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Multiple Well-Posedness of Higher-Order Abstract Cauchy Problem

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Abstract: In this paper, we fulfill some conditions to examine the multiple well-posedness conditions that define the continuous dependence of the solutions and their derivatives on the initial data of the Cauchy problem. Indeed, for the differential operator equation of arbitrary order in a Hilbert space, an appropriate condition is given for the two main operators that assert the multiple well-posedness. Our results are new and complement some previous ones in the literature.

Keywords: *abstract Cauchy problems; asymptotic stability; integrated semi-groups; stability of nonlinear problems in mechanics; well-posedness.*

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1 Introduction

In [21], Vlasenko *et al.* studied the p-fold well-posedness of the higher-order abstract Cauchy problem of the following form:

$$\sum_{j=0}^{n} A_j \frac{d^j u}{dt^j} = 0, \qquad t > 0, \tag{1}$$

$$u^j(0) = u_j, \qquad j = 0, ..., n-1, \tag{2}$$

where A_j $(j = 0, ..., n)$ are linear closed operators from a complex Banach space $\mathcal{E}$ into

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another complex Banach space $\mathcal{F}$ with the domain of definition $\mathcal{D}(A_j) \subset \mathcal{E}$, A_n can be a degenerate operator.

Many real phenomena can be modeled using problems of the form of (1), (2). For example, many initial-value or initial-boundary value problems for partial differential equations arise in mechanics, physics, engineering, control theory, etc., in particular, the description of vibrations (see [5], [13], [2]) or the vibrations of a viscoelastic pipeline [14]. The theory of problems of this kind is also connected with many other branches of mathematics, which translates the importance of their study for both theoretical investigations and practical applications.

Over the past half-century, (1), (2) has been studied extensively by many scholars. Especially, for the first-order abstract Cauchy problem, the theory (or closely related operator semigroup theory) has evolved relatively well since the well-known Hille-Yosida theorem was presented in 1948, and is well documented in the monographs (see [6], [15] and other), the study of the second-order abstract Cauchy problem has also received much attention. For more information, we can refer to [10], [18], [20], and references cited therein. Since integrated semi-groups were introduced at the end of the eighties of the last century, it has become possible to deal with the ill-posed first-order abstract Cauchy problems (see [16], [11], [8]).

In 2004, Vlasenko *et al.* [21] carried out one of the latest and significant research on establishing conditions that ensure the p-fold well-posedness of the Cauchy problem (1), (2). The study imposed that A_n is a bounded operator and $A_{n-1} = F + B$, where F and B are two operators fulfilling certain criteria that lead to guarantee the p-fold well-posedness in the case of the Hilbert space.

Motivated by the aforementioned works, this paper aims to establish new sufficient conditions that ensure the p-fold well-posedness of the problem (1), (2) even when A_n is a non bounded operator and when the requirements on the two main operators A_n and A_{n-1} are different from those in [21]. The theorem presented in this paper is new and extends and improves previously known results.

Due to its great importance and its many applications in various mathematical fields such as nonlinear dynamics and systems theory (see [1], [22], [3]), many scholars have devoted a great deal of attention to this topic. For example, Vlasenko and his collaborators have published several works in this direction, notably in the case of nonlinear operators. As a result, their interesting outcomes inspired us to find new criteria that guarantee the stability of nonlinear dynamics.

The paper is organized as follows. In Section 2, we introduce some necessary definitions and preliminary results that play a crucial role in establishing our main outcomes. In Section 3, we present some previous global existence results which inspired us to investigate problem (1), (2). Finally, we state and prove our main results in Section 4.

2 Preliminaries

Definition 2.1 (see [21]) We call a solution of problem (1), (2) any function u satisfying condition (2) and

$$u \in C^n((0,\infty),\mathcal{E}) \bigcap C^{n-1}([0,\infty),\mathcal{E})$$

so that

$$A_j u \in C^j((0,\infty),\mathcal{F})\bigcap C^{j-1}([0,\infty),\mathcal{F}),\;\; j=1,...,n.$$

Definition 2.2 (see [21]) The Cauchy problem (1), (2) is $p-$fold well-posed ($p \in \{1,...,n\}$) if and only if for any of its solutions u, the following estimate is true:

$$\left\|u^{p-1}(t)\right\| \leq F(t)\sum_{j=0}^{n-1}\|u_j\|,\;\; t \geq 0, \tag{3}$$

where F is a non-negative function from $\mathbb{R}_+$ to $\mathbb{R}_+$.

Definition 2.3 (see [21]) If F is a locally bounded function on $\mathbb{R}_+$ the problem (1), (2) is $p-$fold uniformly well-posed.

Definition 2.4 (see [21]) The problem (1), (2) is said to be $p-$fold exponentially well-posed if $F(t) = Ce^{\omega t}$, where $C \geq 0$ and $\omega \geq 0$.

Remark 2.1 (see [21]) If $p = n$, the problem (1), (2) is full-fold well-posed.

