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Analysis of Problems in Generalized Viscoplasticity under Dynamic Thermal Loading

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Abstract: This paper examines two uncoupled quasistatic problems for thermo-viscoplastic materials, wherein the equation model considers the dependence of mechanical properties on a parameter θ, which represents the absolute temperature. Specifically, both the tensor of viscosity and the plastic deformation depend on this parameter. The boundary conditions for these problems are displacement traction and Signorini conditions. Our analysis establishes the existence of a unique solution to the problems, as well as the continuous dependence of the solution on the parameter θ. To provide a practical demonstration of our findings, we also present two one-dimensional examples that describe the processes involved in these problems.

Keywords: *viscoplastic; temperature; variational equality; Cauchy-Lipschitz method.*

Mathematics Subject Classification (2010): 35D30, 70K75, 74F05, 74M15, 74F05, 74C10, 93A30.

1 Introduction

Our study focuses on the analysis of two models designed for thermo-viscoplastic materials, which exhibit a unique coupling between their mechanical and thermal properties. Over the years, mathematicians, physicists, and engineers have extensively studied thermo-viscoplasticity laws to effectively model the influence of temperature on the behavior of various materials such as metals, magmas, and polymers. To gain further insight, we refer the interested readers to the sources such as [1, 2, 4, 5, 8, 11, 15]. Moreover, practical applications and mechanical interpretations of thermo-viscoplasticity can

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be found in [13, 14]. To accurately describe the behavior of these materials in real-world scenarios, we employ a rate-type constitutive equation of the following form:

$$\dot{\sigma} = \xi\varepsilon(\dot{u}) + \mathcal{G}(\ \sigma, \varepsilon, \theta). \tag{1}$$

Here u, σ represent, respectively, the displacement field, θ is the absolute temperature, ξ is the fourth order elastic tensor and $\mathcal{G}$ is a nonlinear constitutive function, which describes the thermo-plastic behavior of the material and the stress field, $\varepsilon(u) = (\varepsilon_{ij}(u))$ is the linearised strain tensor,

$$\varepsilon_{ij}(u) = \frac{1}{2}(\nabla u\ + \nabla^T u).$$

For the heat flux q, a constitutive classical Fourier law is given by

$$q = K\nabla\theta. \tag{2}$$

In [15], existence and uniqueness results were obtained for problems (1)-(2) under classical displacement traction boundary conditions. However, the research in recent papers has been based on generalized thermo-viscoplastic theories with temperature-independent mechanical properties. In this paper, we aim to investigate the impact of temperature dependence of ξ on the behavior of the solution in generalized thermo-viscoplasticity. To accomplish this, we consider a rate-type constitutive equation of the form

$$\dot{\sigma} = \xi(\theta)\varepsilon\,(\dot{u}) + \mathcal{G}(\sigma, \varepsilon, \theta). \tag{3}$$

The paper is organized as follows. In Section 2, we describe the mathematical model for the problem. And we introduce some notations, list the assumptions on the problem's data, and derive the variational formulation of the model. In Section 3, we state our main existence and uniqueness result which is based on a Cauchy-Lipschitz technique and present the continuous dependence of the solution upon the parameter θ. We give two numerical examples in the last Section 4.

2 Problem Statement

Let Ω be a bounded domain in $\mathbb{R}^d (d = 1, 2, 3)$ with a smooth boundary Γ which is partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $eas\Gamma_1 > 0$. Let $T > 0$ and let $[0, T]$ denote the time interval of interest.

We consider the following mixed problem.

Problem P

Find a displacement field $u : \Omega \times (0, T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0, T) \to \mathbb{S}_d$, a

temperature $\theta : \Omega \to \mathbb{R}$, and the heat flux function $q : \Omega \to \mathbb{R}^d$ such that

$$\dot{\sigma} = \xi(\theta)\varepsilon(\dot{u}) + \mathcal{G}(\sigma, \varepsilon, \theta), \quad \text{in } \Omega \times (0,T), \tag{4}$$
$$Div\sigma + f_0 = 0, \quad \text{in } \Omega \times (0,T), \tag{5}$$
$$divq + r = \dot{\theta}, \quad \text{in } \Omega \times (0,T), \tag{6}$$
$$q = K\nabla\theta, \quad \text{in } \Omega \times (0,T), \tag{7}$$
$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{8}$$
$$\sigma \cdot \nu = f_2, \quad \text{on } \Gamma_2 \times (0,T), \tag{9}$$
$$q \cdot \nu = \chi, \quad \text{on } \Gamma_2 \times (0,T), \tag{10}$$
$$\theta = 0 \quad \text{on } (\Gamma_1 \cup \Gamma_2) \times (0,T), \tag{11}$$
$$u(0) = u_0, \sigma(0) = \sigma_0, \quad \text{in } \Omega, \tag{12}$$
$$\theta(0) = \theta_0, \quad \text{in } \Omega. \tag{13}$$

Here $\mathbb{S}_d$ is the set of second order symmetric tensors on $\mathbb{R}^d$, $\nu = (\nu_i)$ is the unit outward normal to Ω and u_0, σ_0, θ_0 are the initial data.

We consider the following boundary conditions:

$$u_\nu \leq 0, \sigma_\nu \leq 0, \sigma_\tau = 0, \sigma_\nu . u_\nu = 0, \quad \text{on } \Gamma_3 \times (0,T). \tag{14}$$

In this way, we obtain two initial and boundary value problems $(\mathbf{P}_i)$ defined as follows.

