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Properties of Solutions and Stability of a Diffusive Wage-Employment System

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Abstract: This paper focuses on the analysis of the properties of solutions and stability of a diffusive wage-employment system. The system is of a diffusive predator-prey type. By choosing appropriate parameters, the global existence, positivity, uniform boundedness and decay estimates of solutions of the system can be characterized. The stability of the system can be also justified.

Keywords: *wage-employment system; global existence; positivity; predator-prey; stability.*

Mathematics Subject Classification (2010): 35A01, 35A02, 35B0.

1 Introduction

Goodwin [1] has constructed a model of the dynamic relationship between wage and employment. The model incorporates three behaviors of economic systems (the market is in a stable equilibrium, the growth is cyclical and its equilibrium is affected by past changes, the economic relations resemble white noise and the economic motion is random [2]). Goodwin's model is analogous to the Lotka-Volterra predator-prey model, the wage and the employment correspond to the predator and the prey, respectively. The model forms a cyclical pattern. When the employment is at high level, the bargaining power of the employed workers drives up the wages, and so shrinks profits. But when the profits diminish, fewer workers are hired and the employment will decrease leading to the increment of the profits. The more profits, the more workers are hired leading to the increment of the employment.

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Let σ be the capital intensity showing how many years of income have to be tied up to produce a unit of income. The reciprocal value $1/\sigma$ measures the capital productivity, which determines the amount of national income generated by each unit of the invested capital. Goodwin assumes that the labor supply and the labor productivity are exponential processes, which means that the growth rates of both are constants. Let α and β denote the percentage per year of the rise of the labor productivity and the labor supply, respectively.

The change of the real wage (workers' share) depends on the real employment. The rates of both are positively correlated. The employment fluctuates between fairly narrow limits in real life, so the interdependence in a small neighborhood of the equilibrium is permitted to linearize by the linear Phillips curve. Let ρ and γ be the slope and the intercept of the linearization, respectively. Goodwin's real wage - real employment cycle is modeled by [3, 4]

$$\begin{aligned} \dot{u} &= -\eta_1 u + \theta_1 uv, \\ \dot{v} &= \eta_2 v - \theta_2 uv, \end{aligned} \tag{1}$$

where u and v are the rate of real wage and the rate of real employment, respectively, $\eta_1 = \alpha + \gamma$, $\theta_1 = \rho$, $\eta_2 = \frac{1}{\sigma} - (\alpha + \beta)$ and $\theta_2 = \frac{1}{\sigma}$.

In the absence of wages u, the employment rate v in (1) grows exponentially without boundaries. In reality, the employment cannot increase without limits and decreases in productivity since additional workers will not be just as productive as the employed workers. To enhance the more realistic model, a logistic saturation is considered in the second equation in (1) at $u = 0$ by

$$\dot{v} = \eta_2 \left(1 - \frac{v}{K}\right) v.$$

Since the employment cannot surpass total population, the model requires $K = 1$.

The second problem with Goodwin's original model is the reaction of wages to employment since any changes in wages as a result of changes in employment cannot be instantaneous as they are assumed to be. Wage contracts planed ahead do not affect the changes of demand of labor in future, causing a delay in the reaction of wages to employment. This delay can be inserted in the first equation in (1) by

$$w(t) = (h * v)(t) := \int_0^t h(t-s)v(s)\, ds,$$

where h is a nonnegative integrable weight function such that $\int_0^\infty h(s)\, ds = 1$. One of the comfortable weight functions in the economic model is

$$h(t) = ae^{-at}, \quad a > 0. \tag{2}$$

On the other hand, the analysis of stability of equilibriums of the various types of the Lotka-Volterra model has been done. Moreover, many applications including forecasting have also been widely used even in finance and economics, see [5–9] and references therein. This paper focuses on studying the geographical expansion of the wage-employment interaction, as a generalization of (1) obeying the diffusive Lotka-Volterra system

$$\begin{aligned} u_t &= \delta_1 \Delta u - \eta_1 u + \theta_1 u(h * v), \\ v_t &= \delta_2 \Delta v + \eta_2 (1 - v) v - \theta_2 uv, \end{aligned} \tag{3}$$

subject to the initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \quad x \in \Omega, \tag{4}$$

where Δ is the Laplace operator, Ω is the bounded domain in $\mathbb{R}^n$, and δ_1, δ_2 are positive constants of the movement rates of wages and employment, respectively, and h is the weight function in (2). Some results on the dynamics, stability, bifurcation of solutions for the systems of reaction-diffusion systems were established in [10–14] not including the system (3)-(4). Recently, by quasi-semigroup and quasi-group approaches, time-dependent diffusion problems are also being done, see [15, 16].