Examples. Consider the problem of the vibrating string

$$\frac{\partial^2 u}{\partial x^2} - C^2\frac{\partial^2 u}{\partial t^2} = 0,\;\; t>0,\; -\infty < x < \infty,$$

$$u(x,0) = f(x),\quad u_t(x,0) = 0.$$

The solution of this problem is given by

$$u(x,t) = \frac{1}{2}\left[f(x-Ct) + f(x+Ct)\right].$$

Then

$$u(x,t) = \frac{1}{2}\left[G(-Ct)f(x) + G(Ct)f(x)\right],$$

where $\{G(t)\}$ is a semigroup of the operator $\frac{d}{dx}$. We have

$$\begin{aligned}\|u(x,t)\| &= \left\|\frac{1}{2}\left[G(-Ct)f(x) + G(Ct)f(x)\right]\right\| \\ &\leq \frac{1}{2}\|G(-Ct)\|\|f(x)\| + \|G(Ct)\|\|f(x)\|.\end{aligned}$$

$G(-Ct)$ is strongly continuous, then there exist M and ω such that

$$\|G(-Ct)\| \leq M\exp(-\omega Ct),$$

also

$$\|G(Ct)\| \leq M\exp(\omega Ct),$$

which implies that

$$\|u(x,t)\| \leq \tfrac{1}{2}\left(\exp(\omega Ct) + \exp(-\omega Ct)\right)\|f(x)\|.$$

For

$$F(t) = \max\left(\tfrac{1}{2}\left(\exp(\omega Ct) + \exp(-\omega Ct)\right), 1\right),$$

we have

$$\|u(x,t)\| \leq F(t) \sum_{j=0}^{1} \|u_j\|,$$

$$\begin{aligned} u_t(x,t) &= \frac{1}{2}\left[\frac{dG(-Ct)f(x)}{dt} + \frac{dG(Ct)f(x)}{dt}\right] \\ &= \frac{1}{2}\left[AG(-Ct)f(x) + AG(Ct)f(x)\right]. \end{aligned}$$

As the operator A is unbounded, then there is no nonnegative function F such that

$$\|u_t(x,t)\| \leq K(t) \sum_{j=0}^{1} \|u_j\|.$$

Hence, the problem of the vibrating string is well-posed but is not 2-fold well-posed.

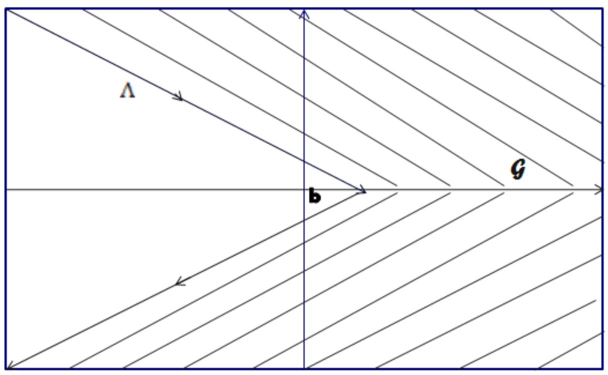
Equation (1) is associated with the characteristic polynomial

$$L(\lambda) = \sum_{j=0}^{n} \lambda^j A_j$$

and its resolvent $\mathcal{R}(\lambda) = L^{-1}(\lambda)$. Let $\mathcal{G}$ be the angle in the complex plane given by

$$\mathcal{G} = \mathcal{G}(b,\theta) = \left\{\lambda = b + re^{i\gamma},\ |\gamma| \leq \pi - \theta,\ r \geq 0\right\}, \quad 0 < \theta < \frac{\pi}{2},$$

and bounded by a pair of rays. The orientation of the boundary contour Λ should be such that the area $\mathcal{G}$ is located on its left side when bypassing below see Figure **A**.



A

Here, the joint properties of the two leading operators A_{n-1} and A_n in terms of the following linear bundle of operators $\mathcal{P}(\lambda)$ and its resolvent $\mathcal{S}(\lambda)$ are

$$\mathcal{P}(\lambda) = \lambda A_n + A_{n-1},$$

$$\mathcal{S}(\lambda) = \mathcal{P}^{-1}(\lambda).$$

Theorem 2.1 *(see [9], [12], [19]) A is a closed linear operator from $\mathcal{E}$ into $\mathcal{E}$ so that $\|A\| \leq q < 1$. Then $(I + A)$ is a reversible and $(I + A)^{-1}$ is a bounded operator.*

Corollary 2.1 *Suppose that $n > 1$, the linear $\mathcal{D}(L)$ is dense in $\mathcal{E}$, the number p belongs to the set $\{1, ..., n\}$ and, in a certain angle $\mathcal{G}(b, \theta)$, $b > 0$, the resolvent $\mathcal{S}(\lambda)$ of the bundle of leading operators satisfies the estimates*

$$\|\mathcal{S}(\lambda) A_j\| \leq C |\lambda|^{n-p}, \quad j = 0, ..., n-1, \tag{4}$$

and

$$\|\mathcal{S}(\lambda) A_j\| \leq C |\lambda|^{q_j}, \quad j = 0, ..., n-2, \tag{5}$$

with the constants $q_j < n - j - 1$. Then the Cauchy problem (1), (2) is full-fold well-posed and $p-$fold exponentially well-posed.

