R}$, $\mathrm{Re}\left(\alpha\right) > 0$, and $\alpha \in \mathbb{C}$, and is defined as

$$E_{\alpha}\left(z\right) = \sum_{k=0}^{\infty} \frac{z^{k}}{\Gamma\left(\alpha k+1\right)}.$$

Definition 2.3 [32] The Caputo fractional derivative (CFD) of f with respect to x and for order $\alpha > 0$ is defined as

$${}_{a}^{C}D_{x}^{\alpha} f\left(x\right) = \frac{1}{\Gamma\left(n-\alpha\right)} \int_{a}^{x} \left(x-s\right)^{n-\alpha-1} f^{(n)}\left(s\right) ds,\ n-1 < \alpha \leq n.$$

Definition 2.4 [33] The Kamal transformation of the CFD is defined as

$$\mathcal{G}\left[{}_{\alpha}^{C}D_{x}^{\alpha} f\left(x\right)\right] = \frac{G\left(v\right)}{v^{\alpha}} - \sum_{k=0}^{n-1} \frac{f^{(k)}\left(0\right)}{v^{\alpha-k-1}},\ n-1 < \alpha \leq n.$$

2.2 The ADM-Kamal method

The fractional PDE is given as

$$\mathcal{G}D_{t}^{\alpha} w\left(x,t\right) + Nw\left(x,t\right) + Rw\left(x,t\right) = g\left(x,t\right). \tag{6}$$

The defined powerpoint is $w\left(x,0\right) = f\left(x\right)$, where w is the function to be determined, g denotes the function that illustrates the homogeneity of the differential equation, R is a linear operator, N is a nonlinear operator, and D_{t}^{α} is the CFD operator with $0 < \alpha \leq 1$, then the approximate solution of equation (6) is

$$\begin{aligned} w_{0} =& f\left(x\right) + \mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[g\left(x,t\right)\right]\right], \\ w_{n+1} =& -\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[A_{n}\right] + v^{\alpha}\mathcal{G}\left[Rw_{n}\right]\right],\ n = 0,1,2,\ldots, \end{aligned} \tag{7}$$

where

$$w = \lim_{k\to 0} \sum_{n=0}^{k} w_{n}.$$

Proof. Equation (6) can be rewritten with $D_{t}^{\alpha} w\left(x,t\right)$ as the subject,

$$D_{t}^{\alpha} w\left(x,t\right) = g\left(x,t\right) - Nw\left(x,t\right) - Rw\left(x,t\right). \tag{8}$$

Using the Kamal transformation in equation (8), we obtain

$$\mathcal{G}[D_{t}^{\alpha} w\left(x,t\right)] = G\left[g(x,t) - Nw(x,t) - Rw(x,t)\right],$$

where α is the order of the CFD, $n-1<\alpha\leq n$, $n\in\mathbb{Z}^{+}$.

$$\frac{w(x,v)}{v^{\alpha}}-\sum_{k=0}^{n-1}\frac{w^{(k)}(x,0)}{v^{\alpha-k-1}}=\mathcal{G}\left[g(x,t)\right]+\mathcal{G}\left[Nw(x,t)\right]+\mathcal{G}\left[Rw(x,t)\right].$$

For $0<\alpha\leq 1$ such that $k=0$, it becomes

$$\begin{aligned}\frac{w(x,v)}{v^{\alpha}}-\frac{w(x,0)}{v^{\alpha-1}}&=\mathcal{G}\left[g(x,t)\right]-\mathcal{G}\left[Nw(x)\right]+\mathcal{G}\left[Rw(x)\right],\\ w(x,v)-vw(x,0)&=v^{\alpha}\mathcal{G}\left[g(x,t)\right]-v^{\alpha}\mathcal{G}\left[Nw(x,t)\right]+v^{\alpha}\mathcal{G}\left[Rw(x,t)\right],\\ w(x,v)&=vw(x,0)+v^{\alpha}\mathcal{G}\left[g(x,t)\right]-v^{\alpha}\mathcal{G}\left[Nw(x,t)\right]-v^{\alpha}\mathcal{G}\left[Rw(x,t)\right].\end{aligned}\tag{9}$$

We use the inverse Kamal transformation in equation (9) to obtain

$$w(x,t)=w(x,0)+\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[g(x,t)\right]\right]-\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[Nw(x,t)\right]\right]-\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[Rw(x,t)\right]\right].\tag{10}$$

