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Global Existence for the 3-D Generalized Micropolar Fluid System in Critical Fourier-Besov Spaces with Variable Exponent

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Abstract: In this work, we study the 3-D generalized Cauchy problem of the incompressible micropolar fluid system (GMFS) in the critical variable exponent Fourier-Besov space $\mathcal{F}\dot{\mathcal{B}}^{4-\frac{3}{p(\cdot)}-2\alpha}_{p(\cdot),q}$. We establish the global well-posedness result with the initial data belonging to $\mathcal{F}\dot{\mathcal{B}}^{4-\frac{3}{p(\cdot)}-2\alpha}_{p(\cdot),q}$, where $p = p(\cdot)$ is a bounded function satisfying $p \in [2, \frac{6}{5-4\alpha}]$, $\alpha \in (\frac{1}{2}, 1]$ and $q \in [1, \frac{3}{2\alpha-1}]$.

Keywords: *global existence; 3-D generalized micropolar fluid system; variable Fourier-Besov space.*

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1 Introduction and Statement of Main Result

We investigate the generalized incompressible micropolar system in the whole space $\mathbb{R}^3$,

$$\begin{cases} \partial_t u + (\chi+\nu)(-\Delta)^{\alpha_1} u + u\cdot\nabla u + \nabla\pi - 2\chi\nabla\times w = 0, & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \partial_t w + \mu(-\Delta)^{\alpha_2} w + u\cdot\nabla w + 4\chi w - \kappa\nabla \operatorname{div} w - 2\chi\nabla\times u = 0, & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ (u,w)|_{t=0} = (u_0, w_0), & \text{in } \mathbb{R}^3. \end{cases} \tag{1}$$

The unknowns are $u = u(x,t), w = w(x,t)$ and $\pi = \pi(x,t)$ representing, respectively, the linear velocity field, the micro-rotation velocity field and the pressure field of the fluid. The nonnegative constants κ, μ, ν and χ represent the viscosity coefficients, which determine fluid physical characteristics and $\alpha_1, \alpha_2 \in (\frac{1}{2}, 1]$ are two positive constants. u_0 and w_0 represent the initial velocities and we assume that $\operatorname{div} u_0 = 0$. Recall that

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the Riesz potential operator $(-\Delta)^k$ is defined as usual through the Fourier transform as $\mathcal{F}[(-\Delta)^k f](\xi) := |\xi|^{2k}\mathcal{F}[f](\xi)$, where $\mathcal{F}[f](\xi) := \hat{f}(\xi) = \frac{1}{(2\pi)^{3/2}}\int_{\mathbb{R}^3} e^{-ix\xi}f(x)dx$. Without loss of generality, throughout this paper, we only consider the situation with $\kappa = \mu = 1$ and $\chi = \nu = 1/2$.

Notice that if $\alpha_1 = \alpha_2 = 1$, then system (1) reduces to the standard micropolar fluid system

$$\begin{cases} \partial_t u - (\chi+\nu)\Delta u + u\cdot\nabla u + \nabla\pi - 2\chi\nabla\times w = 0, & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \partial_t w - \mu\Delta w + u\cdot\nabla w + 4\chi w - \kappa\nabla \operatorname{div} w - 2\chi\nabla\times u = 0, & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ \operatorname{div} u = 0, & \text{in } \mathbb{R}^3\times\mathbb{R}^+, \\ (u,w)|_{t=0} = (u_0,w_0), & \text{in } \mathbb{R}^3, \end{cases} \tag{2}$$

which was created by A.C. Eringen [9] in 1996. It is an essential modification to the classical Navier-Stokes equations in order to better characterize the motion of real-world fluids made up of rigid but randomly oriented particles (such as blood) by investigating the effect of micro-rotation of particles suspended in a viscous medium.

There is a lot of literature devoted to the mathematical theory of the micropolar fluid system. The first result on the existence and uniqueness of solutions of the standard micropolar fluid system was obtained by Galdi and Rionero in [10]. Chen and Miao [5] proved the global existence for the problem (2) with small initial data in the Besov spaces $\dot{\mathcal{B}}^{-1+\frac{3}{p}}_{p,q}$ when $p\in[1,6)$ and $q=\infty$. Inspired by the work of Cannone and Karch [6] on the incompressible Navier-Stokes equations, V.-Roa and Ferreira [8] showed the existence of the solution for the generalized micropolar fluid system in the pseudo-measure space PM^τ which is defined by

$$PM^\tau = \left\{ f\in\mathcal{S}'(\mathbb{R}^n):\ \hat{f}\in L^1_{loc}(\mathbb{R}^n),\ \|f\|_{PM^\tau} = ess\ sup_{x\in\mathbb{R}^n}|\xi|^\tau|\hat{f}(\xi)| < \infty \right\}.$$

Our main result can be stated as follows.