This paper focuses on the global existence, positivity, and uniform boundedness of the solutions of the wage-employment system (3)-(4). Moreover, the stability of the equilibrium points of the systems is also analyzed.

2 Existence of Positive Solution

In what follows, we denote by X the space of bounded and uniformly continuous functions on $\Omega \subset \mathbb{R}$ endowed with the supremum norm. It is well-known that the linear operators $\delta_1\Delta$ and $\delta_2\Delta$ generate the semigroups of contractions T_1 and T_2 on the Banach space X given by

$$[T_i(t)w](x) = \frac{1}{\sqrt{4\pi\delta_i t}} \int_{-\infty}^{\infty} e^{-\frac{|x-s|^2}{4\delta_i t}} w(s)\, ds, \quad T_i(0) = I, \tag{5}$$

respectively, where I is the identity operator on X and $i = 1, 2$. Further, $u(t) = T_1(t)u_0$ and $v(t) = T_2(t)v_0$ are the unique solutions of

$$\begin{aligned} u_t &= \delta_1 \Delta u, \\ v_t &= \delta_2 \Delta v, \end{aligned} \tag{6}$$

subject to the initial conditions (4), respectively.

We also assume that the initial conditions u_0 and v_0 in (4) are the nonnegative elements of X. Following the scheme in [17], we shall show the existence of a global solution to the problem (3)–(4).

Theorem 2.1 *Let $u_0, v_0 \in X$ and $\eta_2 \geq 0$. There exists a unique global classical nonnegative solution (u, v) to the problem* (3)-(4).

Proof. Local existence and uniqueness follow from the solutions to (6) and the Duhamel principle; there exists a $\tau_0 > 0$ such that the problem (3)-(4) has a unique local mild solution $(u, v) \in C([0, \tau_0], X) \times C([0, \tau_0], X)$, i.e.,

$$\begin{aligned} u(t) &= T_1(t)u_0 + \int_0^t T_1(t-s) f(s)\, ds, \quad t \in [0, \tau_0], \\ v(t) &= T_2(t)v_0 + \int_0^t T_2(t-s) g(s)\, ds, \quad t \in [0, \tau_0], \end{aligned}$$

where $f(t) = -\eta_1 u(t) + \theta_1 u(t)(h * v)(t)$ and $g(t) = \eta_2(1 - v(t))v(t) - \theta_2 u(t)v(t)$ for all $t \in [0, \tau_0]$. Since $f, g \in C^\infty((0, \tau_0], X)$, the Lebesgue theory concludes that $u(t), v(t) \in C^\infty(\Omega, \mathbb{R})$ for all $t \in (0, \tau_{\max}]$, where $\tau_{\max} := \tau_0$ is the maximal time of existence of the solution (u, v).

Next, we prove the nonnegativity of the solutions. Let $\lambda_v = \sup\{\|(h*v)(t)\|; 0 \le t \le \tau\}$, where $0 < \tau < \tau_{max}$ and $\lambda_0 = -\eta_1 + \theta_1\lambda_v$. The substitution $u = e^{\lambda_0 t}\varphi$ to the first equation of (3) gives

$$\varphi_t - \delta_1\varphi_{xx} + (\eta_1 - \theta_1(h*v) + \lambda_0)\varphi \equiv 0, \quad x \in \Omega, \quad 0 < t \le \tau,$$

where $\varphi(x,0) = u_0(x)$. Since $v \in C([0,\tau], X)$ and $\eta_1 - \theta_1(h*v) + \lambda_0 \ge 0$, the maximum principle (Theorem 9 on page 43) in [18] implies that φ is nonnegative, which in turn gives the nonnegativity of u.

The substitution $v = e^{\lambda_0 x}\phi(t) + \psi(t)$ to the second equation of (3) yields two Bernoulli's equations in t,

$$\phi'(t) + (\theta_2 u - \lambda_0^2 - \eta_2)\phi(t) = -\eta_2 e^{\lambda_0 x}\phi^2(t), \quad x \in \Omega, \quad 0 < t \le \tau,$$
$$\psi'(t) + (\theta_2 u + 2\eta_2 e^{\lambda_0 x}\phi(t) - \eta_2)\psi(t) = -\eta_2\psi^2(t), \quad x \in \Omega, \quad 0 < t \le \tau.$$

