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Stability and Hopf Bifurcation of a Generalized Differential-Algebraic Biological Economic System with the Hybrid Functional Response and Predator Harvesting

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Abstract: This paper examines the dynamics of a bio-economic predator-prey system that employs harvesting and the hybrid response function. The system includes an algebraic equation because of the economic revenue. We give a thorough mathematical study of the suggested model to highlight some of the significant results. The boundedness and positivity of model's solutions are examined. The coexistence equilibrium of the bio-economic system has been extensively studied, and the behavior of the model around it has been explained using the qualitative theory of dynamical systems (such as local stability and the Hopf bifurcation). The data gained offer a useful framework for understanding the role of economic revenue v. We establish that a positive equilibrium point is locally asymptotically stable when the profit v falls below a particular critical value v^*. Our research shows it to be true. According to our research, economic revenue can stabilize the system, which is the most important of all spaces.

Keywords: *algebraic differential equations; equilibrium point; Hopf bifurcation; predator-prey system; stability.*

Mathematics Subject Classification (2010): 70K42, 70K50, 93A10, 93A30.

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1 Introduction

For humanity's long-term welfare, there is a great deal of interest in comprehending and developing bio-economic models for biodiversity. To preserve the long-term viability and prosperity of the ecosystem, researchers are attempting to generate some possibly beneficial effects.

Numerous studies have focused on understanding these processes. The dynamical behavior of a specific predator-prey ecosystem was explored using a number of differential equations and an algebraic equation [12, 17]. They made important discoveries, including limit cycle, singularity-driven bifurcation, control, and interior equilibrium stability. However, in all of the models examined, only the prey population is harvested. The relationship between the predator and the prey was investigated using a variety of functional responses, including Holling-type I, Holling-type II [10, 11], Holling-type III [13], and Beddington-DeAngelis [19], under the presumption that the isolated predator species had natural mortality.

As far as we know, a dynamical investigation of a predator-prey model with a hybrid functional response has never been done. Since this model exhibits stability and the Hopf bifurcation, we explore it and describe it in this paper [13, 15]. Additionally, we are interested in learning some theoretical guidelines for administering and regulating renewable resources.

We organized the existing information in the manner described below to achieve the predetermined goals: we started our investigation by going through the model-building idea and its biological importance. We sequentially prove the pomposity and boundedness of the model. Following a thorough discussion of the system's stability and the Hopf bifurcation analysis, the existence of a positive equilibrium is then investigated. We conclude by presenting numerical simulation tests that support the theoretical findings.

2 The Model

The study of population dynamics with harvesting has emerged as a fascinating subject in mathematical bio-economics due to the significance of the effective management of renewable resources. When Gordon [7] established a standard property resource economic theory in 1954, he made the following economic proposition. This theory examined the impact of harvest effort on the ecosystem from an ecological perspective.

$$NetEconomicRevenue(NER) = TotalRevenue(TR) - TotalCost(TC). \tag{1}$$

Studying the predator-prey paradigm with the hybrid functional response is both intriguing and crucial:

$$\begin{cases} \frac{dX}{d\tau} = (a_1 - b_1 X - \frac{m_1 Y}{\alpha_1 X + \beta_1 Y + \gamma_1})X, \\ \frac{dY}{d\tau} = (a_2 - \frac{m_2 Y}{X + K_1})Y \end{cases} \tag{2}$$

with the initial values $X(0) > 0$ and $Y(0) > 0$. The constants $a_1,\ a_2,\ b_1,\ b_2,\ m_1,\ m_2,\ \alpha_1,\ \beta_1,\ \gamma_1$ and K_1 are the parameters of the model and are assumed to be nonnegative with β_1 non trivial (if $\beta_1 = 0$, then the model (2) is the same as that in [14]).

These parameters are defined as follows: a_1 (resp.,a_2) describes the growth rate of the prey (resp., of the predator), b_1 measures the strength of competition among the individuals of the prey's species, m_1 is the maximum value which per capita reduction rate of the prey can attain, γ_1 (resp., K_1) measures the extent to which environment provides protection to the prey (resp., to the predator), and m_2 has a similar meaning to m_1. The functional response in (2) was introduced by Beddington [2] and DeAngelis et al. [6].

