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Synchronization of the Restricted Charged Three-Body Problem

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Abstract: This paper deals with the chaos and synchronization behavior of two identical nonlinear dynamical systems of the restricted charged three-body problem. An active control technique is introduced to achieve synchronization between the drive and response systems. Also, an error dynamical system of the drive and response systems has been investigated using active control inputs. Secondly, the Lyapunov theorem on stability and the Routh-Hurwitz criteria have been taken into account for the study of stability of the error dynamical system. Further, a six degree coefficient matrix of the error dynamical system has been investigated. We have concluded by the Lyapunov stability criteria, the error dynamical system is stable. Numerical simulation is taken into account to check the effectiveness of the proposed active control technique.

Keywords: *restricted charged three-body problem; synchronization; Lyapunov stability; Routh-Hurwitz criteria*

Mathematics Subject Classification (2010): 34H10, 37N35, 93C10, 93C15, 93C95.

1 Introduction

The basic theory of chaos and synchronization has a very powerful application in the real world. There are various dynamical systems which have real life applications. There is an opportunity of doing well research and obtaining some new information about real life from the study of the solar dynamical system. There are many forces acting between the celestial bodies of a solar dynamical system. There exist many perturbations such as radiation pressure, oblateness, the Coriolis and centrifugal forces and drag force between the solar system bodies. These perturbations can make new contributions in the study

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of chaos and synchronization of solar system bodies. The basic idea of synchronization between two dynamical systems is one of the important phenomena occurring in nature.

The synchronization of chaotic systems have been studied by many authors and researchers. They have introduced the basic concept and analytical approach to study the chaotic behavior of dynamical systems. Also, they have incorporated some techniques of controlling chaos and studying synchronization in the chaotic systems. Synchronization has been studied by many authors for different physical systems, namely, chaotic systems, the fractional L$\ddot{u}$ system, coupled chaotic systems, between two different chaotic systems via nonlinear feedback control, two Lorenz systems using active control and the Rossler and Chen chaotic dynamical systems [1-6].

In continuation of research on synchronization, chaos synchronization, anti-synchronization, hybrid synchronization, synchronization of the finance chaotic system, synchronization of fractional-order systems, projective synchronization and function projective dual synchronization of chaotic systems, adaptive synchronization, sliding mode control synchronization and adaptive sliding mode control synchronization have been studied by many authors [7-13]. Moreover, the restricted charged three-body problem, low-thrust restricted three-body problem, chaos synchronization in the restricted three-body problem have been studied by many authors [14-18]. In addition, synchronization of fractional-order 3D chaotic system analysis has been incorporated in [19].

In this paper, we have studied the synchronization behavior of two identical systems of the restricted charged three-body problem using the active control technique. This paper is arranged as follows. Section 1 discussed the introductory part of the paper. In Section 2, we have derived the equations of motion of the restricted charged three-body problem. In Section 3, we have used the active control technique for synchronization. In Section 4, we have discussed the numerical simulation for synchronization. Finally, in Section 5, we have concluded the results obtained.

2 Equations of Motion

Let two charged bodies $M_1 = (m_1, q_1)$ and $M_2 = (m_2, q_2)$ with masses m_1 and m_2 ($m_1 > m_2$) and charges q_1, q_2 be moving with angular velocity ω in circular orbits about their center of mass O taken as an origin, and let the third charged infinitesimal body M of mass m_3 be moving in the plane of motion of m_1 and m_2 (see Fig. 1). The motion of the third charged infinitesimal body is affected by the motion of m_1 and m_2 but does not affect them. We shall determine the equations of motion of the charged infinitesimal body of mass m_3 in synodic and dimensionless variables. The angular velocity of the primaries is given by the relation $\omega = \sqrt{\frac{G(m_1+m_2)}{l^3}}$, where l is the distance between the primaries, and G is the gravitational constant. We scale the units by taking the sum of the masses and the distance between the primaries both equal to unity. Therefore $m_1 = 1 - \mu$, $m_2 = \mu\,(0 < \mu \leq 0.5)$ and $\mu = \frac{m_2}{m_1+m_2}$ with $m_1 + m_2 = 1$. Also, the scale of the time is chosen so that the gravitational constant is unity. For the classical case, $q_1 = q_2 = 0$, only the range $0 \leq \mu \leq 0.5$ is of interest since the range $0 < \mu \leq 1$ is the reflection of the previous one with respect to the y-axis. Let us assume that $0 < \mu \leq 0.5$, continuing, we may define $\frac{q_1}{q_1+q_2} = q_1 = 1 - \mu$, $\frac{q_2}{q_1+q_2} = q_2 = \mu$, where $q_1 + q_2 = 1$. The equations of motion of the charged infinitesimal body in the dimensionless co-ordinate system according to [14] can be written as