A direct computation to these equations on $\Omega \times (0,\tau]$ gives a solution

$$v(x,t) = \begin{cases} e^{-\theta_2 ut}(e^{\lambda_0^2 t + \lambda_0 x} + 1), & \eta_2 = 0, \\ \dfrac{e^{-P(t)}}{\eta_2 \int e^{-P(t)}dt} + \dfrac{e^{-Q(t)}}{\eta_2 \int e^{-Q(t)}dt}, & \eta_2 > 0, \end{cases} \tag{7}$$

where

$$P(t) = \int (\theta_2 u - \lambda_0^2 - \eta_2)dt, \quad Q(t) = \int (\theta_2 u + 2w(t) - \eta_2)dt, \quad w(t) = \frac{e^{-P(t)}}{\int e^{-P(t)}dt}.$$

The condition $\eta_2 \ge 0$ implies the nonnegativity of v in (7). Thus, we just proved the existence of a priori bounds for the solution u, v on $[0, \tau_{max})$. From this, we shall prove the global solution of (u, v).

The solution of the problem (3)-(4) can be represented by

$$u(t) = e^{-\eta_1 t}T_1(t)u_0 + \int_0^t e^{-\eta_1(t-s)}T_1(t-s)\theta_1 u(s)(h*v)(s)\,ds, \tag{8}$$

$$v(t) = e^{\eta_2 t}T_2(t)v_0 - \int_0^t e^{\eta_2(t-s)}T_2(t-s)[\eta_2 v^2(s) + \theta_2 u(s)v(s)]\,ds. \tag{9}$$

The contraction of T_2, the nonnegativity of u and v, and (9) give

$$\|v(t)\| \le e^{\eta_2 t}\|v_0\| \quad \text{for all} \quad t \ge 0. \tag{10}$$

Since $\|h*v\| \le \|v\|$, from (8) and (10), we have

$$\|u(t)\| \le \|u_0\| + \theta_1\|v_0\| \int_0^t e^{\eta_2 s}\|u(s)\|\,ds, \quad \text{for all} \quad t \ge 0.$$

Finally, Gronwall's inequality implies

$$\|u(t)\| \le \|u_0\| e^{\theta_1\|v_0\|k(t)} \quad \text{for all} \quad t \ge 0, \tag{11}$$

where

$$k(t) = \begin{cases} \frac{1}{\eta_2}(e^{\eta_2 t} - 1), & \eta_2 > 0, \\ t, & \eta_2 = 0. \end{cases}$$

Results (10) and (11) show that the solutions are global ($\tau_{max} = +\infty$).

Remark 2.1 Theorem 2.1 interprets that if the capital productivity is greater than the sum of the growth rates of labor supply and labor productivity, then the solution (u, v) (wage-employment) of (3)-(4) is positive. This means that the employment and labor power depend on the amount of national income generated by the invested capital.

3 Boundedness of Solution

The solution of the problem (3)-(4) constructed in Theorem 2.1 is not always bounded as is shown in the following lemma.

Lemma 3.1 *If $v_0 \neq 0$ and η_2 is sufficiently large, then the solution (u, v) in Theorem 2.1 is unbounded.*

Proof. Suppose (u, v) is a globally bounded solution, there is a constant $M > 0$ such that $\|u(t)\| \leq M$ and $\|v(t)\| \leq M$ for all $t \geq 0$. Since $v_0 \neq 0$, there exists $\omega > M^2$ such that $T_2(t)v_0 > \omega$ for all $t \geq 0$. By the nonnegativity of u, v, (9) gives

$$v(t) \geq \left(\omega - \frac{M^2(\eta_2 + \theta_2)}{\eta_2}\right) e^{\eta_2 t} + \frac{M^2(\eta_2 + \theta_2)}{\eta_2}.$$

Getting $\eta_2 > \frac{M^2\theta_2}{\omega - M^2}$ gives $\|v(t)\| \to +\infty$ as $t \to +\infty$. We have a contradiction.

Lemma 3.1 states that to get bounded solutions, we need some restrictions either on the coefficients of the system or on the initial data.