Problem $\mathbf{P}_1$ Find the unknowns (u, σ, θ, q) such that (4)-(13) hold. This problem represents a displacement traction problem, in this case, $\Gamma_3 = \phi$.

Problem $\mathbf{P}_2$ Find the unknowns (u, σ, θ, q) such that (4)-(14) hold. This problem models the frictionless contact between the thermo-viscoplastic body and the rigid foundation, (14) represent the Signorini boundary conditions.

2.1 Variational formulation

For a weak formulation, we list the assumptions on the data and derive variational formulations for the contact problems $(\mathbf{P}_i)$. To this end, we need to introduce some notations and preliminary material. For more details, we refer the reader to [3, 12]. We denote by $\mathbb{S}_d$ the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$, while $\|\cdot\|$ denotes the Euclidean norm.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary Γ and let ν denote the unit outer normal on $\partial\Omega = \Gamma$. We shall use the notations

$$\begin{aligned} H = \tilde{H} = & \left[L^2(\Omega)\right]^d, \\ \mathcal{H} = & \tilde{\mathcal{H}} = \left[L^2(\Omega)\right]_s^{d\times d}, \\ Y = & \left[L^2(\Omega)\right]^M, M \in \mathbb{N}, \end{aligned}$$

and

$$\begin{aligned} H_1 = &\quad \{u = (u_i) \in H : \varepsilon(u) \in \mathcal{H}\}, \\ \tilde{H}_1 = &\quad \{\theta \in \tilde{H} : \nabla\theta \in \tilde{\mathcal{H}}\}, \end{aligned}$$

$$\begin{aligned} \mathcal{H}_1 = &\quad \{\sigma \in \mathcal{H} : Div\sigma \in H\}, \\ \mathcal{V} = &\quad \{\sigma \in \mathcal{H}_1 : Div\sigma = 0 \text{ in } \Omega, \sigma\nu = 0 \quad on\ \Gamma_1\}, \end{aligned}$$

$$\begin{aligned} \tilde{\mathcal{H}}_1 = &\quad \left\{q \in \tilde{\mathcal{H}} : divq \in \tilde{H}\right\}, \\ \tilde{\mathcal{V}} = &\quad \left\{q \in \tilde{\mathcal{H}}_1 : divq = 0 \text{ in } \Omega, q\nu = 0 \quad on\ \Gamma_1\right\}, \end{aligned}$$

where $\varepsilon : H \to \mathcal{H}$, $\nabla : \tilde{H} \to \tilde{\mathcal{H}}$, $Div : \mathcal{H} \to H$, and $div : \tilde{\mathcal{H}} \to \tilde{H}$ are the partial derivative operators of the first order, respectively, defined by

$$\begin{aligned} &\varepsilon(u) = (\varepsilon_{ij}(u)), \varepsilon_{ij}(u) = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \\ &\nabla\theta = (\nabla_i\theta), \quad \nabla_i\theta = \tfrac{\partial\theta}{\partial x_i}, \\ &Div\sigma = (\tfrac{\partial\sigma_{ij}}{\partial x_j}), divq = (\tfrac{\partial q_i}{\partial x_i}). \end{aligned}$$

Here and below, the indices i and j run from 1 to d, the spaces H, H_1, $\mathcal{H}$, $\mathcal{H}_1$, $\tilde{H}$, $\tilde{H}_1$, $\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\begin{aligned} (u,v)_H &= \int_\Omega u_i v_i dx, \\ (u,v)_{H_1} &= (u,v)_H + (\varepsilon(u), \varepsilon(v))_{\mathcal{H}}, \\ (\sigma,\tau)_{\mathcal{H}} &= \int_\Omega \sigma_{ij}.\tau_{ij} dx, \\ (\sigma,\tau)_{\mathcal{H}_1} &= (\sigma,\tau)_{\mathcal{H}} + (Div\sigma, Div\tau)_H, \\ (\theta,\varphi)_{\tilde{H}} &= \int_\Omega \theta_i\varphi_i dx, \\ (\theta,\varphi)_{\tilde{H}_1} &= (\theta,\varphi)_{\tilde{H}} + (\nabla\theta, \nabla\varphi)_{\tilde{\mathcal{H}}}, \\ (q,p)_{\tilde{\mathcal{H}}} &= \int_\Omega q_{ij}.p_{ij} dx, \\ (q,p)_{\tilde{\mathcal{H}}_1} &= (q,p)_{\tilde{\mathcal{H}}} + (divq, divp)_{\tilde{H}}. \end{aligned}$$

The associated norms are denoted by $\|\cdot\|_H$, $\|\cdot\|_{H^1}$, $\|\cdot\|_{\mathcal{H}}$, $\|\cdot\|_{\mathcal{H}_1}$, $\|\cdot\|_{\tilde{H}}$, $\|\cdot\|_{\tilde{H}_1}$, $\|\cdot\|_{\tilde{\mathcal{H}}}$, and $\|\cdot\|_{\tilde{\mathcal{H}}_1}$, respectively. Also, for any real normed space X, we denote by X' the strong dual, by $\|\cdot\|_X$, $\|\cdot\|_{X'}$ the norms on X and X', respectively, and by $\langle,\rangle_{X',X}$ the canonical duality pairing between X and X', if in addition, X is a real Hilbert space and $A : X \to X$ is a continuous symmetric and positively definite linear operator, we denote by $\langle,\rangle_{A,X}$ and $\|\cdot\|_{A,X}$ the energetical product and the energetical norm induced by A on X.