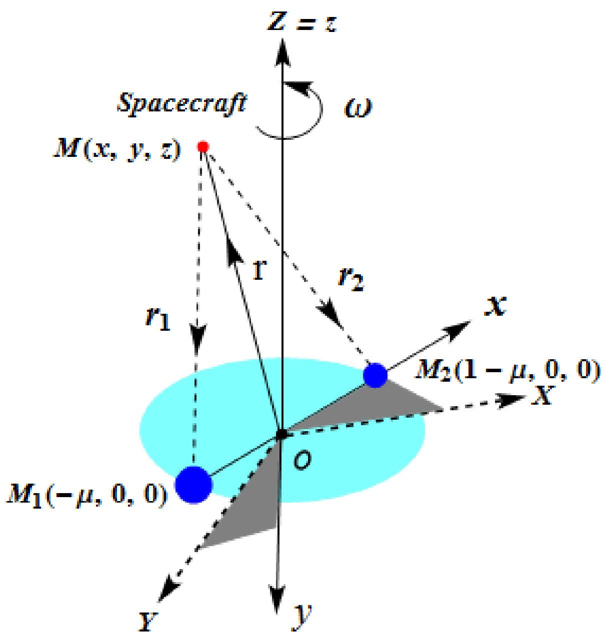

Figure 1: Configuration of the restricted charged three-body problem.

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\left.
\begin{aligned}
\ddot{x} - 2\,\dot{y} &= x - \frac{q-\mu}{r_1^3}(x+\mu) \\
&\quad - \frac{\mu-q}{r_2^3}(x+\mu-1), \\
\ddot{y} + 2\,\dot{x} &= y - \frac{q-\mu}{r_1^3}\,y - \frac{\mu-q}{r_2^3}\,y, \\
\ddot{z} &= -\frac{q-\mu}{r_1^3}\,z - \frac{\mu-q}{r_2^3}\,z,
\end{aligned}
\right\} \tag{1}
$$

where

$$r_1 = \sqrt{(x+\mu)^2 + y^2 + z^2}$$

and

$$r_2 = \sqrt{(x+\mu-1)^2 + y^2 + z^2}.$$

3 Synchronization of the Restricted Charged Three-Body Problem

In this section, we have introduced an active control method to study the synchronization behavior of two identical systems of the restricted charged three-body problem. We have introduced a dynamical system of the restricted charged three-body problem of the solar system bodies. We have formulated the master and slave systems of the solar system bodies with the help of the restricted charged three-body problem. The master system