Theorem 3.1 *If $u_0, v_0 \in X$ and $\eta_2 \geq 0$, then*

$$\|v(t)\| \leq \|v_0\| e^{\eta_2 t} \quad \textit{for all} \quad t \geq 0, \tag{12}$$

$$\|u(t)\| \leq e^{(\theta_1 \|v_0\| c(\tau) - \eta_1)t} \|u_0\| \quad \textit{for all} \quad t \in [0, \tau], \tag{13}$$

where $c(t) := \frac{ae^{\eta_2 t}}{a + \eta_2}$. Further, if $\eta_2 = 0$ and $\eta_1 > \theta_1 \|v_0\|$, then

$$\lim_{t \to \infty} \|u(t)\| = 0.$$

Proof. Substituting $u = \phi e^{-\eta_1 t}$ and $v = \varphi e^{\eta_2 t}$ into (3) and (4), respectively, gives

$$\phi_t - \delta_1 \phi_{xx} = \theta_1 \phi(h * e^{\eta_2 t}\varphi), \tag{14}$$

$$\varphi_t - \delta_2 \varphi_{xx} = -\eta_2 \varphi^2 e^{\eta_2 t} - \theta_2 e^{-\eta_1 t} \phi\varphi \tag{15}$$

with the initial data

$$\phi_0(x) = u_0(x), \qquad \varphi_0(x) = v_0(x). \tag{16}$$

By the nonnegativity of ϕ and φ, (15) with (16) give

$$\varphi(t) = T_2(t)v_0 - \int_0^t T_2(t-s)[\eta_2 \varphi^2(s) e^{\eta_2 s} + \theta_2 e^{-\eta_1 s} \phi(s)\varphi(s)]\, ds \leq T_2(t)v_0 \tag{17}$$

for all $(x, t) \in \Omega \times [0, \infty)$. This implies that

$$\|v(t)\| = \|\varphi(t)\| e^{\eta_2 t} \leq \|v_0\| e^{\eta_2 t}, \quad \text{for all} \quad t \geq 0.$$

Next, by (17) and (14), we obtain

$$\phi_t - \delta_1 \phi_{xx} \le \theta_1 \|v_0\| c(t)\phi.$$

Transforming $\phi = e^{\theta_1 \|v_0\| c(\tau) t} z$ on $\Omega \times [0,\tau]$ gives

$$z_t - \delta_1 z_{xx} \le 0, \quad z(0) = \phi(0) = u_0.$$

This implies that

$$z(t) = T_1(t)u_0, \qquad t \ge 0.$$

Therefore, $\|\phi(t)\| \le e^{\theta_1 \|v_0\| c(\tau) t}\|u_0\|$ and

$$\|u(t)\| = \|\phi(t)\| e^{-\eta_1 t} \le e^{(\theta_1 \|v_0\| c(\tau) - \eta_1)t}\|u_0\| \quad \text{for all} \quad t \in [0,\tau]. \tag{18}$$

Further, for $\eta_2 = 0$ and $\eta_1 > \theta_1\|v_0\|$, (18) implies that

$$\lim_{t\to\infty} \|u(t)\| = 0.$$

Theorem 3.2 *If $\eta_2 = 0$ and $\delta_1 \le \delta_2$, then the solution (u,v) of (3)-(4) is globally bounded. Moreover,*

$$\|v(t)\| \le \|v_0\| \quad \textit{for all} \quad t \ge 0, \tag{19}$$

$$\|u(t)\| \le \|u_0\| + \frac{\theta_1}{\theta_2}\sqrt{\frac{\delta_2}{\delta_1}}\|v_0\| \quad \textit{for all} \quad t \ge 0. \tag{20}$$

Proof. The nonnegativity of v gives $(h * v)(s) \le v(s)$ for all $s \ge 0$. If $\eta_2 = 0$, (12) gives (19). Moreover, from (8) and (9), we have

$$u(t) = e^{-\eta_1 t} T_1(t)u_0 + \theta_1 U(t), \tag{21}$$

$$v(t) = T_2(t)v_0 - \theta_2 V(t), \tag{22}$$

where

$$\begin{aligned} U(t) &= \int_0^t e^{-\eta_1(t-s)} T_1(t-s)u(s)(h*v)(s)\,ds \le \int_0^t T_1(t-s)u(s)v(s)\,ds, \\ V(t) &= \int_0^t T_2(t-s)u(s)v(s)\,ds. \end{aligned} \tag{23}$$

Conditions $\delta_1 \le \delta_2$ and (5) provide

$$T_1(t)w \le \sqrt{\frac{\delta_2}{\delta_1}} T_2(t)w \qquad \text{for all} \quad w \in X, \quad t \ge 0. \tag{24}$$

Moreover, by the nonnegativity of v, (22) implies that

$$V(t) \le \frac{1}{\theta_2} T_2(t)v_0 \qquad \text{for all} \quad t \ge 0. \tag{25}$$

Equations (23), (24) and (25) give

$$U(t) \le \sqrt{\frac{\delta_2}{\delta_1}} V(t) \le \frac{1}{\theta_2}\sqrt{\frac{\delta_2}{\delta_1}} T_2(t)v_0 \qquad \text{for all} \quad t \ge 0. \tag{26}$$

Finally, (26) together with (21) imply (20).

