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An Algorithm for Solving First-Kind Two-Dimensional Volterra Integral Equations Using Collocation Method

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Abstract: The proposed study presents a collocation method to address two types of two-dimensional Volterra integral equations (2D VIEs): nonlinear first kind and linear second kind. The nonlinear equations of the first kind are transformed into the linear second kind equations. A convergent algorithm using the Taylor polynomials is developed to construct a collocation solution that approximates the solution of 2D VIEs of the second kind. The study includes various numerical examples to compare the results of different methods and demonstrate the proposed approach's accuracy and validity. This validation procedure plays a pivotal role in nonlinear dynamics and systems theory, establishing the reliability and stability of novel methods.

Keywords: *two-dimensional Volterra integral equations of the first and second kind; collocation method; Taylor polynomials; error analysis.*

Mathematics Subject Classification (2010): 45D05, 45L05, 65R20, 70K99, 93A99.

1 Introduction

The nonlinear 2D VIE of the first kind, which includes an unknown function u, can be represented in a standard form as follows:

$$\int_0^\tau \int_0^z \kappa(\tau, z, t, s) H(u(t, s)) ds dt = f(\tau, z), \qquad (\tau, z) \in D, \tag{1}$$

where D is a subset of $\mathbb{R}^2$ defined as $[0, T] \times [0, Z]$, f and κ are smooth functions on their corresponding domains. Additionally, H is a continuous inverse function that is

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nonlinear with respect to u. To solve equation (1), we substitute $\omega(t,s) = H(u(t,s))$ and obtain the linear equation

$$\int_0^\tau \int_0^z \kappa(\tau,z,t,s)\omega(t,s)dsdt = f(\tau,z), \qquad (\tau,z) \in D. \tag{2}$$

To obtain an approximation for ω, we transform the first kind VIE (2) into the second kind VIE (3) by differentiating equation (2) with respect to z and τ. This transformation technique aligns with the strategies used in various nonlinear systems analyses. It allows researchers and practitioners to simplify the problem while retaining essential characteristics, thus aiding the analysis and comprehension of complex nonlinear systems. This conversion technique is effective only under the conditions that $f(\tau,0) = f(0,z) = 0$ and $\kappa(\tau,z,\tau,z) \neq 0$ for $(\tau,z) \in D$, and results in a linear 2D VIE of the following form:

$$\begin{aligned}\omega(\tau,z) = g(\tau,z) &+ \int_0^\tau \kappa_1(\tau,z,t)\omega(t,z)dt + \int_0^z \kappa_2(\tau,z,s)\omega(\tau,s)ds \\ &+ \int_0^\tau \int_0^z \kappa_3(\tau,z,t,s)\omega(t,s)dsdt, \qquad (\tau,z) \in D,\end{aligned} \tag{3}$$

where the functions g, κ_1, κ_2 and κ_3 are given smooth functions defined on their corresponding domains by

$$\kappa_1(\tau,z,t) := -\frac{\partial \kappa}{\partial \tau}(\tau,z,t,z)/\kappa(\tau,z,\tau,z), \;\; \kappa_2(\tau,z,s) := -\frac{\partial \kappa}{\partial z}(\tau,z,\tau,s)/\kappa(\tau,z,\tau,z),$$
$$\kappa_3(\tau,z,t,s) := -\frac{\partial^2 \kappa}{\partial \tau \partial z}(\tau,z,t,s)/\kappa(\tau,z,\tau,z), \;\; g(\tau,z) := \frac{\partial^2 f}{\partial \tau \partial z}(\tau,z)/\kappa(\tau,z,\tau,z).$$

The solution for (1) can be approximated as $H^{-1}(\omega(t,s)) = u(t,s)$. The existence and uniqueness of the solution for equation (1), using $H(u(t,s)) = \omega(t,s)$ and equation (3), have been proposed in [1].

The applications of VIEs extend to a diverse range of fields, including physical and engineering domains, population dynamics, economics and finance, fluid dynamics, and heat transfer. By providing a method to efficiently and accurately solve nonlinear integral equations, we contribute to the modeling and analyzing complex nonlinear systems. Our algorithm's ability to handle nonlinearity is directly relevant to nonlinear dynamics, as it provides a means to understand and predict the behaviors of such systems. However, solving these equations has motivated mathematicians to develop reliable methods for their solutions [2–10]. In [2], a method based on applying 2D block-pulse functions was utilized to solve nonlinear 2D VIEs of the first kind. An Euler-type technique was discussed in [1]. The Chelyshkov polynomial strategy for solving (1) was considered in [6]. In [7], the Tau technique was employed to approximate the solution of (2). Nemati and Ordokhani [8] used operational matrices of Legendre polynomials to approximate the solution of a class of (1), specifically when $H = u^n$ and n is a positive integer. In [9], a multi-step method was implemented for the numerical solution of nonlinear 2D VIEs of the first kind. A special case of (3) for $\kappa_1 = \kappa_2 = 0$ is considered in [10].