In [21], for the proof, the authors postulated another Cauchy problem and presented the proof without that assumption. The proof is as follows.

Proof. For $\lambda \in \mathcal{G}(a, \theta)$,

$$\begin{aligned} L(\lambda) &= \lambda^n A_n + \lambda^{n-1} A_{n-1} + \sum_{j=0}^{n-2} \lambda^j A_j \\ &= \lambda^{n-1}(\lambda A_n + A_{n-1}) + \sum_{j=0}^{n-2} \lambda^j A_j \\ &= \lambda^{n-1} P(\lambda) + \sum_{j=0}^{n-2} \lambda^j A_j \\ &= \lambda^{n-1} P(\lambda)(I + \sum_{j=0}^{n-2} \lambda^{j-n+1} P^{-1}(\lambda) A_j), \end{aligned}$$

so

$$L(\lambda) = \lambda^{n-1} P(\lambda)(I + \sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_j).$$

As the estimate (5) is verified, then

$$\begin{aligned} \left\| \sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_j \right\| &\leq \sum_{j=0}^{n-2} |\lambda|^{j-n+1} \|S(\lambda) A_j\| \\ &\leq \sum_{j=0}^{n-2} |\lambda|^{j-n+1} C |\lambda|_j^q \\ &\leq C \sum_{j=0}^{n-2} |\lambda|^{j-n+1+q_j}, \end{aligned}$$

we have $q_j < n - j - 1$, so $j - n + 1 + q_j < 0$, and as $\lambda \in \mathcal{G}(a, \theta)$, this implies that $\left(\frac{1}{|\lambda|}\right)^{-j+n-1-q_j} < \left(\frac{1}{a_0 \sin\theta}\right)^{-j+n-1-q_j}$, $j = 0, \cdots, n-2$. So

$$\begin{aligned}\left\|\sum_{j=0}^{n-2}\lambda^{j-n+1}S(\lambda)A_j\right\| &< C((\frac{1}{|\lambda|})^{n-1-q_0}+(\frac{1}{|\lambda|})^{n-2-q_1}+\cdots+(\frac{1}{|\lambda|})^{1-q_{n-2}}) \\ &< C((\frac{1}{a_0\sin\theta})^{n-1-q_0}+(\frac{1}{a_0\sin\theta})^{n-2-q_1}+\cdots \\ &+ (\frac{1}{a_0\sin\theta})^{1-q_{n-2}}),\end{aligned}$$

for $a_0\sin\theta\leq 1$, we let

$$h=\max\{n-1-q_0,n-2-q_0,\cdots,1-q_{n-2}\},$$

and if $a_0\sin\theta\geq 1$, we let

$$h=\min\{n-1-q_0,n-2-q_0,\cdots,1-q_{n-2}\},$$

in any case, we have

$$\left\|\sum_{j=0}^{n-2}\lambda^{j-n+1}S(\lambda)A_j\right\|<C((n-1)(\tfrac{1}{a_0\sin\theta})^h).$$

For $a_0=\frac{(C(n-1))^{\frac{1}{h}}}{\sin\theta}$, we get

$$C((n-1)(\tfrac{1}{a_0\sin\theta})^h)=1,$$

therefore $\mathcal{G}_0=\mathcal{G}\left(\frac{(C(n-1))^{\frac{1}{h}}}{\sin\theta},\theta\right)\subset\mathcal{G}(a,\theta)$ so that

$$\left\|\sum_{j=0}^{n-2}\lambda^{j-n+1}S(\lambda)A_j\right\|<1.$$

According to Theorem 2.6, we conclude that

$$\gamma(\lambda)=\left(I+\sum_{j=0}^{n-2}\lambda^{j-n+1}S(\lambda)A_j\right)^{-1}$$

exists and is bounded in $\mathcal{G}_0$. We have also

$$\begin{aligned}\left\|L^{-1}(\lambda)A_jx\right\| &= \left\|(\lambda^{n-1}P(\lambda)(I+\sum_{j=0}^{n-2}\lambda^{j-n+1}P^{-1}(\lambda)A_j))^{-1}A_jx\right\| \\ &= \left\|(I+\sum_{j=0}^{n-2}\lambda^{j-n+1}P^{-1}(\lambda)A_j)^{-1}\lambda^{1-n}P^{-1}(\lambda)A_jx\right\| \\ &\leq \left\|(I+\sum_{j=0}^{n-2}\lambda^{j-n+1}P^{-1}(\lambda)A_j)^{-1}\right\||\lambda|^{1-n}\left\|P^{-1}(\lambda)A_j\right\|\|x\| \\ &\leq \left\|(I+\sum_{j=0}^{n-2}\lambda^{j-n+1}P^{-1}(\lambda)A_j)^{-1}\right\||\lambda|^{1-n}\|S(\lambda)A_j\|\|x\| \\ &\leq \left\|(I+\sum_{j=0}^{n-2}\lambda^{j-n+1}P^{-1}(\lambda)A_j)^{-1}\right\||\lambda|^{1-n}C|\lambda|^{n-p}\|x\|,\end{aligned}$$