Theorem 3.3 *Let* $\eta_1 = 0$, $\delta_1 \leq \delta_2$ *and* $0 \leq \eta_2 \leq H(t)$ *for all* $t \geq \tau$, *where* H *is a positively continuous function such that* $\lim_{t\to\infty} tH(t) = 0$ *for some* $\tau > 0$. *The solution* (u, v) *of* (3)-(4) *is globally bounded. Moreover,*

$$\|u(t)\| \leq \|u_0\| + \frac{\theta_1}{\theta_2}\sqrt{\frac{\delta_2}{\delta_1}}\|v_0\| \quad \textit{for all} \quad t \geq 0, \tag{27}$$

$$\|v(t)\| \leq c\|v_0\| \quad \textit{for all} \quad t \geq 0, \quad \textit{for some} \quad c > 0. \tag{28}$$

Proof. For $\eta_1 = 0$, (8) and (9) give

$$u(t) = T_1(t)u_0 + \theta_1 \int_0^t T_1(t-s)u(s)v(s)\, ds, \tag{29}$$

$$v(t) = e^{\eta_2 t}\left[T_2(t)v_0 - \int_0^t e^{-\eta_2 s}T_2(t-s)[\eta_2 v^2(s) + \theta_2 u(s)v(s)]\, ds\right], \tag{30}$$

respectively. Since v is nonnegative, (30) implies that

$$\int_0^t e^{-\eta_2 s}T_2(t-s)[\eta_2 v^2(s) + \theta_2 u(s)v(s)]\, ds \leq T_2(t)v_0. \tag{31}$$

Further, since $\eta_2, \theta_2 > 0$ and the function $f(s) = e^{-\eta_2 s}$ is decreasing on $[0, t]$, (31) gives

$$\int_0^t T_2(t-s)u(s)v(s)\, ds \leq \frac{T_2(t)v_0}{\theta_2}.$$

Therefore, inserting (24) into (29), we obtain

$$u(t) \leq T_1(t)u_0 + \frac{\theta_1}{\theta_2}\sqrt{\frac{\delta_2}{\delta_1}}T_2(t)v_0.$$

This proves (27).

Next, if there exists $\tau > 0$ such that $\eta_2 \leq H(t)$ for all $t \geq \tau$, where H is a positively continuous function such that $\lim_{t\to\infty} tH(t) = 0$, then (12) gives (28), where $c = e^{\tau H(\tau)}$.

Remark 3.1 Theorem 3.2 clarifies that the solution (u, v) of (3)-(4) is globally bounded when the capital productivity is equal to the sum of the growth rates of labor supply and labor productivity. Further, Theorem 3.3 is valid if the rate of labor productivity and the intercept of the linear Phillips curve negate each other. This may occur when the rate of labor productivity is negative.

In particular, if the employment is unlimited, the system (3) has the form

$$\begin{aligned} u_t &= \delta_1 \Delta u - \eta_1 u + \theta_1 uv, \\ v_t &= \delta_2 \Delta v + \eta_2 v - \theta_2 uv \end{aligned} \tag{32}$$

subject to the initial conditions (4), and we have the following theorem.

Theorem 3.4 *If* $\eta_1 = 0$ *and* $u_0(x) > \eta_2/\theta_2$ *for all* $x \in \Omega$, *then the solution* (u, v) *of* (32)-(4) *satisfies*

$$\|v(t)\| \leq \|v_0\| \quad \textit{for all} \quad t \geq 0. \tag{33}$$

Moreover, if there exists $\kappa > \eta_2/\theta_2$ *such that* $u_0(x) > \kappa$ *for all* $x \in \Omega$, *then*

$$\|u(t)\| \leq e^{\frac{\theta_1}{\kappa\theta_2 - \eta_2}\|v_0\|}\|u_0\| \quad \textit{for all} \quad t \geq 0, \tag{34}$$

$$\|v(t)\| \leq e^{-(\kappa\theta_2 - \eta_2)t}\|v_0\| \quad \textit{for all} \quad t \geq 0. \tag{35}$$

Proof. Since $u_0 > \eta_2/\theta_2$, (29) implies that

$$u(t) \geq T_1(t)(\eta_2/\theta_2) \geq \eta_2/\theta_2 \quad \text{for all} \quad t \geq 0.$$

We define a linear operator $B(t) := \eta_2 - \theta_2 u(t)$ on X. From (32), we have

$$v_t(t) = [\delta_2 \Delta + B(t)]v(t). \tag{36}$$

The dissipativity of $B(t)$ for all $t \geq 0$ implies that there exists a contraction quasi semigroup $R(t,s)$ on X generated by $\delta_2\Delta + B(t)$, [19]. Moreover, the problem (36)-(4) has a solution

$$v(t) = R(0,t)v_0 \quad \text{for all} \quad t \geq 0.$$

This proves (33).