The ADM presumes that the function w can be broken down into an infinite series

$$w=\sum_{n=0}^{\infty}w_n,\tag{11}$$

where w_n is recursively determinable. Additionally, this approach presupposes that the infinite polynomial series may decompose the nonlinear operator Nw:

$$Nw=\sum_{n=0}^{\infty}A_n,\tag{12}$$

where $A_n=A_n(w_0,w_1,w_2,\ldots,w_n)$ is the defined Adomian polynomial (AP)

$$A_n(w_0,w_1,w_2,\ldots,w_n)=\frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{k=0}^{n}\lambda^k w_k\right)\right]_{\lambda=0};n\geq 0.$$

With λ denoting a parameter, the AP A_n can be parsed as

$$\begin{aligned}A_0=&\frac{1}{0!}\frac{d^0}{d\lambda^0}\left[N\left(\sum_{k=0}^{0}\lambda^k w_k\right)\right]_{\lambda=0}=N(w_0),\\ A_1=&\frac{1}{1!}\frac{d^1}{d\lambda^1}\left[N\left(\sum_{k=0}^{1}\lambda^k w_k\right)\right]_{\lambda=0}=w_1N\prime(w_0),\\ A_2=&\frac{1}{2!}\frac{d^2}{d\lambda^2}\left[N\left(\sum_{k=0}^{2}\lambda^k w_k\right)\right]_{\lambda=0}=\frac{w_1^2}{2!}N\prime\prime(w_0)+w_2N'(w_0),\\ &\vdots\end{aligned}$$

We substitute the initial conditions, equations (11) and (12) into equation (10):

$$\sum_{n=0}^{\infty}w_n=f(x)+\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[g(x,t)\right]\right]-\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[\sum_{n=0}^{\infty}A_n\right]+v^{\alpha}\mathcal{G}\left[R\sum_{n=0}^{\infty}w_n\right]\right].\tag{13}$$

Describing both sides of (13) gives

$$
\begin{aligned}
w_0 =& f(x) + \mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[g(x,t)\right]\right], \\
w_1 =& -\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[A_0\right] + v^{\alpha}\mathcal{G}\left[Rw_0\right]\right], \\
w_2 =& -\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[A_1\right] + v^{\alpha}\mathcal{G}\left[Rw_1\right]\right], \\
w_3 =& -\mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[A_2\right] + v^{\alpha}\mathcal{G}\left[Rw_2\right]\right]. \\
&\vdots
\end{aligned}
$$

The iterative relation derived from the approximate solution to FPDE (6) is generally defined as

$$
\begin{aligned}
w_0 &= f(x) + \mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[g(x,t)\right]\right], \\
w_{n+1} &= 7 - \mathcal{G}^{-1}\left[v^{\alpha}\mathcal{G}\left[A_n\right] + v^{\alpha}\mathcal{G}\left[Rw_n\right]\right], n = 0,\ 1,\ 2, ..,
\end{aligned} \tag{14}
$$

where

$$
w = \lim_{k\to 0} \sum_{n=0}^{k} w_n.
$$

3 Mean Absolute Error (MAE)

The method that can be used to measure the accuracy of the model in this study is the Mean Absolute Error (MAE). The MAE value represents the average error/error/absolute error between the calculation results/estimated model and the actual value [34]. The MAE formula is defined as

$$
MAE = \frac{1}{n}\sum_{i=1}^{n} \left|\hat{y}_i - y_i\right|, \tag{15}
$$

where n is the number of data, $\hat{y}_i$ is the approximate value, and y_i is the actual value.

4 Solution of the Black-Scholes FPDE for the Option Pricing Model Using the Combined ADM-Kamal Method

This study analyzed the performance of the Black-Scholes FPDE via the combined ADM-Kamal method. The Black-Scholes FPDE defined below follows from (3):

$$
\frac{\partial^{\alpha} v(x.\tau)}{\partial \tau^{\alpha}} = \frac{\partial^2 v(x.\tau)}{\partial x^2} + (k-1)\frac{\partial v(x.\tau)}{\partial x} - kv(x.\tau), \tag{16}
$$

where $0 < \alpha \le 1$ and $v(x,0) = \max\{e^x - 1, 0\}$ represents the initial condition.