Theorem 1.1 *Let $\frac{1}{2} < \alpha = min(\alpha_1,\alpha_2) = \alpha_1 \le 1$, $p(\cdot)\in C^{\log}(\mathbb{R}^n)\cap\mathcal{P}(\mathbb{R}^n)$ such that $2\le p(\cdot)\le\frac{6}{5-4\alpha}$, $1\le\rho<\infty$ and $1\le q<\frac{3}{2\alpha-1}$. Then there exists a small ε_0 such that for any $(u_0,w_0)\in\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}$ satisfying $\nabla\cdot u_0 = 0$ with $\|(u_0,w_0)\|_{\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}} < \varepsilon_0$, the problem (1) admits a unique global mild solution (u,w) in*

$$\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}+\frac{2\alpha}{\rho}}_{p(\cdot),q})\cap\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{2\alpha}{\rho}+\frac{5}{2}-2\alpha}_{2,q})\cap\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha}_{2,q}),$$

such that

$$\|(u,w)\|_{\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}+\frac{2\alpha}{\rho}}_{p(\cdot),q})\cap\mathcal{L}^\rho([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{2\alpha}{\rho}+\frac{5}{2}-2\alpha}_{2,q})\cap\mathcal{L}^\infty([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha}_{2,q})} \le \|(u_0,w_0)\|_{\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.$$

Remark 1.1 Notice that the result of Theorem 1.1 is correct only if the bounded function $p\neq 1$, and then the case $(p,\alpha) = (1,1)$, which corresponds to $\mathcal{F}\dot{\mathcal{B}}^{-1}_{1,q}$, is not included. It is proved in [13] that if $(p,\alpha) = (1,1)$, then the standard micropolar fluid system (SMFS) is well-posed in $\mathcal{F}\dot{\mathcal{B}}^{-1}_{1,q}$, where $1\le q\le 2$, and ill-posed in these spaces for $2 < q\le\infty$, which means that the space $\mathcal{F}\dot{\mathcal{B}}^{-1}_{1,q}$ is optimal. Furthermore, our result is a generalization of the works [4, 12] in which the authors proved that the problem (1) is globally well-posed in the Fourier-Besov spaces $\mathcal{F}\dot{\mathcal{B}}_{p,q}^{4-\frac{3}{p}-2\alpha}$ for $\alpha \in (\frac{1}{2}, 1]$, $1 < p \leq \infty$, $1 \leq q \leq \infty$ and the initial data being small. The aim of this work is to establish this result in the case of variable exponent.

Throughout the paper, we denote $\alpha = min(\alpha_1, \alpha_2)$. Let X, Y be the Banach spaces. We use $(a, b) \in X$ to denote $(a, b) \in X \times X$, $\|(a, b)\|_X$ to denote $\|(a, b)\|_{X\times X}$ and we denote $\|\cdot\|_{X\cap Y} = \|\cdot\|_X + \|\cdot\|_Y$. The notation $a \lesssim b$ means that there exists a positive constant C such that $a \leq Cb$.

2 Preliminaries

In this section, we review the Littlewood-Paley theory and some of the used function spaces and the related properties, we state the microlpolar semigroup and the notion of mild solutions for the system (1), we recall the Banach fixed point theorem which we will apply for proving the existence of a unique mild solution and we present the definition of the Chemin-Lerner type homogeneous Fourier-Besov spaces.

2.1 Littlewood-Paley theory and Fourier-Besov spaces with variable exponent

Let us present some basic properties of the Littlewood-Paley theory and Fourier-Besov spaces with variable exponent.