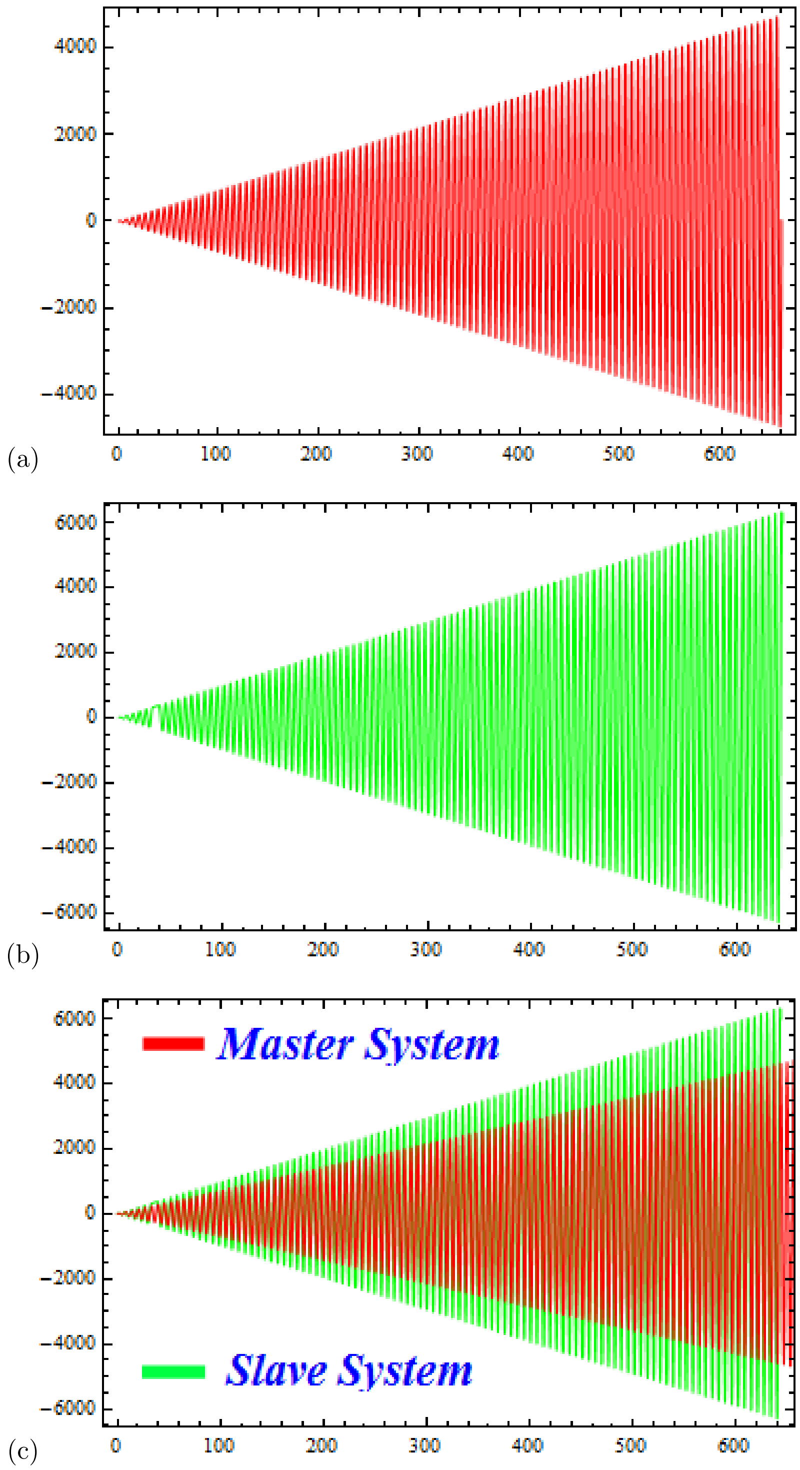

Figure 2: Time series graphs of the master and slave systems for $\mu = 0.05$, $q = 0.15$, (a) For the master system of $x_1(t)$, (b) For the slave system of $y_1(t)$, (c) Combined graphs of the master and slave systems of $x_1(t)$ and $y_1(t)$.

