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A Novel Adaptive Method Based on New Minorant-Majorant Functions Without Line Search for Semidefinite Optimization

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Abstract: A novel robust adaptive method, to solve a semidefinite programming (SDP) problem, is proposed in this study. We are interested in computation of the direction by Newton's method and of the displacement step using minorant-majorant functions instead of line search methods in order to reduce the computation cost. Our new approach is even more beneficial than classical line search methods. We created a MATLAB implementation and ran numerical tests on various sizable instances to validate it. The numerical data gained demonstrate the correctness and effectiveness of our strategy, and are presented in the last section of this paper.

Keywords: *semidefinite optimization, interior point methods, perturbations minorant-majorant functions, general perturbation schemes, line search.*

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1 Introduction

In the last twenty years, Semidefinite Programming (SDP) has evolved as the most exciting and active research area in optimization. Combinatorial optimization, control theory, and conventional convex constrained optimization are only a few of the many disciplines in which SDP has applications. SDP problems arise in several areas of applications such as economic, social, public planning and nonlinear dynamics and systems (see [2, 18]). Most of these applications can often be solved pretty efficiently both in theory and in reality since SDP is solvable through interior-point methods.

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Interior point methods were developed in the sixties by Dikin and Fiacco–McCormick [7], to solve nonlinear mathematical programs with large dimension.

In order to solve the SDP problems, several algorithms have been proposed. Nesterov and Nemirovski [16] and Alizadeh [1] are the researchers who developed interior-point methods (IPMs) for SDP.

To solve SDP, a number of approaches have been put forth, including projective IPMs and their variants [10, 14], central trajectory methods [19], logarithmic barrier methods [5].

The determination and calculation of the displacement step provide an obstacle to establishing an iteration. Unfortunately, computing the displacement step is expensive and difficult in the case of semidefinite problems (particularly when using line search methods [12]).

In this paper, we are interested in solving SDP using a barrier logarithmic method that is simple and effective and is based on new approximate functions (new minorant and new majorant functions). These approximate functions allow the computation of the displacement step easily and quickly, and are more efficient than classical line searches.

We focus on the following SDP problem:

$$\begin{cases} \min b^T x \\ \sum\limits_{i=1}^{m} x_i A_i - C \in S_n^+, \\ x \in \mathbb{R}^m, \end{cases} \tag{1}$$

where $b \in \mathbb{R}^m$, the matrices C, A_i, with $i = 1, \ldots, m$, are the given symmetrical matrices and S_n^+ designs the cone of the symmetrical semidefinite positive $n \times n$ matrix.

The problem (1) is the dual of the following SDP problem:

$$\begin{cases} \max\langle C, Y\rangle \\ \langle A_i, Y\rangle = b_i, \ \forall i = 1, \ldots, m, \\ Y \in S_n^+. \end{cases} \tag{2}$$

Recall that $\langle ., .\rangle$ corresponds to an inner product on the space of $n \times n$ matrices, where the trace of the matrix $(C^T Y)$ is denoted by $\langle C, Y\rangle$.

Their feasible sets involving a non polyhedral convex cone, of positive semidefinite matrices, are called linear semidefinite programs. A priori, one of the advantages of the problem (1) with respect to its dual problem (2) is that the variable of the objective function is a vector instead of being a matrix in the problem (2). Furthermore, under certain convenient hypothesis, the resolution of the problem (1) is equivalent to that of the problem (2) in the sense that the optimal solution of one of the two problems can be reduced directly from the other through the application of the theorem on complementary slackness, see for instance [1, 8, 15].

The problem (1) is approximated by the following perturbed problem $(SDP)_\eta$:

$$\begin{cases} \min \ f_\eta(x) \\ x \in \mathbb{R}^m \end{cases} \tag{$(SDP)_\eta$}$$

with the penalty parameter $\eta > 0$ and $f_\eta : \mathbb{R}^m \to]-\infty, +\infty]$ being the barrier function defined by

$$f_\eta(x) = \begin{cases} b^T x + n\eta \ln \eta - \eta \ln[\det(\sum\limits_{i=1}^{m} x_i A_i - C)] & \text{if } x \in \widehat{X}, \\ +\infty & \text{if not.} \end{cases}$$

The problem $(SDP)_\eta$ can be solved via a classical Newton descent method.