When introducing the following scaling (see [14]): $t = a_1\tau$, $x(t) = (b_1/a_1)X(\tau)$, and $y(t) = (m_2 b_1/a_1 a_2)Y(\tau)$, the hybrid functional response model (2) should take the following nondimensional form:

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{axy}{\alpha x+\beta y+\gamma}, \\ \frac{dy}{dt} = b(1-\frac{y}{x+k})y \end{cases} \tag{3}$$

where $a = (a_1/a_2)(m_1/m_2)$, $b = a_2/a_1$, $\alpha = \alpha_1$, $\beta = \beta_1(a_2/m_2$, $\gamma = \gamma_1(b_1/a_1)$, and $k = K_1(b_1/a_1)$. The model (3) that interests us is introduced in [14].

It is known that the harvest effort is an important factor to construct a useful bioeconomic mathematical model, for this reason, taking (1) into account, we extend the system (3) by considering the following algebraic equation which describes the economic profit v of the harvest effort on the predator:

$$E(t)(py(t) - c) = v, \tag{4}$$

where $0 \leq E(t) \leq E_{max}$ and $y(t) \geq 0$ represent the harvest effort and the density of the predator, respectively. p represents the unit price of the harvested population and c is the cost of harvest effort, the total revenue and total cost are

$$TR = pE(t)y(t), \quad TC = cE(t).$$

Based on (3) and (4), a singular differential-algebraic model that consists of two differential equations and an algebraic equation can be established as follows:

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{axy}{\alpha x+\beta y+\gamma}, \\ \frac{dy}{dt} = b(1-\frac{y}{x+k})y - Ey, \\ 0 = E(py-c) - v \end{cases} \tag{5}$$

which is a semi-explicit differential-algebraic equation of the form

$$\begin{cases} \dot{z} = \frac{dz}{dt} = f(v, X), \\ \quad 0 = g(v, X), \end{cases} \tag{6}$$

where we denote $X = (x, y, E)^T$, with $z = (x, y)^T$ being the differential variable, E being the algebraic variable, v is the bifurcation parameter, f and g are the smooth functions given by

$$f(v, X) = \begin{pmatrix} f_1(v, X) \\ f_2(v, X) \end{pmatrix} = \begin{pmatrix} x((1-x) - \frac{ay}{\alpha x+\beta y+\gamma}) \\ y(b(1-\frac{y}{x+k}) - E) \end{pmatrix},$$

$$g(v, X) = E(py - c) - v.$$

3 Mathematical Analysis

We are just concerned with this model's dynamics, positive octant $\mathbb{R}^3_+$ for biological reasons. Thus, we consider the biologically meaningful initial condition

$$x(0) = x_0 \geq 0, \; y(0) = y_0 \geq 0, \; E(0) = E_0 = \frac{v}{py_0 - c}, \; py_0 - c > 0. \tag{7}$$

3.1 Existence and uniqueness

Proposition 3.1 *The system (5) with the initial conditions (7) has a unique maximal solution* $(x(t), y(t), E(t))$ *in an open subset* U *of* $\Omega = \{(x, y, E)^T \in \mathbb{R}^3_+ / \ py - c > 0\}$ *defined on some maximal interval* $[0, T[$.

Proof. Let $(x, y, E)^T \in U$, then from the algebraic equation $g(x, y, E, v) = 0$, we get $E = \frac{v}{px-c}$, substituting in the first differential equation of (5). The differential-algebraic equation is transformed to the following ordinary differential equation that has the same solution with respect to the differential variables $z = (x, y)^T$:

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{axy}{\alpha x + \beta y + \gamma}, \\ \frac{dy}{dt} = b(1 - \frac{y}{x+k})y - \frac{vy}{py-c}, \end{cases} \tag{8}$$

its vectorial form is $\dot{z} = \frac{dz}{dt} = F(z)$, where

$$F(z) = \begin{pmatrix} x((1-x) - \frac{ay}{\alpha x + \beta y + \gamma}) \\ y(b(1 - \frac{y}{x+k}) - \frac{v}{py-c}) \end{pmatrix}.$$

Clearly, $F \in C^1(U^{'})$, where $U^{'}$ is an open subset of $\Omega^{'} = \{(x, y,)^T \in \mathbb{R}^2_+ / \ py - c > 0\}$. Thus, by applying Cauchy-Lipschitz's theorem for ordinary differential equations [9], we deduce the local existence and uniqueness of the maximal solution $(x, y)^T$ to (8) for any $(x_0, y_0) \in U^{'}$, then the local existence and uniqueness of solution for (5) is straightforward.