If $u_0 \geq \kappa > \eta_2/\theta_2$, again, from (29), we have $u(t) \geq \kappa$. Further, $\eta_2 - \theta_2 u(t) < \eta_2 - \kappa\theta_2 < 0$ for all $t \geq 0$. Therefore, (36) can be rewritten as

$$v_t(t) = [\delta_2\Delta + B(t) + \omega I]v(t) - \omega v(t), \tag{37}$$

where $\omega := \kappa\theta_2 - \eta_2 > 0$. Since $B(t) + \omega I$ is a dissipative operator on X, operator $\delta_2\Delta + B(t) + \omega I$ generates a contraction quasi semigroup $G(t,s)$. Therefore, the contraction quasi semigroup $R(t,s)$ generated by $\delta_2\Delta + B(t)$ has a representation

$$R(t,s) = e^{-\omega s}G(t,s) \quad \text{for all} \quad t, s \geq 0.$$

Thus, the solution of (37)-(4) is given by

$$v(t) = K(0,t)v_0 = e^{-\omega t}G(0,t)v_0 \quad \text{for all} \quad t \geq 0. \tag{38}$$

Equation (38) implies (35). Further, substituting (38) into (29) gives

$$u(t) = T_1(t)u_0 + \theta_1 \int_0^t T_1(t-s)u(s)e^{-\omega s}G(0,s)v_0\, ds.$$

Finally, Gronwall's equation implies (34).

Remark 3.2 Besides Theorem 3.4, we can prove that all the results on the positiveness and (globally) boundedness of solutions in Theorems 2.1, 3.1, 3.2 and 3.3 are valid for the system (32)-(4). The proofs are left to the reader.

4 Stability of Solution

We focus on the system (3)-(4) subject to the no-flux boundary on the regular boundary $\partial\Omega$. To begin with, we will analyze the stability of the equilibrium solution to disclose its vulnerability at parameter variations. System (3) is equivalent to the three-dimensional system

$$\begin{aligned} u_t &= \delta_1 \Delta u - \eta_1 u + \theta_1 uw, \\ v_t &= \delta_2 \Delta v + \eta_2(1-v)v - \theta_2 uv, \\ w_t &= a(v-w), \end{aligned} \tag{39}$$

where w stands for the expectations of the future employment levels based on the past employment levels. The third equation in (39) shows that the expectations change continuously and correct themselves.

Straightforward computation shows that system (39) has three equilibrium points:

$$S_1 = (0,0,0), \qquad S_2 = (0,1,1), \qquad S_3 = \left(\frac{(\theta_1 - \eta_1)\eta_2}{\theta_1\theta_2}, \frac{\eta_1}{\theta_1}, \frac{\eta_1}{\theta_1} \right).$$

After translating the equilibrium point (u^*, v^*, w^*) to the origin by the translation $\bar{u} = u - u^*, \bar{v} = v - v^*$, $\bar{w} = w - w^*$ and still denoting $\bar{u}$, $\bar{v}$ and $\bar{w}$ by u, v and w, respectively, the system (39) reduces to the following system:

$$\begin{aligned} u_t &= \delta_1 \Delta u + (\theta_1 - \eta_1)u + \theta_1 u^* w + f(u,v,w), \\ v_t &= \delta_2 \Delta v - \theta_2 v^* u + (\eta_2 - \theta_2 u^* - 2\eta_2 v^*)v + g(u,v,w), \\ w_t &= a(v - w), \end{aligned} \tag{40}$$

where

$$f(u,v,w) = \theta_1 uw, \qquad g(u,v,w) = -\eta_2 v^2 - \theta_2 uv,$$

subject to the no-flux boundary conditions

$$\partial_\nu u = \partial_\nu v = \partial_\nu w = 0, \quad x \in \partial\Omega, \quad t \geq 0. \tag{41}$$