Let $\theta \in \mathcal{S}(\mathbb{R}^n)$ be a radial positive function such that $0 \leq \theta \leq 1$, $\text{supp}(\theta) \subset \left\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\right\}$ and

$$\sum_{j\in\mathbb{Z}} \theta\left(2^{-j}\xi\right) = 1, \quad \text{for all } \xi \neq 0.$$

Put

$$\theta_j(\xi) = \theta\left(2^{-j}\xi\right), \quad \varphi_j(\xi) = \sum_{k\leq j-1} \theta_k(\xi)$$

and

$$g(x) = \mathcal{F}^{-1}\theta(x), \quad h(x) = \mathcal{F}^{-1}\varphi(x).$$

Now, we present some frequency localization operators

$$\Delta_j u := \mathcal{F}^{-1}\left(\theta_j \mathcal{F}(u)\right) = 2^{nj} \int_{\mathbb{R}^n} g\left(2^j y\right) u(x-y) dy,$$

$$S_j u := \sum_{k\leq j-1} \Delta_k f = \mathcal{F}^{-1}\left(\varphi_j \mathcal{F}(u)\right) = 2^{nj} \int_{\mathbb{R}^n} h\left(2^j y\right) u(x-y) dy,$$

where $\Delta_j = S_j - S_{j-1}$ is a frequency projection to the annulus $\{|\xi| \sim 2^j\}$ and S_j is a frequency to the ball $\{|\xi| \lesssim 2^j\}$.

By using the definition of Δ_j and S_j, we easily prove that

$$\begin{aligned} \Delta_j \Delta_k f = 0, &\quad \text{if } |j-k| \geq 2, \\ \Delta_j\left(S_{k-1} f \Delta_k f\right) = 0, &\quad \text{if } |j-k| \geq 5. \end{aligned}$$

The following Bony para-product decomposition will be applied around the paper:

$$uv = \dot{T}_u v + \dot{T}_v u + R(u,v),$$

where $\dot{T}_u v = \sum_{j\in\mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v$, $\dot{R}(u,v) = \sum_{j\in\mathbb{Z}} \dot{\Delta}_j u \tilde{\Delta}_j v$ and $\tilde{\Delta}_j v = \sum_{|j'-j|\le 1} \dot{\Delta}_{j'} v$.

We define the variable exponent Lebesgue spaces $L^{p(\cdot)}$.

Definition 2.1 ([2]) Let $\mathcal{P}(\mathbb{R}^n)$ denote the set of all measurable functions $p(\cdot) : \mathbb{R}^n \to (0, \infty)$ such that

$$0 < p_- = ess \inf_{\mathrm{x}\in\mathbb{R}^n} p(x) \ , \ ess \sup_{\mathrm{x}\in\mathbb{R}^n} p(x) \ = p_+ < \infty.$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R} \text{ is measurable}, \int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty \right\}$$

equipped with the Luxemburg-Nakano norm

$$\|f\|_{L^{p(\cdot)}} = \inf \left\{ \delta > 0 : \int_{\mathbb{R}^n} (\frac{|f(x)|}{\delta})^{p(x)} dx \le 1 \right\}.$$

The space $L^{p(\cdot)}(\mathbb{R}^n)$ is a Banach space.

Since the $L^{p(\cdot)}$ does not have the same desired properties as $L^p(\mathbb{R}^n)$, we propose the following standard conditions to prove that the Hardy-Littlewood maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$:

i) We say that $p : \mathbb{R}^n \to \mathbb{R}$ is locally log-Hölder continuous, $p \in C^{\log}_{loc}(\mathbb{R}^n)$, if there exists a constant $c_{\log} > 0$ with

$$|p(x) - p(y)| \le \frac{c_{\log}}{\log\left(e + \frac{1}{|x-y|}\right)} \quad \text{for all} \quad x, y \in \mathbb{R}^n and\ x \ne y.$$

ii) We say that p is globally log-Hölder continuous, $p \in C^{\log}(\mathbb{R}^n)$, if $p \in C^{\log}_{loc}(\mathbb{R}^n)$ and there exist a $p_\infty \in \mathbb{R}$ and a constant $c_\infty > 0$ with

$$|p(x) - p_\infty| \le \frac{c_\infty}{\log(e + |x|)} \quad for \ \ all \ \ x \in \mathbb{R}^n.$$

iii) We write $p \in \mathcal{P}^{\log}(\mathbb{R}^n)$ if $0 < p^- \le p(x) \le p^+ \le \infty$ with $1/p \in C^{\log}(\mathbb{R}^n)$.