Based on the defined algorithm of the Black-Scholes fractional partial differential equation with the combined ADM-Kamal method, equation (16) can be rewritten as

$$
D_{\tau}^{\alpha} v(x,\tau) = Rv(x,\tau), \tag{17}
$$

where $Rv = \frac{\partial^2 v}{\partial x^2} + (k-1)\frac{\partial v}{\partial x} - kv$ is a linear operator.

Based on the solution in the form of a recursive relation in equation (14), the solution of equation (17) is

$$\begin{aligned} v_0 &= \max\{e^x - 1.\ 0\}, \\ v_{n+1} &= \mathcal{G}^{-1}\left[v^\alpha \mathcal{G}\left[Rv_n\right]\right],\ n = 0, 1, 2, \ldots. \end{aligned}$$

If the iterative solution is explained, then based on equation (5) and the inverse Kamal transformation, we obtain

$$\begin{aligned} v_1 &= \mathcal{G}^{-1}\left[v^\alpha \mathcal{G}\left[Rv_0\right]\right] \\ &= \mathcal{G}^{-1}\left[v^\alpha \mathcal{G}\left[\frac{\partial^2 v_0}{\partial x^2} + (k-1)\frac{\partial v_0}{\partial x} - kv_0\right]\right] \\ &= \mathcal{G}^{-1}\left[v^\alpha \mathcal{G}\left[k\max\{e^x, 0\} - k\max\{e^x - 1,\ 0\}\ \right]\right] \\ &= \mathcal{G}^{-1}\left[v^{\alpha+1}\left(k\max\{e^x, 0\} - k\max\{e^x - 1,\ 0\}\right)\right] \\ &= \frac{\tau^\alpha}{\Gamma(\alpha+1)}\left(k\max\{e^x, 0\} - k\max\{e^x - 1,\ 0\}\right). \end{aligned}$$

If $\frac{\partial v_1}{\partial x} = \frac{\tau^\alpha}{\Gamma(\alpha+1)}\left(k\max\{e^x, 0\} - k\max\{e^x, 0\}\right) = 0$, we get

$$v_1 == \frac{\tau^{3\alpha}}{\Gamma(3\alpha+1)}\left(k^3k^3\max\{e^x, 0\} - k^3\max\{e^x - 1,\ 0\}\right).$$

Therefore, the approximation solution of the Black-Scholes fractional partial differential equation (16) is obtained as follows:

$$v(x.\tau) = \sum_{n=0}^{\infty} v_n = \max\{e^x - 1,\ 0\}E_\alpha(-k\tau^\alpha) + \max\{e^x, 0\}\left(1 - E_\alpha(-k\tau^\alpha)\right), \quad (18)$$

where $E_\alpha(z)$ is a one-parameter Mittag-Leffler function. Based on Definition 2.2, for $\alpha = 1$, equation (18) can be written as

$$v(x.\tau) = \max\{e^x - 1,\ 0\}e^{-k\tau} + \max\{e^x, 0\}\left(1 - e^{-k\tau}\right). \quad (19)$$

It is obvious that the solution of (19) is similar to the approximate solution of the classical Black-Scholes PDE for order $\alpha = 1$, by using the Sumudu decomposition method. Furthermore, the Black-Scholes PDE defined in (3) has the following exact solution:

$$v(x.\tau) = e^x N(d_1) - e^{-k\tau}N(d_2), \quad (20)$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}, d_2 = d_1 - \sqrt{2\tau}, k = \frac{2r}{\sigma^2},$$

and $N(d)$ is the cumulative distribution function.

5 Numerical Simulation

This section investigates the solution by the Black-Scholes PDE approach based on the combined ADM-Kamal method. Table 1 shows the numerical comparison of the Black-Scholes PDE solution defined in equation (19) for $\alpha = 1$ with the exact solution (equation (20)), for $k = 0.75$, and the magnitude of the error.

x	τ	Approach Solution	Exact Solution	Error
–0.50	0.50	0.189669	0.082462	0.107207
–0.40	1.00	0.353683	0.353395	0.000288
–0.30	1.50	0.500310	0.539833	0.039523
–0.20	2.00	0.636047	0.688686	0.052639
–0.10	2.50	0.766076	0.819751	0.053675
0.00	3.00	0.894601	0.943932	0.049331
0.10	3.50	1.032731	1.068042	0.035311
0.20	4.00	1.171616	1.196726	0.025110
0.30	4.50	1.315641	1.333412	0.017771
0.40	5.00	1.468307	1.480838	0.012531
0.50	5.50	1.632558	1.641369	0.008811

Table 1: Comparison of the numerical solution of the Black-Scholes PDE with the exact solution.