Henceforth, we will only focus on the special case when $\Omega := (0,\pi)$ and $X := H^2(\Omega)$ is the standard Sobolev space, so we consider the following system:

$$\begin{aligned} u_t &= \delta_1 u_{xx} + (\theta_1 - \eta_1)u + \theta_1 u^* w + f(u,v,w), \\ v_t &= \delta_2 v_{xx} - \theta_2 v^* u + (\eta_2 - \theta_2 u^* - 2\eta_2 v^*)v + g(u,v,w), \\ w_t &= a(v - w) \end{aligned} \tag{42}$$

subject to the Neumann boundary condition

$$u_x(0,t) = u_x(\pi,t) = 0, \quad v_x(0,t) = v_x(\pi,t) = 0, \quad w_x(0,t) = w_x(\pi,t) = 0, \quad t \geq 0. \tag{43}$$

The linearized system (42) at (u^*, v^*, w^*) can be written as

$$\begin{pmatrix} u_t \\ v_t \\ w_t \end{pmatrix} = L \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where

$$L = \begin{pmatrix} \delta_1 \dfrac{\partial^2}{\partial x^2} + \theta_1 - \eta_1 & 0 & \theta_1 u^* \\ -\theta_2 v^* & \delta_2 \dfrac{\partial^2}{\partial x^2} + \eta_2 - \theta_2 u^* - 2\eta_2 v^* & 0 \\ 0 & a & -a \end{pmatrix}$$

on the domain

$$\mathcal{D}(L) = \{(u,v,w) \in [H^2(\Omega)]^3 : u'(0) = u'(\pi) = 0,\ v'(0) = v'(\pi) = 0,\ w'(0) = w'(\pi)\}.$$

It is well-known that the eigenvalue problem

$$z'' = \mu z, \quad x \in (0,\pi), \quad z'(0) = z'(\pi) = 0,$$

has the eigenvalues $\mu_n = -n^2$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, with the corresponding eigenfunctions $\phi_n(x) = \cos nx$. Let

$$\begin{pmatrix} \phi \\ \psi \\ \varphi \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} \cos nx,$$

where a_n, b_n, c_n are constants, be an eigenfunction of L with the eigenvalue λ, that is,

$$L \begin{pmatrix} \phi \\ \psi \\ \varphi \end{pmatrix} = \lambda \begin{pmatrix} \phi \\ \psi \\ \varphi \end{pmatrix}.$$

The orthogonality of the function sequence (ϕ_n) implies that

$$L_n \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix} = \lambda \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}, \quad n \in \mathbb{N}_0,$$

where

$$L_n = \begin{pmatrix} -n^2\delta_1 + \theta_1 - \eta_1 & 0 & \theta_1 u^* \\ -\theta_2 v^* & -n^2\delta_2 + \eta_2 - \theta_2 u^* - 2\eta_2 v^* & 0 \\ 0 & a & -a \end{pmatrix}.$$

Lemma 3.1 of [13] implies that λ is an eigenvalue for L if and only if λ is an eigenvalue for L_n for some $n \in \mathbb{N}_0$. The characteristic equation of L_n at (u^*, v^*, w^*) is

$$\lambda^3 + \alpha_n \lambda^2 + \beta_n \lambda + \gamma_n = 0, \quad n \in \mathbb{N}_0, \tag{44}$$

where

$$\begin{aligned} \alpha_n &= n^2(\delta_1 + \delta_2) + a + \eta_1 - \eta_2 - \theta_1 + u^*\theta_2 + 2v^*\eta_2, \\ \beta_n &= n^4\delta_1\delta_2 + n^2(a(\delta_1 + \delta_2) + \delta_2\eta_1 - \delta_1\eta_2 - \delta_2\theta_1) + a(\eta_1 - \eta_2 - \theta_1) \\ &\quad + \eta_2(\theta_1 - \eta_1) + (n^2\delta_1 + a + \eta_1 - \theta_1)(\theta_2 u^* + 2\eta_2 v^*), \\ \gamma_n &= an^4\delta_1\delta_2 + an^2(\delta_2\eta_1 - \delta_1\eta_2 - \delta_2\theta_1) + a\eta_2(\theta_1 - \eta_1) \\ &\quad + (n^2\delta_1 + \eta_1 - \theta_1)(a\theta_2 u^* + 2a\eta_2 v^*) + au^*v^*\theta_1\theta_2. \end{aligned} \tag{45}$$

The standard linear operator theory provides that if all the eigenvalues of the operator L have negative real parts, then (u^*, v^*, w^*) is asymptotically stable, and if some eigenvalues have positive real parts, then (u^*, v^*, w^*) is unstable. For (44), we have the following lemma.