Let

$$\begin{aligned} H_\Gamma &= (H^{1/2}(\Gamma))^d,\ \tilde{H}_\Gamma = (\tilde{H}^{1/2}(\Gamma))^d \\ \text{and } \gamma &: H_1(\Gamma)^d \to H_\Gamma,\ \tilde{\gamma} : \tilde{H}^1(\Gamma)^d \to \tilde{H}_\Gamma, \end{aligned}$$

be the trace map. We introduce the following closed sub-spaces of H_1 and $\tilde{H}_1$:

$$\begin{aligned} V = &\quad \{u \in H_1 : \gamma u = 0 \quad on\ \ \Gamma_1\}, \\ \tilde{V} = &\quad \left\{\theta \in \tilde{H}_1 : \tilde{\gamma}\theta = 0 \quad on\ \ \tilde{\Gamma}_1\right\}, \\ Q = &\quad \left\{\eta \in H^1 : \eta = 0 \quad on\ \ \Gamma_1 \cup \Gamma_2\right\}. \end{aligned}$$

We introduce the following notations for the problems (P_i):

$$L(t,v) = \langle f_0(t), v\rangle + \langle f_2(t), \gamma v\rangle_{L^2(\Gamma_2)}$$

U_{ad}, and $\sum\limits_{ad}(t,v)$. For the problem P_1, we have

$$\begin{gathered} U_{ad} = V, \\ \sum_{ad}(t,v) = \{\tau \in \mathcal{H};\ \langle \tau, \varepsilon(w)\rangle = L(t,w); \quad \forall w \in V\} \\ (\sum_{ad} \text{ does not depend on } V). \end{gathered}$$

For the problem P_2, we have

$$\begin{gathered} U_{ad} = \{v \in V;\ v_\nu \le 0 \text{ on } \Gamma_3\} \\ \sum_{ad}(t,v) = \{\tau \in \mathcal{H};\ \langle \tau, \varepsilon(w) - \varepsilon(v)\rangle \ge L(t, w-v); \quad \forall w \in V\}. \end{gathered}$$

In the study of the Problem $(\mathbf{P}_i)$, we consider the following assumptions. The *operator* $\xi : \mathbb{R}^d \times \mathbb{S}_d \to \mathbb{S}_d$ satisfies

$$\begin{cases} \text{(a) There exists } L_\xi > 0 \text{ such that} \\ \|\xi(\theta_1) - \xi(\theta_2)\| \leqslant L_\xi \|\theta_1 - \theta_2\| \text{ for all } \theta_1, \theta_2 \in \mathbb{R}^d, \\ \text{(b) } \xi(\theta).\sigma.\tau = \sigma.\xi(\theta).\tau, \quad \forall \theta \in \mathbb{R}^d, \quad \forall \sigma, \tau \in \mathbb{S}_d, \\ \text{(c) There exists } \alpha > 0 \text{ such that } \ \xi(\theta).\sigma.\sigma \geqslant \alpha\|\sigma\|^2, \\ \quad \forall \theta \in \mathbb{R}^d, \forall \sigma \in \mathbb{S}_d, \\ \text{(d) } \xi(\theta) \text{ is Lebesgue measurable on } \Omega, \\ \text{(e) There exists } \beta > 0 \text{ such that } \ \|\xi(\theta)\| \le \beta. \end{cases} \tag{15}$$

The *operator* $\mathcal{G} : \mathbb{S}_d \times \mathbb{S}_d \times \mathbb{R}^d \to \mathbb{S}_d$ satisfies

$$\begin{cases} \text{(a) There exists } L_{\mathcal{G}} > 0 \text{ such that} \\ \|\mathcal{G}(\sigma_1, \varepsilon_1, \theta_1) - \mathcal{G}(\sigma_2, \varepsilon_2, \theta_2)\| \leqslant L_{\mathcal{G}} \left(\|\sigma_1 - \sigma_2\| + \|\varepsilon_1 - \varepsilon_2\| + \|\theta_1 - \theta_2\|\right) \\ \text{for all } \sigma_1, \sigma_2 \in \mathbb{S}_d, \quad \varepsilon_1, \varepsilon_2 \in \mathbb{S}_d, \quad \theta_1, \theta_2 \in \mathbb{R}^d, \\ \text{(b) The mapping } \mathcal{G}(\sigma, \varepsilon, \theta) \text{ is Lebesgue measurable on } \Omega, \end{cases} \tag{16}$$

K is a symmetric and positively definite bounded tensor, i.e., the *tensor* $K : \Omega \times \mathbb{S}_d \to \mathbb{S}_d$ satisfies

$$\begin{cases} \text{(a) } K(x).q.p = q.K(x).p, \quad \forall q, p \in \mathbb{S}_d, \quad \text{a.e in } \Omega, \\ \text{(b) There exists } \lambda > 0 \text{ such that } \ K(x).q.q \geqslant \lambda\|q\|^2 \\ \ \text{for all } q \in \mathbb{S}_d, \quad \text{a.e in } \Omega, \\ \text{(c)} K_{ij} \in L^\infty(\Omega) \ \text{ for all } i, j \in 1, 2, 3. \end{cases} \tag{17}$$