This paper presents an extension of the collocation method proposed in previous works such as [11–14], to solve equations (1) and (3) by utilizing Taylor's theorem in two variables. Additionally, the method is straightforward to implement, and the iterative formulas used to obtain the approximate solution do not require solving any algebraic equations. This showcases the possibility of our method to address broader challenges in nonlinear dynamics that involve integral equations, paving the way for its adaptation in various related problem domains.

The remainder of this paper is structured as follows: the next section outlines our approach to approximating the solution of equation (3) through the Taylor polynomials. Section 3 focuses on the convergence analysis. To demonstrate the validity of our theoretical results, we provide several numerical examples in Section 4. Finally, in Section 5, we present our conclusion and offer suggestions for future research.

2 Description of the Method

In this section, we approximate solutions of 2D VIE (3) in the space
$S^{(-1)}_{p-1,p-1}(\Pi_{N,M}) = \{v : v_{n,m} = v|_{D_{n,m}} \in \pi_{p-1,p-1}, n = 0, 1, ..., N-1; m = 0, 1, ..., M-1\}$ of the real bivariate polynomial spline functions of degree (at most) $p-1$ in τ and z. Its dimension is NMp^2. Here, $\Pi_N = \{\tau_i = ih, i = 0, 1, ..., N\}$ and $\Pi_M = \{z_j = jk, j = 0, 1, ..., M\}$ denote, respectively, uniform partitions of the intervals $[0, T]$ and $[0, Z]$ with the step sizes given by $h = \frac{T}{N}$ and $k = \frac{Z}{M}$. These partitions defined a grid for D
$\Pi_{N,M} = \Pi_N \times \Pi_M = \{(\tau_n, z_m), 0 \leq n \leq N, 0 \leq m \leq M\}$. Set the subintervals
$\sigma_n = [\tau_n; \tau_{n+1}), n = 0, 1, ..., N-2;$ $\quad \sigma_{N-1} = [\tau_{N-1}, \tau_N]$,
$\delta_m = [z_m; z_{m+1}), m = 0, 1, ..., M-2;$ $\quad \delta_{M-1} = [z_{M-1}, z_M]$, and $D_{n,m} := \sigma_n \times \delta_m$ for all $n = 0, 1, ..., N-1; m = 0, 1, ..., M-1$.

To defined the collocation solution, we use the Taylor polynomial on each rectangle $D_{n,m}; n = 0, 1, ..., N-1; m = 0, 1, ..., M-1$. Note that the solution ω of (3) is known at the point$(0, 0)$: $\omega(0, 0) = g(0, 0)$.

2.1 Taylor collocation solution in $D_{0,0}$

We approximate ω in the rectangle $D_{0,0}$ by the polynomial

$$v_{0,0}(\tau, z) = \sum_{i+j=0}^{p-1} \frac{1}{i!j!} \frac{\partial^{i+j}\omega(0, 0)}{\partial\tau^i \partial z^j} \tau^i z^j, \quad (\tau, z) \in D_{0,0}, \tag{4}$$

where $\dfrac{\partial^{i+j}\omega(0, 0)}{\partial\tau^i\partial z^j}$ is the exact value of $\dfrac{\partial^{i+j}\omega}{\partial\tau^i\partial z^j}$ at the point $(0, 0)$. We differentiate equation (3) j times with respect to z and i times with respect to τ, we obtain

$$\begin{aligned}
&\frac{\partial^{i+j}\omega(0, 0)}{\partial\tau^i\partial z^j} = \partial_1^{(i)}\partial_2^{(j)} g(0, 0) \\
&+ \sum_{l=0}^{j}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q} \binom{j}{l}\binom{q}{\eta} \frac{\partial^{q-\eta}}{\partial\tau^{q-\eta}} \left[\partial_1^{(i-1-q)}\partial_2^{(j-l)}\kappa_1(\tau, z, \tau)\right]_{z=0}^{\tau=0} \frac{\partial^{\eta+l}\omega(0, 0)}{\partial\tau^\eta\partial z^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i} \binom{r}{l}\binom{i}{\eta} \frac{\partial^{i-\eta}}{\partial\tau^{i-\eta}} \left[\frac{\partial^{r-l}}{\partial z^{r-l}}\left(\partial_2^{(j-1-r)}\kappa_2(\tau, z, z)\right)\right]_{z=0}^{\tau=0} \frac{\partial^{\eta+l}\omega(0, 0)}{\partial\tau^\eta\partial z^l} \\
&+ \sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q} \binom{r}{l}\binom{q}{\eta} \frac{\partial^{q-\eta}}{\partial\tau^{q-\eta}} \left[\left.\frac{\partial^{i-1-q}}{\partial\tau^{i-1-q}}\right|_{t=\tau} \left(\frac{\partial^{r-l}}{\partial z^{r-l}}\left[\partial_2^{(j-1-r)}\kappa_3(\tau, z, t, z)\right]\right)\right]_{z=0}^{\tau=0} \\
&\times \frac{\partial^{\eta+l}\omega(0, 0)}{\partial\tau^\eta\partial z^l}.
\end{aligned}$$

