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# Solvability of Nonlinear Elliptic Problems with Degenerate Coercivity in Weighted Sobolev Space

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**Abstract:** In this paper, we investigate the existence of our entropy solution for the nonlinear elliptic equation

$$-\operatorname{div}[\omega(x)a(x,u,\nabla u)] = f - \operatorname{div} F, \quad \text{in } \Omega,$$

in the setting of the weighted Sobolev space $W_0^{1,p}(\Omega,\omega)$. We focus on the case where the operator has a degenerate coercivity and $f \in L^1(\Omega)$, $F \in [L^{p'}(\Omega,\omega^{1-p'})]^N$.

**Keywords:** *nonlinear elliptic equations; degenerate coercivity; entropy solutions; weighted Sobolev spaces.*

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## 1 Introduction

Partial differential equations have many applications in various areas of engineering, mathematics, physics, and other applied sciences (see for instance [8, 20]). In the last years, there has been an increasing interest in the study of various mathematical problems in weighted Sobolev spaces motivated by many considerations in applications (see [1, 2, 5, 10, 11] and the references therein).

Let $\Omega$ be a bounded smooth subset of $\mathbb{R}^N$ with $N \geq 2$ and $1 < p < \infty$. We are interested in proving the existence of entropy solutions to the following elliptic Dirichlet problem:

$$(\mathcal{P}) \begin{cases} -\operatorname{div}[\omega(x)a(x,u,\nabla u)] = f - \operatorname{div} F, & \text{in } \Omega, \\ u(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

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Here $a(x,s,\xi) : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathédory function satisfying the following conditions:

$$a(x,s,\xi).\xi \geq \frac{\alpha}{(1+|s|)^{\theta(p-1)}}|\xi|^p \tag{1}$$

for some $\alpha > 0$ and some real number $\theta$ such that $0 \leq \theta < 1$.

As far as the datum $f$ and $F$ are concerned, we will assume that $f$ belongs to the space $L^1(\Omega)$, and $F \in [L^{p'}(\Omega, \omega^{1-p'})]^N$ with $(1/p + 1/p' = 1)$.

Problems like $(\mathcal{P})$ have been studied by many authors in the non-weighted case. In [18], Leone and Porretta studied the nonlinear elliptic problem

$$Bu = f(x) - \operatorname{div}(F) \quad \text{ in } \Omega$$

in the setting of Sobolev spaces, where $Bu = -\operatorname{div}(a(x,u,\nabla u))$ is a Leray-Lions operator from $W_0^{1,p}(\Omega)$ to $W^{-1,p'}(\Omega)$, they demonstrated the existence of entropy solutions. In addition, Alvino et al. [4] have proved that the nonlinear elliptic equations $-\operatorname{div}(a(x,u,\nabla u)) = f$ admit the entropy solutions under assumption (1).

Notice that the existence of a weak solution for the Dirichlet problem $(\mathcal{P})$ has been obtained by Cavalheiro in [12] under the condition

$$a(x,s,\xi).\xi \geq \alpha|\xi|^p, \tag{2}$$

and by assuming that $f/\omega \in L^{p'}(\Omega,\omega)$. Also, he discussed in [11] the existence of our entropy solution when $f \in L^1(\Omega)$.

Our objective in this work is to study the problem $(\mathcal{P})$ when the operator satisfies assumption (1) instead of (2). The main difficulties that arise in our study are due, on the one hand, to the fact that the differential operator $A(u) = -\operatorname{div}(\omega(x)a(x,u,\nabla u))$, which is well defined between $W_0^{1,p}(\Omega,\omega)$ and its dual $W^{-1,p'}(\Omega,\omega^{1-p'})$, may not be coercive on $W_0^{1,p}(\Omega,\omega)$ when $u$ is large. This imposes that, even if the datum is extremely regular, the classical methods used to demonstrate the existence of a solution to problem $(\mathcal{P})$ cannot be used.

On the other hand, $f$ only belongs to $L^1(\Omega)$, making it difficult to show the existence of a weak solution. To get around this difficulty, we will use in this paper the concept of entropy solutions. This concept was introduced in [7], and then used by many authors to study elliptic equations (see [2–4, 6, 10, 18]).

The structure of this paper is as follows. In Section 2, we recall some preliminary results which will be used later. In Section 3, we give the assumptions on the data, then we state the main results which will be proved in Section 4.

## 2 Preliminaries

In this section, we provide a brief facts about the weighted Sobolev space as well as some $\mathcal{A}_p$-weight features. Let $\omega = \omega(x)$ be a weight function, that is, $0 < \omega < \infty$, and a locally integrable function on $\mathbb{R}^N$. By integration, each weight $\omega$ generates a measure on the measurable subsets of $\mathbb{R}^N$. This measure is denoted by $\mu$ and defined as follows:

$$\mu(\mathcal{S}) = \int_{\mathcal{S}} \omega(x)dx$$

for a measurable set $\mathcal{S} \subset \mathbb{R}^N$.

**Definition 2.1** Let $\omega$ be a weight and $1 < p < \infty$. We say that $\omega$ belongs to $\mathcal{A}_p$-weight if there exists a positive constant $C_{\omega,p}$ such that, for every ball $\mathcal{B} \subset \mathbb{R}^N$,

$$\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \omega dx\right)\left(\frac{1}{|\mathcal{B}|}\int_{\mathcal{B}} \omega^{1/(1-p)} dx\right)^{p-1} \leq C_{\omega,p} \quad \text{if } p > 1,$$

where $|.|$ denotes the $N$-dimensional Lebesgue measure in $\mathbb{R}^N$.