$K^{-1}A$ is a symmetric and positively definite bounded tensor, i.e.,

$$\begin{cases} \text{(a) } K^{-1}A(x).q.p = q.K^{-1}A(x).p, \quad \forall q, p \in \mathbb{S}_d, \quad \text{a.e in } \Omega, \\ \text{(b) There exists } \delta > 0 \text{ such that } \ K^{-1}A(x).q.q \geqslant \delta\|q\|^2 \\ \ \text{for all } q \in \mathbb{S}_d \quad \text{a.e in } \Omega, \\ \text{(c) } \left(K^{-1}A\right)_{ij} \in L^\infty(\Omega) \ \text{ for all } i, j \in 1, 2, 3. \end{cases} \tag{18}$$

We also suppose that

$$f_0 \in C^1(0,T;H), \tag{19}$$
$$r \in L^2(0,T;L^2(\Omega)), \tag{20}$$
$$f_2 \in C^1(0,T;H^1_\Gamma), \tag{21}$$
$$\chi \in L^2(\Gamma_2), \tag{22}$$
$$Div\boldsymbol{\sigma}_0 + f_0(0) = 0 \text{ in } \Omega, \tag{23}$$
$$\boldsymbol{\sigma}_0.\nu = f_0(0) \text{ on } \Gamma_2, \tag{24}$$
$$u_0 \in H_1, \tag{25}$$
$$\boldsymbol{\sigma}_0 \in \mathcal{H}_1, \tag{26}$$
$$\theta_0 \in L^2(\Omega). \tag{27}$$

By using standard arguments, we obtain the following variational formulation of the problem ((4)-(14)).

2.2 Problem $\mathcal{P}_V$

Find the displacement field $u : [0,T] \to \mathbb{R}^d$, the stress field $\boldsymbol{\sigma} : [0,T] \to \mathbb{S}_d$, the temperature function $\theta : [0,T] \to \mathbb{R}$, and the heat flux $q : [0,T] \to \mathbb{R}^d$ such that

$$u(t) = U_{ad}, \sigma(t) \in \sum_{ad}(t, w(t)), \quad \forall t \in [0,T], \tag{28}$$
$$\dot{\sigma}(t) = \xi(\theta(t)).\ \varepsilon(\dot{u}(t)) + \mathcal{G}\big(\sigma(t), \varepsilon(u(t)),\ \theta(t)\big), \tag{29}$$
$$u(0) = u_0, \quad \sigma(0) = \sigma_0, \tag{30}$$
$$(K\nabla\theta, \nabla\eta)_H + (\dot{\theta}, \eta)_H = (r, \eta) + (\chi, \gamma\eta), \tag{31}$$
$$\theta(0) = \theta_0. \tag{32}$$

We notice that the variational formulation PV is formulated in terms of a displacement field, a stress field, a temperature and heat. The existence of the unique solution of the problem PV is stated and proved in the next section.

3 Existence and Uniqueness of a Solution

Now, we propose our existence and uniqueness result.

Theorem 3.1 *Assume that (*(15)-(27)*) hold. Then there exists a unique weak solution of the problem (*(28)-(32)*) such that*

$$\theta \in L^2(0,T;Q) \cap C(0,T;L^2(\Omega)), \tag{33}$$
$$u \in C^1(0,T;H_1), \tag{34}$$
$$\sigma \in C^1(0,T;\mathcal{H}_1). \tag{35}$$

The proof of Theorem 3.1 will be carried out in several steps. It is based on parabolic equations and a Cauchy-Lipschitz technique.

In the first step, we consider the following variational problem.

Problem $\mathbf{PV}_\theta$
Find a temperature function $\theta : [0,T] \to \mathbb{R}$ such that

$$(K\nabla\theta, \nabla\eta)_H + (\dot{\theta}, \eta)_H = (r, \eta) + (\chi, \gamma\eta), \tag{36}$$
$$\theta(0) = \theta_0. \tag{37}$$

The existence and uniqueness of the functions θ satisfying (33) can be obtained by using classical results concerning parabolic equations. For the problem $\mathbf{PV}_\theta$, we have the following lemma.

Lemma 3.1 $\mathbf{PV}_\theta$ *has a unique solution satisfying*

$$\theta \in L^2(0,T;Q) \cap C(0,T;L^2(\Omega)). \tag{38}$$

Proof Applying classical results concerning parabolic equations, we get the existence and uniqueness of the solution of ((36)-(37)) with the regularity

$$\theta \in L^2(0,T;Q) \cap C(0,T;L^2(\Omega)).$$

Now, the existence and uniqueness of the solution (u, σ) of the mechanical problem with the regularity(34), (35) can be proved by considering θ as a known function and the existence and uniqueness of the solution of the mechanical problem is proved by reducing the problem under consideration to an ordinary differential one in a Hilbert space.

In the second step, we consider the following variational problem.

Problem $\mathbf{PV_u}$
Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$, and the stress $\sigma : \Omega \times [0,T] \to \mathbb{S}_d$ such that

$$u(t) = U_{ad}, \sigma(t) \in \sum_{ad}(t, w(t)), \quad \forall t \in [0,T], \tag{39}$$
$$\dot{\sigma}(t) = \xi(\theta(t)).\ \varepsilon(\dot{u}(t)) + \mathcal{G}(\sigma(t), \varepsilon(u(t)),\ \theta(t)), \tag{40}$$
$$u(0) = u_0, \quad \sigma(0) = \sigma_0. \tag{41}$$

The existence and uniqueness of the functions (u, σ) satisfying ((35),(36)) is given by the following lemma.