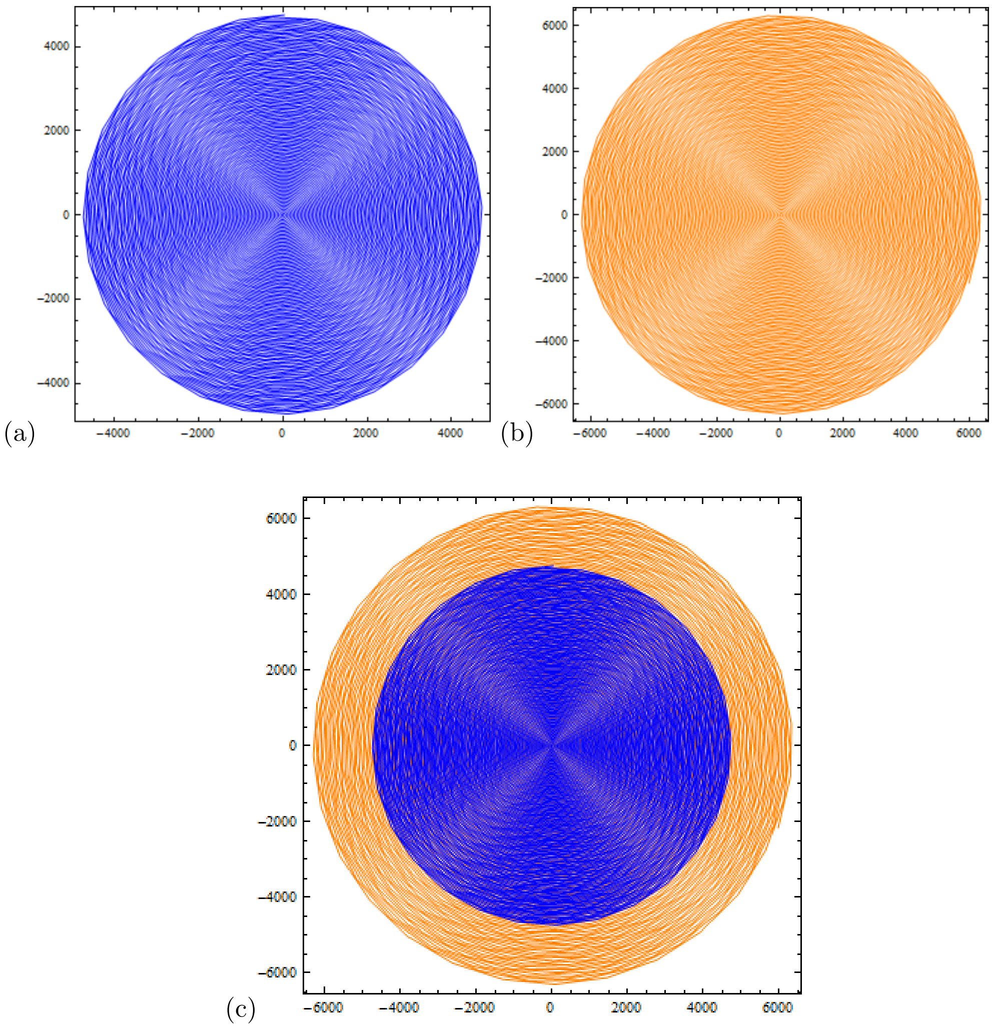

Figure 3: Phase portrait graphs of the master and slave systems for $\mu = 0.05$, $q = 0.15$, (a) For the master system of $(x_1(t), x_2(t))$, (b) For the slave system of $(y_1(t), y_2(t))$, and (c) Combined graphs of the master and slave systems of $(x_1(t), x_2(t))$ and $(y_1(t), y_2(t))$.