3.2 Positivity and boundedness

Regarding the positivity of solution for the system (5), we introduce the following proposition.

Proposition 3.2 *Any smooth solution of (5), defined on the maximal interval* $[0, T[$, *with positive initial condition (7), remains positive for all* $t \in [0, T[$.

Proof. From the system (8), it follows that $x = 0$ implies $\frac{dx}{dt} = 0$ and $y = 0$ implies $\frac{dy}{dt} = 0$, thus $x = 0$ and $y = 0$ are invariant sets showing that $x(t) \geq 0$ and $y(t) \geq 0$ whenever $x(0) > 0$ and $y(0) > 0$.

From the second equation of (8), we deduce that for all $t \in [0, T[$,

$$py(t) - c \neq 0. \tag{9}$$

Suppose that there exists $t^* \in [0, T[$ such that $E(t^*) < 0$, it follows that $px(t^*) - c < 0$, then by applying the intermediate value theorem to the continuous function $py(t) - c$ on the interval $[0, t^*]$, we deduce the existence of $\tilde{t} \in]0, t^*[$ such that $py(\tilde{t}) - c = 0$, which contradicts (9), thus, $E(t) \geq 0$ for all $t \in [0, T[$.

Clearly, when the prey biomass x approaches to the critical value $y_c = \frac{c}{p}$, the finishing effort E will being unbounded is not realistic.

To answer the boundedness of the solution for system (5), we impose a realistic ecological contraint in the context that the economic policy requires a minimum level $y_{min} > 0$ for the resource given by

$$y(t) \geq y_{min} > \frac{c}{p}, \quad \forall t \geq 0. \tag{10}$$

This constraint will affect the fishing effort E that will be constrained by a fixed production capacity. We denote this limit capacity by E_{max}, then

$$0 < E(t) \leq E_{max} = \frac{v}{py_{min} - c}, \quad \forall t \geq 0. \tag{11}$$

Next, we will show that, under some assumptions, the solutions of system (5), which start in $\mathbb{R}^3_+$, are ultimately bounded. First, let us give the following comparison result.

Definition 3.1 A solution $\phi(t, t_0, x_0, y_0, E_0)$ of system (5) is said to be ultimately bounded with respect to $\mathbb{R}^3_+$ if there exists a compact region $A \subset \mathbb{R}^3_+$ and a finite time T $(T = T(t_0, x_0, y_0, E_0)$ such that, for any $(t_0, x_0, y_0, E_0) \in \mathbb{R} \times \mathbb{R}^3_+$,

$$\phi(t, t_0, x_0, y_0, E_0) \in A, \quad \forall t \geq T. \tag{12}$$

Proposition 3.3 *All solutions of the system (5) subject to the initial conditions (7) and constraint (11) are bounded in $\mathbb{R}^3_+$ with an ultimate bound.*

Proof. (1) We have for all $t \geq 0$, $0 \leq x(t) \leq 1$ and $0 \leq x + y \leq L_1$, see [14]

$$L_1 = \frac{1}{4b}(5b + (1+b)^2(1+k).$$

Then

$$(x(t), y(t), E(t)) \in A = \{(x, y, E) \in \mathbb{R}^3_+ : \ 0 \leq x \leq 1, \ 0 \leq x + y \leq L_1, \ 0 \leq E \leq E_{max}\}.$$

(2) We have to prove that, for $(x(0), y(0), E(0)) \in \mathbb{R}^3_+$, $(x(t), y(t), E(t)) \in A$ when $t \to +\infty$. We will show that $\overline{lim}_{t\to+\infty} x(t) \leq 1$, $\overline{lim}_{t\to+\infty}(x(t) + y(t)) \leq L_1$, and $\overline{lim}_{t\to+\infty} E(t) \leq E_{max}$, see [14]. Then we conclude that system (5) is dissipative in $\mathbb{R}^3_+$.