Lemma 3.2 $\mathbf{PV_u}$ *has a unique solution satisfying*

$$u \in C^1(0,T;H_1), \tag{42}$$
$$\sigma \in C^1(0,T;\mathcal{H}_1). \tag{43}$$

Proof. In order to prove Lemma 3.2, we need some preliminaries given by the following lemma, whose proof can be easily obtained.

Lemma 3.3 *Let* (15), $\theta \in C(0,T;L^2(\Omega))$ *hold, then for all* $t \in [0,T]$, *we have*

$$\begin{array}{rcl} \|\xi(\theta(t)).\sigma\|_{\mathcal{H}} & \leq & \beta\|\sigma\|_{\mathcal{H}}, \\ \langle \xi(\theta(t))\sigma, \sigma\rangle_{\mathcal{H}} & \geqslant & \alpha\|\sigma\|^2_{\mathcal{H}}, \\ \left\|\xi^{-1}(\theta(t)).\sigma\right\|_{\mathcal{H}} & \leq & \frac{1}{\alpha}\|\sigma\|_{\mathcal{H}}, \\ \left\langle \xi^{-1}(\theta(t))\sigma, \sigma\right\rangle_{\mathcal{H}} & \geqslant & \frac{\alpha}{\beta^2}\|\sigma\|^2_{\mathcal{H}}. \end{array}$$

Let now $X = V \times \mathcal{V}$. Using the properties of the trace maps, from (19) and (21), we obtain the existence of the function $\tilde{\sigma} \in W^{1,\infty}(0,T;\tilde{H})$ such that

$$\begin{array}{ll} \tilde{\sigma}\nu(t) = f_2(t) & \text{on } \Gamma_2 \times [0,T], \\ Div\ \tilde{\sigma}(t) + f_0 = 0 & \text{in } \Omega \times [0,T]. \end{array}$$

Let now

$$\begin{array}{lll} a: & [0,T] \times X \times X & \to \mathbb{R}, \\ F: & [0,T] \times X & \to X \end{array}$$

be given by

$$a(t,x,y) = \langle \xi(\theta(t))\varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}} + \left\langle \xi^{-1}(\theta(t))\sigma, \tau\right\rangle_{\mathcal{H}}, \tag{44}$$

$$\begin{aligned} \langle F(t,x), y\rangle_X &= \left\langle \xi^{-1}(\theta(t))\mathcal{G}(\sigma + \tilde{\sigma}(t), \varepsilon(u), \theta(t)), \tau\right\rangle_{\mathcal{H}}, \\ &\quad - \langle \mathcal{G}(\sigma + \tilde{\sigma}(t), \varepsilon(u)), \theta(t), \varepsilon(v)\rangle_{\mathcal{H}}, \\ &\quad - \left\langle \dot{\tilde{\sigma}}(t), \varepsilon(v)\right\rangle_{\mathcal{H}} - \left\langle \xi^{-1}(\theta(t))\dot{\tilde{\sigma}}(t), \tau\right\rangle_{\mathcal{H}}, \\ \text{for all } x &= (u,\sigma),\ y = (v,\tau) \in X \text{ and } t \in [0,T]. \end{aligned} \tag{45}$$

Let us now denote

$$\sigma = \bar{\sigma} + \tilde{\sigma},\ x = (u, \bar{\sigma}), \tag{46}$$

$$\sigma_0 = \bar{\sigma}_0 + \tilde{\sigma}(0),\ x_0 = (u_0, \bar{\sigma}_0). \tag{47}$$

We have the following result

Lemma 3.4 *The pair $(u,\sigma) \in C^1(0,T;H_1 \times \mathcal{H}_1)$ is a solution of the problem P_u if and only if $x \in C^1(0,T;X)$ is a solution of the problem*

$$a(t, \dot{x}(t), y) = \langle F(t,x), y\rangle_X, \tag{48}$$

$$x(0) = x_0, \tag{49}$$

where $X = V \times \mathcal{V}$, $x = (u, \bar{\sigma})$.

Proof. Using (46), (47), it is easy to see that $(u,\sigma) \in C^1(0,T;H_1 \times \mathcal{H}_1)$ is a solution of the viscoplastic problem if, and only if $x \in C^1(0,T;X)$ and

$$\dot{\bar{\sigma}} = \xi(\theta).\varepsilon(\dot{u}) + \mathcal{G}(\bar{\sigma} + \tilde{\sigma}, \varepsilon(u), \theta) - \dot{\tilde{\sigma}}, \tag{50}$$

$$u(0) = u_0, \quad \bar{\sigma}(0) = \bar{\sigma}_0. \tag{51}$$

Let us suppose (50)-(51) are fulfilled. Using the fact that $\boldsymbol{\varepsilon}(v)$ is the orthogonal complement of $\mathcal{V}$ in $\mathcal{H}$, we have (48).

Conversely, let (48) hold and let

$$z(t) = \dot{\bar{\sigma}}(t) - \xi(t)\varepsilon(\dot{u}) - \mathcal{G}(\bar{\sigma} + \tilde{\sigma}, \varepsilon(u), \theta) - \dot{\tilde{\sigma}}. \tag{52}$$

Taking $y = (v, 0) \in X$ in (57) and using the orthogonality of $\varepsilon(v)$ and v, we get

$$\langle z(t), \boldsymbol{\varepsilon}(v)\rangle_{\mathcal{H}} = 0. \tag{53}$$

Now, we take $y = (0, \tau) \in X$ in (48) and using the orthogonality of $\varepsilon(v)$ and v, we get

$$\left\langle \xi^{-1}(\theta\,(t))z\,(t)\,, \tau \right\rangle_{\mathcal{H}} = 0. \tag{54}$$

Due to the orthogonality of $\boldsymbol{\varepsilon}(v)$ in $\mathcal{H}$ to v, from (53), we get $z\,(t) \in V$, thus we may put $\tau = z\,(t)$ in (54), and thus from (39), we deduce $z\,(t) = 0$. Hence, we proved that (50) is equivalent to (49).