The example of Table 1 with a caption is given below.

Based on the numerical simulations presented in Table 1, the results of the comparison of the Black-Scholes PDE solution with the exact solution using equation (15) is 3.66%.

Referring to equation (2), then we get

$$x = \ln\left(\frac{S}{E}\right)\tau = \frac{\sigma^2}{2}\left(T - t\right)v\left(x\tau\right) = \frac{C\left(St\right)}{E}k = \frac{2r}{\sigma^2},$$

where T denotes the time or maturity date, r is the risk-free interest rate, E is the exercise price, σ is the volatility of the asset price, t is the time, and S is the price of the asset. Based on (18), the price model for the call option C of fractional order is

$$C\left(S.t\right) = \max\left\{S - E, 0\right\}E_\alpha\left(\zeta\right) + \max\left\{S, 0\right\}\left(1 - E_\alpha\left(\zeta\right)\right), \tag{21}$$

where $\zeta = -\frac{2^{2-\alpha}r}{\sigma^{2-2\alpha}}\left(T - t\right)^\alpha$. Next, the formula for the price of the put option P of fractional order, which is based on the put-call parity formula, is given as

$$P\left(S.t\right) = \max\left\{S - E,\ 0\right\}E_\alpha\left(\zeta\right) + \max\left\{S, 0\right\}\left(1 - E_\alpha\left(\zeta\right)\right) + Ee^{-r(T-t)} - S. \tag{22}$$

Suppose the stock price is represented by the price of asset S in this study. Figure 1 shows the call option price $C(S.t)$ against the stock price S of the Black-Scholes partial differential equation solution based on the combined ADM-Kamal method, with dissimilar values of α, where the exercise price is $E = 5$ and the risk-free interest rate (RIR) is $r = 5\%$ for a one-year option contract.

Figure 2 shows the put option price $P(S.t)$ against the stock price variable S from the Black-Scholes PDE solution based on the combined ADM-Kamal method with dissimilar values of α, where the exercise price is $E = 5$ and the RIR is $r = 5\%$ for a one-year contract.

We fix $\alpha = \{0.2, 0.5, 0.8, 0.9, 1\}$. In Fig. 1, an increase in α value will lower the call option price. Meanwhile, in Fig.2, an increase in alpha value will lower the put option price. We have calculated the price of the call option C in equation (21) which is

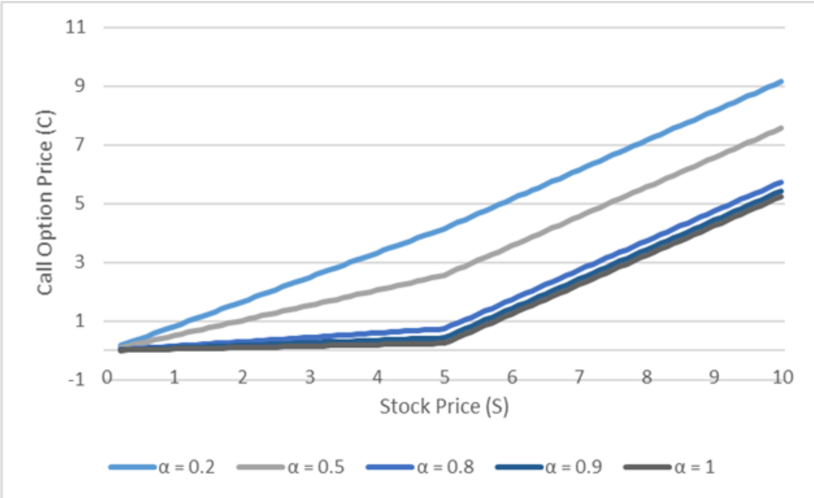

Figure 1: The price $C(S.t)$ with a fractional order of α against the stock price S.