4 Existence and Positivity Equilibrium Points

In this section, we aim to inspect the existence of the positive equilibrium points and to study their stability.
An equilibrium point of the system (5) is a solution of the following equations:

$$\begin{cases} f_1(v, X) = 0, \\ f_2(v, X) = 0, \\ g(v, X) = 0. \end{cases} \tag{13}$$

By the analysis of the roots for (13), it follows that

(i) If $v = 0$, then there exist at least three boundary equilibrium points

$$X_{e1} = (0, 0, 0), \ X_{e2} = (1, 0, 0), \ X_{e3} = (0, 0, k),$$

and if $k(\alpha - \beta) \leq \gamma$, the system (5) has a unique equilibrium $P^*(x^*, y^*, 0)$, where x^* is the root of the equation

$$a(a(x + k) = (1 - x)((\alpha + \beta)x + \beta k + \gamma) \tag{14}$$

or, equivalently, the quadratic equation

$$(\alpha+\beta)x^2+(\beta k+\gamma+a-\alpha-\beta)x+(a-\beta)k-\gamma=0 \tag{15}$$

satisfying $0<x^*\leq 1$, and

$$y^*=x^*+k. \tag{16}$$

For the proof see [14].

(ii) If $v>0$, the interior equilibrium points $P^*(x^*,y^*,E^*)$ are defined by the system

$$\begin{cases} 1-x-\frac{ay}{\alpha x+\beta y+\gamma}=0, \\ b(1-\frac{y}{x+k})=\frac{v}{py-c} \end{cases} \tag{17}$$

or, equivalently, the system

$$\begin{cases} 1-x-\frac{ay}{\alpha x+\beta y+\gamma}=0, \\ x=\frac{by(py-c)}{b(py-c)-v}-k \end{cases} \tag{18}$$

satisfying $0<x^*\leq 1$, and y^* is a solution of the fourth degree equation

$$y^4+By^3+Cy^2+Dy+E=0, \tag{19}$$

where A,B,C,D,E are given by

$$A=b^2p^2(\alpha+\beta),$$

$$B=bp[bp(a+\beta(k-1)+\gamma-\alpha)-2\alpha bc-\beta(bc+v)]/A,$$

$$C=b[2\alpha bpck+pv(\alpha k-\gamma-2a)-2bpc(a+\gamma)+(\alpha+\beta)bc^2+$$

$$+\beta cv+p(1-k)(2bc(\alpha+\beta)+pb(\alpha k-\gamma))]/A,$$

$$D=(bv+v)[(k-1)(bp(2\alpha k-\gamma)+bc(\alpha k-\gamma)+\beta v)+a(bc+v)^2]/A,$$

$$E=(cb+v)[(1-k)(bp(\gamma-2\alpha k)-bc(\alpha+\beta-\beta v)+b(ac-\alpha k)+B\gamma+av]/A.$$

The equation (19) is equivalent by the change of variable $y=Y-\frac{B}{4}$ of the equation

$$Y^4+PY^2+QY+R=0, \tag{20}$$

where P,Q,R are given by

$$P=-\frac{3B^2}{8}+C,$$

$$Q=(\frac{B}{2})^3-\frac{BC}{2}+D,$$

$$R=-3(\frac{B}{4})^4-\frac{B^2C}{16}-\frac{BD}{4}+E.$$

We use Ferrari's method to solve the equation (19). If $Q=0$ if and only if $B^3-4BC+8D=0$, the equation reduces to a bisquare equation which is easy to solve.
We assume that the equation does not reduce to a bisquare equation ($8Q=B^3-4BC+8D\neq 0$), the equation (20) is rewrite as

$$(Y^2+\frac{B}{2})^2=(\frac{B^2}{4}-C)Y^2-DY-E,$$

or

$$(Y^2+\frac{B}{2}+\lambda)^2=(\frac{B^2}{4}-C+2\lambda)Y^2+(B\lambda-D)Y+\lambda^2-E.$$

The second member is a square if and only if

$$(B\lambda-D)^2=4(\frac{B^2}{4}-C+2\lambda)(\lambda^2-E),$$

or

$$8\lambda^3-4C\lambda^2+(2BD-8E)\lambda+B^2E+4CE-D^2=0.$$

Let λ_0 be a solution of this cubic solvent and let μ_0 be the square root of $2\lambda_0-c+\frac{B^2}{4}$ which is necessarily nonzero according to the hypothesis $Q\neq 0$. The equation is then written as