The difficulty in line search is the presence of the determinant in the definition of the logarithmic barrier function which leads to a very high cost in the classical exact or approximate procedures of line search. In our approach, instead of minimizing f_η, along the descent direction at a current point x, we propose the minorant $\breve{G}$ and majorant $\widetilde{G}$ functions for which the optimal solution of the displacement step α is obtained explicitly.

Let us minimize the function G so that

$$\begin{aligned} \frac{1}{\eta}[f_\eta(x+\alpha d) - f_\eta(x)] &= G(\alpha) \geq \breve{G}(\alpha),\ \forall \alpha > 0, \\ \frac{1}{\eta}[f_\eta(x+\alpha d) - f_\eta(x)] &= G(\alpha) \leq \widetilde{G}(\alpha),\ \forall \alpha > 0, \end{aligned}$$

with $G(0) = \breve{G}(0) = 0,\ G'(0) = \breve{G}'(0) < 0$ and $G(0) = \widetilde{G}(0) = 0,\ G'(0) = \widetilde{G}'(0) < 0$.

The criterion $G''(0) = \breve{G}''(0)$ and $G''(0) = \widetilde{G}(0)$ guarantees that the approximations $\breve{G}$ and $\widetilde{G}$ of G are of the highest quality.

This novel strategy's key idea is to present a unique method for computing the displacement step based on minorant-majorant functions. In contrast to the conventional methods of line search, we then achieve an explicit approximation that reduces the objective and is both inexpensive and reliable.

The main advantage of $(SDP)_\eta$ resides in the strict convexity of its objective function and convexity of its feasible domain. As a result, the prerequisites for optimality are both necessary and sufficient. This encourages theoretical and numerical research of the problem.

Six sections make up the remainder of this paper. In Section 2, we briefly recall some results in linear semidefinite programming and give some preliminary results. The convergence findings of the perturbed problem into the initial one are presented in Section 3. In Section 4, we provide the solution of the associated perturbed problem and the important crucial result of the paper by introducing new approximate functions (minorant and majorant functions). The effectiveness of the approximations as compared to classical line-searches is illustrated by numerical tests in Section 5. Section 6 contains some concluding remarks.

2 Background and Preliminary Results

This section provides the necessary background for the upcoming development. In Subsection 2.1, we review some results in linear semidefinite programming. In Subsection 2.2, we review some statistical inequalities.

2.1 Backdrop and brief information on linear semidefinite programming

In what follows, we denote by

1. $X = \{x \in \mathbb{R}^m : \sum_{i=1}^m x_i A_i - C \in S_n^+\}$ the set of feasible solutions of (1).

2. $\widehat{X} = \{x \in \mathbb{R}^m : \sum_{i=1}^m x_i A_i - C \in int(S_n^+)\}$ the set of strictly feasible solutions of (1).

3. $F = \{Y \in S_n^+ : \langle A_i, Y\rangle = b_i,\ \forall i = 1, \ldots, m\}$ the set of feasible solutions of (2).

4. $\widehat{F} = \{Y \in F : Y \in int(S_n^+)\}$ the set of strictly feasible solutions of (2).

Here, $int(S_n^+)$ is the set of the symmetrical definite positive $n \times n$ matrices.
Let us state the following necessary assumptions.

- (**A1**) The system of equations $\langle A_i, Y \rangle = b_i$, $i = 1, \dots, m$ is of rank m.

- (**A2**) The sets $\widehat{X}$ and $\widehat{F}$ are non empty.

We know that (see [1, 3])

1. The sets of optimal solutions of problems (2) and (1) are non empty convex and compact.
2. If $\bar{x}$ is an optimal solution of (1), then $\bar{Y}$ is an optimal solution of (2) if and only if $\bar{Y} \in F$ and $\left(\sum_{i=1}^{m} \bar{x}_i A_i - C \right) \bar{Y} = 0$.
3. If $\bar{Y}$ is an optimal solution of (2), then $\bar{x}$ is an optimal solution of (1) if and only if $\bar{x} \in X$ and $\left(\sum_{i=1}^{m} \bar{x}_i A_i - C \right) \bar{Y} = 0$.

According to assumptions (A1) and (A2), the solution of problem (1) permits to give the solution of problem (2) and vice-versa.