and as $(I+\sum\limits_{j=0}^{n-2}\lambda^{j-n+1}S(\lambda)A_j)^{-1}$ exists and is bounded in $\mathcal{G}_0$, then there is $K>0$ such that

$$\left\|(I+\sum_{j=0}^{n-2}\lambda^{j-n+1}S(\lambda)A_j)^{-1}\right\|\leq K$$

so that

$$\left\|L(\lambda)^{-1}A_jx\right\|\leq KC\left|\lambda\right|^{1-n+n-p}\left\|x\right\|.$$

This implies that there exists $K_1=CK$ such that

$$\left\|L(\lambda)^{-1}A_jx\right\|\leq K_1\left|\lambda\right|^{1-p}\left\|x\right\|,$$

and hence

$$\left\|R(\lambda)A_jx\right\|\leq\frac{K_1}{\left|\lambda\right|^{1-p}}\left\|x\right\|.$$

According to Theorem 4 in [21], the result is proved.

We conclude that two conditions (4), (5) are substitutes for condition (19) in [21]. By this corollary, we get the full-fold uniform well-posedness of the initial boundary-value problems that describe small vibrations of an elastic bar in [13].

Theorem 2.2 *(see [9], [12], [19]) A is a self-adjoint operator and A^{-1} exists and is a bounded operator on $R(A)$. Then $\overline{R(A)}=\mathcal{H}$ and A^{-1} is a self-adjoint operator.*

Theorem 2.3 *(see [9], [12], [19]) Suppose $\mathcal{D}(A)$ and $R(A)$ are dense in $\mathcal{H}$ and A^{-1} exists. Then $(A^*)^{-1}$ exists and $(A^*)^{-1}=\left(A^{-1}\right)^*$.*

3 Global Existence Result

In this part, we display some existing theorems [21] that set some conditions which guarantee two inequalities (4) and (5) so that to achieve the full-fold exponentially well-posedness in the case of the Hilbert space $\mathcal{E}=\mathcal{F}=\mathcal{H}$ and for the closed operators A_j with the domains of definitions $\mathcal{D}(A_j)$ being dense in $\mathcal{H}$.

Theorem 3.1 *Suppose that $n>1$, A_n is a nonnegative bounded operator and $A_{n-1}=F+B$, where F is a self-adjoint operator and B is a symmetric or skew-symmetric operator. Suppose that the linear $\bigcap\limits_{j=0}^{n-1}\mathcal{D}(A_j)$ is dense in H, $\mathcal{D}F\subset\bigcap\limits_{j=0}^{n-2}\mathcal{D}\left(A_j^*\right)\cap\mathcal{D}(B)$, for certain $b\geq 0$, the operator $F+bA_n$ is positive-definite (with a positive lower bound), and*

$$q=\left\|B\left(F+bA_n\right)^{-1}\right\|<1. \tag{6}$$

Then the Cauchy problem (1), (2) is full-fold well-posed and (n-1)-fold exponentially well-posed. In addition, if A_n is positive-definite, then the problem is full-fold exponentially well-posed.

We can find another condition that guarantees the same results as the previous theorem.

Theorem 3.2 *Suppose that* $n > 1$, $D(L)$ *is dense in* $\mathcal{H}$, $A_n = A_n^* \geq 0$, $A_{n-1} = F + B$, *where* $F = F^*$, B *is a symmetric or skew-symmetric operator, and for a certain number* $b \geq 0$, *estimate* (6) *is true, furthermore,* $F + bA_n \geq mI > 0$,

$$\mathcal{D}\left((F + bA_n)^{\frac{1}{2}}\right) \subset \mathcal{D}(A_n) \cap \mathcal{D}(B) \cap \left(\bigcap_{j=0}^{n-2} \mathcal{D}(A_n^*)\right).$$

Then the Cauchy problem (1), (2) *is full-fold well-posed and (n-1)-fold exponentially well-posed. In addition, if the operator* A_n *is positive-definite and bounded, then the problem is full-fold exponentially well-posed.*

4 Main Result

In this section, we give the main result of the paper. We will study the case where A_n is an unbounded and non self-adjoint operator.

