# Multiple Well-Posedness of Higher-Order Abstract Cauchy Problem 

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#### Abstract

In this paper, we fulfill some conditions to examine the multiple wellposedness conditions that define the continuous dependence of the solutions and their derivatives on the initial data of the Cauchy problem. Indeed, for the differential operator equation of arbitrary order in a Hilbert space, an appropriate condition is given for the two main operators that assert the multiple well-posedness. Our results are new and complement some previous ones in the literature.


Keywords: abstract Cauchy problems; asymptotic stability; integrated semi-groups; stability of nonlinear problems in mechanics; well-posedness.

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## 1 Introduction

In [21, Vlasenko et al. studied the p-fold well-posedness of the higher-order abstract Cauchy problem of the following form:

$$
\begin{gather*}
\sum_{j=0}^{n} A_{j} \frac{d^{j} u}{d t^{j}}=0, \quad t>0,  \tag{1}\\
u^{j}(0)=u_{j}, \quad j=0, \ldots, n-1, \tag{2}
\end{gather*}
$$

where $A_{j}(j=0, \ldots, n)$ are linear closed operators from a complex Banach space $\mathcal{E}$ into

[^0]another complex Banach space $\mathcal{F}$ with the domain of definition $\mathcal{D}\left(A_{j}\right) \subset \mathcal{E}, A_{n}$ can be a degenerate operator.

Many real phenomena can be modeled using problems of the form of (1), (2). For example, many initial-value or initial-boundary value problems for partial differential equations arise in mechanics, physics, engineering, control theory, etc., in particular, the description of vibrations (see [5], [13], [2]) or the vibrations of a viscoelastic pipeline [14]. The theory of problems of this kind is also connected with many other branches of mathematics, which translates the importance of their study for both theoretical investigations and practical applications.

Over the past half-century, (11), (2) has been studied extensively by many scholars. Especially, for the first-order abstract Cauchy problem, the theory (or closely related operator semigroup theory) has evolved relatively well since the well-known Hille-Yosida theorem was presented in 1948, and is well documented in the monographs (see [6], 15] and other), the study of the second-order abstract Cauchy problem has also received much attention. For more information, we can refer to 10$],[18, ~[20$, and references cited therein. Since integrated semi-groups were introduced at the end of the eighties of the last century, it has become possible to deal with the ill-posed first-order abstract Cauchy problems (see [16], [11], [8]).

In 2004, Vlasenko et al. 21] carried out one of the latest and significant research on establishing conditions that ensure the p-fold well-posedness of the Cauchy problem (1), (2). The study imposed that $A_{n}$ is a bounded operator and $A_{n-1}=F+B$, where $F$ and $B$ are two operators fulfilling certain criteria that lead to guarantee the p-fold well-posedness in the case of the Hilbert space.

Motivated by the aforementioned works, this paper aims to establish new sufficient conditions that ensure the p-fold well-posedness of the problem (1), (2) even when $A_{n}$ is a non bounded operator and when the requirements on the two main operators $A_{n}$ and $A_{n-1}$ are different from those in 21]. The theorem presented in this paper is new and extends and improves previously known results.

Due to its great importance and its many applications in various mathematical fields such as nonlinear dynamics and systems theory (see [1] [22, [3]), many scholars have devoted a great deal of attention to this topic. For example, Vlasenko and his collaborators have published several works in this direction, notably in the case of nonlinear operators. As a result, their interesting outcomes inspired us to find new criteria that guarantee the stability of nonlinear dynamics.

The paper is organized as follows. In Section 2, we introduce some necessary definitions and preliminary results that play a crucial role in establishing our main outcomes. In Section 3, we present some previous global existence results which inspired us to investigate problem (11), (2). Finally, we state and prove our main results in Section 4.

## 2 Preliminaries

Definition 2.1 (see 21) We call a solution of problem (1), (2) any function $u$ satisfying condition (2) and

$$
u \in C^{n}((0, \infty), \mathcal{E}) \bigcap C^{n-1}([0, \infty), \mathcal{E})
$$

so that

$$
A_{j} u \in C^{j}((0, \infty), \mathcal{F}) \cap C^{j-1}([0, \infty), \mathcal{F}), \quad j=1, \ldots, n
$$

Definition 2.2 (see [21]) The Cauchy problem (1), (22) is $p$-fold well-posed $(p \in\{1, \ldots, n\})$ if and only if for any of its solutions $u$, the following estimate is true:

$$
\begin{equation*}
\left\|u^{p-1}(t)\right\| \leq F(t) \sum_{j=0}^{n-1}\left\|u_{j}\right\|, \quad t \geq 0 \tag{3}
\end{equation*}
$$

where $F$ is a non-negative function from $\mathbb{R}_{+}$to $\mathbb{R}_{+}$.
Definition 2.3 (see 21) If $F$ is a locally bounded function on $\mathbb{R}_{+}$the problem (1), (2) is $p$-fold uniformly well-posed.

Definition 2.4 (see (21) The problem (1), (2) is said to be $p$-fold exponentially well-posed if $F(t)=C e^{\omega t}$, where $C \geq 0$ and $\omega \geq 0$.