$$(Y^2+\frac{B}{2}+\lambda)^2=(\mu_0Y+\frac{B\lambda_0-D}{2\mu_0})^2.$$

Therefore, it is equivalent to

$$Y^2+(\mu_0+\frac{B}{2})Y+\lambda_0+\frac{B\lambda_0-D}{2\mu_0}=0 \tag{21}$$

or

$$Y^2+(-\mu_0+\frac{B}{2})Y+\lambda_0-\frac{B\lambda_0-D}{2\mu_0}=0. \tag{22}$$

For (21), the discriminant is

$$\Delta_+=-2\lambda_0-C+2\frac{D-B\lambda_0}{\mu_0}+B\mu_0+\frac{B^2}{2},$$

$$y_{e,0}=\frac{-\mu_0+\sqrt{\Delta_+}}{2}-\frac{B}{4}\quad and\quad y_{e,1}=\frac{-\mu_0-\sqrt{\Delta_+}}{2}-\frac{B}{4}.$$

(22) is solved in the same way as (21) by replacing everywhere μ_0 by $-\mu_0$,

$$\Delta_-=-2\lambda_0-C-2\frac{D-B\lambda_0}{\mu_0}-B\mu_0+\frac{B^2}{2},$$

$$y_{e,2}=\frac{\mu_0+\sqrt{\Delta_-}}{2}-\frac{B}{4}\quad and\quad y_{e,3}=\frac{\mu_0-\sqrt{\Delta_-}}{2}-\frac{B}{4}.$$

5 Dynamic Analysis near the Coexistence Equilibria

In this section, we study the stability of the interior equilibrium X_e and analyse the bifurcation through it using the bifurcation theory and normal form theory.

5.1 Local stability analysis

For the analysis of the local stability of X_e, we let $X=Q\bar{X}$, here

$$\bar{X}=(x,y,\bar{E})^T,\quad Q=\begin{pmatrix}1&0&0\\0&1&0\\0&-\frac{E_ep}{px_e-c}&1\end{pmatrix}.$$

Then we get

$$D_X g(X_e)Q = (0, 0, py_e - c),$$

and

$$\bar{E} = E + \frac{E_e py}{py_e - c}.$$

Then the system can be expressed as follows:

$$\begin{cases} \frac{dx}{dt} = x(1 - x - \frac{ay}{\alpha x + \beta y + \gamma}), \\ \frac{dy}{dt} = y(b(1 - \frac{y}{x+k}) - \bar{E} + \frac{E_e py}{py_e - c}), \\ 0 = (\bar{E} - \frac{E_e py}{py_e - c})(py - c) - v. \end{cases} \tag{23}$$

We denote also

$$f(v, \bar{X}) \begin{pmatrix} f_1(v, \bar{X}) \\ f_2(v, \bar{X}) \end{pmatrix} = \begin{pmatrix} x(1 - x - \frac{ay}{\alpha x + \beta y + \gamma}) \\ y(b(1 - \frac{y}{x+k}) - \bar{E} + \frac{E_e py}{py_e - c}) \end{pmatrix},$$

$$g(v, \bar{X}) = (\bar{E} - \frac{E_e py}{py_e - c})(py - c) - v, \quad \bar{X} = (x, y, \bar{E})^T,$$

and

$$D_X g(\bar{X}_e)Q = (0, 0, py_e - c).$$

For system (23), we consider the following local parametrization:

$$\bar{X} = \phi(v, Y) = \bar{X}_e + U_0 Y + v_0 h(v, Y), \quad g(v, \phi(v, Y)) = 0.$$

Here, $Y = (y_1, y_2)$, $U_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, $V_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a smooth mapping. More information about the local parametrization can be found in [4,8]. Then we can deduce that the parametric system of (23) takes the form

$$\begin{cases} \dot{y_1} = \frac{dy_1}{dt} = f_1(v, \phi(v, Y)), \\ \dot{y_2} = \frac{dy_2}{dt} = f_2(v, \phi(v, Y)). \end{cases} \tag{24}$$