Theorem 4.1 *Suppose that* $n > 1$, $A_n = K + C$, *where* K *is a positive self-adjoint operator and* C *is a symmetric operator, and* A_{n-1} *is a symmetric operator such that the linear* $\bigcap_{j=0}^{n-1} \mathcal{D}(A_j)$ *is dense in* $\mathcal{H}$ *so that*

$$\mathcal{D}(C^*) \subset \bigcap_{j=0}^{n-2} \mathcal{D}\left(A_j^*\right) \cap \mathcal{D}(A_{n-1}),\ \mathcal{D}(K) \subset \mathcal{D}(C), \tag{7}$$

and for a certain $b \geq 0$, *the operator* $A_{n-1} + bK$ *is positive-definite, we have*

$$q = \left\|C(A_{n-1} + bK)^{-1}\right\| \leq \frac{1}{|\lambda|} \tag{8}$$

and

$$\left\|(A_{n-1} + bK) K^{-1}\right\| \leq \frac{1}{b}. \tag{9}$$

Then the Cauchy problem (1), (2) is full-fold well-posed and full-fold exponentially well-posed.

Proof. For $\lambda \in G(b, \theta)$, we have

$$\begin{aligned} \mathcal{D}\left((\lambda A_n + A_{n-1})^*\right) &= \mathcal{D}\left((\lambda K + \lambda C + A_{n-1})^*\right) \\ &= \mathcal{D}(\lambda K) \cap \mathcal{D}(\lambda C) \cap \mathcal{D}\left(A_{n-1}^*\right). \end{aligned} \tag{10}$$

Because C is a symmetric operator, we have $\mathcal{D}(C) \subset \mathcal{D}(C^*)$ and $\mathcal{D}(K) \subset \mathcal{D}(C)$, which means that

$$\mathcal{D}(K) \subset \mathcal{D}(C) \subset \mathcal{D}(C^*).$$

Then

$$\mathcal{D}\left((\lambda K + \lambda C + A_{n-1})^*\right) = \mathcal{D}(K) \cap \mathcal{D}\left(A_{n-1}^*\right)$$

and

$$\mathcal{D}(K) \subset \mathcal{D}(C^*) \subset \mathcal{D}\left(A_{n-1}^*\right).$$

Thus, we have

$$\mathcal{D}\left((\lambda K + \lambda C + A_{n-1})^*\right) = \mathcal{D}(K) \tag{11}$$

and

$$\begin{aligned} \mathcal{D}((\lambda K + \lambda C^* + A_{n-1})) &= \mathcal{D}(K) \cap \mathcal{D}(C^*) \cap \mathcal{D}(A_{n-1}) \\ &= \mathcal{D}(K) \cap \mathcal{D}(A_{n-1}). \end{aligned}$$

As C is a symmetric operator and using (7), we have

$$\mathcal{D}(K) \subset \mathcal{D}(C) \subset \mathcal{D}(C^*) \subset \mathcal{D}(A_{n-1}),$$

which implies

$$\mathcal{D}((\lambda K + \lambda C^* + A_{n-1})) = \mathcal{D}(K). \tag{12}$$

Furthermore, according to (11) and (12), we have

$$\mathcal{D}\left((\lambda K + \lambda C + A_{n-1})^*\right) = \mathcal{D}(\lambda K + \lambda C^* + A_{n-1}).$$

In addition, $\forall x \in \mathcal{D}(\lambda K + \lambda C^* + A_{n-1})$, C and A_{n-1} are symmetric operators and K is a self-adjoint operator, we can therefore write

$$\begin{aligned} \langle (\lambda K + \lambda C^* + A_{n-1}) x, x\rangle &= \left\langle \left(\lambda K + \lambda C^* + A_{n-1}^*\right) x, x\right\rangle \\ &= \left\langle (\lambda K + \lambda C + A_{n-1})^* x, x\right\rangle. \end{aligned}$$

As $A_{n-1} + bK$ is a positive operator, we have

$$\begin{aligned} \mathcal{D}\left((A_{n-1} + bK)^*\right) &= \mathcal{D}\left(A_{n-1}^*\right) \cap \mathcal{D}(K) \\ &= \mathcal{D}(K). \end{aligned}$$

We have also

$$\begin{aligned} \mathcal{D}((A_{n-1} + bK)) &= \mathcal{D}(A_{n-1}) \cap \mathcal{D}(K) \\ &= \mathcal{D}(K). \end{aligned}$$

Therefore

$$\mathcal{D}((A_{n-1} + bK)^*) = \mathcal{D}(A_{n-1} + bK).$$

Let us remind that A_{n-1} are symmetric operators and K is a self-adjoint operator, then $\forall x,\ y \in \mathcal{D}\left((A_{n-1} + bK)^*\right)$,

$$\langle (A_{n-1} + bK) x, y\rangle = \langle x, (A_{n-1} + bK) y\rangle.$$

This implies that $A_{n-1} + bK$ is a self-adjoint operator, and as C is a symmetric operator, we get

$$\mathcal{D}((A_{n-1} + bK)) = \mathcal{D}(K).$$

As $\mathcal{D}(K) \subset \mathcal{D}(C)$, we have

$$\mathcal{D}\left(\left(A_{n-1}+bK\right)\right)\subset\mathcal{D}\left(C\right)$$

and

$$\begin{aligned}\|C\| &= \left\|C\left(A_{n-1}+bK\right)^{-1}\left(A_{n-1}+bK\right)\right\| \\ &\leq \left\|C\left(A_{n-1}+bK\right)^{-1}\right\|\left\|\left(A_{n-1}+bK\right)\right\| \\ &\leq q\left\|\left(A_{n-1}+bK\right)\right\|.\end{aligned}$$