Remark 2.1 (see 21]) If $p=n$, the problem (1], 22) is full-fold well-posed.
Examples. Consider the problem of the vibrating string

$$
\begin{gathered}
\frac{\partial^{2} u}{\partial x^{2}}-C^{2} \frac{\partial^{2} u}{\partial t^{2}}=0, \quad t>0,-\infty<x<\infty \\
u(x, 0)=f(x), \quad u_{t}(x, 0)=0
\end{gathered}
$$

The solution of this problem is given by

$$
u(x, t)=\frac{1}{2}[f(x-C t)+f(x+C t)]
$$

Then

$$
u(x, t)=\frac{1}{2}[G(-C t) f(x)+G(C t) f(x)]
$$

where $\{G(t)\}$ is a semigroup of the operator $\frac{d}{d x}$. We have

$$
\begin{aligned}
\|u(x, t)\| & =\left\|\frac{1}{2}[G(-C t) f(x)+G(C t) f(x)]\right\| \\
& \leq \frac{1}{2}\|G(-C t)\|\|f(x)\|+\|G(C t)\|\|f(x)\|
\end{aligned}
$$

$G(-C t)$ is strongly continuous, then there exist $M$ and $\omega$ such that

$$
\|G(-C t)\| \leq M \exp (-\omega C t)
$$

also

$$
\|G(C t)\| \leq M \exp (\omega C t)
$$

which implies that

$$
\|u(x, t)\| \leq \frac{1}{2}(\exp (\omega C t)+\exp (-\omega C t))\|f(x)\|
$$

For

$$
F(t)=\max \left(\frac{1}{2}(\exp (\omega C t)+\exp (-\omega C t)), 1\right),
$$

we have

$$
\begin{aligned}
& \|u(x, t)\| \leq F(t) \sum_{j=0}^{1}\left\|u_{j}\right\| \\
u_{t}(x, t)= & \frac{1}{2}\left[\frac{d G(-C t) f(x)}{d t}+\frac{d G(C t) f(x)}{d t}\right] \\
= & \frac{1}{2}[A G(-C t) f(x)+A G(C t) f(x)]
\end{aligned}
$$

As the operator $A$ is unbounded, then there is no nonnegative function $F$ such that

$$
\left\|u_{t}(x, t)\right\| \leq K(t) \sum_{j=0}^{1}\left\|u_{j}\right\|
$$

Hence, the problem of the vibrating string is well-posed but is not 2-fold well-posed.
Equation (11) is associated with the characteristic polynomial

$$
L(\lambda)=\sum_{j=0}^{n} \lambda^{j} A_{j}
$$

and its resolvent $\mathcal{R}(\lambda)=L^{-1}(\lambda)$. Let $\mathcal{G}$ be the angle in the complex plane given by

$$
\mathcal{G}=\mathcal{G}(b, \theta)=\left\{\lambda=b+r e^{i \gamma}, \quad|\gamma| \leq \pi-\theta, \quad r \geq 0\right\}, \quad 0<\theta<\frac{\pi}{2}
$$

and bounded by a pair of rays. The orientation of the boundary contour $\Lambda$ should be such that the area $\mathcal{G}$ is located on its left side when bypassing below see Figure $\mathbf{A}$.


A
Here, the joint properties of the two leading operators $A_{n-1}$ and $A_{n}$ in terms of the following linear bundle of operators $\mathcal{P}(\lambda)$ and its resolvent $\mathcal{S}(\lambda)$ are

$$
\begin{gathered}
\mathcal{P}(\lambda)=\lambda A_{n}+A_{n-1} \\
\mathcal{S}(\lambda)=\mathcal{P}^{-1}(\lambda)
\end{gathered}
$$

Theorem 2.1 (see [9], [12], [19]) $A$ is a closed linear operator from $\mathcal{E}$ into $\mathcal{E}$ so that $\|A\| \leq q<1$. Then $(I+A)$ is a reversible and $(I+A)^{-1}$ is a bounded operator.

Corollary 2.1 Suppose that $n>1$, the linear $\mathcal{D}(L)$ is dense in $\mathcal{E}$, the number $p$ belongs to the set $\{1, \ldots, n\}$ and, in a certain angle $\mathcal{G}(b, \theta), b>0$, the resolvent $\mathcal{S}(\lambda)$ of the bundle of leading operators satisfies the estimates

$$
\begin{equation*}
\left\|\mathcal{S}(\lambda) A_{j}\right\| \leq C|\lambda|^{n-p}, \quad j=0, \ldots, n-1 \tag{4}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\mathcal{S}(\lambda) A_{j}\right\| \leq C|\lambda|^{q_{j}}, \quad j=0, \ldots, n-2 \tag{5}
\end{equation*}
$$

with the constants $q_{j}<n-j-1$. Then the Cauchy problem (1), (2) is full-fold well-posed and $p$-fold exponentially well-posed.

In 21, for the proof, the authors postulated another Cauchy problem and presented the proof without that assumption. The proof is as follows.