**Lemma 2.1** *[16]. Let $\mathcal{B}$ be a ball in $\mathbb{R}^N$ and $\mathcal{S}$ be a measurable subset of $\mathcal{B}$. If $\omega \in \mathcal{A}_p, 1 < p < \infty$, then*

$$\left(\frac{|\mathcal{S}|}{|\mathcal{B}|}\right)^p \leq C_{\omega,p}\frac{\mu(\mathcal{S})}{\mu(\mathcal{B})}.$$

**Remark 2.1** If $\mu(\mathcal{S}) = 0$, then $|\mathcal{S}| = 0$. Thus, for every sequence $(u_n)$ in $\mathcal{B}$ that converges $\mu$-a.e. to some $u$, we have $u_n \longrightarrow u$ a.e.

The weighted Lebesgue space $L^p(\Omega, \omega)$ is defined for every weight $\omega$ and $1 \leq p < \infty$ by

$$L^p(\Omega, \omega) = \left\{u = u(x) : u\omega^{1/p} \in L^p(\Omega)\right\},$$

and it is endowed with the norm

$$\|u\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |u(x)|^p \omega(x) dx\right)^{1/p}.$$

**Definition 2.2** Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$, $1 < p < \infty$ and let $\omega$ be an $\mathcal{A}_p$-weight. The weighted Sobolev space $W^{1,p}(\Omega, \omega)$ is defined as the set of all functions $u \in L^p(\Omega, \omega)$ with weak derivatives $\frac{\partial u}{\partial x_i} \in L^p(\Omega, w)$, for all $i = 1, \dots, N$.

The norm of $u$ in $W^{1,p}(\Omega, \omega)$ is given by

$$\|u\|_{W^{1,p}(\Omega,\omega)} = \left(\int_\Omega |u|^p \omega(x) \mathrm{d}x + \int_\Omega |\nabla u|^p \omega(x) \mathrm{d}x\right)^{\frac{1}{p}}. \tag{3}$$

The space $W_0^{1,p}(\Omega, \omega)$ is defined as the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(\Omega,\omega)} = \left(\int_\Omega |\nabla u|^p \omega(x) \mathrm{d}x\right)^{\frac{1}{p}}. \tag{4}$$

A compact imbedding is required because we are working with compactness methods to find solutions to nonlinear elliptic equations. As a result, we assume also that the domain $\Omega$ is smooth.

**Theorem 2.1** *[13]. Let $\Omega$ be a bounded smooth domain. For $\omega \in \mathcal{A}_p$, we have the compact embedding*

$$W_0^{1,p}(\Omega, \omega) \hookrightarrow\hookrightarrow L^p(\Omega, \omega).$$

**Theorem 2.2** *[14]. Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$. Take $1 < p < \infty$ and a function $\omega \in \mathcal{A}_p$. There exist positive constants $C_\Omega$ and $\delta$ such that for all $u \in C_0^\infty(\Omega)$ and all $\eta$ satisfying $1 \leq \eta \leq \frac{N}{N-1} + \delta$,*

$$\|u\|_{L^{\eta p}(\Omega,\omega)} \leq C_\Omega \||\nabla u|\|_{L^p(\Omega,\omega)}. \tag{5}$$

**Definition 2.3** Let $\omega$ be a weight function and let $q$ be a positive real number. The weighted Marcinkievicz space $\mathcal{M}^q(\Omega,\omega)$ is the set of all measurable functions $f:\Omega\to\mathbb{R}$ such that the function

$$\Phi_f(k)=\mu(\{x\in\Omega:|f(x)|>k\})\quad k>0,$$

satisfies, for some positive constant $C$, an estimate of the form $\Phi_f(k)\le Ck^{-q}$.

**Remark 2.2** It follows from [17] that if $1\le q<p$ and $\Omega\subset\mathbb{R}^N$ is a bounded set, then

$$L^p(\Omega,\omega)\subset\mathcal{M}^p(\Omega,\omega)\text{ and }\mathcal{M}^p(\Omega,\omega)\subset L^q(\Omega,\omega).$$

## 3 Basic Assumptions and Main Result

### 3.1 Basic assumptions

Let $\Omega$ be an open bounded smooth domain of $\mathbb{R}^N(N\ge 2)$, $p>1$ and $\omega\in\mathcal{A}_p$. Let $a:\Omega\times\mathbb{R}\times\mathbb{R}^N\to\mathbb{R}^N$ be a Carathéodry function (that is, $a(.,s,\xi)$ is measurable on $\Omega$ for every $(t,\xi)$ in $\mathbb{R}\times\mathbb{R}^N$, and $a(x,.,.)$ is continuous on $\mathbb{R}\times\mathbb{R}^N$ for almost every $x$ in $\Omega$). Assume that

$$a(x,s,\xi)\cdot\xi\ge b(|s|)|\xi|^p \tag{6}$$

for almost every $x$ in $\Omega$ and for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where

$$b(s)=\frac{\alpha}{(1+s)^{\theta(p-1)}} \tag{7}$$

for some $0\le\theta<1$ and some $\alpha>0$;

$$|a(x,s,\xi)|\le l_0(x)+l_1(x)|s|^{p-1}+l_2(x)|\xi|^{p-1} \tag{8}$$

for almost every $x$ in $\Omega$ and for every $(s,\xi)\in\mathbb{R}\times\mathbb{R}^N$, where $l_0$, $l_1$ and $l_2$ are non-negative functions, with $l_0\in L^{p'}(\Omega,\omega)$ and $l_1,l_2\in L^\infty(\Omega)$;

$$[a(x,s,\xi)-a(x,s,\xi')]\cdot(\xi-\xi')>0 \tag{9}$$

for almost every $x$ in $\Omega$ and for every $s\in\mathbb{R}$, for every $\xi,\xi'$ for every $\xi,\xi'$ in $\mathbb{R}^N$ with $\xi\ne\xi'$. As regards the source term, we assume that

$$f\in L^1(\Omega)\quad\text{ and }\quad F\in[L^{p'}(\Omega,\omega^{1-p'})]^N. \tag{10}$$

