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A Novel Fractional-Order Chaotic System and Its Synchronization via Adaptive Control Method

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Abstract: In this paper, the synchronization of a new fractional-order chaotic system via the adaptive control method is introduced. Firstly, the novel fractional-order system is presented and its dynamics is investigated throughout the Lyapunov exponents spectrum. Secondly, based on the stability theory of fractional-order systems, synchronization of the fractional-order system with fully uncertain parameters is realized by designing appropriate adaptive controllers and estimation laws. Finally, numerical simulations are implemented to demonstrate the effectiveness and flexibility of the synchronization controllers and the estimation laws for the unknown parameters.

Keywords: *chaotic system; strange attractor; Lyapunov exponent; Lyapunov stability theory; adaptive control; synchronization.*

Mathematics Subject Classification (2010): 34D08, 34C28, 37B55, 37B25, 37D45, 70K20, 93D05, 93D21

1 Introduction

Fractional chaotic dynamical systems are non-linear systems that allow sensitivity to initial conditions, those chaotic systems are widely used in several fields such as physics, chemistry, biology, economics, especially in secure communication and cryptography[1-2].

In 1990, Pecora and Carol [3] introduced for the first time a method of synchronization of two integer-order chaotic systems. After that, many research papers have addressed synchronization between integer-integer order chaotic systems [4-6]. Recently,

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the synchronization of chaos within fractional chaotic systems has attracted a considerable attention and different techniques have been used to achieve different types of synchronization for these systems, among them, active control [7-8], backstepping control [10], and the modified projective synchronization [11], the hyprid projective synchronization [12], the full state hybrid projective synchronization [13], the inverse full state hybrid projective synchronization [14]. An adaptive control method [15-17] is one of the robust control methods used for chaos synchronization when parameters are unknown or initially uncertain. In an adaptive method, the control law and the parameter update law are designed so that the chaotic response system behaves like chaotic drive systems. Thus, the adaptive scheme maintains a system's consistent performance in the presence of uncertainty and variation in plant parameters. This control method differs from other control methods because it does not require advance information about the limits on these uncertain or time-varying parameters because it relates to a control law that changes itself. Several research papers study adaptive control in the case of integer-order systems, see F. Hannachi [15] and V. Sundarapandian [16-17]. However, the works presented and related to the synchronization between fractional-order chaotic systems remain limited, and finding new results using this method of control is required because of its important applications, especially in the fields of secure communication and encryption [18-20]. In this work, a novel fractional order system is presented and its chaoticity is confirmed using the Lyapunouv exponents tool. Moreover, the synchronization of the fractional-order system with unknown parameters is realized by designing appropriate adaptive controllers and estimation laws using the stability theory of fractional-order systems. Finally, the effectiveness of the proposed scheme for this type of synchronization is demonstrated by an illustrative example with numerical simulation in Matlab.

This paper is organized as follows. In Sections 2 and 3, basic tools of fractional calculus and a description of the novel fractional-order chaotic system are given. In Section 4, using the new chaotic system, chaos synchronization between fractional-fractional chaotic systems for commensurate orders via the adaptive control method and stability theory of fractional-order systems is investigated. In Section 5, numerical simulations are given to demonstrate the effectiveness and flexibility of the synchronization controllers and the estimation laws for the unknown parameters. The conclusion is given in Section 6.

2 Basics of Fractional Calculus

Definition 2.1 ([21]){Caputo fractional derivative}
The Caputo fractional derivative of order $q \in \mathbb{R}^+$ on the half axis $\mathbb{R}^+$ is defined as follows:

$$ {}_a^C D_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{q-n+1}} d\tau, t > a, \tag{1} $$

with $n = \min\{k \in \mathbb{N}/k > q\}, q > 0$.

To simplify the notation, we replace ${}_a^C D_t^q f(t)$ by ${}^C D_t^q f(t)$.

Lemma 2.1 *The trivial solution of the following fractional-order system:*

$$ D_t^q X(t) = F(X(t)), $$

where D_t^q *is the Caputo fractional derivative of order* $q, 0 < q \leq 1, F : R^n \longrightarrow R^n$, *is asymptotically stable if there exists a positive definite function* $V(X(t))$ *such that* $D_t^q V(X(t)) < 0$ *for all* $t > 0$.