Proof. For $\lambda \in \mathcal{G}(a, \theta)$,

$$
\begin{aligned}
L(\lambda) & =\lambda^{n} A_{n}+\lambda^{n-1} A_{n-1}+\sum_{j=0}^{n-2} \lambda^{j} A_{j} \\
& =\lambda^{n-1}\left(\lambda A_{n}+A_{n-1}\right)+\sum_{j=0}^{n-2} \lambda^{j} A_{j} \\
& =\lambda^{n-1} P(\lambda)+\sum_{j=0}^{n-2} \lambda^{j} A_{j} \\
& =\lambda^{n-1} P(\lambda)\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} P^{-1}(\lambda) A_{j}\right)
\end{aligned}
$$

so

$$
L(\lambda)=\lambda^{n-1} P(\lambda)\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_{j}\right)
$$

As the estimate (5) is verified, then

$$
\begin{aligned}
\left\|\sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_{j}\right\| & \leq \sum_{j=0}^{n-2}|\lambda|^{j-n+1}\left\|S(\lambda) A_{j}\right\| \\
& \leq \sum_{j=0}^{n-2}|\lambda|^{j-n+1} C|\lambda|_{j}^{q} \\
& \leq C \sum_{j=0}^{n-2}|\lambda|^{j-n+1+q_{j}}
\end{aligned}
$$

we have $q_{j}<n-j-1$, so $j-n+1+q_{j}<0$, and as $\lambda \in \mathcal{G}(a, \theta)$, this implies that $\left(\frac{1}{|\lambda|}\right)^{-j+n-1-q_{j}}<\left(\frac{1}{a_{0} \sin \theta}\right)^{-j+n-1-q_{j}}, j=0, \cdots, n-2$. So

$$
\begin{aligned}
\left\|\sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_{j}\right\| & <C\left(\left(\frac{1}{|\lambda|}\right)^{n-1-q_{0}}+\left(\frac{1}{|\lambda|}\right)^{n-2-q_{1}}+\cdots+\left(\frac{1}{|\lambda|}\right)^{1-q_{n-2}}\right) \\
& <C\left(\left(\frac{1}{a_{0} \sin \theta}\right)^{n-1-q_{0}}+\left(\frac{1}{a_{0} \sin \theta}\right)^{n-2-q_{1}}+\cdots\right. \\
& \left.+\left(\frac{1}{a_{0} \sin \theta}\right)^{1-q_{n-2}}\right)
\end{aligned}
$$

for $a_{0} \sin \theta \leq 1$, we let

$$
h=\max \left\{n-1-q_{0}, n-2-q_{0}, \cdots, 1-q_{n-2}\right\}
$$

and if $a_{0} \sin \theta \geq 1$, we let

$$
h=\min \left\{n-1-q_{0}, n-2-q_{0}, \cdots, 1-q_{n-2}\right\},
$$

in any case, we have

$$
\left\|\sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_{j}\right\|<C\left((n-1)\left(\frac{1}{a_{0} \sin \theta}\right)^{h}\right)
$$

For $a_{0}=\frac{(C(n-1))^{\frac{1}{h}}}{\sin \theta}$, we get

$$
C\left((n-1)\left(\frac{1}{a_{0} \sin \theta}\right)^{h}\right)=1,
$$

therefore $\mathcal{G}_{0}=\mathcal{G}\left(\frac{(C(n-1))^{\frac{1}{h}}}{\sin \theta}, \theta\right) \subset \mathcal{G}(a, \theta)$ so that

$$
\left\|\sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_{j}\right\|<1
$$

According to Theorem 2.6, we conclude that

$$
\gamma(\lambda)=\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_{j}\right)^{-1}
$$

exists and is bounded in $\mathcal{G}_{0}$. We have also

$$
\begin{aligned}
\left\|L^{-1}(\lambda) A_{j} x\right\| & =\left\|\left(\lambda^{n-1} P(\lambda)\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} P^{-1}(\lambda) A_{j}\right)\right)^{-1} A_{j} x\right\| \\
& =\left\|\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} P^{-1}(\lambda) A_{j}\right)^{-1} \lambda^{1-n} P^{-1}(\lambda) A_{j} x\right\| \\
& \leq\left\|\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} P^{-1}(\lambda) A_{j}\right)^{-1}\right\||\lambda|^{1-n}\left\|P^{-1}(\lambda) A_{j}\right\|\|x\| \\
& \leq\left\|\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} P^{-1}(\lambda) A_{j}\right)^{-1}\right\||\lambda|^{1-n}\left\|S(\lambda) A_{j}\right\|\|x\| \\
& \leq\left\|\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} P^{-1}(\lambda) A_{j}\right)^{-1}\right\||\lambda|^{1-n} C|\lambda|^{n-p}\|x\|
\end{aligned}
$$

and as $\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_{j}\right)^{-1}$ exists and is bounded in $\mathcal{G}_{0}$, then there is $K>0$ such that

$$
\left\|\left(I+\sum_{j=0}^{n-2} \lambda^{j-n+1} S(\lambda) A_{j}\right)^{-1}\right\| \leq K
$$

so that

$$
\left\|L(\lambda)^{-1} A_{j} x\right\| \leq K C|\lambda|^{1-n+n-p}\|x\|
$$

This implies that there exists $K_{1}=C K$ such that

$$
\left\|L(\lambda)^{-1} A_{j} x\right\| \leq K_{1}|\lambda|^{1-p}\|x\|
$$

and hence

$$
\left\|R(\lambda) A_{j} x\right\| \leq \frac{K_{1}}{|\lambda|^{1-p}}\|x\|
$$

According to Theorem 4 in 21, the result is proved.
We conclude that two conditions (4), (5) are substitutes for condition (19) in 21]. By this corollary, we get the full-fold uniform well-posedness of the initial boundary-value problems that describe small vibrations of an elastic bar in 13 .