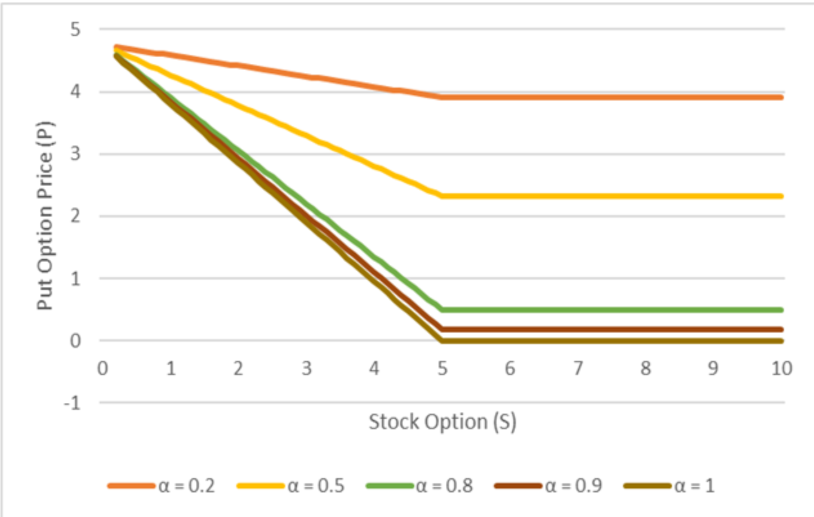

Figure 2: The price $P(S.t)$ with a fractional order of α against the stock price S.

simplified to

$$\begin{aligned} C\left(S.t\right) &= \max\left\{S-E,\ 0\right\}E_{\alpha}\left(\zeta\right)+\max\left\{S,0\right\}\left(1-E_{\alpha}\left(\zeta\right)\right) \\ &= \left(S-E\right)E_{\alpha}\left(\zeta\right)+S\left(1-E_{\alpha}\left(\zeta\right)\right) \\ &= SE_{\alpha}\left(\zeta\right)-EE_{\alpha}\left(\zeta\right)+S-SE_{\alpha}\left(\zeta\right) \\ &= S-EE_{\alpha}\left(\zeta\right). \end{aligned} \tag{23}$$

Meanwhile, for the put option price P, the equation (22) becomes

$$\begin{aligned} P(S.t) &= \max\{S-E,\ 0\}E_\alpha(\zeta) + \max\{S,0\}(1-E_\alpha(\zeta)) + Ee^{-r(T-t)} - S \\ &= (S-E)E_\alpha(\zeta) + S(1-E_\alpha(\zeta)) + Ee^{-r(T-t)} - S \\ &= SE_\alpha(\zeta) - EE_\alpha(\zeta) + S - SE_\alpha(\zeta) + Ee^{-r(T-t)} - S \\ &= E\left(e^{-r(T-t)} - E_\alpha(\zeta)\right). \end{aligned} \tag{24}$$

The price of $P(S.t)$ for the stock price is higher than the exercise price ($S > E$) and is not affected by the stock price S. However, it is only affected by the exercise price E, $e^{-r(T-t)}$ and ζ is constant. Also, if $S > E$, then the price of the put option tends to be constant, regardless of the stock price.

To calculate the price of the call option C, if the share price is less than or equal to the exercise price ($S \leq E$), we obtained

$$C(S.t) = \max\{S-E,\ 0\}E_\alpha(\zeta) + \max\{S,0\}(1-E_\alpha(\zeta)) = S(1-E_\alpha(\zeta)). \tag{25}$$

The call option price for the stock price is less than or equal to the exercise price ($S \leq E$) and is not affected by the exercise price E, but only influenced by the stock prices S and ζ is constant.

Meanwhile, we get

$$\begin{aligned} P(S.t) &= \max\{S-E,\ 0\}E_\alpha(\zeta) + \max\{S,0\}(1-E_\alpha(\zeta)) + Ee^{-r(T-t)} - S \\ &= S(1-E_\alpha(\zeta)) + Ee^{-r(T-t)} - S \\ &= S - SE_\alpha(\zeta) + Ee^{-r(T-t)} - S \\ &= Ee^{-r(T-t)} - SE_\alpha(\zeta). \end{aligned} \tag{26}$$

Then, with using Definition 2.2, for $\alpha = 1$, equations (21) and (22) become

$$C(S.t) = \max\{S-E,\ 0\}e^{-r(T-t)} + \max\{S,0\}\left(1-e^{-r(T-t)}\right), \tag{27}$$

and

$$C(S.t) = SN(d_1) - Ee^{-r(T-t)}N(d_2). \tag{28}$$

Based on equations (27) and (28), it can be seen that the determination of the price of buy and sell options, respectively, using the Black-Scholes model with a fractional order for $\alpha = 1$ is not affected by the stock price volatility because there is no parameter. Equations (27) and (28) are equivalent to the formula for the call and put option prices obtained from the results of the classical Black-Scholes equation (not fractional order) via the method of the Adomian-Laplace decomposition [35].