2.2 Preliminary inequalities

The following result is due to H. Wolkowicz et al. [20], see also J. P. Crouzeix et al. [6] for additional results.

Proposition 2.1 *[20]*

$$\begin{aligned} \bar{x} - \sigma_x \sqrt{n-1} &\leq \min_i x_i \leq \bar{x} - \frac{\sigma_x}{\sqrt{n-1}}, \\ \bar{x} + \frac{\sigma_x}{\sqrt{n-1}} &\leq \max_i x_i \leq \bar{x} + \sigma_x \sqrt{n-1}. \end{aligned}$$

Let us recall that B. Merikhi et al. [5] proposed some useful inequalities related to the maximum and minimum of $x_i > 0$ for any $i = 1, ..., n$.

$$n \ln(\bar{x} - \sigma_x \sqrt{n-1}) \leq A \leq \sum_{i=1}^{n} \ln(x_i) \leq B \leq n \ln(\bar{x}) \tag{7}$$

with

$$\begin{aligned} A &= (n-1)\ln(\bar{x} + \frac{\sigma_x}{\sqrt{n-1}}) + \ln(\bar{x} - \sigma_x \sqrt{n-1}), \\ B &= \ln(\bar{x} + \sigma_x \sqrt{n-1}) + (n-1)\ln(\bar{x} - \frac{\sigma_x}{\sqrt{n-1}}) \end{aligned}$$

so that $\bar{x}$ and σ_x are respectively, the mean and the standard deviation of a statistical series $\{x_1, x_2, ..., x_n\}$ of n real numbers. These quantities are defined as follows:

$$\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i \quad \text{and} \quad \sigma_x^2 = \frac{1}{n} \sum_{i=1}^{n} x_i^2 - \bar{x}^2 = \frac{1}{n} \sum_{i=1}^{n} (x_i - \bar{x})^2.$$

The main advantage of $(SDP)_\eta$ resides in the strict convexity of its objective function and its feasible domain. Consequently, the conditions of optimality are necessary and sufficient. This fosters theoretical and numerical studies of the problem.

Before this, it is necessary to show that $(SDP)_\eta$ has at least an optimal solution.

3 Theoretical Aspects of Perturbed Problem

3.1 Existence of solution of the perturbed problem

For $x \in \widehat{X}$, let us introduce the symmetrical positive definite matrix $B(x)$ of rank m, and the lower triangular matrix $L(x)$ such that

$$B(x) = \sum_{i=1}^{m} x_i A_i - C = L(x)L^T(x),$$

and let us define, for $i, j = 1, ..., m$,

$$\begin{aligned}
&\widehat{A}_i(x) = [L(x)]^{-1} A_i [L^T(x)]^{-1},\\
&b_i(x) = \operatorname{trace}(\widehat{A}_i(x)) = \operatorname{trace}(A_i B^{-1}(x)),\\
&\Delta_{ij}(x) = \operatorname{trace}(B^{-1}(x) A_i B^{-1}(x) A_j) = \operatorname{trace}(\widehat{A}_i(x)\widehat{A}_j(x)).
\end{aligned}$$

Thus, $b(x) = (b_i(x))_{i=1,...,m}$ is a vector of $\mathbb{R}^m$ and $\Delta(x) = (\Delta_{ij}(x))_{i,j=1,...,m}$ is a symmetrical matrix of rank m.

The previous notation will be used in the expressions of the gradient and the Hessian H of f_η. To show that problem $(SDP)_\eta$ has a solution, it is sufficient to show that f_η is inf-compact.

Theorem 3.1 *[5] The function f_η is twice continuously differentiable on $\widehat{X}$. Actually, for all $x \in \widehat{X}$, we have*
(a) $\nabla f_\eta(x) = b - \eta b(x)$.
(b) $H = \nabla^2 f_\eta(x) = \eta \Delta(x)$.
(c) The matrix $\Delta(x)$ is positive definite.

Since f_η is strictly convex, $(SDP)_\eta$ has at most one optimal solution.

For the existence of solution of the perturbed problem, firstly, we start with the following definition.

Definition 3.1 Let f be a function defined from $\mathbb{R}^m$ to $\mathbb{R} \cup \{\infty\}$, f is called inf-compact if for all $\eta > 0$, the set $S_\eta(f) = \{x \in \mathbb{R}^m : f(x) \leq \eta\}$ is compact, which implies its cone of recession is reduced to zero.