Theorem 2.2 (see [9], 12], [19]) A is a self-adjoint operator and $A^{-1}$ exists and is a bounded operator on $R(A)$. Then $\overline{R(A)}=\mathcal{H}$ and $A^{-1}$ is a self-adjoint operator.

Theorem 2.3 (see [9], [12], [19]) Suppose $\mathcal{D}(A)$ and $R(A)$ are dense in $\mathcal{H}$ and $A^{-1}$ exists. Then $\left(A^{*}\right)^{-1}$ exists and $\left(A^{*}\right)^{-1}=\left(A^{-1}\right)^{*}$.

## 3 Global Existence Result

In this part, we display some existing theorems 21 that set some conditions which guarantee two inequalities (4) and (5) so that to achieve the full-fold exponentially wellposedness in the case of the Hilbert space $\mathcal{E}=\mathcal{F}=\mathcal{H}$ and for the closed operators $A_{j}$ with the domains of definitions $\mathcal{D}\left(A_{j}\right)$ being dense in $\mathcal{H}$.

Theorem 3.1 Suppose that $n>1, A_{n}$ is a nonnegative bounded operator and $A_{n-1}=$ $F+B$, where $F$ is a self-adjoint operator and $B$ is a symmetric or skew-symmetric operator. Suppose that the linear $\bigcap_{j=0}^{n-1} \mathcal{D}\left(A_{j}\right)$ is dense in $H, \mathcal{D} F \subset \bigcap_{j=0}^{n-2} \mathcal{D}\left(A_{j}^{*}\right) \cap \mathcal{D}(B)$, for certain $b \geq 0$, the operator $F+b A_{n}$ is positive-definite (with a positive lower bound), and

$$
\begin{equation*}
q=\left\|B\left(F+b A_{n}\right)^{-1}\right\|<1 \tag{6}
\end{equation*}
$$

Then the Cauchy problem (1), (2) is full-fold well-posed and ( $n$-1)-fold exponentially wellposed. In addition, if $A_{n}$ is positive-definite, then the problem is full-fold exponentially well-posed.

We can find another condition that guarantees the same results as the previous theorem.

Theorem 3.2 Suppose that $n>1, D(L)$ is dense in $\mathcal{H}, A_{n}=A_{n}^{*} \geq 0, A_{n-1}=$ $F+B$, where $F=F^{*}, B$ is a symmetric or skew-symmetric operator, and for a certain number $b \geq 0$, estimate (6) is true, furthermore, $F+b A_{n} \geq m I>0$,

$$
\mathcal{D}\left(\left(F+b A_{n}\right)^{\frac{1}{2}}\right) \subset \mathcal{D}\left(A_{n}\right) \cap \mathcal{D}(B) \cap\left(\bigcap_{j=0}^{n-2} \mathcal{D}\left(A_{n}^{*}\right)\right)
$$

Then the Cauchy problem (1), (2) is full-fold well-posed and (n-1)-fold exponentially wellposed. In addition, if the operator $A_{n}$ is positive-definite and bounded, then the problem is full-fold exponentially well-posed.

## 4 Main Result

In this section, we give the main result of the paper. We will study the case where $A_{n}$ is an unbounded and non self-adjoint operator.

Theorem 4.1 Suppose that $n>1, A_{n}=K+C$, where $K$ is a positive self-adjoint operator and $C$ is a symmetric operator, and $A_{n-1}$ is a symmetric operator such that the linear $\bigcap_{j=0}^{n-1} \mathcal{D}\left(A_{j}\right)$ is dense in $\mathcal{H}$ so that

$$
\begin{equation*}
\mathcal{D}\left(C^{*}\right) \subset \bigcap_{j=0}^{n-2} \mathcal{D}\left(A_{j}^{*}\right) \cap \mathcal{D}\left(A_{n-1}\right), \mathcal{D}(K) \subset \mathcal{D}(C) \tag{7}
\end{equation*}
$$

and for a certain $b \geq 0$, the operator $A_{n-1}+b K$ is positive-definite, we have

$$
\begin{equation*}
q=\left\|C\left(A_{n-1}+b K\right)^{-1}\right\| \leq \frac{1}{|\lambda|} \tag{8}
\end{equation*}
$$

and

$$
\begin{equation*}
\left\|\left(A_{n-1}+b K\right) K^{-1}\right\| \leq \frac{1}{b} \tag{9}
\end{equation*}
$$

Then the Cauchy problem (1), (2) is full-fold well-posed and full-fold exponentially wellposed.