### 3.2 Main result

We first give the definition of an entropy solution of problem $(\mathcal{P})$. For a given constant $k>0$, the truncation function $T_k:\mathbb{R}\to\mathbb{R}$ is defined by

$$T_k(s)=\max\{-k,\min\{k,s\}\}.$$

We denote by $\mathcal{T}_0^{1,p}(\Omega,\omega)$ the set of all measurable functions $u:\Omega\to\mathbb{R}$ such that for every $k>0$, the truncated function $T_k(u)$ belongs to $W_0^{1,p}(\Omega,\omega)$.

Let $u\in\mathcal{T}_0^{1,p}(\Omega,\omega)$, then there exists a unique measurable function $v:\Omega\to\mathbb{R}^N$ such that

$$\nabla T_k(u)=v\chi_{\{|u|<k\}}. \tag{11}$$

If $u\in\mathcal{T}_0^{1,p}(\Omega,\omega)$, the weak gradient $\nabla u$ of $u$ is defined as the unique function $v$ which satisfies (11).

**Definition 3.1** A function $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$ is an entropy solution of the problem $(\mathcal{P})$ if

$$\int_\Omega \omega a(x,u,\nabla u)\cdot \nabla T_k(u-v)dx = \int_\Omega f T_k(u-v)dx + \int_\Omega F\cdot \nabla T_k(u-v)dx \tag{12}$$

for every $k>0$ and for every $v \in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$.

The main result proved in this paper is the following.

**Theorem 3.1** *Assume that the Carathéodry function a satisfies* (6)-(9). *Then there exists an entropy solution u of the problem* $(\mathcal{P})$.

**Proposition 3.1** *The entropy solution u of the problem* $(\mathcal{P})$ *satisfies*

$$u \in \mathcal{M}^r(\Omega,\omega) \quad \textit{with } r = \eta(p-1)(1-\theta), \tag{13}$$

$$|\nabla u| \in \mathcal{M}^s(\Omega,\omega) \quad \textit{with } s = \frac{p\eta(p-1)(1-\theta)}{r+\theta(p-1)+1}, \tag{14}$$

*where* $\eta$ *is a constant such that* $1 \le \eta \le \frac{N}{N-1}+\delta$.

## 4 Proof of the Main Result

### 4.1 Useful lemmas

**Lemma 4.1** *Let* $u \in \mathcal{T}_0^{1,p}(\Omega,\omega)$, *where* $\omega \in \mathcal{A}_p$, $1<p<\infty$. *Let* $1 \le \lambda < p$, *and suppose that u satisfies*

$$\int_{\{|u|<k\}} |\nabla u|^p \omega \mathrm{d}x \le M k^\lambda, \quad \forall k>0. \tag{15}$$

*Then u belongs to* $\mathcal{M}^r(\Omega,\omega)$ *with* $r=\eta(p-\lambda)$ *(where* $1\le \eta \le \frac{N}{N-1}+\delta$*). More precisely, there exists* $C>0$ *such that*

$$\Phi_u(k) \le C M^\eta k^{-r}.$$

***Proof.*** For $0<\varepsilon\le k$, we have $\{x\in\Omega : |u|\ge \varepsilon\} = \{x\in\Omega : |T_k(u)|\ge\varepsilon\}$. Thus

$$\begin{aligned}
\mu(\{x\in\Omega : |u|>\varepsilon\}) &= \mu(\{x\in\Omega : |T_k(u)|\ge\varepsilon\}) \\
&= \int_{\{|T_k(u)|\ge\varepsilon\}} \omega(x)\mathrm{d}x \\
&= \frac{1}{\varepsilon^{\eta p}} \int_{\{|T_k(u)|\ge\varepsilon\}} \varepsilon^{\eta p}\omega(x)\mathrm{d}x \\
&\le \frac{1}{\varepsilon^{\eta p}} \int_{\{|T_k(u)|\ge\varepsilon\}} |T_k(u)|^{\eta p}\omega(x)\mathrm{d}x \\
&\le \frac{1}{\varepsilon^{np}} \|T_k(u)\|^{\eta p}_{L^{\eta p}(\Omega,\omega)} .
\end{aligned}$$

By Theorem 2.2 and inequality (15), we get

$$\|T_k(u)\|_{L^{\eta p}(\Omega,\omega)} \le C_\Omega \, \||\nabla T_k(u)\|_{L^p(\Omega,\omega)} \le C_\Omega (Mk^\lambda)^{1/p},$$

which implies that

$$\mu(\{x \in \Omega : |u| > \varepsilon\}) \leq \frac{1}{\varepsilon^{\eta p}} C(Mk^{\lambda})^{\eta}.$$

Therefore, by taking $\varepsilon = k$, we have

$$\mu(\{x \in \Omega : |u| > k\}) \leq \frac{1}{k^{\eta p}} C(Mk^{\lambda})^{\eta} = CM^{\eta} k^{-\eta(p-\lambda)} = CM^{\eta} k^{-r}.$$

**Lemma 4.2** *Assume that the hypothesis in Lemma 4.1 holds true. Then* $|\nabla u| \in \mathcal{M}^s(\Omega, \omega)$*, where* $s = pr/(r+\lambda)$ *(with* $r$ *as in Lemma 4.1 ). More precisely, there exists* $C > 0$ *such that*

$$\Phi_{\nabla u}(k) \leq CM^{(r+\eta\lambda)/(r+\lambda)} k^{-s}.$$

***Proof.*** We set for every $k, \rho > 0$, the function

$$\Psi(k, \rho) = \mu\left(\{x \in \Omega : |\nabla u|^p > \rho, |u| > k\}\right).$$