Lemma 2.2 *Let $x(t) \in \mathbb{R}$ be a continuous and derivable function. Then, for any time instant $t \geqslant t_0$,*

$$\frac{1}{2}{}^C D_t^\alpha x^2(t) \leqslant x(t){}_{t_0}^C D_t^\alpha x(t), \quad \forall \alpha \in (0,1).$$

3 Presentation of the New Fractional-Order System

Several works [24-26] have reported on a 3-D dynamic model of finance (the Ma system) and offered its simplified system equations. In this paper, we consider the fractional version of this simplified finance system [26] given by

$$\begin{cases} {}^C D_t^{q_1} x_1(t) = -a(x_1 + x_2), \\ {}^C D_t^{q_2} x_2 = -x_2 - c x_1 x_3, \\ {}^C D_t^{q_3} x_3 = b + a x_1 x_2, \end{cases} \tag{2}$$

where D^q is the Caputo derivative operator, a, b, c are positive reals parameters. The model describes the time variations of three state variables $x_i, i = \overline{1,3}$: the interest rate x_1, the investment demand x_2, and the price index x_3. With the chosen parameters $a = 4, b = 20, c = 4$ and with the orders $(q_1; q_2; q_3) = (0.9; 0.95; 0.98)$;$(x_1(0); x_2(0); x_3(0)) = (0.4; 0.5; 0.2)$, the Lyapunov exponents are computed using the Benettin–Wolf algorithm [23] in Matlab and given as follows:

$$L_1 = 1.2790; L_2 = -0.0001 \simeq 0; L_3 = -6.7482. \tag{3}$$

So, we have $L_1 > 0$, then the novel fractional system (2) is chaotic. The chaotic attractor and the Lyapynov spectrum are depicted in Fig.1. Indeed, the Kaplan-Yorke dimension for this chaotic system is calculated as

$$D_{KY} = 2 + \frac{L_1 + L_2}{|L_3|} = 2.1895. \tag{4}$$

4 Adaptive Synchronization of the Novel Fractional System with Commensurate Order

This section aims to design an adaptive control law for globally synchronizing the identical novel fractional chaotic system with unknown system parameters. The master system is given by

$$\begin{cases} {}^C D_t^{q} x_1(t) = -a(x_1 + x_2), \\ {}^C D_t^{q} x_2(t) = -x_2 - d x_1 x_3, \\ {}^C D_t^{q} x_3(t) = b + a x_1 x_2. \end{cases} \tag{5}$$

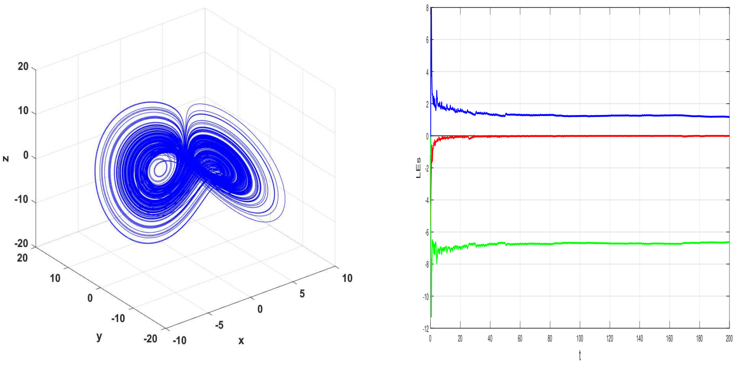

Figure 1: Lyapunov exponents spectrum.

The slave system is obtained as follows:

$$\begin{cases} {}^{C}D_t^q y_1(t) = -a(y_1 + y_2) + u_1, \\ \\ {}^{C}D_t^q y_2(t) = -y_2 - cy_1 y_3 + u_2, \\ \\ {}^{C}D_t^q y_3(t) = b + ay_1 y_2 + u_3. \end{cases} \tag{6}$$

In both systems (5) and (6), we have parameters a, b, c, which are unknown, and their estimates $a_1(t), b_1(t), c_1(t)$, respectively. We will later search for an adaptive feedback controls u_1, u_2, u_3. Define now the synchronization error between the systems (5) and (6) as

$$e_i = y_i - x_i, i = \overline{1,3}. \tag{7}$$

Equation (7) implies

$${}^{C}D_t^q e_i(t) = {}^{C}D_t^q y_i(t) - {}^{C}D_t^q x_i(t), i = \overline{1,3}. \tag{8}$$

Thus, the synchronization error dynamics between (5) and (6) is obtained as

$$\begin{cases} {}^{C}D_t^q e_1(t) = a(e_1 + e_2) + u_1, \\ \\ {}^{C}D_t^q e_2(t) = -e_2 + cx_1 x_3 - cy_1 y_3 + u_2, \\ \\ {}^{C}D_t^q e_3(t) = a(y_1 y_2 - x_1 x_2) + u_3. \end{cases} \tag{9}$$

We take the adaptive control law as follows:

$$\begin{cases} u_1 = -a_1(e_2 + e_2) - k_1 e_1, \\ u_2 = e_2 - c_1 x_1 x_3 + c_1 y_1 y_3 - k_2 e_2, \\ u_3 = -a_1(y_1 y_2 - x_1 x_2) - k_3 e_3, \end{cases} \tag{10}$$

where $k_i > 0$, i=1,...,3, are the gains constants.