Based on equation (20), thus obtained the classical Black-Scholes model (CBLM) for the call option price is as follows:

$$P(S.t) = \max\{S-E,\ 0\}e^{-r(T-t)} + \max\{S,0\}\left(1-e^{-r(T-t)}\right) + Ee^{-r(T-t)} - S. \tag{29}$$

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, d_2 = d_1 - \sigma\sqrt{T-t}.$$

Furthermore, we use the put-call parity formula, which is $S+P-C = Ee^{-r(T-t)}$, and the identity formula for the cumulative distribution function for the normal distribution, which is $N(d) + N{-}d = 1$. So, the CBLM is obtained for the put option price as follows:

$$P(S.t) = Ee^{-r(T-t)}N(-d_2) - SN(-d_1), \tag{30}$$

where σ is the volatility of the stock price, r is the risk-free interest rate, E is the exercise price, S is the stock price, and T is the expiration date of the option contract.

Figure 3 shows the $C(S.t)$ against the stock price S of the Black-Scholes model with a fractional order for $\alpha = 1$ (see equation (27)) compared to the classical Black-Scholes model (see equation (29)), where the exercise price $E = 5$ and the interest rate $r = 5\%$ for a one-year option contract.

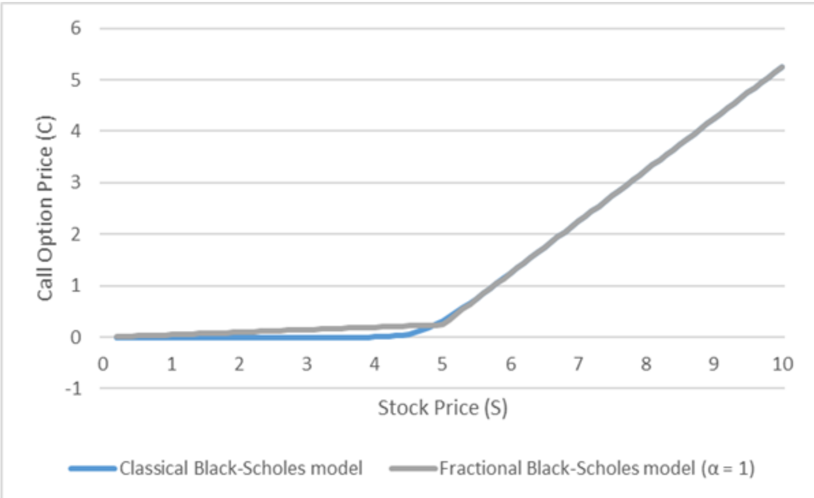

Figure 3: Comparison of the fractional order Black-Scholes model for $\alpha = 1$ with the classical Black-Scholes model for call option prices over a one-year period.

Furthermore, Figure 4 shows the $P(S.t)$ against the stock price S of the Black-Scholes model with a fractional order for $\alpha = 1$ (see equation (28)) compared to the classical Black-Scholes model (see equation (30)), where the exercise price $E = 5$ and the interest rate $r = 5\%$ for a one-year option contract.

Based on the numerical simulation presented in Figure 3, the comparison of the fractional Black-Scholes model for $\alpha = 1$ with the classical Black-Scholes model for call options prices over a one-year period is 7.80%. Meanwhile, in Figure 4, the comparison of the Black-Scholes model with a fractional order for $\alpha = 1$ with the CBSM for put option prices over a one-year period is 7.80%.

For the stock price higher than the exercise price ($S > E$), the Black-Scholes model has a fractional order with $\alpha = 1$ and the calcullation of the call option price C in