It is clear that the function $\rho \to \Psi(k, \rho)$ is decreasing. Thus for $k > 0$ and $\rho > 0$, we obtain

$$\Psi(0, \rho) \leq \frac{1}{\rho} \int_0^{\rho} \Psi(0, s) \mathrm{d}s \leq \Psi(k, 0) + \frac{1}{\rho} \int_0^{\rho} (\Psi(0, s) - \Psi(k, s)) \mathrm{d}s. \tag{16}$$

From Lemma 4.1, we have

$$\Psi(k, 0) \leq CM^{\eta} k^{-r}. \tag{17}$$

Now, observe that

$$\begin{aligned} \Psi(0, s) - \Psi(k, s) &= \mu\left(\{x \in \Omega : |\nabla u|^p > s, |u| > 0\}\right) - \mu\left(\{x \in \Omega : |\nabla u|^p > s, |u| > k\}\right) \\ &= \mu\left(\{x \in \Omega : |\nabla u|^p > s, |u| \leq k\}\right). \end{aligned}$$

Hence, we get by using Proposition 6.24 in [15] that

$$\begin{aligned} \int_0^{\infty} (\Psi(0, s) - \Psi(k, s)) ds &= \int_0^{\infty} \mu\left(\{x \in \Omega : |\nabla u|^p > s, |u| \leq k\}\right) ds \\ &= \int_{\{|u| < k\}} |\nabla u(x)|^p \omega(x) dx \\ &\leq Mk^{\lambda}. \end{aligned} \tag{18}$$

Going back to (16) and using (17) and (18), we obtain

$$\Psi(0, \rho) \leq CM^{\eta} k^{-r} + \frac{1}{\rho} Mk^{\lambda}. \tag{19}$$

A minimization of the right-hand side of (19) in $k$ gives

$$\Psi(0, \rho) \leq CM^{(r+\eta\lambda)/(r+\lambda)} \rho^{-r/(r+\lambda)}.$$

Setting $\rho = h^p$, we obtain

$$\Psi(0, h^p) = \mu\left(\{x \in \Omega : |\nabla u| > h\}\right) \leq CM^{(r+\eta\lambda)/(r+\lambda)} h^{-rp/(r+\lambda)} = CM^{(r+\eta\lambda)/(r+\lambda)} h^{-s},$$

where $s = rp/(r+\lambda)$.

Let $n \in \mathbb{N}$, and define, for $u$ in $W_0^{1,p}(\Omega, \omega)$, the differential operator

$$A_n(u) = -\operatorname{div}[\omega(x) a(x, T_n(u), \nabla u)].$$

**Lemma 4.3** *The operator $A_n$ maps $W_0^{1,p}(\Omega,\omega)$ into its dual $W^{-1,p'}(\Omega,\omega^{1-p'})$. Moreover, $A_n$ is bounded, pseudomonotone and coercive in the following sense:*

$$\frac{< A_n v, v >}{\|v\|_{W_0^{1,p}(\Omega,\omega)}} \longrightarrow +\infty \quad \text{if} \quad \|v\|_{W_0^{1,p}(\Omega,\omega)} \longrightarrow +\infty, v \in W_0^{1,p}(\Omega,\omega).$$

***Proof.*** By (8), we deduce that for every $u$ in $W_0^{1,p}(\Omega,\omega)$,

$$\begin{aligned}\int_\Omega \Big(\omega|a(x,T_n(u),\nabla u)|\Big)^{p'}\omega^{1-p'}dx \\ &\leq \int_\Omega \Big(l_0 + l_1|T_n(u)|^{p/p'} + l_2|\nabla u|^{p/p'}\Big)^{p'}\omega dx \\ &\leq C\left[\|l_0\|^{p'}_{L'(\Omega,\omega)} + \Big(C_\Omega\|l_1\|^{p'}_{L^\infty(\Omega)} + \|l_2\|^{p'}_{L^\infty(\Omega)}\Big)\|u\|^p_{W_0^{1,p}(\Omega,\omega)}\right],\end{aligned}$$

which means that $\omega a(x,T_n(u),\nabla u)$ belongs to $\Big(L^{p'}(\Omega,\omega^{1-p'})\Big)^N$. Therefore

$$A_n(u) \in W^{-1,p'}(\Omega,\omega^{1-p'}), \quad \forall n \in \mathbb{N}.$$

Thanks to Hölder's inequality and (8), we have for all $u, v \in W_0^{1,p}(\Omega,\omega)$,

$$\begin{aligned}|\langle A_n u, v\rangle| \leq &\left(C\left[\|l_0\|^{p'}_{L'(\Omega,\omega)} + \Big(C_\Omega\|l_1\|^{p'}_{L^\infty(\Omega)} + \|l_2\|^{p'}_{L^\infty(\Omega)}\Big)\|u\|^p_{W_0^{1,p}(\Omega,\omega)}\right]\right)^{1/p'} \\ &\times \|v\|_{W_0^{1,p}(\Omega,\omega)}.\end{aligned}$$

Thus $A_n$ is bounded from $W_0^{1,p}(\Omega,\omega)$ to $W^{-1,p'}(\Omega,\omega^{1-p'})$.

For the coercivity, by using (6), we get for every $u \in W_0^{1,p}(\Omega,\omega)$,

$$\begin{aligned}\langle A_n u, u\rangle &= \int_\Omega \omega a(x,T_n(u),\nabla u)\cdot\nabla u dx \\ &\geq \int_\Omega \omega b(|T_n(u)|)|\nabla u|^p dx \\ &\geq b(n)\|u\|^p_{W_0^{1,p}(\Omega,\omega)}.\end{aligned}$$

Hence, the operator $A_n$ is coercive.