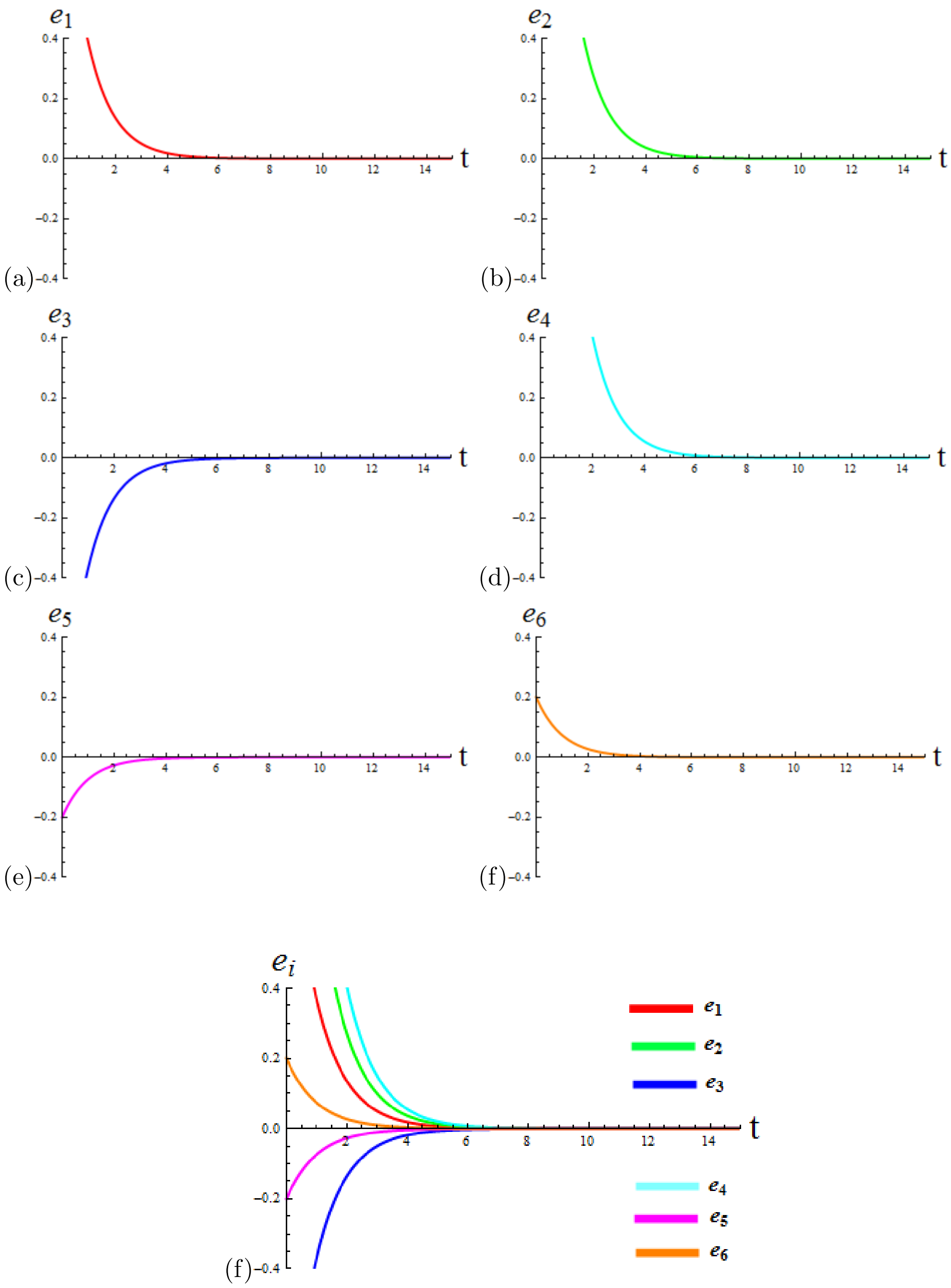

Figure 4: The convergence of the error dynamical system, (a) for $e_1(t)$, (b) for $e_2(t)$, (c) for $e_3(t)$, (d) for $e_4(t)$, (e) for $e_5(t)$, (f) for $e_6(t)$, and (g) the combined convergence of errors $e_1(t)$, $e_2(t)$, $e_3(t)$, $e_4(t)$, $e_5(t)$, $e_6(t)$.

for the restricted charged three-body problem is defined by $\dot{x} = f(x, y, z)$ and the slave system $\dot{y} = g(x, y, z)$, where $x(t), y(t)$ and $z(t)$ are the phase space (state variables), and $\dot{x} = f(x, y, z)$, $\dot{y} = g(x, y, z)$, and $\dot{z} = h(x, y, z)$ are the corresponding nonlinear functions. We want to study the synchronization behavior of the master and slave systems for the restricted charged three-body problem.