As K is a self-adjoint operator, $\mathcal{D}\left(A_{n-1}+bK\right)=\mathcal{D}\left(K\right)$ and

$$\begin{aligned}\left\|\left(A_{n-1}+bK\right)\right\| &= \left\|\left(A_{n-1}+bK\right)K^{-1}K\right\| \\ &\leq \left\|\left(A_{n-1}+bK\right)K^{-1}\right\|\left\|K\right\| \\ &\leq \frac{1}{b}\left\|K\right\| \\ &\leq \left\|bK\right\|.\end{aligned}$$

Then, according to Theorem 4.12 in [9], we obtain

$$\left|\left\langle Cx,x\right\rangle\right|\leq q\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle, \tag{13}$$

$$\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle\leq\left\langle bKx,x\right\rangle \tag{14}$$

for all $x\in\mathcal{D}\left(K\right)$ and $\lambda\in G\left(b,\theta\right).$

Using (13) and (14), for all $x\in\mathcal{D}\left(K\right)$ and for all $\lambda\in G\left(b,\theta\right)$, we get

$$\begin{aligned}&\left|\left\langle\left(\lambda A_n+A_{n-1}\right)x,x\right\rangle\right| \\ &= \left|\left\langle\left(\lambda K+\lambda C+A_{n-1}\right)x,x\right\rangle\right| \\ &\geq \left|\left\langle\left(\lambda K+A_{n-1}\right)x,x\right\rangle\right|-\left|\lambda\right|\left|\left\langle Cx,x\right\rangle\right| \\ &\geq \left|\left\langle\left(\lambda K+A_{n-1}-bK+bK\right)x,x\right\rangle\right|-\left|\lambda\right|q\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle \\ &\geq \left|\left\langle\lambda Kx,x\right\rangle\right|-\left\langle\left(bK-\left(A_{n-1}+bK\right)\right)x,x\right\rangle-\left|\lambda\right|q\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle \\ &\geq \left|\left\langle\lambda Kx,x\right\rangle\right|-\left\langle bKx,x\right\rangle+\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle-\left|\lambda\right|q\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle.\end{aligned}$$

As $\left|\lambda\right|\geq b$, then $-\left\langle bKx,x\right\rangle\geq-\left|\lambda\right|\left\langle Kx,x\right\rangle$, which implies that

$$\begin{aligned}&\left|\left\langle\left(\lambda A_n+A_{n-1}\right)x,x\right\rangle\right| \\ &\geq \left|\lambda\right|\left\langle Kx,x\right\rangle-\left|\lambda\right|\left\langle Kx,x\right\rangle+\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle-\left|\lambda\right|q\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle \\ &\geq \left(1-\left|\lambda\right|q\right)\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle.\end{aligned}$$

Since $q<\frac{1}{\left|\lambda\right|}$, we have $q\left|\lambda\right|<1$, this gives $0<1-q\left|\lambda\right|<1$, hence $\exists\alpha_0\in\left]0,1\right[$ with

$$1-q\left|\lambda\right|\geq\alpha_0,$$

$A_{n-1}+bK$ is a positive-definite operator, i.e.,

$$\forall x\in\mathcal{D}\left(A_{n-1}+bK\right),\ \exists C_0>0:\ \left\langle\left(A_{n-1}+bK\right)x,x\right\rangle\geq C_0\left\langle x,x\right\rangle,$$

which implies

$$\begin{aligned}\left|\left\langle\left(\lambda A_n+A_{n-1}\right)x,x\right\rangle\right| &\geq \left(1-\left|\lambda\right|q\right)\left\langle\left(A_{n-1}+bK\right)x,x\right\rangle \\ &\geq \alpha_0C_0\left\langle x,x\right\rangle.\end{aligned}$$

Hence

$$\left|\left\langle\left(\lambda A_n+A_{n-1}\right)x,x\right\rangle\right|\geq C_1\left\langle x,x\right\rangle, \tag{15}$$

$$\lambda \in \mathcal{G}, \quad x \in \mathcal{D}(K),$$

with $C_1 = 0 + \alpha_0 C_0$. Now, we have

$$(\lambda A_n + A_{n-1})^* = \lambda K + \lambda C^* + A_{n-1}.$$

Then $A_{n-1} + bK$ is a positive self-adjoint operator and C^* is a symmetric operator, therefore

$$\mathcal{D}(A_{n-1} + bK) = \mathcal{D}(K) \subset \mathcal{D}(C^*)$$

and

$$\begin{aligned} \|C^*\| &= \left\| C^* (A_{n-1} + bK)^{-1} (A_{n-1} + bK) \right\| \\ &\leq \left\| C^* (A_{n-1} + bK)^{-1} \right\| \|(A_{n-1} + bK)\| \\ &\leq q \|(A_{n-1} + bK)\|. \end{aligned}$$