Let $p, q \in \mathcal{P}(\mathbb{R}^n)$, we denote by $\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)$ the space consisting of all sequences $\{h_i\}_i$ of measurable functions on $\mathbb{R}^n$ such that

$$\varrho_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\big((h_i)_i\big) := \sum_{i\ge 0} \inf \left\{ \delta_i > 0 | \varrho_{p(\cdot)}\left(h_i/\delta_i^{\frac{1}{q(\cdot)}}\right) \le 1 \right\}$$

with the convention $\delta^{1/\infty} = 1$. Also, the norm is defined as usual:

$$\|(h_i)_i\|_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)} := \inf \left\{ \lambda > 0 | \varrho_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\left(\frac{1}{\lambda}(h_i)_i\right) \leqslant 1 \right\}.$$

If $q^+ < \infty$, then

$$\varrho_{\ell^{q(\cdot)}\left(L^{p(\cdot)}\right)}\left((h_i)_i\right) = \sum_{i\geq 0} \| \, |h_i|^{q(\cdot)} \, \|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Then we define the variable exponent homogeneous Fourier-Besov space $\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$.

Definition 2.2 ([3]) Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$ and $p(\cdot),\ q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$ with $0 < p_- \leq p(\cdot) \leq \infty$. The homogeneous Fourier-Besov space with variable exponent $\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}$ is defined by the set of all $f \in \mathcal{Z}'(\mathbb{R}^n)$ such that

$$\|f\|_{\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q(\cdot)}} := \|\{2^{js(\cdot)}\theta_j \hat{f}\}_{-\infty}^{\infty}\|_{l^{q(\cdot)}(L^{p(\cdot)})} < \infty.$$

The space $\mathcal{Z}'(\mathbb{R}^n)$ is the dual space of

$$\mathcal{Z}(\mathbb{R}^n) = \{f \in \mathcal{S}(\mathbb{R}^n) : (D^\alpha f)(0) = 0, \forall \alpha \text{ multi-index}\}.$$

Next proposition describes some useful assertions we use in this work related to $L^{p(\cdot)}$ spaces and Besov spaces with variable exponent.

Proposition 2.1 *([7]). (1) (Hölder inequality) Let* $p_1(\cdot),\ p_2(\cdot), \in \mathcal{P}(\mathbb{R}^n)$*, and define* $p \in \mathcal{P}(\mathbb{R}^n)$ *by* $\frac{1}{p(x)} = \frac{1}{p_1(x)} + \frac{1}{p_2(x)}$. *Then there exists a constant* C *depending only on* p_- *and* p_+ *such that*

$$\|fg\|_{L^{p(\cdot)}} \leq C\|f\|_{L^{p_1(\cdot)}}\|g\|_{L^{p_2(\cdot)}}$$

holds for every $f \in L^{p_1(\cdot)}$ *and* $g \in L^{p_2(\cdot)}$.
(2) ([2]) Let $p_0(\cdot),\ p_1(\cdot),\ q(\cdot) \in \mathcal{P}(\mathbb{R}^n)$*, and* $s_0(\cdot),\ s_1(\cdot) \in L^\infty \cap C^{log}(\mathbb{R}^n)$ *with* $s_0(\cdot) \geq s_1(\cdot)$. *If* $\dfrac{1}{q_0(\cdot)}, \dfrac{1}{q_1(\cdot)}$ *and* $s_0(x) - \frac{n}{p_0(x)} = s_1(x) - \frac{n}{p_1(x)}$ *are locally log-Hölder continuous, then*

$$\dot{\mathcal{B}}^{s_0(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \dot{\mathcal{B}}^{s_1(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

The following result deals with the product of two functions in the Chemin-Lerner space.

Proposition 2.2 *([1]). Let* $s > 0,\ 1 \leq \gamma,\ \rho,\ \rho_1,\ \rho_2,\ p, q, r \leq \infty$ *such that* $\frac{1}{\rho} = \frac{1}{\rho_1} + \frac{1}{\rho_2}$ *and* $\frac{1}{\gamma} = \frac{1}{r} + \frac{1}{p}$. *Then we have*

$$\|ab\|_{\mathcal{L}^\rho \dot{B}^s_{\gamma,q}} \lesssim \|a\|_{\mathcal{L}^{\rho_1}\dot{B}^s_{\gamma,q}}\|b\|_{\mathcal{L}^{\rho_2}L^r} + \|b\|_{\mathcal{L}^{\rho_1}\dot{B}^s_{\gamma,q}}\|a\|_{\mathcal{L}^{\rho_2}L^r}.$$