As the function f_η takes the value $+\infty$ on the boundary of X and is differentiable on $\widehat{X}$, then it is lower semi-continuous. In order to prove that $(SDP)_\eta$ has one optimal solution, it suffices to prove that the recession cone of f_η

$$S_0\left((f_\eta)_\infty\right) = \{d \in \mathbb{R}^m,\ (f_\eta)_\infty(d) \leq 0\}$$

is reduced to zero, i.e., $d = 0$ if $(f_\eta)_\infty(d) \leq 0$, where $(f_\eta)_\infty$ is defined for $x \in \widehat{X}$ as

$$(f_\eta)_\infty(d) = \lim_{\alpha \to +\infty} \left[\xi(\alpha) = \frac{f_\eta(x + \alpha d) - f_\eta(x)}{\alpha}\right].$$

This leads to the following proposition.

Proposition 3.1 *[5] If $b^T d \leq 0$ and $\sum_{i=1}^{m} d_i A_i \in \widehat{X}$, then $d = 0$.*

3.2 Uniqueness of the solution of the perturbed problem

As f_η is inf-compact and strictly convex, therefore the problem $(SDP)_\eta$ admits a unique optimal solution. We denote by $x(\eta)$ or x_η the unique optimal solution of $(SDP)_\eta$.

3.3 Convergence of perturbed problem to (1)

Proposition 3.2 *[5] For $\eta > 0$, let x_η be an optimal solution of the problem $(SDP)_\eta$, then there exists $x \in X$ being an optimal solution of (1) such that*

$$\lim_{\eta \to 0} x_\eta = x.$$

Remark 3.1 We know that if one of the problems (1) and (2) has an optimal solution, and the values of their objective functions are equal and finite, the other problem has an optimal solution.

4 The Numerical Aspects of Perturbed Problem

4.1 Newton descent direction

With the presence of the barrier function, the problem $(SDP)_\eta$ can be considered as the one without constraints. So, one can solve it by a classical slope method. As f_η takes the $+\infty$ value on the boundary of X, then the iterates x are in $\widehat{X}$. Thus, the new proposed method is an interior point method.

Let $x \in \widehat{X}$ be the actual iterate. As a slope direction in x, let us take Newton's direction d as a solution of the linear system

$$\nabla^2 f_\eta(x) d = -\nabla f_\eta(x).$$

By virtue of Theorem 1, the precedent linear system is equivalent to the system

$$\Delta(x) d = b(x) - \frac{1}{\eta} b, \tag{3}$$

where $b(x)$ and $\Delta(x)$ are defined in Subsection 3.1.

The matrix $\Delta(x)$ being symmetrical, positive definite, the linear system (3) can be effectively solved through the Cholesky decomposition.

Evidently, one can admit $\nabla f(x) \neq 0$ (otherwise, the optimum is reached). It follows that $d \neq 0$. The direction d being calculated, we search $\bar{\alpha} > 0$ giving a significant decrease to f_η over the semi-line $x + \alpha d$, $\alpha > 0$, with the conservation of the positive definiteness of the matrix $B(x + \bar{\alpha} d)$. Then, the next iterate will be taken equal to $x + \bar{\alpha} d$. Thus, we can consider the function

$$\begin{aligned} G(\alpha) &= \frac{1}{\eta}[f_\eta(x + \alpha d) - f_\eta(x)], \ x + \alpha d \in \widehat{X}, \\ G(\alpha) &= \frac{1}{\eta} b^T d\alpha - \ln\det(B(x + \alpha d)) + \ln\det(B(x)). \end{aligned}$$

Since $\nabla^2[f_\eta(x)]d = -\nabla f_\eta(x)$, we have

$$d^T\nabla^2 f_\eta(x)d = -d^T\nabla f_\eta(x) = d^T b(x) - \eta d^T b.$$

To simplify the notations, we consider

$$B = B(x) = \sum_{i=1}^{m} x_i A_i - C \text{ and } H = \sum_{i=1}^{m} d_i A_i,$$

B being symmetrical and positive definite, there exists a lower triangular matrix L such that $B = LL^T$.