Consequently, the Jacobian matrix $A(v)$ of the parametric system (24) at $Y = 0$ takes the form

$$\begin{aligned} A(v) &= \begin{pmatrix} D_{y_1} f_1(v, \phi(v, Y)) & D_{y_2} f_1(v, \phi(v, Y)) \\ D_{y_1} f_2(v, \phi(v, Y)) & D_{y_2} f_2(v, \phi(v, Y)) \end{pmatrix}, \\ &= \begin{pmatrix} D_{\bar{X}} f_1(v, \bar{X}_e) \\ D_{\bar{X}} f_2(v, \bar{X}_e) \end{pmatrix} \begin{pmatrix} D_{\bar{X}} g(v, \bar{X}_e) \\ U_0^T \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ I_2 \end{pmatrix}, \\ &= \begin{pmatrix} D_x f_1(v, \bar{X}_e(v)) & D_y f_1(v, \bar{X}_e(v)) \\ D_x f_2(v, \bar{X}_e(v)) & D_y f_2(v, \bar{X}_e(v)) \end{pmatrix}, \\ &= \begin{pmatrix} x_e(-1 + \frac{a\alpha y_e}{(\alpha x_e + \beta y_e + \gamma)^2}) & -\frac{a x_e(\alpha x_e + \gamma)}{(\alpha x_e + \beta y_e + \gamma)^2} \\ \frac{b y_e^2}{(x_e + k)^2} & y_e(\frac{-b}{x_e + k} + \frac{pE_e}{py_e - c}) \end{pmatrix}. \end{aligned}$$

Therefore, the characteristic equation of the matrix $A(v)$ can be expressed as

$$\lambda^2 + T_1 \lambda + T_2 = 0, \tag{25}$$

where

$$T_1 = x_e(1 - \frac{a\alpha y_e}{(\alpha x_e + \beta y_e + \gamma)^2}) + y_e(\frac{b}{x_e + k} - \frac{pE_e}{py_e - c}),$$

$$T_2 = x_e y_e(1 - \frac{a\alpha y_e}{(\alpha x_e + \beta y_e + \gamma)^2})(\frac{-b}{x_e + k} + \frac{pE_e}{py_e - c}) + \frac{aby_e^2 x_e(\alpha x_e + \gamma)}{(x_e + k)^2(\alpha x_e + \beta y_e + \gamma)^2}.$$

Remark 5.1 *The positive equilibrium point $\bar{X}_e$ of the system (5) corresponds to the equilibrium point $Y = 0$ of system (24).*

Corollary 5.1 *For the positive equilibrium point $\bar{X}_e$ of the system (23), we have*
(i) If $T_1^2(v) \geq 4T_2(v)$ and $T_2(v) > 0$, then when $T_1(v) > 0$, $\bar{X}_e$ is a locally asymptotically stable node. When $T_1(v) < 0$, $\bar{X}_e$ is an unstable node.
(ii) If $T_2(v) < 0$, then $\bar{X}_e$ is an unstable saddle point.
(iii) If $T_1^2(v) < 4T_2(v)$, then when $T_1(v) > 0$, $\bar{X}_e$ is a locally asymptotically focus. When $T_1(v) < 0$, $\bar{X}_e$ is an unstable focus.

5.2 Hopf bifurcation analysis

The Hopf bifurcation is a very interesting type of bifurcation of systems. It refers to the local birth or death of a periodic solution from an equilibrium point as a parameter crosses a critical value named a bifurcation value.

We discuss the Hopf bifurcation in the system (23) from the equilibrium point $\bar{X}_e$ by considering the economic profit v as a bifurcation value.
If $T_1^2(v) < 4T_2(v)$, then the equation (25) has a pair of conjugate complex roots

$$\lambda_{1,2} = -\frac{1}{2}T_1(v) \pm i\sqrt{T_2(v) - \frac{T_1^2(v)}{4}} = \eta(v) \pm i\theta(v).$$

Let $2\eta(v) = T_1(v) = 0$, we get the bifurcation value v^* that satisfies

$$v^* = \frac{(py_e - c)^2}{p}(\frac{x_e}{y_e}(1 - \frac{a\alpha y_e}{(\alpha x_e + \beta y_e + \gamma)^2}) + \frac{b}{x_e + k})$$

if

$$\frac{a\alpha y_e}{(\alpha x_e + \beta y_e + \gamma)^2} = 1.$$

Moreover,

$$\eta(v^*) = 0, \quad \theta^* = \theta(v^*) = \frac{y_e}{(x_e + k)(\alpha x_e + \beta y_e + \gamma)}\sqrt{abx_e(\alpha x_e + \gamma)},$$