Synchronization is defined as a process in which two or more systems interact with each other. We can obtain a combined effect of the dynamical properties using the synchronization phenomenon. Mathematically, we can define the synchronization by the expression $|x(t)-y(t)| \to 0$ as $t \to 0$. When the above expression holds, then the systems of six equations are said to be completely synchronized. According to the control theory of synchronization, a dynamical system depends on the design of control laws for the slave system using the known information of the master system so as to ensure that the controlled receiver synchronizes with the master system. Therefore, the slave chaotic system completely investigates the dynamics of the master system with respect to time. The model of the restricted charged three-body problem defined by Eq. (1) can be written as a system of six first-order differential equations. We have introduced six variables such as $x_1 = x$, $x_2 = \dot{x}$, $x_3 = y$, $x_4 = \dot{y}$, $x_5 = z$ and $x_6 = \dot{z}$. Therefore, the master system of the Eq.(1) is defined as

$$\left.\begin{aligned} \dot{x_1} &= x_2, \\ \dot{x_2} &= x_1 + 2\,x_4 - \left(\frac{q-\mu}{r_1^3}(x_1+\mu) + \frac{\mu-q}{r_2^3}(x_1+\mu-1)\right), \\ \dot{x_3} &= x_4, \\ \dot{x_4} &= x_3 - 2\,x_2 - \left(\frac{q-\mu}{r_1^3} + \frac{\mu-q}{r_2^3}\right) x_3, \\ \dot{x_5} &= x_6, \\ \dot{x_6} &= -\left(\frac{q-\mu}{r_1^3} + \frac{\mu-q}{r_2^3}\right) x_5, \end{aligned}\right\} \tag{2}$$

where $r_1^2 = (x_1+\mu)^2 + x_3^2 + x_5^2$ and $r_2^2 = (x_1+\mu-1)^2 + x_3^2 + x_5^2$. We can write the identical slave system of six first-order differential equations corresponding to Eq.(2) as

$$\left.\begin{aligned} \dot{y_1} &= y_2 + u_1(t), \\ \dot{y_2} &= y_1 + 2\,y_4 - \frac{q-\mu}{d_1^3}(y_1+\mu) - \frac{\mu-q}{d_2^3}(y_1+\mu-1) + u_2(t), \\ \dot{y_3} &= y_4 + u_3(t), \\ \dot{y_4} &= y_3 - 2\,y_2 - \left(\frac{q-\mu}{d_1^3} + \frac{\mu-q}{d_2^3}\right) y_3 + u_4(t), \\ \dot{y_5} &= y_6 + u_5(t), \\ \dot{y_6} &= -\left(\frac{q-\mu}{d_1^3} + \frac{\mu-q}{d_2^3}\right) y_5 + u_6(t) \end{aligned}\right\} \tag{3}$$

with $d_1^2 = (y_1+\mu)^2 + y_3^2 + y_5^2$ and $d_2^2 = (y_1+\mu-1)^2 + y_3^2 + y_5^2$, where $u_i(t) = 1, 2, 3, 4, 5, 6$ are called the control functions. We can define the error functions in such a way that in the synchronization state

$$\lim_{t\to\infty} e_i(t) \to 0, i = 1, 2, 3, 4, 5, 6,$$

$$\begin{aligned} e_1 &= y_1 - x_1, \\ e_2 &= y_2 - x_2, \\ e_3 &= y_3 - x_3, \\ e_4 &= y_4 - x_4, \\ e_5 &= y_5 - x_5, \\ e_6 &= y_6 - x_6. \end{aligned}$$

We can transform the above error dynamical system in the derivative form of the input variables. After taking the derivative of the above error dynamical system, a new form can be written as

$$\left.\begin{aligned} \dot{e_1} &= \dot{y_1} - \dot{x}_1, \\ \dot{e_2} &= \dot{y_2} - \dot{x}_2, \\ \dot{e_3} &= \dot{y_3} - \dot{x}_3, \\ \dot{e_4} &= \dot{y_4} - \dot{x}_4, \\ \dot{e_5} &= \dot{y_5} - \dot{x}_5, \\ \dot{e_6} &= \dot{y_6} - \dot{x}_6. \end{aligned}\right\} \tag{4}$$