Proof. For $\lambda \in G(b, \theta)$, we have

$$
\begin{align*}
\mathcal{D}\left(\left(\lambda A_{n}+A_{n-1}\right)^{*}\right) & =\mathcal{D}\left(\left(\lambda K+\lambda C+A_{n-1}\right)^{*}\right) \\
& =\mathcal{D}(\lambda K) \cap \mathcal{D}(\lambda C) \cap \mathcal{D}\left(A_{n-1}^{*}\right) . \tag{10}
\end{align*}
$$

Because $C$ is a symmetric operator, we have $\mathcal{D}(C) \subset \mathcal{D}\left(C^{*}\right)$ and $\mathcal{D}(K) \subset \mathcal{D}(C)$, which means that

$$
\mathcal{D}(K) \subset \mathcal{D}(C) \subset \mathcal{D}\left(C^{*}\right)
$$

Then

$$
\mathcal{D}\left(\left(\lambda K+\lambda C+A_{n-1}\right)^{*}\right)=\mathcal{D}(K) \cap \mathcal{D}\left(A_{n-1}^{*}\right)
$$

and

$$
\mathcal{D}(K) \subset \mathcal{D}\left(C^{*}\right) \subset \mathcal{D}\left(A_{n-1}^{*}\right)
$$

Thus, we have

$$
\begin{equation*}
\mathcal{D}\left(\left(\lambda K+\lambda C+A_{n-1}\right)^{*}\right)=\mathcal{D}(K) \tag{11}
\end{equation*}
$$

and

$$
\begin{aligned}
\mathcal{D}\left(\left(\lambda K+\lambda C^{*}+A_{n-1}\right)\right) & =\mathcal{D}(K) \cap \mathcal{D}\left(C^{*}\right) \cap \mathcal{D}\left(A_{n-1}\right) \\
& =\mathcal{D}(K) \cap \mathcal{D}\left(A_{n-1}\right)
\end{aligned}
$$

As $C$ is a symmetric operator and using (7), we have

$$
\mathcal{D}(K) \subset \mathcal{D}(C) \subset \mathcal{D}\left(C^{*}\right) \subset \mathcal{D}\left(A_{n-1}\right)
$$

which implies

$$
\begin{equation*}
\mathcal{D}\left(\left(\lambda K+\lambda C^{*}+A_{n-1}\right)\right)=\mathcal{D}(K) . \tag{12}
\end{equation*}
$$

Furthermore, according to (11) and (12), we have

$$
\mathcal{D}\left(\left(\lambda K+\lambda C+A_{n-1}\right)^{*}\right)=\mathcal{D}\left(\lambda K+\lambda C^{*}+A_{n-1}\right)
$$

In addition, $\forall x \in \mathcal{D}\left(\lambda K+\lambda C^{*}+A_{n-1}\right), C$ and $A_{n-1}$ are symmetric operators and $K$ is a self-adjoint operator, we can therefore write

$$
\begin{aligned}
\left\langle\left(\lambda K+\lambda C^{*}+A_{n-1}\right) x, x\right\rangle & =\left\langle\left(\lambda K+\lambda C^{*}+A_{n-1}^{*}\right) x, x\right\rangle \\
& =\left\langle\left(\lambda K+\lambda C+A_{n-1}\right)^{*} x, x\right\rangle .
\end{aligned}
$$

As $A_{n-1}+b K$ is a positive operator, we have

$$
\begin{aligned}
\mathcal{D}\left(\left(A_{n-1}+b K\right)^{*}\right) & =\mathcal{D}\left(A_{n-1}^{*}\right) \cap \mathcal{D}(K) \\
& =\mathcal{D}(K)
\end{aligned}
$$

We have also

$$
\begin{aligned}
\mathcal{D}\left(\left(A_{n-1}+b K\right)\right) & =\mathcal{D}\left(A_{n-1}\right) \cap \mathcal{D}(K) \\
& =\mathcal{D}(K)
\end{aligned}
$$

Therefore

$$
\mathcal{D}\left(\left(A_{n-1}+b K\right)^{*}\right)=\mathcal{D}\left(A_{n-1}+b K\right)
$$

Let us remind that $A_{n-1}$ are symmetric operators and $K$ is a self-adjoint operator, then $\forall x, y \in \mathcal{D}\left(\left(A_{n-1}+b K\right)^{*}\right)$,

$$
\left\langle\left(A_{n-1}+b K\right) x, y\right\rangle=\left\langle x,\left(A_{n-1}+b K\right) y\right\rangle .
$$

This implies that $A_{n-1}+b K$ is a self-adjoint operator, and as $C$ is a symmetric operator, we get

$$
\mathcal{D}\left(\left(A_{n-1}+b K\right)\right)=\mathcal{D}(K)
$$

As $\mathcal{D}(K) \subset \mathcal{D}(C)$, we have

$$
\mathcal{D}\left(\left(A_{n-1}+b K\right)\right) \subset \mathcal{D}(C)
$$

and

$$
\begin{aligned}
\|C\| & =\left\|C\left(A_{n-1}+b K\right)^{-1}\left(A_{n-1}+b K\right)\right\| \\
& \leq\left\|C\left(A_{n-1}+b K\right)^{-1}\right\|\left\|\left(A_{n-1}+b K\right)\right\| \\
& \leq q\left\|\left(A_{n-1}+b K\right)\right\|
\end{aligned}
$$