2.2 Taylor collocation solution in $D_{n,0}$

We approximate ω in the rectangles $D_{n,0}, n = 1, ..., N-1$ by the polynomials

$$v_{n,0}(\tau,z)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}_{n,0}(\tau_n,0)}{\partial\tau^i\partial z^j}(\tau-\tau_n)^i z^j\ ,\quad (\tau,z)\in D_{n,0}, \tag{5}$$

where $\hat{v}_{n,0}$ is the exact solution of the integral equation

$$\begin{aligned}\hat{v}_{n,0}(\tau,z)&=g(\tau,z)+\int_0^z\kappa_2(\tau,z,s)\hat{v}_{n,0}(\tau,s)ds\\&+\sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\kappa_1(\tau,z,t)v_{\xi,0}(t,z)dt+\int_{\tau_n}^{\tau}\kappa_1(\tau,z,t)\hat{v}_{n,0}(t,z)dt\\&+\sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\int_0^z\kappa_3(\tau,z,t,s)v_{\xi,0}(t,s)dsdt+\int_{\tau_n}^{\tau}\int_0^z\kappa_3(\tau,z,t,s)\hat{v}_{n,0}(t,s)dsdt.\end{aligned} \tag{6}$$

We differentiate (6) j times with respect to z and i times with respect to τ, we obtain

$$\begin{aligned}&\frac{\partial^{i+j}\hat{v}_{n,0}(\tau_n,0)}{\partial\tau^i\partial z^j}=\partial_1^{(i)}\partial_2^{(j)}g(\tau_n,0)\\&+\sum_{\xi=0}^{n-1}\sum_{l=0}^{j}\binom{j}{l}\int_{\tau_\xi}^{\tau_{\xi+1}}\Big[\partial_1^{(i)}\partial_2^{(j-l)}\kappa_1(\tau,z,t)\Big]_{z=0}^{\tau=\tau_n}\frac{\partial^l v_{\xi,0}(t,0)}{\partial z^l}dt\\&+\sum_{l=0}^{j}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial\tau^{q-\eta}}\Big[\partial_1^{(i-1-q)}\partial_2^{(j-l)}\kappa_1(\tau,z,\tau)\Big]_{z=0}^{\tau=\tau_n}\frac{\partial^{\eta+l}\hat{v}_{n,0}(\tau_n,0)}{\partial\tau^\eta\partial z^l}\\&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i}\binom{r}{l}\binom{i}{\eta}\frac{\partial^{i-\eta}}{\partial\tau^{i-\eta}}\left[\frac{\partial^{r-l}}{\partial z^{r-l}}[\partial_2^{(j-1-r)}\kappa_2(\tau,z,z)]\right]_{z=0}^{\tau=\tau_n}\frac{\partial^{\eta+l}\hat{v}_{n,0}(\tau_n,0)}{\partial\tau^\eta\partial z^l}\\&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{\tau_\xi}^{\tau_{\xi+1}}\frac{\partial^i}{\partial\tau^i}\left[\frac{\partial^{r-l}}{\partial z^{r-l}}[\partial_2^{(j-1-r)}\kappa_3(\tau,z,t,z)]\right]_{z=0}^{\tau=\tau_n}\frac{\partial^l v_{\xi,0}(t,0)}{\partial z^l}dt\\&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial\tau^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial\tau^{i-1-q}}\Big|_{t=\tau}\left(\frac{\partial^{r-l}}{\partial z^{r-l}}\Big[\partial_2^{(j-1-r)}\kappa_3(\tau,z,t,z)\Big]\right)\right]_{z=0}^{\tau=\tau_n}\\&\times\frac{\partial^{\eta+l}\hat{v}_{n,0}(\tau_n,0)}{\partial\tau^\eta\partial z^l}.\end{aligned}$$