2.2 Fractional micropolar semigroup and mild solutions

The following system is the corresponding linear system of (1):

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha_1} u - \nabla \times w = 0, \\ \partial_t w + (-\Delta)^{\alpha_2} w + 2w - \nabla \operatorname{div} w - \nabla \times u = 0, \\ \operatorname{div} u = 0, \\ (u, w)|_{t=0} = (u_0, w_0). \end{cases} \tag{3}$$

The solution operator of the above problem is denoted by the notation $G(t)$, i.e., for specified initial data (u_0, w_0) in suitable function space, if we denote by

$(u,w)^T = G(t)\,(u_0,w_0)^T$ the unique solution of the problem (3), then

$$(\widehat{G(t)f})(\xi) = e^{-\mathcal{A}(\xi)t}\hat{f}(\xi) \quad \text{for } f(x) = (f_1(x), f_2(x))^T,$$

where

$$\mathcal{A}(\xi) = \begin{bmatrix} |\xi|^{2\alpha_1} I & \mathcal{B}(\xi) \\ \mathcal{B}(\xi) & \left(|\xi|^{2\alpha_2}+2\right) I + \mathcal{C}(\xi) \end{bmatrix}$$

with

$$\mathcal{B}(\xi) = i \begin{bmatrix} 0 & \xi_3 & -\xi_2 \\ -\xi_3 & 0 & \xi_1 \\ \xi_2 & -\xi_1 & 0 \end{bmatrix} \text{ and } \mathcal{C}(\xi) = \begin{bmatrix} {\xi_1}^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & {\xi_2}^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & {\xi_3}^2 \end{bmatrix}.$$

Moreover, by applying the Leray projection $\mathbf{P}$ to both sides of the first equations of (1), one can eliminate the pressure π and we get

$$\begin{cases} \partial_t u + (-\Delta)^{\alpha_1} u + \mathbf{P}(u\cdot\nabla u) - \nabla\times w = 0, \\ \partial_t w + (-\Delta)^{\alpha_2} w + u\cdot\nabla w + 2w - \nabla\,\mathrm{div}\,w - \nabla\times u = 0, \\ \mathrm{div}\,u = 0, \\ (u,w)|_{t=0} = (u_0,w_0), \end{cases} \tag{4}$$

where $\mathbf{P} = I + \nabla(-\Delta)^{-1}$ div is the 3×3 matrix pseudo-differential operator in $\mathbb{R}^3$ with the symbol $\left(\delta_{ij} - \frac{\xi_i\xi_j}{|\xi|^2}\right)_{i,j=1}^3$. We denote

$$U(x,t) = \begin{pmatrix} u(x,t) \\ w(x,t) \end{pmatrix},\ U_0 = \begin{pmatrix} u(x,0) \\ w(x,0) \end{pmatrix} = \begin{pmatrix} u_0 \\ w_0 \end{pmatrix},$$
$$U_i(x,t) = \begin{pmatrix} u_i(x,t) \\ w_i(x,t) \end{pmatrix},\ i = 1,2,$$

and

$$U_1\tilde{\otimes}U_2 = \begin{pmatrix} u_1\otimes u_2 \\ u_1\otimes w_2 \end{pmatrix}, \quad \widetilde{\mathbf{P}}\nabla\cdot\left(U_1\tilde{\otimes}U_2\right) = \begin{pmatrix} \mathbf{P}\nabla\cdot(u_1\otimes u_2) \\ \nabla\cdot(u_1\otimes w_2) \end{pmatrix}.$$

Solving system (4) can be reduced to finding a solution U to the following integral equations:

$$U(t) = G(t)U_0 - \int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U\otimes U)(\tau)d\tau. \tag{5}$$

A solution of (5) *is called a mild solution of* (1). Now, we present a property of the semigroup $G(\cdot)$.