$$v^* = \frac{b(px_e - c)^2}{p(x_e + k)},$$

which implies that if

$$\eta^{'}(v^*) = \frac{1}{2}\frac{d}{dv}(x_e(1 - \frac{a\alpha y_e}{(\alpha x_e + \beta y_e + \gamma)^2}) + y_e(\frac{b}{x_e + k} - \frac{pv_e}{(py_e - c)^2})_{v=v^*} =$$

$$= -\frac{py_e}{2(py_e - c)^2} \neq 0,$$

then the Hopf bifurcation occurs at the value v^*. The signal of the number σ is given by

$$16\sigma = \frac{1}{\theta^*}(a^1_{11}(a^2_{11} - a^1_{12}) + a^2_{22}(a^2_{12} - a^1_{22}) + (a^2_{11}a^2_{12} - a^1_{12}a^1_{22})) + (a^1_{111} + a^1_{122} + a^2_{112}a^2_{222}),$$

where
$a^1_{11} = \tau_1 f_{1y_1y_1}$, $a^1_{12} = \tau_2 f_{1y_1y_2}$, $a^1_{22} = \frac{\tau_2^2}{\tau_1} f_{1y_2y_2}$, $a^1_{111} = \tau_1^2 f_{1y_1y_1y_1}$, $a^1_{112} = \tau_1\tau_2 f_{1y_1y_1y_2}$,
$a^1_{122} = \tau_2^2 f_{1y_1y_2y_2}$, $a^1_{222} = \frac{\tau_2^3}{\tau_1} f_{1y_2y_2y_2}$.
$a^2_{11} = \frac{\tau_1^2}{\tau_2} f_{2y_1y_1}$, $a^2_{12} = \tau_1 f_{2y_1y_2}$, $a^2_{22} = \tau_2 f_{2y_2y_2}$, $a^2_{111} = \frac{\tau_1^3}{\tau_2} f_{2y_1y_1y_1}$, $a^2_{112} = \tau_1^2 f_{2y_1y_1y_2}$,
$a^2_{122} = \tau_1\tau_2 f_{2y_1y_2y_2}$, $a^2_{222} = \tau_2^2 f_{2y_2y_2y_2}$,
which determines the direction of the Hopf bifurcation through the interior equilibrium $X_e(v)$ of the system (2.5), as stated in the following theorem.

Theorem 5.1 *For the system (5), there exist a positive constant $0 < \varepsilon \ll 1$ and two small neighborhoods of the positive equilibrium point $X_e(v)$: O_1 and O_2, where $O_1 \subset O_2$.*

Case 1 : *If $\sigma > 0$, then*

1. *When $v^* < v < v^* + \varepsilon$, $X_e(v)$ rejects all the points in O_2, so it is unstable.*
2. *When $v^* - \varepsilon < v < v^*$, sytem (5) has at least a periodic solution located in $\bar{O}_1$ (the cloture of O_1), one of them rejects all the points in $\bar{O}_1 \setminus X_e(v)$, at the same time another periodic solution (may be the same one) rejects all points in $O_2 \setminus \bar{O}_1$, and $X_e(v)$ is locally asympototic stable.*

Case 2 : *If $\sigma < 0$, then*

1. *When $v^* - \varepsilon < v < v^*$, $X_e(v)$ attracts all the points in O_2, and $X_e(v)$is locally asymptotic stable.*
2. *When $v^* < v < v^* - \varepsilon$, system (5) has at least a periodic solution located in $\bar{O}_1$, one of them attracts all the points in $\bar{O}_1 \setminus X_e(v)$, at the same time another periodic solution (may be the same one) attracts all points in $O_2 \setminus \bar{O}_1$, and $X_e(v)$ is unstable.*