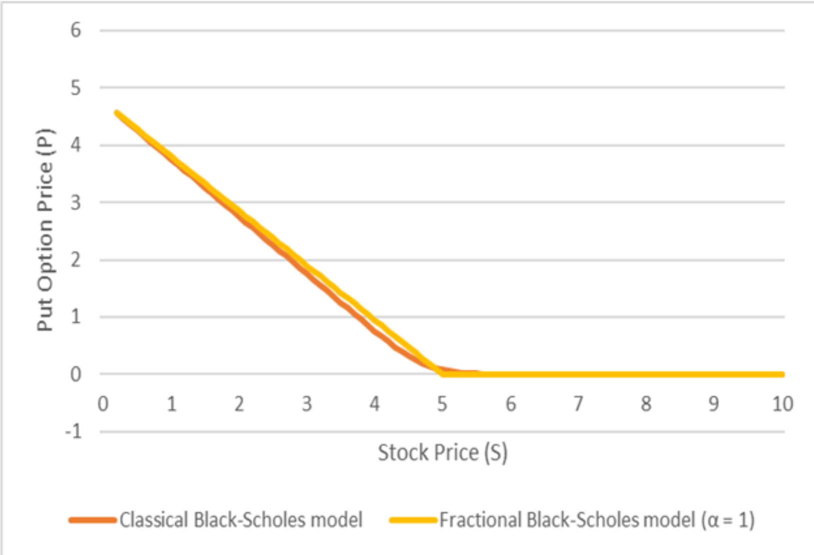

Figure 4: Comparison of the fractional order Black-Scholes model for $\alpha = 1$ with the classic Black-Scholes model for put option prices over a one-year period.

equation (27) can be simplified to

$$\begin{aligned} C(S.t) &= \max\{S-E,\ 0\}e^{-r(T-t)} + \max\{S,0\}\left(1-e^{-r(T-t)}\right) \\ &= (S-E)\,e^{-r(T-t)} + S\left(1-e^{-r(T-t)}\right) \\ &= Se^{-r(T-t)} - Ee^{-r(T-t)} + S - Se^{-r(T-t)} \\ &= S - Ee^{-r(T-t)}. \end{aligned} \tag{31}$$

Meanwhile, for the put option price P, equation (28) becomes

$$\begin{aligned} P(S.t) &= \max\{S-E,\ 0\}e^{-r(T-t)} + \max\{S,0\}\left(1-e^{-r(T-t)}\right) + Ee^{-r(T-t)} - S \\ &= (S-E)\,e^{-r(T-t)} + S\left(1-e^{-r(T-t)}\right) + Ee^{-r(T-t)} - S \\ &= Se^{-r(T-t)} - Ee^{-r(T-t)} + S - Se^{-r(T-t)} + Ee^{-r(T-t)} - S \\ &= 0. \end{aligned} \tag{32}$$

If the stock price is higher than the exercise price ($S > E$), then the put price P is equal to 0. This is in accordance with the illustration of the numerical simulation in Figure 4. For the stock price less than or equal to the exercise price ($S \leq E$) and the Black-Scholes model of fractional order with $\alpha = 1$, the calculation of the call option price can be simplified to

$$C(S.t) = \max\{S-E,\ 0\}e^{-r(T-t)} + \max\{S,0\}\left(1-e^{-r(T-t)}\right) = S\left(1-e^{-r(T-t)}\right). \tag{33}$$

The call option price for the stock price less than or equal to the exercise price ($S \leq E$) is not affected by the exercise price E, only influenced by the stock price S and $e^{-r(T-t)}$ which are constant.

Meanwhile, we get

$$\begin{aligned} P\left(S.t\right) &= \max\left\{S-E,\ 0\right\}e^{-r(T-t)} + \max\left\{S,0\right\}\left(1-e^{-r(T-t)}\right) + Ee^{-r(T-t)} - S \\ &= S\left(1-e^{-r(T-t)}\right) + Ee^{-r(T-t)} - S \\ &= \left(E-S\right)e^{-r(T-t)}. \end{aligned} \tag{34}$$

The price of put options for the stock prices less than or equal to the exercise price ($S \leq E$) is affected by the difference between the exercise price and the stock price (E–S) and $e^{-r(T-t)}$ which is constant.

Next, suppose $T = t$, the option transaction is exercised at maturity, thus equations (27) and (28) become

$$C\left(S,T\right) = \max\left\{S-E,0\right\}, \tag{35}$$

and

$$P\left(S,T\right) = \max\left\{S-E,0\right\} + E - S. \tag{36}$$

Equations (35) and (36) are equivalent to the payoff obtained from buying call and put options without taking into account the premium.

6 Conclusion

The main finding of this study is the investigation of a combined approach for solving the Black-Scholes Fractional Partial Differential Equation (FPDE) for the Option Pricing Model using the Adomian Decomposition Method (ADM) and the Kamal Integral Transform (KIT). In conclusion, the ADM-Kamal method is a very effective and powerful way to obtain both approximate and numerical solutions.

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