Next, let us put $E = L^{-1}H(L^{-1})^T$, since $d \neq 0$, the assumption (A1) implies that $H \neq 0$ and then $E \neq 0$.

There are two main techniques used for computing the displacement step α_k.

1) Line search methods such as the Goldstein-Armijo method, Wolfe method, Fibonacci method, etc. These methods are based on the minimization of the one-dimensional function

$$\varphi(\alpha) = \min_{\alpha>0} f_\eta(x + \alpha d).$$

Unfortunately, they are very delicate and time consuming.

2) The approximate function (majorant and minorant function) method is a sophisticated technique introduced by Crouzeix and Merikhi [5] to solve a positive semidefinite programming problem. The goal of this technique consists in approximating the function $G(\alpha)$ defined by

$$G(\alpha) = \frac{1}{\eta}[f_\eta(x + \alpha d) - f_\eta(x)].$$

Contrarily to the line search method, the approximate function is simple, and one can easily compute its minimum. This allows the computation of the displacement step without complications and in a short time.

In the following proposition, we give a simple form of the function $G(\alpha)$.

Proposition 4.1 *[5] With this notation, for any $\alpha > 0$ such that $I + \alpha E$ is positive definite,*

$$G(\alpha) = \alpha[trace(E) - trace(E^2)] - \ln\det(I + \alpha E). \tag{4}$$

Let us denote by λ_i the eigenvalues of the symmetric matrix E, then the function G can be written as follows:

$$G(\alpha) = \sum_{i=1}^{n}[\alpha(\lambda_i - \lambda_i^2) - \ln(1 + \alpha\lambda_i)],\ \alpha \in [0, \widehat{\alpha}[\,, \tag{5}$$

with

$$\widehat{\alpha} = \sup[\alpha : 1 + \alpha\lambda_i > 0 \text{ for all } i\,] = \sup[\alpha : x + \alpha d \in \widehat{X}]. \tag{6}$$

Let us observe that $\widehat{\alpha} = +\infty$ if E is positive semidefinite, and $0 < \widehat{\alpha} < \infty$ otherwise. It is clear that G is convex on $[0, \widehat{\alpha}[\,$, $G(0) = 0$ and

$$0 < \sum_i \lambda_i^2 = G''(0) = -G'(0).$$

Besides, $G(\alpha) \to +\infty$ when $\alpha \to \widehat{\alpha}$. It follows that there exists a unique point α_{opt} such that $G'(\alpha_{opt}) = 0$, where G reaches its minimum at this point.

Unfortunately, there is no an explicit formula that gives α_{opt}, and the resolution of the equation $G'(\alpha_{opt}) = 0$ through iterative methods needs at each iteration the computation of G and G'. These computations are too expensive because the expression of G in (4) contains the determinant which is difficult to calculate and the expression of (5) necessitates the knowledge of the eigenvalues of E. It is a numerical problem of large size. These difficulties make us look for other new alternatives approaches. Once E is calculated, it is easy to calculate the following quantities:

$$trace(E) = \sum_i e_{ii} = \sum_i \lambda_i \quad \text{and} \quad trace(E^2) = \sum_{i,j} e_{ij}^2 = \sum_i \lambda_i^2.$$

Based on this proposition, we give, in the following section, new notions of non expensive approximate functions for G that offer some variable displacement steps to every iteration with a simple technique. We prove the efficiency of one of them by numerical experiments that we will present at the end of this work.

Now, we give the crucial result of the paper.

4.2 New minorant and majorant functions of G

Let us go back to the equations (5) and (6), denote by $\bar{\lambda}$ and σ_λ, respectively, the mean and the standard deviations of λ_i, and by $\|\lambda\|$ the Euclidean norm of the vector λ. So,

$$\|\lambda\|^2 = n(\bar{\lambda}^2 + \sigma_\lambda^2) = G''(0) = -G'(0),$$

and

$$G(\alpha) = n\bar{\lambda}\alpha - \|\lambda\|^2\alpha - \sum_{i=1}^{n} \ln(1 + \alpha\lambda_i). \tag{8}$$