The following lemma can be easily obtained.

Lemma 3.5 *For every $x \in X$ and $t \in [0, T]$, there exists a unique element $z \in X$ such that*

$$a\,(t, z, y) = \langle F(t, x), y \rangle_X\,, \tag{55}$$

where $X = V \times \mathcal{V}$, $x = (u, \bar{\sigma})$.

Proof. Let $x \in X$ and $t \in [0, T]$, using the properties of ξ, ξ^{-1} and Korn's equality, we get that $a\,(t, ., .)$ is bilinear continuous and coercive, hence the existence and uniqueness of z which satisfies (55) following from Lax-Miligram's Lemma.

The previous lemma allows us to consider the operator

$$A: \quad [0, T] \times X \quad \to \quad X$$

defined as $A\,(t, x) = z$, moreover, we have the following result.

Lemma 3.6 *The operator A is continuous and there exists $C > 0$ such that*

$$|A\,(t, x_1) - A\,(t, x_2)|_X \leq C\,|x_1 - x_2|_X\,,\ \forall x_1, x_2 \in X. \tag{56}$$

Proof. *Let us consider $t_1, t_2 \in [0, T]$; $\quad x_i = (u_i, \sigma_i) \in X$ and let $z_i = (w_i, \tau_i) \in X$ be defined by $z_i = A\,(t_i, x_i)$.*

Using (55), we have

$$a(t_1, z_1, z_1 - z_2) - a(t_2, z_2, z_1 - z_2) = \langle F(t_1, z_1) - F(t_2, z_2), z_1 - z_2 \rangle_X\,, \tag{57}$$

and from (39) and Korn's inequality, we get

$$\begin{aligned} a\,(t_1, z_1, z_{1-}z_2) \quad & -a\,(t_2, z_2, z_{1-}z_2) \\ & \geqslant C\,\|z_{1-}z_2\|_X^2 - \|[\xi(\theta\,(t_1)) - \xi(\theta\,(t_2))]\,\varepsilon\,(w_2)\|_H\,\|z_{1-}z_2\|_X \\ & \quad - \left\|\left[\xi^{-1}(\theta\,(t_1)) - \xi^{-1}(\theta\,(t_2))\right]\tau_2\right\|_H\,\|z_{1-}z_2\|_X\,. \end{aligned} \tag{58}$$

In a similar way, from (36), (37), we get

$$\begin{aligned} \langle F(t_1, x_1) \quad & -F(t_2, x_2), z_{1-}z_2 \rangle_X \\ & \leq C(\|x_{1-}x_2\|_X + \|\tilde{\sigma}\,(t_1) - \tilde{\sigma}\,(t_2)\|_H \\ & \quad + \|\tilde{\sigma}\,(t_1) - \tilde{\sigma}\,(t_2)\|_H + \|\theta\,(t_1) - \theta\,(t_2)\|_{L^2(\Omega)} \\ & \quad + \left\|\left[\xi^{-1}(\theta\,(t_1)) - \xi^{-1}(\theta\,(t_2))\right]\right\|_H F\,(\sigma_2 + \tilde{\sigma}\,(t)\,, \varepsilon\,(u)\,, \theta\,(t_2)) \\ & \quad + \left\|\left[\xi^{-1}(\theta\,(t_1)) - \xi^{-1}(\theta\,(t_2))\right]\sigma\,(t_2)\right\|_H).\,\|z_{1-}z_2\|_X\,. \end{aligned} \tag{59}$$

So, from (57)-(58), it results

$$\begin{aligned}
&\|z_1 - z_2\|_X \\
&\qquad \leq \mathcal{C}\big(\|[\xi(\theta(t_1)) - \xi(\theta(t_2))]\, \varepsilon(w_2)\|_H \\
&\qquad\quad + \left\| \left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right] \tau_2 \right\|_H \\
&\qquad\quad + \|x_{1-}x_2\|_X + \|\tilde{\sigma}(t_1) - \tilde{\sigma}(t_2)\|_H + \|\theta(t_1) - \theta(t_2)\|_{L^2(\Omega)} \\
&\qquad\quad + \left\| \left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right] \right\|_H F(\sigma_2 + \tilde{\sigma}(t), \varepsilon(u), \theta(t_2)) \\
&\qquad\quad + \left\| \left[\xi^{-1}(\theta(t_1)) - \xi^{-1}(\theta(t_2))\right] \sigma(t_2) \right\|_H \big) . \|z_{1-}z_2\|_X . \qquad (60)
\end{aligned}$$

Using the properties of ξ, ξ^{-1} and the regularity of $\tilde{\sigma}$, u, from (60), we get $z_1 \longrightarrow z_2$ in X when $t_1 \longrightarrow t_2$ in $[0,t]$, and $x_1 \longrightarrow x_2$ in X. Hence A is a continuous operator, moreover, taking $t_1 = t_2$ in (60), we get (56).

Now, we have all the ingredients needed to prove Theorem 3.1.