It remains to show that $A_n$ is pseudomonotone. We thus consider a sequence $u_j$ in $W_0^{1,p}(\Omega,\omega)$ such that

$$\begin{cases} u_j \rightharpoonup u, & \text{in } W_0^{1,p}(\Omega,\omega), \\ A_n u_j \rightharpoonup \psi^n, & \text{in } W^{-1,p'}(\Omega,\omega^{1-p'}), \\ \limsup_{j\to\infty}\langle A_n u_j, u_j\rangle & \leq \langle \psi^n, u\rangle. \end{cases} \tag{20}$$

We shall prove that

$$\psi^n = A_n u \text{ and } \langle A_n u_j, u_j\rangle \to \langle A_n u, u\rangle.$$

Firstly, since $W_0^{1,p}(\Omega,\omega) \hookrightarrow\hookrightarrow L^p(\Omega,\omega)$, one has

$$u_j \to u \quad \text{in } L^p(\Omega,\omega) \text{ for a subsequence denoted again by } (u_j)_j.$$

Due to the boundedness of the sequence $(u_j)_j$ in $W_0^{1,p}(\Omega,\omega)$, and using the growth assumption (8), we have that $\omega a(x,T_n(u_j),\nabla u_j)$ is bounded in $\left(L^{p'}(\Omega,\omega^{1-p'})\right)^N$. Therefore, there exists a function $\phi\in L^{p'}(\Omega,\omega^{1-p'})$ such that

$$\omega a(x,T_n(u_j),\nabla u_j)\rightharpoonup\phi \text{ weakly in } \left(L^{p'}(\Omega,\omega^{1-p'})\right)^N. \tag{21}$$

It is clear that, for all $v\in W_0^{1,p}(\Omega,\omega)$,

$$\begin{aligned}\langle\psi^n,v\rangle&=\lim_{j\to+\infty}\langle A_nu_j,v\rangle\\&=\lim_{j\to+\infty}\int_\Omega\omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla v dx\\&=\int_\Omega\phi\cdot\nabla v dx.\end{aligned}\tag{22}$$

Hence, by hypotheses, we have

$$\begin{aligned}\limsup_{j\to+\infty}<A_nu_j,u_j>&=\limsup_{j\to+\infty}\int_\Omega\omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u_j dx\\&\le\int_\Omega\phi\cdot\nabla u dx.\end{aligned}\tag{23}$$

On the other hand, by (9),

$$[a(x,T_n(u_j),\nabla u_j)-a(x,T_n(u_j),\nabla u))]\cdot\nabla(u_j-u)>0.$$

Hence

$$\begin{aligned}\int_\Omega\omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u_j dx>&\int_\Omega\omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u dx\\&+\int_\Omega\omega a(x,T_n(u_j),\nabla u)\cdot(\nabla u_j-\nabla u)dx.\end{aligned}\tag{24}$$

Using (21), we get

$$\liminf_{j\to+\infty}\int_\Omega\omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u_j dx\ge\int_\Omega\phi\cdot\nabla u. \tag{25}$$

By using (23) and (25), we get

$$\lim_{j\to\infty}\int_\Omega\omega a(x,T_n(u_j),\nabla u_j)\cdot\nabla u_j dx=\int_\Omega\phi\cdot\nabla u dx. \tag{26}$$

As a result of (22) and (26), we get

$$\langle A_nu_j,u_j\rangle\to\langle\psi^n,u\rangle \text{ as } j\longrightarrow+\infty.$$

Yet, due to (26) and the strong convergence $\omega a(x,T_n(u_j),\nabla u)\to\omega a(x,T_n(u),\nabla u)$ in $\left(L^{p'}(\Omega,\omega^{1-p'})\right)^N$, we deduce that

$$\lim_{j\to+\infty}\int_\Omega(\omega a(x,T_n(u_j),\nabla u_j)-\omega a(x,T_n(u_j),\nabla u))\cdot(\nabla u_j-\nabla u)\,dx=0,$$

and so, by virtue of Lemma 3.2 in [1],

$$\nabla u_j \to \nabla u \text{ a.e. in } \Omega,$$

we deduce then that $\omega a(x, T_n(u_j), \nabla u_j)$ converges to $\omega a(x, T_n(u), \nabla u)$ weakly in $\left(L^{p'}(\Omega, \omega^{1-p'})\right)^N$. This implies that $\psi^n = A_n u$.

### 4.2 Approximate problem

We consider the sequence of approximate problems

$$(P_n)\begin{cases} A_n(u_n) = f_n - \operatorname{div} F & in \;\; \Omega, \\ \quad u_n = 0 & on \;\; \partial\Omega, \end{cases}$$

where $(f_n)$ is a sequence of functions in $C_0^\infty(\Omega)$ which is strongly convergent to $f$ in $L^1(\Omega)$ such that $\|f_n\|_{L^1(\Omega)} \leq \|f\|_{L^1(\Omega)}$.

Since the source term $f_n - \operatorname{div}(F)$ belongs to the dual space $W^{-1,p'}(\Omega, \omega^{1-p'})$, in view of Lemma 4.3, the operator $A_n$ is pseudomonotone, and by Theorem 2.7 in [19], there exists at least one solution $u_n \in W_0^{1,p}(\Omega, \omega)$ of problem $(P_n)$ in the sense that

$$\int_\Omega \omega a(x, T_n(u_n), \nabla u_n).\nabla v dx = \int_\Omega f_n v dx + \int_\Omega F.\nabla v dx \tag{27}$$

for every $v \in W_0^{1,p}(\Omega, w)$.