The problem consists in looking for $\bar{\alpha} \in \left]0, \widehat{\alpha}\right[$ with $\widehat{\alpha} = \min\limits_{\lambda_i < 0} \left\{ \frac{-1}{\lambda_i} \right\}$ to give a significant decrease of the convex function G. Let us insist that the best natural choice $\bar{\alpha} = \alpha_{opt}$, where $G'(\alpha_{opt}) = 0$, presents numerical complications. However, one can find approximately $\bar{\alpha}$, but this procedure necessitates, also, too many computations of G and G'. However, if we use a line search, it becomes convenient to know the superior born $\breve{\alpha}$ of the G domain, which is numerically difficult to solve. Consequently, we will take the upper borne of $\widetilde{\alpha}$ given in Proposition 2.1.

$$\begin{aligned} \breve{\alpha} &= \sup[\alpha : 1 + \alpha\gamma > 0] \text{ with } \gamma = \bar{\lambda} + \sigma_\lambda\sqrt{n-1}, \\ \widetilde{\alpha} &= \sup[\alpha : 1 + \alpha\beta > 0] \text{ with } \beta = \bar{\lambda} + \frac{\sigma_\lambda}{\sqrt{n-1}}. \end{aligned}$$

This strategy consists in minimizing a minorant and majorant approximation $\breve{G}$ and $\widetilde{G}$ of G instead of minimizing G over $\left[0, \widehat{\alpha}\right[$. To be efficient, this minorant and majorant approximation needs to be simple and sufficiently near G. In our case, it requires

$$\begin{aligned} 0 &= \breve{G}(0),\ \|\lambda\|^2 = \breve{G}''(0) = -\breve{G}'(0), \\ 0 &= \widetilde{G}(0),\ \|\lambda\|^2 = \widetilde{G}''(0) = -\widetilde{G}'(0). \end{aligned}$$

The following lemma introduces two new approximate functions for G.

Lemma 4.1 *For all* $\alpha \in [0, \widehat{\alpha}[\cap [0, \breve{\alpha}[$, *we have*

$$\breve{G}_{Min}(\alpha) \leq G(\alpha),$$

and for all $\alpha \in [0, \widehat{\alpha}[\cap [0, \widetilde{\alpha}[$, *we have*

$$G(\alpha) \leq \widetilde{G}_{Maj}(\alpha),$$

where

$$\breve{G}_{Min}(\alpha) = \frac{\|\lambda\|^2}{\gamma}\alpha - q \ln\left(1 + \frac{\|\lambda\|^2}{\gamma}\alpha\right), \ \forall \alpha \geq 0, \ 0 < q < 1,$$

and

$$\widetilde{G}_{Maj}(\alpha) = \frac{\|\lambda\|^2}{\beta}\alpha - p \ln\left(1 + \frac{\|\lambda\|^2}{\beta}\alpha\right), \ \forall \alpha \geq 0, \ 0 < p < 1.$$

Proof. 1. We start by proving that $\breve{G}_{Min}(\alpha) \leq G(\alpha)$.
We have $G(\alpha) = n\bar{\lambda}\alpha - \|\lambda\|^2\alpha - \sum_{i=1}^{n} \ln(1 + \alpha\lambda_i)$. Then we put

$$H(\alpha) = G(\alpha) - \breve{G}_{Min}(\alpha).$$

Then $H(0) = H'(0) = 0$ and we have, for all $\alpha > 0$,

$$H''(\alpha) = \sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + \alpha\lambda_i)^2} - \frac{\lambda_i^2}{\left(1 + \alpha\frac{\|\lambda\|^2}{\gamma}\right)^2} \geq 0.$$

Because $|\lambda_i| \leq \|\lambda\|$ and $\gamma \leq \|\lambda\|$, it gives $H(\alpha) \geq 0, \ \forall \alpha \geq 0$.
So $\breve{G}_{Min}(\alpha) \leq G(\alpha)$.

2. Now we prove that $G(\alpha) \leq \widetilde{G}_{Maj}(\alpha)$. We put: $K(\alpha) = \widetilde{G}_{Maj}(\alpha) - G(\alpha)$. Then $K(0) = K'(0) = 0$ and we have, for all $\alpha > 0$,

$$K''(\alpha) = \frac{\|\lambda\|^4}{(\beta + \alpha\|\lambda\|^2)^2} + \sum_{i=1}^{n} \frac{\lambda_i^2}{(1 + \alpha\lambda_i)^2} \geq 0.$$

This gives $K(\alpha) \geq 0, \ \forall \alpha \geq 0$. So $G(\alpha) \leq \widetilde{G}_{Maj}(\alpha)$.