Lemma 2.1 *([8]) Let $\frac{1}{2} < \alpha \le 1$. Then for $|\xi| \ne 0$ and $t \ge 0$, there exists $C = C(\alpha_1,\alpha_2) > 0$ (independent of ξ) such that*

$$|e^{-t\mathcal{A}(\xi)}| \le \begin{cases} e^{-|\xi|^{\alpha_1}t} & \text{if } |\xi| \le 1, \\ e^{-C|\xi|^{\alpha}t} & \text{if } |\xi| > 1. \end{cases} \tag{6}$$

In particular, if $\alpha = \alpha_1$, then

$$\|e^{-t\mathcal{A}(\xi)}\| \le e^{-|\xi|^{2\alpha}t} \text{ for all } |\xi| > 0. \tag{7}$$

2.3 Banach fixed point theorem and Chemin-Lerner type homogeneous Fourier-Besov spaces

We recall an existence and uniqueness result for an abstract operator equation in a Banach space, which will be used to prove the main result.

Lemma 2.2 *([13]) Let E be a Banach space with the norm $\|\cdot\|$ and $B : E \longrightarrow E$ be a bilinear operator such that for any $x_1, x_2 \in E$, $\|B(x_1, x_2)\| \leq \eta\|x_1\|\|x_2\|$, then for any $y \in E$ such that $\|y\| < \dfrac{1}{4\eta}$, the equation $x = y + B(x, x)$ has a solution $x \in E$. In particular, the solution is such that $\|x\| \leq 2\|y\|$ and it is the only one such that $\|x\| < \dfrac{1}{2\eta}$.*

Let us observe that if $y = G(t)U_0$ and

$$B(U, U) = -\int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla \cdot (U \otimes U)(\tau)d\tau,$$

then the integral equation (5) has the form $U = y + B(U, U)$ required in Lemma 2.2.

Now, we give the definition of the Chemin-Lerner type homogeneous Fourier-Besov spaces with variable exponent.

Definition 2.3 ([11]) Let $s(\cdot) \in C^{log}(\mathbb{R}^n)$, $p(\cdot),\ q(\cdot) \in \mathcal{P}(\mathbb{R}^n) \cap C^{log}(\mathbb{R}^n)$, $T \in [0, \infty)$ and $1 \leq q,\ \rho \leq \infty$. We define the Chemin-Lerner type homogeneous Fourier-Besov space with variable exponent $\mathcal{L}^\rho([0, T); \mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q})$ by

$$\mathcal{L}^\rho\ ([0, T); \mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q}) = \left\{ g \in \mathcal{Z}'(\mathbb{R}^n);\ \|g\|_{\mathcal{L}^\rho([0,T);\cdot\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q})} < \infty \right\}$$

with the norm

$$\|g\|_{\mathcal{L}^\rho([0,T);\mathcal{F}\dot{\mathcal{B}}^{s(\cdot)}_{p(\cdot),q})} = \left(\sum_{j\in\mathbb{Z}} \|2^{js(\cdot)}\theta_j \hat{g}\|^q_{L^\rho([0,T);L^{p(\cdot)})} \right)^{\frac{1}{q}}.$$

3 A Priori Estimates

Thanks to Lemma 2.2, the key to the proof of Theorem 1.1 is to make a priori estimates for (1). In the lemma given below, we prove the linear estimate for equation (5).

Lemma 3.1 (Linear estimate) *Let $\frac{1}{2} < \alpha = \alpha_1 \leq 1$, $1 \leq \rho, q \leq +\infty$ and $p(\cdot), p_1(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ such that $p_1(\cdot) \leq p(\cdot)$. Assume that $U_0 \in \mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}$ and $\rho_1 \in [\rho, +\infty]$, then the following inequality holds:*

$$\|G(t)U_0\|_{\mathcal{L}^{\rho_1}_T \mathcal{F}\dot{\mathcal{B}}^{4-2\alpha+\frac{2\alpha}{\rho_1}-\frac{3}{p_1(\cdot)}}_{p_1(\cdot),q}} \lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.$$