In the error dynamical system of Eqs. (4), the dot over e_i, x_i, y_i, $i = 1, 2, 3, 4, 5, 6$ indicates the differentiation with respect to time. After using Eqs. (2), (3) and (4), the above error dynamical system can be transformed into a new form and can be written as

$$\left.\begin{aligned} \dot{e_1}(t) &= e_2(t) + u_1(t), \\ \dot{e_2}(t) &= e_1(t) + 2\,e_4(t) - e_1(t)\left(\frac{q-\mu}{r_1^3} + \frac{\mu-q}{r_2^3}\right) + u_2(t), \\ \dot{e_3}(t) &= e_4(t) + u_3(t), \\ \dot{e_4}(t) &= e_3(t) - 2\,e_2(t) - e_3(t)\left(\frac{q-\mu}{r_1^3} + \frac{\mu-q}{r_2^3}\right) + u_4(t), \\ \dot{e_5}(t) &= e_6(t) + u_5(t), \\ \dot{e_6}(t) &= -\,e_5(t)\left(\frac{q-\mu}{r_1^3} + \frac{\mu-q}{r_2^3}\right) + u_6(t). \end{aligned}\right\} \tag{5}$$

In Eq.(5), we have introduced the input variables to study the behavior of a dynamical system. The error dynamical system defined by Eq. (5) to be controlled must be a linear system with control inputs. Hence, we have incorporated the control functions after eliminating the non-linear terms in $e_1(t)$, $e_2(t)$, $e_3(t)$, $e_4(t)$, $e_5(t)$ and $e_6(t)$ of equation (5) as given below

$$\left.\begin{aligned} u_1(t) &= v_1(t), \\ u_2(t) &= e_1(t)\left(\frac{q-\mu}{r_1^3} + \frac{\mu-q}{r_2^3}\right) + v_2(t), \\ u_3(t) &= v_3(t), \\ u_4(t) &= e_3(t)\left(\frac{q-\mu}{r_1^3} + \frac{\mu-q}{r_2^3}\right) + v_4(t), \\ u_5(t) &= v_5(t), \\ u_6(t) &= e_5(t)\left(\frac{q-\mu}{r_1^3} + \frac{\mu-q}{r_2^3}\right) + v_6(t). \end{aligned}\right\} \tag{6}$$

The control inputs have been introduced to study the behavior of a dynamical system. We can take one or more input variables to control a dynamical system, which depend on our choice. We want to control an output of a dynamical system. We can obtain the desired output of a dynamical system using these control inputs. There are two control inputs which are used in Eqs. (5) and (6). Using Eqs. (5) and (6), we can write the linear error dynamical system as given below

$$\left.\begin{aligned} \dot{e_1}(t) &= e_2(t) + v_1(t),\\ \dot{e_2}(t) &= e_1(t) + 2\,e_4(t) + v_2(t),\\ \dot{e_3}(t) &= e_4(t) + v_3(t),\\ \dot{e_4}(t) &= e_3(t) - 2\,e_2(t) + v_4(t),\\ \dot{e_5}(t) &= e_6(t) + v_5(t),\\ \dot{e_6}(t) &= v_6(t). \end{aligned}\right\} \tag{7}$$

Equation (7) represents the error dynamical system with new control inputs. The formulated Eq. (7) is the error dynamics, which can be interpreted as a control problem where the system to be controlled is a linear system with control inputs $v_1(t)$, $v_2(t)$, $v_3(t)$, $v_4(t)$, $v_5(t)$ and $v_6(t)$. We have introduced some new active control variables $v_1(t)$, $v_2(t)$, $v_3(t)$, $v_4(t)$, $v_5(t)$ and $v_6(t)$ which are given by the relation

$$\left.\begin{pmatrix} v_1(t)\\ v_2(t)\\ v_3(t)\\ v_4(t)\\ v_5(t)\\ v_6(t) \end{pmatrix} = M \begin{pmatrix} e_1(t)\\ e_2(t)\\ e_3(t)\\ e_4(t)\\ e_5(t)\\ e_6(t) \end{pmatrix}\right\}. \tag{8}$$