We deduce that the functions $\breve{G}_{Min}$ and $\widetilde{G}_{Maj}$ reach their minimum at one unique point:

$$\bar{\alpha}_{Min} = (q - 1)\frac{\gamma^2}{\|\lambda\|^2}, \quad \bar{\alpha}_{Maj} = (p - 1)\frac{\beta^2}{\|\lambda\|^2}.$$

5 Description of the Algorithm and Numerical Results

In this section, we present the algorithm of our approach to obtain an optimal solution $\bar{x}$ of the problem (1) and some numerical results to demonstrate the performance of our methods.

5.1 The algorithm

Begin algorithm

Initialization

We have to decide on the strategy of the displacement step. $\varepsilon > 0$ is a given precision, $\eta > 0$, $\rho > 0$ and $\sigma \in]0, 1[$ are fixed parameters. Start with $x^k \in \widehat{X}$ and $k = 0$.

Iteration

1. Take $B = B(x^k) = \sum_{i=1}^{m} x_i^k A_i - C$ and L such that $B = LL^T$.
2. Compute
$$\begin{cases} \widehat{A}_i(x^k) = [L(x^k)]^{-1} A_i [L^T(x^k)]^{-1}, \quad b(x^k) = \operatorname{trace}(\widehat{A}_i(x^k)), \\ \Delta_{ij}(x^k) = \operatorname{trace}(\widehat{A}_i(x^k)\widehat{A}_j(x^k)), \quad H = \eta\Delta(x^k). \end{cases}$$
3. Solve the linear system $Hd = \eta b(x) - b$.
4. Calculate $E = L^{-1}H(L^{-1})^T$, trace(E) and trace(E^2).
5. Take the new iterate $x^{k+1} = x^k + \bar{\alpha}d$ such that $\bar{\alpha}$ is obtained by the use of the displacement step strategy of $\breve{G}_i,\ i = 1, ..., 3$.
6. If $n\eta > \varepsilon$, do $x^k = x^{k+1}$, $\eta = \sigma\eta$ and go to (1).
7. If $|b^T x^{k+1} - b^T x^k| > n\rho\eta$, do $x^k = x^{k+1}$ and go to (1).
8. Take $k = k + 1$.
9. **Stop:** x^{k+1} is an approximate solution of the problem (1).

End algorithm

We know that the optimal solution of $(SDP)_\eta$ is an approximation of the solution of problem (1). The closer η is to zero, the better the approximation. Unfortunately, when η approaches zero; the problem $(SDP)_\eta$ becomes ill-conditioned. For this reason, we use at the beginning of the iteration the values of η that are not near to zero, and verify $n\eta < \varepsilon$. We can explain the interpretation of the update η as follows: if $x(\eta)$ is an exact solution of $(SDP)_\eta$, so $b^T x(\eta) \in [m(0), m(0) + n\eta]$. It is then not necessary to keep on the calculus of the iterates when $|b^T x^{k+1} - b^T x^k| \leq n\rho\eta$.

The displacement step $\bar{\alpha}$ will be determined by the classical line search of Armijo-Goldstein-Price type or by one of three following strategies St i, by minimizing the majorant function $\widetilde{G}$ and the minorant function $\breve{G}$.

In the next subsection, we present comparative numerical tests to prove the effectiveness of our approach over the line search method.

5.2 Numerical tests

The following examples are taken from the literature, see for instance [4, 5, 9], and implemented in MATLAB. We have taken $\varepsilon = 1.0e - 004$, $\sigma = 0.125$ and two values of ρ, $\rho = 1$ or $\rho = 2$.

In the table of results, (exp (m, n)) represents the size of the example, (Itrat) represents the number of iterations necessary to obtain an optimal solution, (Time) represents the time of computation in seconds (s), (LS) represents the classical line search of the Armijo-Goldstein method and (St Maj) and (St Min) represent the strategies which use the minorant functions $\breve{G}$ and the majorant function $\widetilde{G}$, respectively.

5.2.1 Examples with fixed size

In the following examples, $diag(x)$ is the $n \times n$ diagonal matrix with the components of x as the diagonal entries.