Proof. Thanks to Hölder's inequality, Lemma 2.1 and the hypothesis $p_1(\cdot) \leq p(\cdot)$, we have

$$\|G(t)U_0\|_{\mathcal{L}^{\rho_1}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}}_{p_1(\cdot),q})} \leq \|\|2^{j\left(4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}\right)}\theta_j e^{-\mathcal{A}(\xi)t}\widehat{U_0}\|_{L^{\rho_1}([0,\infty),L^{p_1(\cdot)})}\|_{\ell^q}$$
$$\leq \| \sum_{k=0;\pm1} \|2^{j(4-2\alpha-\frac{3}{p(\cdot)})}\theta_j\hat{U_0}\|_{L^{p(\cdot)}}\|2^{j(\frac{2\alpha}{\rho_1}+\frac{3}{p_1'(\cdot)}-\frac{3}{p'(\cdot)})}\theta_{j+k}e^{-t|.|^{2\alpha}}\|_{L^{\rho_1}([0,\infty),L^{\frac{p(\cdot)p_1(\cdot)}{p(\cdot)-p_1(\cdot)}})}\|_{\ell^q}$$
$$\lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}}.$$

Consequently, one obtains

$$\|G(t)U_0\|_{\mathcal{L}^{\rho_1}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}}_{p_1(\cdot),q})} \lesssim \|U_0\|_{\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p(\cdot)}}_{p(\cdot),q}}. \tag{8}$$

For the bilinear estimate, we have the following lemma.

Lemma 3.2 (Bilinear estimate) *Let* $\frac{1}{2} < \alpha = \alpha_1 \leq 1$, $p_1 \in \mathcal{P}(\mathbb{R}^n)$ *such that* $p_1(\cdot) \leq \frac{6}{5-4\alpha}$, $1 \leq \rho \leq \infty$ *and* $\rho_1 \in [\rho, \infty]$. *Then we have the following inequality:*

$$\|\int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\otimes U_2)(\tau)d\tau\|_{\mathcal{L}^{\rho_1}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}}_{p_1(\cdot),q})}$$
$$\lesssim \|U_1\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,q})}\|U_2\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha}_{2,q})}$$
$$\times \|U_2\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,q})}\|U_1\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha}_{2,q})}.$$

Proof. Thanks to Hölder's inequality, Hausdorf-Young's inequality, and Young's inequality, we have

$$\|\int_0^t G(t-\tau)\widetilde{\mathbf{P}}\nabla\cdot(U_1\otimes U_2)(\tau)d\tau\|_{\mathcal{L}^{\rho_1}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}}_{p_1(\cdot),q})}$$
$$\lesssim \Big\| \| \int_0^t 2^{j\left(4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}\right)}\theta_j e^{-(t-\tau)\mathcal{A}(\xi)}\,\mathrm{div}\left(\widehat{U_1\otimes U_2}\right)\mathrm{d}\tau\|_{L^{\rho_1}([0,\infty),L^{p_1(\cdot)})}\Big\|_{\ell^q}$$
$$\lesssim \Big\| \| \int_0^t 2^{j\left(4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}\right)}\theta_j e^{-(t-\tau)|\cdot|^{2\alpha}}\,\mathrm{div}\left(\widehat{U_1\otimes U_2}\right)\mathrm{d}\tau\|_{L^{\rho_1}([0,\infty),L^{p_1(\cdot)})}\Big\|_{\ell^q}$$
$$\lesssim \Big\| \int_0^t \|2^{j\left(5-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}\right)}\theta_j e^{-(t-\tau)|\cdot|^{2\alpha}}\|_{L^{\frac{6p_1(\cdot)}{6-(5-4\alpha)p_1(\cdot)}}}\|\widehat{U_1\otimes U_2}\|_{L^{\frac{6}{5-4\alpha}}}\mathrm{d}\tau\|_{L^{\rho_1}([0,\infty))}\Big\|_{\ell^q}$$
$$\lesssim \Big\|\|2^{j\left(\frac{2\alpha}{\rho}+\frac{5}{2}-2\alpha\right)}\|\dot{\Delta}_j(U_1\otimes U_2)\|_{L^{\frac{6}{4\alpha+1}}}\|_{L^{\rho}([0,\infty))}\Big\|_{\ell^q}.$$

Consequently, by using Proposition 2.2, we obtain the result

$$B(U_1\otimes U_2)\|_{\mathcal{L}^{\rho_1}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{4-2\alpha-\frac{3}{p_1(\cdot)}+\frac{2\alpha}{\rho_1}}_{p_1(\cdot),q})} \lesssim \|U_1\otimes U_2\|_{\mathcal{L}^{\rho}([0,\infty),\dot{\mathcal{B}}^{\frac{2\alpha}{\rho}+\frac{5}{2}-2\alpha}_{\frac{6}{4\alpha+1},q})}$$
$$\lesssim \|U_1\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,q})}\|U_2\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha}_{2,q})}$$
$$+ \|U_2\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}_{2,q})}\|U_1\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}^{\frac{5}{2}-2\alpha}_{2,q})}.$$