Lemma 4.1 *[20] The real parts of the roots of the equation $x^3 + \alpha x^2 + \beta x + \gamma = 0$ are all negative if and only if $\alpha > 0$, $\alpha\beta - \gamma > 0$ and $\gamma > 0$.*

Evaluation (44) at the equilibrium S_1 together with Lemma 4.1 give that S_1 is asymptotically stable if $\eta_2 < n^2\delta_2$ for some n. However, this stability is not significant in the economic sense since S_1 provides the absence of wages and employment. The equilibrium S_2 is asymptotically stable if $\eta_1 + n^2\delta_1 > \theta_1$ for some n implying a decrease in wages. Since S_2 represents the absence of wages corresponding to full employment, it is meaningless in this case.

We note that the position of the equilibrium S_3 does not depend on the delay μ, but its stability does. The stability conditions of the equilibrium S_3 are summarized in the following theorem.

Theorem 4.1 *Let $\mu = 1/a$ be a delay and $\delta_n = n^2(\delta_1 + \delta_2)$. The stability of S_3 is considered in three cases:*

(a) If $\theta_1(\theta_1 - \eta_1 - \delta_n) - \eta_1\eta_2 < 0$ for some n, then S_3 is asymptotically stable, regardless of the delay.

(b) If $\theta_1(\theta_1 - \eta_1 - \delta_n) - \eta_1\eta_2 > 0$ and $\mu(\theta_1 - \eta_1 - \delta_n - \frac{\eta_1\eta_2}{\theta_1}) < 1$ for some n, then S_3 is asymptotically stable.

(c) If $\mu(\theta_1 - \eta_1 - \delta_n - \frac{\eta_1\eta_2}{\theta_1}) > 1$ for some n, then S_3 is unstable.

Proof. An application of Lemma 4.1 to the roots of (44) at S_3 implies that S_3 is asymptotically stable if

$$\theta_1 - \eta_1 - \delta_n - \frac{\eta_1\eta_2}{\theta_1} < a, \tag{46}$$

where $\delta_n = n^2(\delta_1 + \delta_2)$ for some $n \in \mathbb{N}_0$.

(a) Since $a > 0$, the inequality in (46) is valid if the left-hand side of (46) is negative (regardless of μ), i.e.,

$$\theta_1(\theta_1 - \eta_1 - \delta_n) - \eta_1\eta_2 < 0. \tag{47}$$

Since $\delta_n > 0$ for all $n \in \mathbb{N}_0$, the left-hand side of (47) is valid if $\eta_1 > \theta_1$ or $\eta_1 + \delta_n < \theta_1$ for some n and η_2 is large enough.

(b) For a small delay μ, inequality (46) gives that $\mu(\theta_1 - \eta_1 - \delta_n - \frac{\eta_1\eta_2}{\theta_1}) < 1$ and the left-hand side is positive.

(c) The condition $\mu(\theta_1 - \eta_1 - \delta_n - \frac{\eta_1\eta_2}{\theta_1}) > 1$ for some n negates the inequality in (46). This implies that characteristic equation (44) may have eigenvalues with positive real parts.

Remark 4.1 (a) The condition (47) is valid if $\eta_1 > \theta_1$ or $\eta_1 + \delta_n < \theta_1$ for some n together with η_2 being large enough. The stability due to both conditions is regardless of the delay μ and this case rarely happens in real economic life. The first hypothesis confirms that wage-employment system (42)-(43) is stable if the growth rate of the labor productivity is greater than the difference from the slope of the linear Phillips curve to its intercept. At this point, all solutions must approach the positive stable equilibrium when $t \to \infty$.

(b) In particular, for $n = 0$, S_3 is a locally asymptotically stable equilibrium for system (42)-(43) without the diffusion Δ (or system (1)).

(c) If $\eta_2 = 0$ and $\delta_1 \leq \delta_2$, Theorem 3.2 implies that the equilibrium $\left(0, \frac{\eta_1}{\theta_1}\right)$ is global asymptotically stable on the uv-plane. However, similar to the equilibrium S_2, the equilibrium is not meaningful in economic sense.

5 Conclusions

In this paper, we extend the original wage-employment system to the diffusive system. The system is of a diffusive predator-prey type. The properties of solutions to the system including global existence, positivity, uniform boundedness and decay estimates depend on the parameters being varied. The system has three equilibrium points, one of which is asymptotically stable for the appropriate parameters and the stability is economically meaningful.

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