Proof of Theorem 3.1 Using the hypothesis on u_0, σ_0, we get that $x_0 \in X$, and by Lemma 3.5 and the classical Cauchy-Lipschitz theorem, we get that there exists a unique solution $x \in C^1(0,T;X)$ of the Cauchy problem

$$\begin{aligned}
\dot{x}(t) &= A(t, x(t)), \\
x(0) &= x_0.
\end{aligned}$$

Theorem **3.1** follows now from the definition of the operator A, and Lemma 3.3 .

4 The Continuous Dependence with respect to Parameter θ and Numerical Examples

In this section, we prove the continuous dependence of the solution (u, σ) upon the data θ. Moreover, we give two one-dimensional examples to illustrate the results.

4.1 The continuous dependence of the solution with respect parameter θ

We consider the case when ξ in (4) does not depend on θ and we replace (4) by ξ which is a symmetric and positively definite tensor.
We have the following result.

Theorem 4.1 *Let ξ be a symmetric and positively definite tensor,* (15), (18), (20)-(24) *hold, and let (u_i, σ_i) be the solutions of the problem* (28)-(32) *for $\theta = \theta_i, i = 1, 2$. Then there exists $C > 0$ such that*

$$\|u_1 - u_2\|_{C^1(0,T;H_1)} + \|\sigma_1 - \sigma_2\|_{C^1(0,T;\mathcal{H}_1)} \leq C \|\theta_1 - \theta_2\|_{C^0(0,T;Y)} . \qquad (61)$$

Proof. Let $\tilde{\sigma}$ be a function which satisfies

$$\begin{gathered}
\tilde{\sigma} \cdot \nu = f_2, \\
Div\tilde{\sigma} + f_0 = 0, \\
\bar{\sigma_i} = \sigma_i - \tilde{\sigma}, \quad x_i = (u_i, \sigma_i), \quad i = 1, 2. \qquad (62)
\end{gathered}$$

From Theorem 3.1, we have

$$\dot{x_i}(t) = A_i(t, x_i(t)), \qquad (63)$$

$$x_i(0) = x_0, \qquad (64)$$

where x_0 is given by $x_0 = (u_0, \sigma_0)$ and the operators A_i are defined by Lemma 3.6 with replacing θ by θ_i in (44). In a similar way as in (60), we obtain

$$\|A(t, x_1(t)) - A(t, x_2(t))\|_X \leq C\left(\|x_1(t) - x_2(t)\|_X + \|\theta_1(t) - \theta_2(t)\|_Y\right) \tag{65}$$

for all $t \in [0, T]$, hence from (63) and (64), using a standard technique, we get

$$(\|x_1(t) - x_2(t)\|_X) \leq C \int_0^t \|\theta_1(s) - \theta_2(s)\|_X \, ds \tag{66}$$

$$(\|\dot{x_1}(t) - \dot{x_2}(t)\|_X) \leq C\left(\|x_1(t) - x_2(t)\|_X + \|\theta_1(t) - \theta_2(t)\|_Y\right) \tag{67}$$

for all $t \in [0, T]$. Theorem 4.1 follows now from (62), (66) and (67).

4.2 One-dimensional Examples

In this section, we give two one-dimensional examples to illustrate the main results.

One-dimensional model
We consider a thermoviscoplastic body $\Omega =]0, 1[$ whose boundary is divided in three parts Γ_1, Γ_2 and Γ_3. We suppose that the body is fixed at $x = 0$ and is subject to the action of a body force of density $f_0(x, t) = 10$. On the part Γ_2, known tractions act on the body. We suppose then the volume heat $r = 0$ and the thermal boundary conditions are

$$\theta(0, t) = \theta_0, \quad \theta(1, t) = a,$$

where θ_0 and a are given.

We also use a thermoviscoplastic law, i.e.,

$$\dot{\sigma} = E(\theta)\dot{\varepsilon} - \sigma + E(\theta)\varepsilon. \tag{68}$$

Here $\varepsilon = \frac{\partial u}{\partial x}$, $\dot{\varepsilon} = \frac{\partial \varepsilon}{\partial t}$, $\dot{\sigma} = \frac{\partial \sigma}{\partial t}$ and $E(\theta)\dot{\varepsilon}$ is the modulus of elasticity. For this consideration, we have the following.

Example 1. Let us consider a thermoviscoplastic problem of the form (4)-(15) in the following context:

$\Omega =]0, 1[, \Gamma_1 = \{0\}, \Gamma_2 = \{1\}, \Gamma_3 = \emptyset, f_0(x, t) = 10, u(x, 0) = 0, u(0, t) = 0$ and $\sigma(1, t) = 0, \sigma(x, 0) = 10 - 10x, r = 0$.

In this case, the problem is the classical displacement-traction formulated as follows.

Find a displacement field $u : \Omega \times (0, T) \to \mathbb{R}^d$, a stress field $\sigma : \Omega \times (0, T) \to \mathbb{S}_d$, a

temperature $\theta : \Omega \to \mathbb{R}$, and the heat flux function $q : \Omega \to \mathbb{R}^d$ such that

$$\frac{\partial q}{\partial x}(x,t) = 0, \tag{69}$$

$$q = \frac{\partial \theta}{\partial x}, \tag{70}$$

$$\theta(0,T) = \theta_0, \tag{71}$$

$$\theta(1,T) = a, \tag{72}$$

$$\frac{\partial \sigma}{\partial x}(x,t) + 10 = 0, \tag{73}$$

$$\dot{\sigma} = E(\theta)\dot{\varepsilon}(u(x,t)) - \sigma(x,t) + E(\theta)\varepsilon(u(x,t)), \tag{74}$$

$$u(0,t) = 0, \tag{75}$$

$$u(x,0) = 0, \tag{76}$$

$$\sigma(1,t) = 0, \tag{77}$$

$$\sigma(x,0) = 10 - 10x, \tag{78}$$

where u, σ, θ, and q are unknowns.