By choosing $v = T_k(u_n)$ in (27), we can take $n > k$, then applying (6) in the first term and using Hölder's then Young's inequalities in the last one, we obtain

$$b(k) \int_\Omega |\nabla T_k(u_n)|^p \omega dx \leq k\|f\|_{L^1(\Omega)} + \frac{b(k)}{2} \int_\Omega |\nabla T_k(u_n)|^p \omega dx + C_1 \int_\Omega \left|\frac{F}{\omega}\right|^{p'} \omega dx.$$

Thus we have the estimate

$$\int_\Omega |\nabla T_k(u_n)|^p \omega dx \leq C_2(1+k)^{\theta(p-1)+1}. \tag{28}$$

### 4.3 Local convergence of $u_n$ in $\mu$-measure

Combining the previous estimation with Lemma 4.1, we conclude that $u_n \in \mathcal{M}^r(\Omega, \omega)$ with $r = \eta(p-1)(1-\theta)$, we also have that $\mu(\{x \in \Omega : |u_n(x)| > k\})$ is bounded uniformly in $n$ for every $k > 0$, that is,

$$\mu(\{x \in \Omega : |u_n(x)| > k\}) \leq Ck^{-r}, \tag{29}$$

and by Lemma 4.2, we have that the sequence $(|\nabla u_n|)$ is bounded in $\mathcal{M}^s(\Omega, \omega)$ with $s = \frac{pr}{r+\theta(p-1)+1}$. Our objective now is to prove that $u_n \to u$ locally in $\mu-$measure. For that, we will use the same reasoning as in the proof of Theorem 2.11 in [10] (see also Theorem 6.1 in [7]). Let $\rho > 0$, we have

$$\{|u_n - u_m| > \rho\} \subset \{|u_n| > k\} \cup \{|u_m| > k\} \cup \{|T_k(u_n) - T_k(u_m)| > \rho\}$$

so that

$$\mu(\{|u_n-u_m|>\rho\}) \leq \mu(\{|u_n|>k\})+\mu(\{|u_m|>k\})+\mu(\{|T_k(u_n)-T_k(u_m)|>\rho\}). \quad (30)$$

Fix $\varepsilon > 0$. From (29), there exists $k_\varepsilon = k$ such that

$$\mu(\{|u_n|>k\}) + \mu(\{|u_m|>k\}) \leq \frac{\varepsilon}{2}.$$

Since $(\nabla T_k(u_n))$ is bounded in $L^p_{loc}(\Omega,\omega)$ for all $k>0$ and $T_k(u_n)$ belongs to $W_0^{1,p}(\Omega,\omega)$, we can assume that $(T_k(u_n))$ is a Cauchy sequence in $L^q(\Omega\cap B_R,\omega)$ for any $q<p\eta=p^*$ and any $R>0$ and

$$T_k(u_n)\to T_k(u) \quad \text{in } L^p_{loc}(\Omega,\omega) \text{ and a.e.}$$

Then

$$\begin{aligned}\mu(\{|T_k(u_n)-T_k(u_m)|>\rho\}\cap B_R) &= \int_{\{|T_k(u_n)-T_k(u_m)|>\rho\}\cap B_R} \omega dx\\ &\leq k^{-q}\int_{\Omega\cap B_R} |T_k(u_n)-T_k(u_m)|^q \omega dx \leq \frac{\varepsilon}{2}\end{aligned}$$

for all $n,m\geq n_0(k,\rho,R)$. It follows that $(u_n)$ is a Cauchy sequence in $\mu$-measure. As a consequence, there exist a function $u$ and a subsequence, still denoted by $u_n$, such that

$$u_n \to u \quad \mu\text{-a.e. in } \Omega, \quad (31)$$

then by Remark 2.1, one has

$$u_n\to u \quad \text{a.e. in } \Omega. \quad (32)$$

Using (31) and (28), we have

$$\begin{aligned}&T_k(u_n) \rightharpoonup T_k(u) \text{ weakly in } W_0^{1,p}(\Omega,\omega) \text{ for every } k>0,\\ &T_k(u_n)\to T_k(u) \text{ strongly in } L^p(\Omega,\omega) \text{ and } \mu\text{ -a.e. in } \Omega \text{ for every } k>0.\end{aligned} \quad (33)$$

Hence, $T_k(u)\in W_0^{1,p}(\Omega,\omega)$.

Furthermore, by the weak lower semicontinuity of the norm $W_0^{1,p}(\Omega,\omega)$, estimate (28) still holds for $u$, that is,

$$\int_\Omega |\nabla T_k(u)|^p \omega dx \leq C(1+k)^{\theta(p-1)+1}, \qquad \forall k>0.$$

Applying again Lemma 4.1 and Lemma 4.2, we find that $u\in\mathcal{M}^r(\Omega,\omega)$ and $|\nabla u|\in\mathcal{M}^s(\Omega,\omega)$.