4 Proof of Theorem 1.1

In the following, we consider the Banach space

$$\mathcal{E} = \mathcal{L}^{\rho}([0,\infty)\,, \mathcal{F}\dot{\mathcal{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}+\frac{2\alpha}{\rho}}\left(\mathbb{R}^3\right)) \cap \mathcal{L}^{\rho}([0,\infty)\,, \mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}}\left(\mathbb{R}^3\right)) \cap \mathcal{L}^{\infty}([0,\infty)\,, \mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha}\left(\mathbb{R}^3\right)),$$

and define mappings as $\Theta(U) = G(t)U_0 + B(U,U)$. Then, to solve (1), it suffices to find the fixed point of the mapping θ. First, from Lemma 3.1, we have

$$\|G(t)U_0\|_{\mathcal{E}} \le C_1 \|U_0\|_{\mathcal{F}\dot{\mathcal{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}}}. \tag{9}$$

By Lemma 3.2, we obtain

$$\begin{aligned}\|B(U_1\otimes U_2)\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{p(\cdot),q}^{4-2\alpha-\frac{3}{p(\cdot)}+\frac{2\alpha}{\rho}})} &\lesssim \|U_1\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}})}\|U_2\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha})}\\ &\quad + \|U_2\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}})}\|U_1\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha})},\end{aligned}$$

$$\begin{aligned}\|B(U_1\otimes U_2)\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha})} &\lesssim \|U_1\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}})}\|U_2\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha})}\\ &\quad + \|U_2\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}})}\|U_1\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha})},\end{aligned}$$

and

$$\begin{aligned}\|B(U_1\otimes U_2)\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}})} &\lesssim \|U_1\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}})}\|U_2\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha})}\\ &\quad + \|U_2\|_{\mathcal{L}^{\rho}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha+\frac{2\alpha}{\rho}})}\|U_1\|_{\mathcal{L}^{\infty}([0,\infty),\mathcal{F}\dot{\mathcal{B}}_{2,q}^{\frac{5}{2}-2\alpha})}.\end{aligned}$$

Consequently,

$$\|B(U_1\otimes U_2)\|_{\mathcal{E}} \le C_2\|U_1\|_{\mathcal{E}}\|U_2\|_{\mathcal{E}}. \tag{10}$$

Combining (9) and (10), we get

$$\begin{aligned}\|\Theta(U)\|_{\mathcal{E}} &\le \|G(t)U_0\|_{\mathcal{E}} + \|\int_0^t G(t-\tau)\tilde{\mathbb{P}}\tilde{\nabla}\cdot(U\otimes U)d\tau\|_{\mathcal{E}}\\ &\le \|G(t)U_0\|_{\mathcal{E}} + \|B(U\otimes U)\|_{\mathcal{E}}\\ &\le C_1\|U_0\|_{\mathcal{F}\dot{\mathcal{B}}_{p(\cdot),q}^{4-\frac{3}{p(\cdot)}-2\alpha}} + C_2\|U\|_{\mathcal{E}}\|U\|_{\mathcal{E}}.\end{aligned}$$

Then, for any $U_0 \in \mathcal{F}\dot{\mathcal{B}}_{p(\cdot),q}^{4-\frac{3}{p(\cdot)}-2\alpha}$ if $\|U_0\|_{\mathcal{F}\dot{\mathcal{B}}_{p(\cdot),q}^{4-\frac{3}{p(\cdot)}-2\alpha}} < \dfrac{1}{4C_1C_2}$, by Lemma 2.2, we conclude that the problem (1) admits a unique global mild solution $U \in \mathcal{E}$ such that $\|U\|_{\mathcal{E}} \le \frac{1}{2C_2}$.

5 Conclusion

In this paper, we considered the 3-D generalized micropolar fluid system which can describe many phenomena that occur in a large number of complex fluids, including animal blood and liquid crystals. By using the Littlewood-Paley decomposition theory and Fourier localization technique, we prove the global existence for the system (1) in variable exponent Fourier-Besov spaces and our result can be seen as a complement to the corresponding result of Zhu and Zhao [12].

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