As $K$ is a self-adjoint operator, $\mathcal{D}\left(A_{n-1}+b K\right)=\mathcal{D}(K)$ and

$$
\begin{aligned}
\left\|\left(A_{n-1}+b K\right)\right\| & =\left\|\left(A_{n-1}+b K\right) K^{-1} K\right\| \\
& \leq\left\|\left(A_{n-1}+b K\right) K^{-1}\right\|\|K\| \\
& \leq \frac{1}{b}\|K\| \\
& \leq\|b K\|
\end{aligned}
$$

Then, according to Theorem 4.12 in [9], we obtain

$$
\begin{gather*}
|\langle C x, x\rangle| \leq q\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle  \tag{13}\\
\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle \leq\langle b K x, x\rangle \tag{14}
\end{gather*}
$$

for all $x \in \mathcal{D}(K)$ and $\lambda \in G(b, \theta)$.
Using (13) and (14), for all $x \in \mathcal{D}(K)$ and for all $\lambda \in G(b, \theta)$, we get

$$
\begin{aligned}
\mid\left\langle\left(\lambda A_{n}\right.\right. & \left.\left.+A_{n-1}\right) x, x\right\rangle \mid \\
& =\left|\left\langle\left(\lambda K+\lambda C+A_{n-1}\right) x, x\right\rangle\right| \\
& \geq\left|\left\langle\left(\lambda K+A_{n-1}\right) x, x\right\rangle\right|-|\lambda||\langle C x, x\rangle| \\
& \geq\left|\left\langle\left(\lambda K+A_{n-1}-b K+b K\right) x, x\right\rangle\right|-|\lambda| q\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle \\
& \geq|\langle\lambda K x, x\rangle|-\left\langle\left(b K-\left(A_{n-1}+b K\right)\right) x, x\right\rangle-|\lambda| q\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle \\
& \geq|\langle\lambda K x, x\rangle|-\langle b K x, x\rangle+\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle-|\lambda| q\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle .
\end{aligned}
$$

As $|\lambda| \geq b$, then $-\langle b K x, x\rangle \geq-|\lambda|\langle K x, x\rangle$, which implies that $\left|\left\langle\left(\lambda A_{n}+A_{n-1}\right) x, x\right\rangle\right|$

$$
\begin{aligned}
& \geq|\lambda|\langle K x, x\rangle-|\lambda|\langle K x, x\rangle+\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle-|\lambda| q\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle \\
& \geq(1-|\lambda| q)\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle
\end{aligned}
$$

Since $q<\frac{1}{|\lambda|}$, we have $q|\lambda|<1$, this gives $0<1-q|\lambda|<1$, hence $\left.\exists \alpha_{0} \in\right] 0,1[$ with

$$
1-q|\lambda| \geq \alpha_{0}
$$

$A_{n-1}+b K$ is a positive-definite operator, i.e.,

$$
\forall x \in \mathcal{D}\left(A_{n-1}+b K\right), \exists C_{0}>0:\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle \geq C_{0}\langle x, x\rangle
$$

which implies

$$
\begin{aligned}
\left|\left\langle\left(\lambda A_{n}+A_{n-1}\right) x, x\right\rangle\right| & \geq(1-|\lambda| q)\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle \\
& \geq \alpha_{0} C_{0}\langle x, x\rangle
\end{aligned}
$$

Hence

$$
\begin{equation*}
\left|\left\langle\left(\lambda A_{n}+A_{n-1}\right) x, x\right\rangle\right| \geq C_{1}\langle x, x\rangle \tag{15}
\end{equation*}
$$

$$
\lambda \in \mathcal{G}, \quad x \in \mathcal{D}(K)
$$

with $C_{1}=0+\alpha_{0} C_{0}$. Now, we have

$$
\left(\lambda A_{n}+A_{n-1}\right)^{*}=\lambda K+\lambda C^{*}+A_{n-1}
$$

Then $A_{n-1}+b K$ is a positive self-adjoint operator and $C^{*}$ is a symmetric operator, therefore

$$
\mathcal{D}\left(A_{n-1}+b K\right)=\mathcal{D}(K) \subset \mathcal{D}\left(C^{*}\right)
$$

and

$$
\begin{aligned}
\left\|C^{*}\right\| & =\left\|C^{*}\left(A_{n-1}+b K\right)^{-1}\left(A_{n-1}+b K\right)\right\| \\
& \leq\left\|C^{*}\left(A_{n-1}+b K\right)^{-1}\right\|\left\|\left(A_{n-1}+b K\right)\right\| \\
& \leq q\left\|\left(A_{n-1}+b K\right)\right\|
\end{aligned}
$$

Using Theorem 4.12 in (9), we can easily get

$$
\begin{equation*}
\left|\left\langle C^{*} x, x\right\rangle\right| \leq q\left\langle\left(A_{n-1}+b K\right) x, x\right\rangle \tag{16}
\end{equation*}
$$

In the same way as in the proof of inequality 15 and using the relation (16) we can prove the following:

$$
\begin{equation*}
\left|\left\langle\left(\lambda A_{n}+A_{n-1}\right)^{*} x, x\right\rangle\right| \geq C_{2}\langle x, x\rangle, \quad \lambda \in \mathcal{G}, \quad x \in \mathcal{D}(K) \tag{17}
\end{equation*}
$$

In addition, using (15) and 17), we get the existence and the boundedness of $\left(\lambda A_{n}+A_{n-1}\right)^{-1}$.