In Eq. (8), M is a 6×6 constant matrix to be determined. The error dynamical system (7) can be re-written as given below

$$\left.\begin{pmatrix} \dot{e_1}(t)\\ \dot{e_2}(t)\\ \dot{e_3}(t)\\ \dot{e_4}(t)\\ \dot{e_4}(t)\\ \dot{e_6}(t) \end{pmatrix} = N \begin{pmatrix} e_1(t)\\ e_2(t)\\ e_3(t)\\ e_4(t)\\ e_5(t)\\ e_6(t) \end{pmatrix}\right\}. \tag{9}$$

In equation (9), N is a 6×6 coefficient matrix. According to the Lyapunov stability theory and Routh-Hurwitz criteria, the eigenvalues of the coefficient matrix of the error system must be real or complex with negative real parts. We can choose the elements of the matrix arbitrarily, there are several ways to choose in order to satisfy the Lyapunov and Routh-Hurwitz criteria. Therefore, the matrix corresponding to Eq. (7) can be defined as

$$M = \begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 0\\ -1 & -1 & 0 & -2 & 0 & 0\\ 0 & 0 & -1 & -1 & 0 & 0\\ 0 & 2 & -1 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & -1\\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

and the matrix corresponding to Eq. (9) given by

$$N = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{10}$$

becomes a matrix with the eigenvalues having negative real parts. After using Eqs. (9) and (10), we have obtained an expression which is given below

$$\left.\begin{aligned} \dot{e_1}(t) &= -e_1(t) \\ \dot{e_2}(t) &= -e_2(t) \\ \dot{e_3}(t) &= -e_3(t) \\ \dot{e_4}(t) &= -e_4(t) \\ \dot{e_5}(t) &= -e_5(t) \\ \dot{e_6}(t) &= -e_6(t). \end{aligned}\right\} \tag{11}$$

The system of Eqs.(11) indicates an equation of the error dynamical system of the restricted charged three-body problem. We shall study the stability of the above error dynamical system given by Eq.(11). In order to study the stability of the error dynamical system, we shall determine the solution of Eq.(11) using the Lyapunov stability criteria. It is concluded by the Lyapunov stability theory, the above error dynamical system is stable.

4 Analysis of Numerical Simulation

We have introduced two parameters, namely, the mass ratio μ and the charge q of the second primary. We shall discuss the effects of these two parameters μ and q. We have taken the numerical values of the given parameters and the initial conditions for studying simulation. For the parameters included in the system under investigation $\mu = 0.05$, $q = 0.15$ and with the initial conditions for the master and slave systems $[x_1(0), x_2(0), x_3(0), x_4(0), x_5(0), x_6(0)] = [3.5, -4.75, -2.5, 3.35, 0.85, 0.75]$ and $[y_1(0), y_2(0), y_3(0), y_4(0), y_5(0), y_6(0)] = [4.5, -2.75, -3.5, 6.35, 0.65, 0.95]$, respectively. We have simulated the system under consideration using mathematica. The phase portraits and time series analysis of the master and slave systems show the irregular behavior of the dynamical system (see Figures 2 and 3). And for $[e_1(0), e_2(0), e_3(0), e_4(0), e_5(0), e_6(0)] = [1, 2, -1, 3, -0.2, 0.2]$, the convergence diagrams of errors are the proof of achieving synchronization between the master and slave systems (see Figure 4).

5 Conclusion

In this paper, we have studied the synchronization behavior of two identical nonlinear systems of the restricted charged three-body problem using different initial conditions through active control technique depending on the Lyapunov stability theory and the

Routh-Hurwitz criteria. We have observed that the master and slave systems of the restricted charged three-body problem are completely synchronized. Also, we have observed that the error propagation between the master and slave systems of the restricted charged three-body problem is tending to zero. The obtained results were certified by numerical simulations using the latest version 12.0 of Wolfram Mathematica®. In this paper, the present work is applicable for the study of some perturbations on the artificial satellite in space. This paper is applicable in various astrophysical systems. There are three examples such as Sun-Earth-Satellite system, Sun-Jupiter-Satellite system and Earth-Moon-Satellite system.

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