Using Theorem 4.12 in [9], we can easily get

$$|\langle C^* x, x \rangle| \leq q \langle (A_{n-1} + bK) x, x \rangle. \tag{16}$$

In the same way as in the proof of inequality (15) and using the relation (16), we can prove the following:

$$\left| \langle (\lambda A_n + A_{n-1})^* x, x \rangle \right| \geq C_2 \langle x, x \rangle, \quad \lambda \in \mathcal{G}, \ x \in \mathcal{D}(K). \tag{17}$$

In addition, using (15) and (17), we get the existence and the boundedness of $(\lambda A_n + A_{n-1})^{-1}$.

Furthermore, we have

$$\mathcal{D}\left((\lambda A_n + A_{n-1})^*\right) = \mathcal{D}(K)$$

and

$$\mathcal{D}(\lambda A_n + A_{n-1}) = \mathcal{D}(K)$$

with

$$\mathcal{D}\left((\lambda A_n + A_{n-1})^*\right) = \mathcal{D}(\lambda A_n + A_{n-1}).$$

So, $\forall x \in \mathcal{D}(\lambda A_n + A_{n-1})$,

$$\begin{aligned} \langle (\lambda A_n + A_{n-1}) x, x \rangle &= \langle (\lambda K + \lambda C + A_{n-1}) x, x \rangle \\ &= |\lambda| \langle Kx, x \rangle + |\lambda| \langle Cx, x \rangle + \langle A_{n-1} x, x \rangle. \end{aligned}$$

As K is a self-adjoint operator and A_{n-1} is a symmetric operator, we can write

$$\begin{aligned} \langle (\lambda A_n + A_{n-1}) x, x \rangle &= |\lambda| \langle x, Kx \rangle + |\lambda| \langle x, Cx \rangle + \langle x, A_{n-1} x \rangle, \\ &= \langle x, (\lambda A_n + A_{n-1}) x \rangle. \end{aligned}$$

This implies that $(\lambda A_n + A_{n-1})$ is a self-adjoint operator and $(\lambda A_n + A_{n-1})^{-1}$ exists and is bounded on $R(\lambda A_n + A_{n-1})$. According to Theorem 2.9, we conclude that

$R(\lambda A_n + A_{n-1})$ is dense in H and $\mathcal{D}(\lambda A_n + A_{n-1})$ is dense in H. From Theorem 2.9, we find that $\left((\lambda A_n + A_{n-1})^*\right)^{-1}$ exists with

$$\left((\lambda A_n + A_{n-1})^*\right)^{-1} = \left((\lambda A_n + A_{n-1})^{-1}\right)^* = (\lambda A_n + A_{n-1})^{-1}. \tag{18}$$

According to (15), (17) and (18), and for $\mathcal{Q}(\lambda) = \left((\lambda A_n + A_{n-1})^*\right)^{-1}$, there exists a constant $C_3 > 0$ such that

$$\|\mathcal{Q}(\lambda)\| \leq C_3, \quad \lambda \in \mathcal{G}. \tag{19}$$

Use $\mathcal{D}(C^*) \subset \bigcap_{j=0}^{n-2} \mathcal{D}(A^*)_j$ and $\mathcal{D}(K) \subset \mathcal{D}(C)$, where C is a symmetric operator and any $C \subset C^*$, so $\mathcal{D}(K) \subset \bigcap_{j=0}^{n-2} \mathcal{D}(A^*)_j$ and

$$\mathcal{D}(\mathcal{P}^*(\lambda)) = \mathcal{D}(\lambda K + \lambda C + A_{n-1})^* = \mathcal{D}(\lambda K + \lambda C^* + A_{n-1}) = \mathcal{D}(K).$$

Hence

$$\mathcal{D}(\mathcal{P}^*(\lambda)) \subset \bigcap_{j=0}^{n-2} \mathcal{D}(A^*)_j.$$

As $\mathcal{P}^*(\lambda)$ and A_j^* are closed, and according to Remark 1.5 in [9], we have A_j^* are $\mathcal{P}^*(\lambda)$-bounded, where $j = 0, ..., n-2$, this gives

$$\left\|A_j^*\right\| \leq a + b\,\|\mathcal{P}^*(\lambda)\|. \quad \lambda \in \mathcal{G} \;\; a > 0, \;\; b > 0, \;\; j = 0, ..., n-2, \tag{20}$$

and thus

$$\begin{aligned} \left\|A_j^* \mathcal{Q}(\lambda)\right\| &\leq \left\|A_j^*\right\| \|\mathcal{Q}(\lambda)\| \\ &\leq (a + b\,\|\mathcal{P}^*(\lambda)\|)\|\,\mathcal{Q}(\lambda)\|, \\ &\leq a\,\|\mathcal{Q}(\lambda)\| + b\,\|\mathcal{P}^*(\lambda)\|\,\|\mathcal{Q}(\lambda)\|, \end{aligned}$$

where $j = 0, ..., n-2$. We have

$$\mathcal{Q}(\lambda) = (\mathcal{P}^*(\lambda))^{-1},$$

so

$$\|\mathcal{P}^*(\lambda)\|\,\|\mathcal{Q}(\lambda)\| = 1.$$

Therefore

$$\left\|A_j^* \mathcal{Q}(\lambda)\right\| \leq a\,\|\mathcal{Q}(\lambda)\| + b, \quad \lambda \in \mathcal{G}. \tag{21}$$