Furthermore, we have

$$
\mathcal{D}\left(\left(\lambda A_{n}+A_{n-1}\right)^{*}\right)=\mathcal{D}(K)
$$

and

$$
\mathcal{D}\left(\lambda A_{n}+A_{n-1}\right)=\mathcal{D}(K)
$$

with

$$
\mathcal{D}\left(\left(\lambda A_{n}+A_{n-1}\right)^{*}\right)=\mathcal{D}\left(\lambda A_{n}+A_{n-1}\right)
$$

So, $\forall x \in \mathcal{D}\left(\lambda A_{n}+A_{n-1}\right)$,

$$
\begin{aligned}
\left\langle\left(\lambda A_{n}+A_{n-1}\right) x, x\right\rangle & =\left\langle\left(\lambda K+\lambda C+A_{n-1}\right) x, x\right\rangle \\
& =|\lambda|\langle K x, x\rangle+|\lambda|\langle C x, x\rangle+\left\langle A_{n-1} x, x\right\rangle
\end{aligned}
$$

As $K$ is a self-adjoint operator and $A_{n-1}$ is a symmetric operator, we can write

$$
\begin{aligned}
\left\langle\left(\lambda A_{n}+A_{n-1}\right) x, x\right\rangle & =|\lambda|\langle x, K x\rangle+|\lambda|\langle x, C x\rangle+\left\langle x, A_{n-1} x\right\rangle \\
& =\left\langle x,\left(\lambda A_{n}+A_{n-1}\right) x\right\rangle .
\end{aligned}
$$

This implies that $\left(\lambda A_{n}+A_{n-1}\right)$ is a self-adjoint operator and $\left(\lambda A_{n}+A_{n-1}\right)^{-1}$ exists and is bounded on $R\left(\lambda A_{n}+A_{n-1}\right)$. According to Theorem 2.9, we conclude that
$R\left(\lambda A_{n}+A_{n-1}\right)$ is dense in $H$ and $\mathcal{D}\left(\lambda A_{n}+A_{n-1}\right)$ is dense in $H$. From Theorem 2.9, we find that $\left(\left(\lambda A_{n}+A_{n-1}\right)^{*}\right)^{-1}$ exists with

$$
\begin{equation*}
\left(\left(\lambda A_{n}+A_{n-1}\right)^{*}\right)^{-1}=\left(\left(\lambda A_{n}+A_{n-1}\right)^{-1}\right)^{*}=\left(\lambda A_{n}+A_{n-1}\right)^{-1} \tag{18}
\end{equation*}
$$

According to 15, 17 and 18), and for $\mathcal{Q}(\lambda)=\left(\left(\lambda A_{n}+A_{n-1}\right)^{*}\right)^{-1}$, there exists a constant $C_{3}>0$ such that

$$
\begin{equation*}
\|\mathcal{Q}(\lambda)\| \leq C_{3}, \quad \lambda \in \mathcal{G} \tag{19}
\end{equation*}
$$

Use $\mathcal{D}\left(C^{*}\right) \subset \bigcap_{j=0}^{n-2} \mathcal{D}\left(A^{*}\right)_{j}$ and $\mathcal{D}(K) \subset \mathcal{D}(C)$, where $C$ is a symmetric operator and any $C \subset C^{*}$, so $\mathcal{D}(K) \subset \bigcap_{j=0}^{n-2} \mathcal{D}\left(A^{*}\right)_{j}$ and

$$
\mathcal{D}\left(\mathcal{P}^{*}(\lambda)\right)=\mathcal{D}\left(\lambda K+\lambda C+A_{n-1}\right)^{*}=\mathcal{D}\left(\lambda K+\lambda C^{*}+A_{n-1}\right)=\mathcal{D}(K) .
$$

Hence

$$
\mathcal{D}\left(\mathcal{P}^{*}(\lambda)\right) \subset \bigcap_{j=0}^{n-2} \mathcal{D}\left(A^{*}\right)_{j}
$$

As $\mathcal{P}^{*}(\lambda)$ and $A_{j}^{*}$ are closed, and according to Remark 1.5 in $\left[9\right.$, we have $A_{j}^{*}$ are $\mathcal{P}^{*}(\lambda)$ bounded, where $j=0, \ldots, n-2$, this gives

$$
\begin{equation*}
\left\|A_{j}^{*}\right\| \leq a+b\left\|\mathcal{P}^{*}(\lambda)\right\| . \quad \lambda \in \mathcal{G} a>0, b>0, j=0, \ldots, n-2 \tag{20}
\end{equation*}
$$

and thus

$$
\begin{aligned}
\left\|A_{j}^{*} \mathcal{Q}(\lambda)\right\| & \leq\left\|A_{j}^{*}\right\|\|\mathcal{Q}(\lambda)\| \\
& \leq\left(a+b\left\|\mathcal{P}^{*}(\lambda)\right\|\right)\|\mathcal{Q}(\lambda)\| \\
& \leq a\|\mathcal{Q}(\lambda)\|+b\left\|\mathcal{P}^{*}(\lambda)\right\|\|\mathcal{Q}(\lambda)\|
\end{aligned}
$$

where $j=0, \ldots, n-2$. We have

$$
\mathcal{Q}(\lambda)=\left(\mathcal{P}^{*}(\lambda)\right)^{-1}
$$

so

$$
\left\|\mathcal{P}^{*}(\lambda)\right\|\|\mathcal{Q}(\lambda)\|=1
$$

Therefore

$$
\begin{equation*}
\left\|A_{j}^{*} \mathcal{Q}(\lambda)\right\| \leq a\|\mathcal{Q}(\lambda)\|+b, \quad \lambda \in \mathcal{G} \tag{21}
\end{equation*}
$$