For the proof see [19], where we use
$f_{1y_1}(v^*, \bar{X}_e) = 0$, $f_{2y_2}(v^*, \bar{X}_e) = 0$, $f_{1y_2}(v^*, \bar{X}_e) =, \quad -\frac{ax_e(\alpha x_e+\gamma)}{(\alpha x_e+\beta y_e+\gamma)^2}$,
$f_{2y_1}(v^*, \bar{X}_e) = \frac{by_e^2}{(x_e+k)^2}$, $f_{1y_1y_1}(v^*, \bar{X}_e) = -2 + 2\frac{a\alpha^2 y_e(\beta y_e+\gamma)}{(\alpha x_e+\beta y_e+\gamma)^3}$,
$f_{1y_1y_2}(v^*, \bar{X}_e) = f_{1y_2y_1}(v^*, \bar{X}_e) = \frac{-a\gamma(\alpha x_e+\beta y_e+\gamma)-2a\alpha\beta x_e y_e}{(\alpha x_e+\beta y_e+\gamma)^3}$,
$f_{1y_2y_2}(v^*, \bar{X}_e) = \frac{2a\beta x_e(\alpha x_e+\gamma)}{(\alpha x_e+\beta y_e+\gamma)^3}$, $f_{2y_1y_1}(v^*, \bar{X}_e) = \frac{-2by_e^2}{(x_e+k)^3}$,
$f_{2y_1y_2}(v^*, \bar{X}_e) = f_{2y_2y_1}(v^*, \bar{X}_e) = \frac{2by_e}{(x_e+k)^2}$, $f_{2y_2y_2}(v^*, \bar{X}_e) = -2\frac{by_e}{(x_e+k)(py_e-c)}$,
$f_{1y_1y_1y_1}(v^*, \bar{X}_e) = \frac{-4a\alpha^2 y_e(\beta y_e+\gamma)+2a\alpha^3 x_e y_e}{(\alpha x_e+\beta y_e+\gamma)^4}$,
$f_{1y_1y_1y_2}(v^*, \bar{X}_e) = f_{1y_1y_2y_1}(v^*, \bar{X}_e) = f_{1y_2y_1y_1}(v^*, \bar{X}_e) = \frac{2a\alpha(-\beta y_e+\gamma)(\alpha x_e+\beta y_e+\gamma)+6a\alpha^2\beta x_e y_e}{(\alpha x_e+\beta y_e+\gamma)^4}$,
$f_{1y_1y_2y_2}(v^*, \bar{X}_e) = f_{1y_2y_2y_1}(v^*, \bar{X}_e) =$
$= f_{1y_2y_1y_2}(v^*, \bar{X}_e) = \frac{2a\beta(2\alpha x_e+\gamma)(\alpha x_e+\beta y_e+\gamma)-6a\alpha\beta x_e(\alpha x_e+\gamma)}{(\alpha x_e+\beta y_e+\gamma)^4}$,
$f_{1y_2y_2y_2}(v^*, \bar{X}_e) = \frac{-6\alpha\beta^2 x_e(\alpha x_e+\gamma)}{(\alpha x_e+\beta y_e+\gamma)^4}$, $\quad f_{2y_1y_1y_1}(v^*, \bar{X}_e) = \frac{6by_e^2}{(x_e+k)^4}$,

$f_{2y_1y_1y_2}(v^*,\bar{X}_e)=f_{2y_1y_2y_1}(v^*,\bar{X}_e)=f_{2y_2y_1y_1}(v^*,\bar{X}_e)=\frac{-4by_e}{(x_e+k)^3}$,
$f_{2y_1y_2y_2}(v^*,\bar{X}_e)=f_{2y_2y_2y_1}(v^*,\bar{X}_e)=f_{2y_2y_1y_2}(v^*,\bar{X}_e)=\frac{2b}{(x_e+k)^2}$,
$f_{2y_2y_2y_2}(v^*,\bar{X}_e)=\frac{6p^2cE_e}{(py_e-c)^3}$.
and
$\tau_1=\frac{\sqrt{ax_e(\alpha x_e+\gamma)}}{\alpha x_e+\beta y_e+\gamma}$, $\tau_2=-\frac{y_e\sqrt{b}}{x_e+k}$.

6 Conclusion

This paper examines the stability and the Hopf bifurcation of a differential-algebraic biological economic system with a hybrid functional response. A dynamical investigation of a predator-prey model with a hybrid functional response equiped with an algebraic equation has never been done. We consider the system's dynamic behavior when only the prey is vulnerable to harvesting. Only the positive equilibrium points are of interest from a biological standpoint. By examining their associated characteristic equation and applying the new normal form theorem, the local stability of the inner equilibrium is determined. When the economic revenue v is changed, the inner equilibrium's stability property changes. Additionally, a one-parameter bifurcation analysis of the economic revenue is performed. When the system bifurcates, the properties of periodic solutions in the system are obtained by computing the parameter σ. The qualitative analysis, that is, the foundation of the revised model will be completed by future research. Additionally, it will include the numerical simulations used to support the outcomes.

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