From (19) and (21), we find that $\exists C_4 > 0$, where

$$\left\|A_j^* \mathcal{Q}(\lambda)\right\| \leq C_4, \quad \lambda \in \mathcal{G} \quad j = 0, ..., n-2, \tag{22}$$

and

$$\left\|A_j^* \mathcal{Q}(\lambda)\right\| = \left\|A_j^* \left(\mathcal{P}^{-1}(\lambda)\right)^*\right\| = \left\|\left(\mathcal{P}^{-1}(\lambda) A_j\right)^*\right\| = \left\|\left(\mathcal{P}^{-1}(\lambda) A_j\right)\right\|$$

so that

$$\left\|\left(\mathcal{P}^{-1}(\lambda) A_j\right)\right\| \leq C_4, \ \ \lambda \in \mathcal{G}(b, \theta), \ j=0, ..., n-2. \tag{23}$$

We have $\mathcal{D}(K) \subset \mathcal{D}(A_{n-1})$ and $\mathcal{D}(\mathcal{P}^*(\lambda)) = \mathcal{D}(K)$. If A_{n-1} is a symmetric operator, then

$$\mathcal{D}(\mathcal{P}^*(\lambda)) \subset \mathcal{D}(A_{n-1}) \subset \mathcal{D}\left(A_{n-1}^*\right).$$

As A_{n-1}^* is closable, so A_{n-1}^* is $\mathcal{P}^*(\lambda)$-bounded, therefore

$$\left\|A_{n-1}^*\right\| \leq c + d\left\|\mathcal{P}^*(\lambda)\right\|, \ \ \lambda \in \mathcal{G}, c > 0, \ d > 0. \tag{24}$$

This implies

$$\begin{aligned} \left\|A_{n-1}^* \mathcal{Q}(\lambda)\right\| & \leq \left\|A_{n-1}^*\right\| \left\|\mathcal{Q}(\lambda)\right\| \\ & \leq (c + d\left\|\mathcal{P}^*(\lambda)\right\|) \left\|\mathcal{Q}(\lambda)\right\|, \\ & \leq c\left\|\mathcal{Q}(\lambda)\right\| + d\left\|\mathcal{P}^*(\lambda)\right\| \left\|\mathcal{Q}(\lambda)\right\|, \end{aligned}$$

which we can rewrite as

$$\left\|A_{n-1}^* \mathcal{Q}(\lambda)\right\| \leq C_5, \ \ \lambda \in \mathcal{G},$$

where $C_5 = c\left\|\mathcal{Q}(\lambda)\right\| + d$. Therefore $\lambda \in G(b, \theta)$, and we have

$$\left\|\mathcal{P}^{-1}(\lambda) A_{n-1}\right\| \leq C_5, \ \ \lambda \in \mathcal{G}. \tag{25}$$

Using (23) and (25), we find that

$$\left\|\left(\mathcal{P}^{-1}(\lambda) A_j\right)\right\| \leq C_6, \ \ \lambda \in G(b, \theta), \ j=0, ..., n-1, \tag{26}$$

where $C_6 = \max(C_4, C_5)$. Moreover, the use of (23) leads to the estimate

$$\left\|\left(\mathcal{P}^{-1}(\lambda) A_j\right)\right\| \leq C_4 \left|\lambda\right|^{q_j}, \ \ \lambda \in G(b, \theta), \ j=0, ..., n-2, \tag{27}$$

where $q_j = 0$. Using (26), we get the estimate

$$\left\|\left(\mathcal{P}^{-1}(\lambda) A_j\right)\right\| \leq C_6 \left|\lambda\right|^{n-p}, \ \ \lambda \in G(b, \theta), \ j=0, ..., n-1. \tag{28}$$

For $n = p$, according to (27), (28) and Corollary 2.4, we obtain the full-fold exponential well-posedness and full-fold well-posedness of the Cauchy problem.

5 Conclusion

In this paper, we found the sufficient conditions for the operators Aj in (1), in the case of the Hilbert space, that guarantee the conditions (4), (5) even if A_n is not bounded and self-adjoint. As a result, the problem (1), (2) is full-fold exponentially well-posed. There will always be attempts to find the least possible sufficient conditions to fulfill the multi well-posedness.

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