From (19) and 21, we find that $\exists C_{4}>0$, where

$$
\begin{equation*}
\left\|A_{j}^{*} \mathcal{Q}(\lambda)\right\| \leq C_{4}, \quad \lambda \in \mathcal{G} \quad j=0, \ldots, n-2 \tag{22}
\end{equation*}
$$

and

$$
\left\|A_{j}^{*} \mathcal{Q}(\lambda)\right\|=\left\|A_{j}^{*}\left(\mathcal{P}^{-1}(\lambda)\right)^{*}\right\|=\left\|\left(\mathcal{P}^{-1}(\lambda) A_{j}\right)^{*}\right\|=\left\|\left(\mathcal{P}^{-1}(\lambda) A_{j}\right)\right\|
$$

so that

$$
\begin{equation*}
\left\|\left(\mathcal{P}^{-1}(\lambda) A_{j}\right)\right\| \leq C_{4}, \quad \lambda \in \mathcal{G}(b, \theta), j=0, \ldots, n-2 \tag{23}
\end{equation*}
$$

We have $\mathcal{D}(K) \subset \mathcal{D}\left(A_{n-1}\right)$ and $\mathcal{D}\left(\mathcal{P}^{*}(\lambda)\right)=\mathcal{D}(K)$. If $A_{n-1}$ is a symmetric operator, then

$$
\mathcal{D}\left(\mathcal{P}^{*}(\lambda)\right) \subset \mathcal{D}\left(A_{n-1}\right) \subset \mathcal{D}\left(A_{n-1}^{*}\right)
$$

As $A_{n-1}^{*}$ is closable, so $A_{n-1}^{*}$ is $\mathcal{P}^{*}(\lambda)$-bounded, therefore

$$
\begin{equation*}
\left\|A_{n-1}^{*}\right\| \leq c+d\left\|\mathcal{P}^{*}(\lambda)\right\|, \quad \lambda \in \mathcal{G}, c>0, d>0 \tag{24}
\end{equation*}
$$

This implies

$$
\begin{aligned}
\left\|A_{n-1}^{*} \mathcal{Q}(\lambda)\right\| & \leq\left\|A_{n-1}^{*}\right\|\|\mathcal{Q}(\lambda)\| \\
& \leq\left(c+d\left\|\mathcal{P}^{*}(\lambda)\right\|\right)\|\mathcal{Q}(\lambda)\| \\
& \leq c\|\mathcal{Q}(\lambda)\|+d\left\|\mathcal{P}^{*}(\lambda)\right\|\|\mathcal{Q}(\lambda)\|
\end{aligned}
$$

which we can rewrite as

$$
\left\|A_{n-1}^{*} \mathcal{Q}(\lambda)\right\| \leq C_{5}, \quad \lambda \in \mathcal{G}
$$

where $C_{5}=c\|\mathcal{Q}(\lambda)\|+d$. Therefore $\lambda \in G(b, \theta)$, and we have

$$
\begin{equation*}
\left\|\mathcal{P}^{-1}(\lambda) A_{n-1}\right\| \leq C_{5}, \quad \lambda \in \mathcal{G} \tag{25}
\end{equation*}
$$

Using (23) and 25), we find that

$$
\begin{equation*}
\left\|\left(\mathcal{P}^{-1}(\lambda) A_{j}\right)\right\| \leq C_{6}, \quad \lambda \in G(b, \theta), j=0, \ldots, n-1 \tag{26}
\end{equation*}
$$

where $C_{6}=\max \left(C_{4}, C_{5}\right)$. Moreover, the use of 23 leads to the estimate

$$
\begin{equation*}
\left\|\left(\mathcal{P}^{-1}(\lambda) A_{j}\right)\right\| \leq C_{4}|\lambda|^{q_{j}}, \quad \lambda \in G(b, \theta), j=0, \ldots, n-2 \tag{27}
\end{equation*}
$$

where $q_{j}=0$. Using 26 , we get the estimate

$$
\begin{equation*}
\left\|\left(\mathcal{P}^{-1}(\lambda) A_{j}\right)\right\| \leq C_{6}|\lambda|^{n-p}, \quad \lambda \in G(b, \theta), j=0, \ldots, n-1 \tag{28}
\end{equation*}
$$

For $n=p$, according to $27,(28)$ and Corollary 2.4, we obtain the full-fold exponential well-posedness and full-fold well-posedness of the Cauchy problem.

## 5 Conclusion

In this paper, we found the sufficient conditions for the operators $A j$ in (1), in the case of the Hilbert space, that guarantee the conditions (4), (5) even if $A_{n}$ is not bounded and self-adjoint. As a result, the problem (17), (2) is full-fold exponentially well-posed. There will always be attempts to find the least possible sufficient conditions to fulfill the multi well-posedness.

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