Example 1: $m = 2$, $n = 3$,
$$C = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 2 & -2 \\ 1 & -2 & 2 \end{pmatrix}, A_1 = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b = (0, 1)^t.$$

Example 2: $m = 3$, $n = 5$,
$$C = \begin{pmatrix} -4 & 0 & 0 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, b = (8, 7, 3)^t.$$

Example 3: $m = 3, n = 6$,
$$C = \begin{pmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, A_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$
$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, b = (0, 0, 1)^t.$$

Example 4: $m = 6$, $n = 12$,
$C = diag(-4, -5, -1, -3, 5, -8, 0, 0, 0, 0, 0, 0)^t$,
$A_1 = diag(1, 0, -4, 3, 1, 1, 1, 0, 0, 0, 0, 0)^t$,
$A_2 = diag(5, 3, 1, 0, -1, 3, 0, 1, 0, 0, 0, 0)^t$, $A_3 = diag(4, 5, -3, 3, -4, 1, 0, 0, 1, 0, 0, 0,)^t$,
$A_4 = diag(0, -1, 0, 2, , 1, -5, 0, 0, 0, 1, 0, 0)^t$, $A_5 = diag(-2, 1, 1, 1, 2, 2, 0, 0, 0, 0, 1, 0)^t$,
$A_6 = diag(2, -3, 2, -1, 4, 5, 0, 0, 0, 0, 0, 1)^t$, $b = (1, 4, 4, 5, 7, 5)^t$.
The obtained results are given in the following table.

exp (m, n)	St Min		St Maj		LS	
	Itrat	Time	Itrat	Time	Itrat	Time
exp $1(2, 3)$	4	0.032	3	0.024	5	0.25
exp $2(3, 5)$	5	0.056	2	0.0022	7	0.36
exp $3(3, 6)$	5	0.094	4	0.023	6	0.36
exp $5(6, 12)$	3	0.0016	1	0.0002	3	0.087

5.2.2 Example with variable size

Example 1: (Example Cube)

$n = 2m,\ C$ is the $n \times n$ identity matrix, $b = (2, ..., 2)^T \in \mathbb{R}^m,\ a \in \mathbb{R}$, and the entries of the $n \times n$ matrix $A_k,\ k = 1, \dots, m$, are given by

$$A_k[i,j] = \begin{cases} 1 & \text{if } i = j = k & \text{or } i = j = k+m, \\ a^2 & \text{if } i = j = k+1 & \text{or } i = j = k+m+1, \\ -a & \text{if } i = k,\ j = k+1 & \text{or } i = k+m,\ j = k+m+1, \\ -a & \text{if } i = k+1,\ j = k & \text{or } i = k+m+1,\ j = k+m, \\ 0 & \text{otherwise.} \end{cases}$$

Test: $a = 0$ and $C = -I$.

The following table resumes the obtained results.

Size (m, n)	St Min		St Maj		LS	
	Itrat	Time	Itrat	Time	Itrat	Time
$(50, 100)$	2	102	1	65	dvg	
$(100, 200)$	3	402	2	214	dvg	
$(200, 400)$	3	798	2	685	dvg	

dvg means that the algorithm does not terminate within a finite time.

Commentary. We notice that the two strategies converge to the optimal solution. These tests show clearly that our two strategies offer an optimal solution of (1) and (2) in a reasonable time and with a small number of iterations. We conclude the proposed method is more effective than the line search, and it can improve the results obtained by the line search method. When the instances get larger, this is especially true. Additionally, the reduction in time is substantial because it is clear that the suggested technique takes at least twice as long as the line searches method to arrive at the best answer.

6 Conclusion

In order to solve a linear semidefinite problem, a logarithmic barrier technique based on novel majorant and minorant functions is presented in this study. These two novel approximations provide displacement steps more quickly, cheaply, and easily than the line search. The effectiveness of the majorant and minorant function methodology in comparison to the line search method is demonstrated by numerical data. Our important result is applicable and very important in different problems of nonlinear dynamics in practice. As allways, we arrived to problem of optimization after we solve these problems, then we choose our approach for solving it. The idea of introducing our new majorant and minorant functions appears to be a topic worth exploring in the future in the nonlinear dynamics problems and other problems.

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