### 4.4 Strong convergence of truncations

Our aim now is prove that

$$T_k(u_n)\to T_k(u) \quad \text{strongly in } W_0^{1,p}(\Omega,\omega) \text{ for all } k>0. \quad (34)$$

For $n > k$, we write

$$\begin{aligned} I(n) =& \int_\Omega \omega[a(x, T_k(u_n), \nabla T_k(u_n)) - a(x, T_k(u_n), \nabla T_k(u))] \cdot \nabla(T_k(u_n) - T_k(u))dx \\ =& \int_\Omega \omega a(x, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla(T_k(u_n) - T_k(u))dx \\ &- \int_\Omega \omega a(x, T_k(u_n), \nabla T_k(u)) \cdot \nabla(T_k(u_n) - T_k(u))dx \\ =& I_1(n) - I_2(n). \end{aligned}$$

Keeping in mind that (6) implies that $a(x, s, 0) = 0$, we get

$$\begin{aligned} I_1(n) =& \int_{\{|u_n|<k\}} \omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u))dx \\ =& \int_\Omega \omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u))dx \\ &- \int_{\{|u_n|\geq k\}} \omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u))dx. \end{aligned}$$

Observing that $\nabla T_k(u_n) = 0$ on the set $\{|u_n| \geq k\}$, we obtain

$$\begin{aligned} I(n) =& \int_\Omega \omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u))dx \\ &+ \int_{\{|u_n|\geq k\}} \omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla T_k(u)dx \\ &- \int_\Omega \omega a(x, T_k(u_n), \nabla T_k(u)) \cdot \nabla(T_k(u_n) - T_k(u))dx. \end{aligned}$$

We take $T_k(u_n) - T_k(u)$ as a test function in (27) and we get

$$\begin{aligned} &\int_\Omega \omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u))dx \\ &\quad = \int_\Omega f_n(T_k(u_n) - T_k(u))dx + \int_\Omega F \cdot \nabla(T_k(u_n) - T_k(u))dx. \end{aligned}$$

By the almost convergence of $u_n$ and using the strong convergence of $f_n$ in $L^1(\Omega)$, we obtain

$$\lim_{n\to\infty} \int_\Omega f_n(T_k(u_n) - T_k(u))dx = 0.$$

Also, since $F$ belongs to $\left(L^{p'}(\Omega, \omega^{1-p'})\right)^N$ and by (33), we obtain

$$\lim_{n\to\infty} \int_\Omega F \cdot \nabla(T_k(u_n) - T_k(u))dx = 0.$$

Therefore

$$\lim_{n\to\infty} \int_\Omega \omega a(x, T_n(u_n), \nabla u_n) \cdot \nabla(T_k(u_n) - T_k(u))dx = 0. \tag{35}$$

Using the growth assumption (8), for every $u$ in $W_0^{1,p}(\Omega,\omega)$, we have that $\omega|a(x,T_n(u),\nabla u)|$ is bounded $L^{p'}(\Omega,\omega^{1-p'})$. Therefore, it converges weakly to some $g$ in $L^{p'}(\Omega,\omega^{1-p'})$ and we have

$$\lim_{n\to\infty}\int_{\{|u_n|\geq k\}}\omega a(x,T_n(u_n),\nabla u_n)\cdot\nabla T_k(u)dx=\int_{\{|u|\geq k\}}g\cdot\nabla T_k(u)=0. \tag{36}$$

By virtue of Vitali's theorem, we obtain

$$\omega(x)a(x,T_n(u_n),\nabla T_k(u))\to\omega(x)a(x,u,\nabla T_k(u))\text{ strongly in }\left(L^{p'}(\Omega,\omega^{1-p'})\right)^N.$$

It follows from (33) that

$$\lim_{n\to\infty}\int_\Omega\omega a(x,T_k(u_n),\nabla T_k(u))\cdot\nabla(T_k(u_n)-T_k(u))dx=0. \tag{37}$$

Bringing together (35)-(37), we conclude that

$$\lim_{n\to\infty}I(n)=0.$$

Now we can apply Lemma 3.2 in [1] to get (34). Hence, for every fixed $k>0$, we have

$$\omega a(x,T_k(u_n),\nabla T_k(u_n))\to\omega a(x,T_k(u),\nabla T_k(u))\quad\text{in }(L^{p'}(\Omega,\omega^{1-p'}))^N. \tag{38}$$

### 4.5 Passage to the limit

We will now demonstrate that $u$ satisfies (12). Let $v\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$. Testing (27) with $\psi_n=T_k\left(u_n-v\right)$, we get

$$\int_\Omega\omega a\left(x,T_n(u_n),\nabla u_n\right)\cdot\nabla\psi_n dx=\int_\Omega f_n\psi_n dx+\int_\Omega F\cdot\nabla\psi_n dx.$$

If $M=k+\|v\|_{L^\infty(\Omega)}$ and $n>M$, then

$$\begin{aligned}\int_\Omega\omega a\left(x,T_n(u_n),\nabla u_n\right)\cdot\nabla T_k\left(u_n-v\right)dx&=\int_\Omega\omega a(x,T_n(u_n),\nabla T_M(u_n))\cdot\nabla T_k(u_n-v)dx\\&=\int_\Omega\omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla T_k(u_n-v)dx.\end{aligned}$$

Thus, we can write

$$\int_\Omega\omega a(x,T_M(u_n),\nabla T_M(u_n))\cdot\nabla T_k(u_n-v)dx=\int_\Omega f_nT_k(u_n-v)dx+\int_\Omega F\cdot\nabla T_k(u_n-v)dx. \tag{39}$$

Hence we can pass to the limit as n tends to infinity, using (33) and (38), we obtain

$$\int_\Omega\omega a(x,u,\nabla u)\cdot\nabla T_k(u-v)dx=\int_\Omega fT_k(u-v)dx+\int_\Omega F\cdot\nabla T_k(u-v)dx$$

for every $v\in W_0^{1,p}(\Omega,\omega)\cap L^\infty(\Omega)$ and for every $k>0$.