By substituting (10) into (9), we obtain the closed-loop error as

$$\begin{cases} {}^{C}D_t^q e_1(t) = (a-a_1)(e_2+e_1) - k_1 e_1, \\ {}^{C}D_t^q e_2(t) = (c-c_1)(x_1x_3 - y_1y_3) - k_2 e_2, \\ {}^{C}D_t^q e_3(t) = (a-a_1)(y_1y_2 - x_1x_2) - k_3 e_3. \end{cases} \tag{11}$$

We can define the estimation errors for the parameters as

$$\begin{cases} e_a(t) = a - a_1(t), \\ e_b(t) = b - b_1(t), \\ e_c(t) = c - c_1(t). \end{cases} \tag{12}$$

Applying the Caputo derivative of order q in (12), we obtain

$$\begin{cases} {}^{C}D_t^q e_a(t) = -{}^{C}D_t^q a_1(t), \\ {}^{C}D_t^q e_b(t) = -{}^{C}D_t^q b_1(t), \\ {}^{C}D_t^q e_c(t) = -{}^{C}D_t^q c_1(t). \end{cases} \tag{13}$$

By using (12), we rewrite the closed-loop system (11) as

$$\begin{cases} {}^{C}D_t^q e_1(t) = e_a(e_2+e_1) - k_1 e_1, \\ {}^{C}D_t^q e_2(t) = e_c(x_1x_3 - y_1y_3) - k_2 e_2, \\ {}^{C}D_t^q e_3(t) = e_a(y_1y_2 - x_1x_2) - k_3 e_3. \end{cases} \tag{14}$$

Our goal now is to lead the system (14) to zero. Let us choose the Lyapunov candidate function as follows:

$$V(e_1, e_2, e_3, e_a, e_c) = \frac{1}{2}\left(k_1 e_1^2 + k_2 e_2^2 + k_3 e_3^2 + e_a^2 + e_c^2\right), \tag{15}$$

which is a positive definite function on $\mathbb{R}^5$. Applying the Caputo operator of differentiation in V along the trajectories of the systems (13) and (14) and using Lemma 2.2, we obtain

$$\begin{cases} {}^{C}D_t^q V(e_1, e_2, e_3, e_a, e_c) = \frac{1}{2}D^q k_1 e_1^2 + \frac{1}{2}k_2 D^q e_2^2 + \frac{1}{2}k_3 D^q e_3^2 + \frac{1}{2}D^q e_a^2 + \frac{1}{2}D^q e_c^2 \\ \leq k_1 e_1 D^q e_1 + k_2 e_2 D^q e_2 + k_3 e_3 D^q e_3 + e_a D^q e_a + e_c D^q e_c \\ \leq -k_1^2 e_1^2 - k_2^2 e_2^2 - k_3^2 e_3^2 + e_a\left(k_1 e_1^2 + k_1 e_1 e_2 + k_3 e_3 y_1 y_2 - k_3 e_3 x_1 x_2 - D^q a_1(t)\right) \\ +e_c\left(k_2 e_2 x_1 x_3 - k_2 e_2 y_1 y_3 - D^q c_1(t)\right). \end{cases} \tag{16}$$

In view of (16), we take the parameter update law as follows:

$$\begin{cases} {}^{C}D_t^q a_1(t) = k_1 e_1^2 + k_1 e_1 e_2 + k_3 e_3 y_1 y_2 - k_3 e_3 x_1 x_2, \\ {}^{C}D_t^q c_1(t) = k_2 e_2 x_1 x_3 - k_2 e_2 y_1 y_3. \end{cases} \tag{17}$$

Substituting (17) into (16), we obtain

$${}^{C}D_t^q V(e_1, e_2, e_3, e_a, e_c) \leq -k_1^2 e_1^2 - k_2^2 e_2^2 - k_3^2 e_3^2 < 0.$$

Hence, the zero solution of the error system (16) is assymptoticly stable and the following theorem is proved.