Example 2. In this case, we suppose that the body is in frictionless contact with a rigid foundation. Then problem (4)-(15) is the following Signorini contact problem:

$$\frac{\partial q}{\partial x}(x,t) = 0, \tag{79}$$

$$q = \frac{\partial \theta}{\partial x}, \tag{80}$$

$$\theta(0,T) = \theta_0, \tag{81}$$

$$\theta(1,T) = a, \tag{82}$$

$$\frac{\partial \sigma}{\partial x}(x,t) + 10 = 0, \tag{83}$$

$$\dot{\sigma} = E(\theta)\dot{\varepsilon}(u(x,t)) - \sigma(x,t) + E(\theta)\varepsilon(u(x,t)), \tag{84}$$

$$u(0,t) = 0, \tag{85}$$

$$u(x,0) = 0, \tag{86}$$

$$\sigma(x,0) = 10 - 10x, \tag{87}$$

$$u(1,t) \leq \frac{1}{4}, \sigma(1,t) \leq 0, \sigma(1,t)\left(u(1,t) - \frac{1}{4}\right) = 0, \tag{88}$$

where u, σ, θ, and q are unknowns.

In Section 2, we considered the Signorini contact problem with a zero gap. The results there can be extended straightforward to the situation with a nonzero initial gap g, here $g = \frac{1}{4}$.

Now, we present the exact solution for the example below. For the thermal problem, we can easily find the solution (θ, q). For the isotherm mechanical problem, we have the following.

For Example 1: From the equilibrium equation, we have

$$\sigma(x,t) = -10x + k(t). \tag{89}$$

Substituting for the equation law and boundary conditions, we have

$$\varepsilon(x,t) = C(t)e^{-t} + \frac{\sigma(x,t)}{E(\theta)}. \tag{90}$$

Using the initial conditions, we obtain

$$\varepsilon(u(x,t)) = \left(\frac{10}{E(\theta)}x - \frac{10}{E(\theta)}\right)e^{-t} + \frac{-10x + k(t)}{E(\theta)}, \tag{91}$$

$$u(x,t) = \left(\frac{5}{E(\theta)}x - \frac{10}{E(\theta)}\right)e^{-t} + \frac{-5x^2 + k(t)x}{E(\theta)}. \tag{92}$$

Using the boundary conditions of σ, we have $k(t) = 10$, the exact solution of the displacement traction problem is

$$\sigma(x,t) = -10x + 10, \tag{93}$$

$$u(x,t) = \frac{10}{E(\theta)}\left(\frac{x^2}{2} - x\right)\left(e^{-t} - 1\right). \tag{94}$$

For Example 2: Using the same technique as Example 1, we have

$$\sigma(x,t) = -10x + k(t), \tag{95}$$

$$u(x,t) = \left(\frac{5x^2}{E(\theta)} - \frac{10}{E(\theta)}\right)e^{-t} + \frac{-5x^2 + k(t)x}{E(\theta)}. \tag{96}$$

At $x = 1$, the body is in contact with the foundation, then $u(1,t) \leq \frac{1}{4}$.

- When $u(1,t) = \frac{1}{4}$, there is a contact, then we obtain

$$k(t) = \frac{5}{2}\left[\frac{E(\theta)}{10} + 2 + 2e^{-t}\right]$$

and $\sigma(1,t) < 0$ gives $k(t) < 10$ and $t > log\left[\frac{20}{20-E(\theta)}\right]$.

The exact solution of the Signorini problem is as follows.

- In the case when $t \in \left[0; log\left(\frac{20}{20-E(\theta)}\right)\right]$,

  there is no contact, we have $\sigma(1,t) = 0$, then $k(t) = 10$.

  The solution is

$$\begin{cases} \sigma(x,t) = -10x + 10, \\ u(x,t) = \frac{10}{E(\theta)}\left(\frac{x^2}{2} - x\right)\left(e^{-t} - 1\right). \end{cases} \tag{97}$$

- In the case when $t > log\left(\frac{20}{20-E(\theta)}\right)$,

  there is a contact and we have $u(1,t) = \frac{1}{4}$, $\sigma(1,t) < 0$.

  The solution is

$$\begin{cases} \sigma(x,t) = -10x - \frac{5}{2}\left(\frac{E(\theta)}{10} + 2 + 2e^{-t}\right), \\ u(x,t) = \frac{10}{E(\theta)}\left[\frac{x^2}{2}\left(e^{-t} - 1\right) + \frac{1}{4}\left(-2e^{-t} + E(\theta) + 2\right)\right]. \end{cases} \tag{98}$$

Remark. From these two one-dimensional examples, we deduce that the solution (u, σ) is dependent on $E(\theta)$.

5 Conclusion

This study addresses two uncoupled quasistatic problems in thermo-viscoplastic materials. A model is proposed where the mechanical properties of the problem are dependent on a parameter θ, which can be interpreted as the absolute temperature. The boundary conditions include displacement traction and the Signorini conditions. The existence of a unique solution to the problems is proven, and two examples in one-dimensional study are presented to describe the problem processes.

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