**Example 4.1** We put ourselves in the situation $N = 2$, $p = 3$. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$, the weight function $\omega(x,y) = (x^2+y^2)^{-1/2}$ is such that $\omega \in \mathcal{A}_3$. And the function $f(x,y) = \frac{\cos(xy)}{(x^2+y^2)^{1/3}} \in \mathrm{L}^1(\Omega)$ and $F(x,y) = \Big((x^2+y^2)\sin(xy), (x^2+y^2)^{-1/3}\cos(xy)\Big) \in \Big[L^{\frac{3}{2}}(\Omega, \omega^{-\frac{1}{2}})\Big]^2$. The Carathéodory function is defined as follows: $a : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2, a((x,y),s,\xi) = \frac{\xi}{\sqrt{1+|s|}}$. Therefore, by virtue of Theorem 3.1, the problem

$$\begin{cases} -\operatorname{div}[\omega(x,y)a((x,y),u,\nabla u)] = f(x,y) - \operatorname{div} F(x,y) & \text{in } \Omega, \\ u(x,y) = 0, & \text{on } \partial\Omega, \end{cases}$$

has an entropy solution.

## 5 Conclusion

Through this work, we were able to demonstrate the existence and regularity of solutions for some nonlinear elliptic equations of the form $-\operatorname{div}[\omega(x)a(x,u,\nabla u)] = f - \operatorname{div} F$, in the framework of the weighted Sobolev spaces. The novelty here is that the operator $A(u) =$ - $\operatorname{div}[\omega(x)a(x,u,\nabla u)]$ is a nonlinear degenerate elliptic operator in the sense that the Carathéodory function $a(\cdot,\cdot,\cdot)$ satisfies the degenerate coercivity (6) instead of the case where $A$ is a uniformly elliptic operator, that is, when $b$ is the constant function. Let us point out that this work can be seen as a generalization of the work in [11] and [18].

## References


[1] Y. Akdim, E. Azroul and A. Benkirane. Existence of solutions for quasilinear degenerate ellipti equations. *Electronic Journal of Differential Equations* **2001** (71) (2001) 1–19.

[2] Y. Akdim, E. Azroul and M. Rhoudaf. Entropy solutions of nonlinear elliptic equations with measurable boundary conditions and without strict monotonocity conditions. *Eur. J. Pure Appl. Math.* **1** (4) (2008) 56–71.

[3] Y. Akdim, M. Rhoudaf and A. Salmani. Existence of solution for nonlinear anisotropic degenerated elliptic unilateral problems. *Nonlinear Dynamics and Systems Theory* **18** (3) (2018) 213–224.

[4] A. Alvino, L. Boccardo, V. Ferone, L. Orsina and G. Trombetti. Existence results for nonlinear elliptic equations with degenerate coercivity. *Annali di Matematica* **182** (2003) 53–79.

[5] K. Ammar and H. Redwane. Renormalized solutions for nonlinear degenerate elliptic problems with $L^1$-data. *Rev. Mat. Complut.* **22** (2009) 37-52.

[6] E. Azroul, M. B. Benboubker and S. Ouaro. The obstacle problem associated with nonlinear elliptic equations in generalized sobolev spaces. *Nonlinear Dynamics and Systems Theory* **14** (3) (2014) 224–243.

[7] P. Bélinan, L. Boccardo, T. Gallouët, R. Gariepy, M. Pierre and J. L. Vasquez. An $L^1$ theory of existence and uniqueness of solutions of nonlinear elliptic equations. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **22** (2) (1995) 241–273.

[8] S. Bentouati, M.Islem Soualhi, A. Abdellah El-Hadj and H. Yeklef. Design and Analysis of Continuous Positive Airway Pressure Valve Using a 3D Printing and Computational Fluid Dynamic. *Nonlinear Dynamics and Systems Theory* **21** (3) (2021) 229–237.

[9] L. Boccardo, F. Murat and J. P. Puel. Existence of bounded solutions for nonlinear elliptic unilateral problems. *Ann. Mat. Pura Appl.* **152** (1988) 183–196.

[10] A. C. Cavalheiro. The solvability of Dirichlet problem for a class of degenerate elliptic equations with $L^1$-data. *Applicable Analysis* **85** (8) (2006) 941–961.

[11] A. C. Cavalheiro. Existence of entropy solutions for degenerate quasilinear elliptic equations in $L^1$. *Communications in Mathematics* **22** (2014) 57–69.

[12] A. C. Cavalheiro. Existence of solutions for some degenerate quasilinear elliptic equations. *Le Matematiche* **LXIII** (II) (2008) 101–112.

[13] S.-K. Chua, S. Rodney and R. Wheeden. A compact embedding theorem for generalized Sobolev spaces. *Pacific J. Math.* **265** (1) (2013) 17–57.

[14] E. Fabes, C. Kenig and R. Serapioni. The local regularity of solutions of degenerate elliptic equations. *Comm. PDEs.* **7** (1982) 77–116.

[15] G. B. Folland. *Real Analysis*. Wiley-Interscience, New York, 1984.

[16] J. Heinonen, T. Kilpeläinen and O. Martio *Nonlinear Potential Theory of Degenerate Elliptic Equations*. Oxford Math. Monographs, Clarendon Press, 1993.

[17] A. Kufner, O. John and S. Fučík. *Function Spaces*. Noordhoof International Publishing, Leyden, 1977.

[18] C. Leone and A. Porretta. Entropy solutions for nonlinear elliptic equations in $L^1$. *Nonlinear Anal.* **32** (1998) 325–334.

[19] J. L. Lions. *Quelques méthodes de resolution de problèmes aux limites non linéaires*. Dunod, Gauthier-Villars, Paris, 1969.

[20] K. Oktafianto, A. Z. Arifin, E. F. Kurniawati, T.Tafrikan, T. Herlambang and F. Yudianto. Tsunami Wave Simulation in the Presense of a Barrier. *Nonlinear Dynamics and Systems Theory* **23** (1) (2023) 89–78.