Theorem 4.1 *The novel fractional chaotic systems (5) and (6) with unknown parameters are globally synchronized for all initial conditions by the adaptive control law (10) and the parameter update law (17) with $k_i > 0$ gains constants.*

5 Numerical Simulation

In order to verify our results, we use the Adams-Bashforth-Moulton algorithm [22] for the fractional-order system to solve the systems of differential equations (5), (6), (14) and (17). The initial conditions are chosen as $(x_1(0), x_2(0), x_3(0)) = (0.4, 0.5, 0.2)$, $(y_1(0), y_2(0), y_3(0)) = (5.4, -4.5, 2.2)$, respectively. $(e_1(0), e_2(0), e_3(0)) = (5, -5, 2)$, $(a_1(0), c_1(0)(0)) = (5, 4.5)$. In Fig.2, the synchronization between the states x_i and y_i, $i = \overline{1,3}$, is depicted. In Fig.3, the time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$ is depicted.

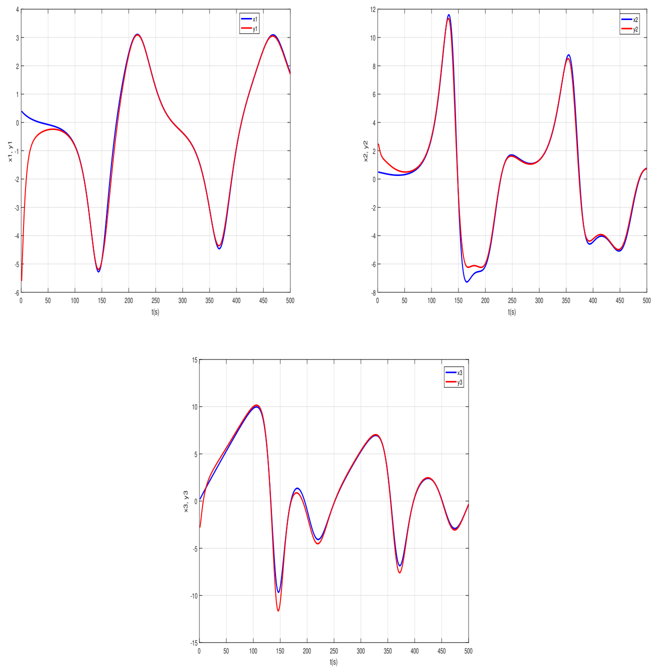

Figure 2: Synchronization between $x_i, y_i, i = 1, 2, 3$.

6 Conclusion

This paper reports on a new fractional-order chaotic system and its synchronization via the adaptive control method. The novel fractional-order system is presented and its chaoticity is confirmed using the Lyapunov exponents tool. Moreover, the synchronization of the fractional-order system with unknown parameters is realized by designing appropriate adaptive controllers and estimation laws using the stability theory of fractional-order systems. Numerical simulations are implemented to demonstrate the effectiveness and flexibility of the synchronization controllers and the estimation laws for the unknown parameters.

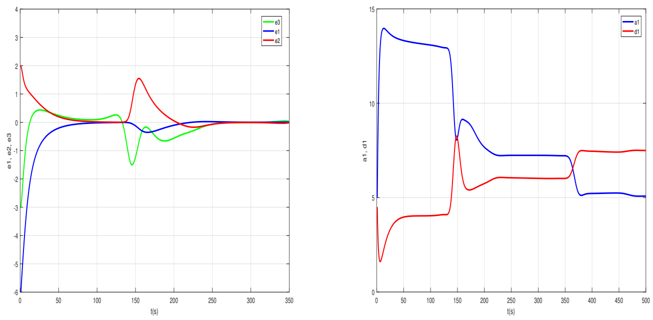

Figure 3: (a) The time-history of the synchronization errors $e_1(t), e_2(t), e_3(t)$ and (b) Parameters estimation.

7 Concluding remarks

The main important points in this work are:

- A novel fractional-order financial system is presented and its chaoticity is confirmed using the Lyapunov exponents.
- The synchronization between identical 3-D fractional-order financial systems with commensurate order and unknown parameters is achieved by designing appropriate adaptive controllers and estimation laws using the stability theory of fractional-order systems.

The results achieved through the study of the novel fractional-order financial system have wide-ranging applications. These include areas such as secure communication and signal encryption, making further research of the system particularly relevant. As such, they will be given due consideration in future work.

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