2.3 Taylor collocation solution in $D_{n,m}$

We approximate ω by $v_{n,m}$ in $D_{n,m}, n = 0, 1, ..., N-1$ and $m = 1, ..., M-1$, so that

$$v_{n,m}(\tau,z)=\sum_{i+j=0}^{p-1}\frac{1}{i!j!}\frac{\partial^{i+j}\hat{v}_{n,m}(\tau_n,z_m)}{\partial\tau^i\partial z^j}(\tau-\tau_n)^i(z-z_m)^j\ ,\quad (\tau,z)\in D_{n,m}, \tag{7}$$

where $\hat{v}_{n,m}$ is the exact solution of the integral equation

$$\begin{aligned}
\hat{v}_{n,m}(\tau,z) &= g(\tau,z)+\sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\kappa_1(\tau,z,t)v_{\xi,m}(t,z)dt+\int_{\tau_n}^{\tau}\kappa_1(\tau,z,t)\hat{v}_{n,m}(t,z)dt\\
&+\sum_{\rho=0}^{m-1}\int_{z_\rho}^{z_{\rho+1}}\kappa_2(\tau,z,s)v_{n,\rho}(\tau,s)ds+\int_{z_m}^{z}\kappa_2(\tau,z,s)\hat{v}_{n,m}(\tau,s)ds\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\int_{z_\rho}^{z_{\rho+1}}\kappa_3(\tau,z,t,s)v_{\xi,\rho}(t,s)dsdt\\
&+\sum_{\xi=0}^{n-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\int_{z_m}^{z}\kappa_3(\tau,z,t,s)v_{\xi,m}(t,s)dsdt+\sum_{\rho=0}^{m-1}\int_{\tau_n}^{\tau}\int_{z_\rho}^{z_{\rho+1}}\kappa_3(\tau,z,t,s)v_{n,\rho}(t,s)dsdt\\
&+\int_{\tau_n}^{\tau}\int_{z_m}^{z}\kappa_3(\tau,z,t,s)\hat{v}_{n,m}(t,s)dsdt.
\end{aligned}\tag{8}$$

We differentiate (8) j times with respect to z and i times with respect to τ, we obtain

$$\begin{aligned}
&\frac{\partial^{i+j}\hat{v}_{n,m}(\tau_n,z_m)}{\partial\tau^i\partial z^j}=\partial_1^{(i)}\partial_2^{(j)}g(\tau_n,z_m)\\
&+\sum_{\xi=0}^{n-1}\sum_{l=0}^{j}\binom{j}{l}\int_{\tau_\xi}^{\tau_{\xi+1}}\partial_1^{(i)}\partial_2^{(j-l)}\kappa_1(\tau_n,z_m,t)\frac{\partial^l v_{\xi,m}(t,z_m)}{\partial z^l}dt\\
&+\sum_{l=0}^{j}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{j}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial\tau^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j-l)}\kappa_1(\tau,z,\tau)\right]_{z=z_m}^{\tau=\tau_n}\frac{\partial^{\eta+l}\hat{v}_{n,m}(\tau_n,z_m)}{\partial\tau^\eta\partial z^l}\\
&+\sum_{\rho=0}^{m-1}\sum_{\eta=0}^{i}\binom{i}{\eta}\int_{z_\rho}^{z_{\rho+1}}\partial_1^{(i-\eta)}\partial_2^{(j)}\kappa_2(\tau_n,z_m,s)\frac{\partial^\eta v_{n,\rho}(\tau_n,s)}{\partial\tau^\eta}ds\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{\eta=0}^{i}\binom{r}{l}\binom{i}{\eta}\frac{\partial^{i-\eta}}{\partial\tau^{i-\eta}}\left[\frac{\partial^{r-l}}{\partial z^{r-l}}[\partial_2^{(j-1-r)}\kappa_2(\tau,z,z)]\right]_{z=z_m}^{\tau=\tau_n}\frac{\partial^{\eta+l}\hat{v}_{n,m}(\tau_n,z_m)}{\partial\tau^\eta\partial z^l}\\
&+\sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1}\int_{\tau_\xi}^{\tau_{\xi+1}}\int_{z_\rho}^{z_{\rho+1}}\partial_1^{(i)}\partial_2^{(j)}\kappa_3(\tau_n,z_m,t,s)v_{\xi,\rho}(t,s)dsdt\\
&+\sum_{\xi=0}^{n-1}\sum_{r=0}^{j-1}\sum_{l=0}^{r}\binom{r}{l}\int_{\tau_\xi}^{\tau_{\xi+1}}\frac{\partial^i}{\partial\tau^i}\left[\frac{\partial^{r-l}}{\partial z^{r-l}}[\partial_2^{(j-1-r)}\kappa_3(\tau,z,t,z)]\right]_{z=z_m}^{\tau=\tau_n}\frac{\partial^l v_{\xi,m}(t,z_m)}{\partial z^l}dt\\
&+\sum_{\rho=0}^{m-1}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{q}{\eta}\int_{z_\rho}^{z_{\rho+1}}\frac{\partial^{q-\eta}}{\partial\tau^{q-\eta}}\left[\partial_1^{(i-1-q)}\partial_2^{(j)}\kappa_3(\tau,z,\tau,s)\right]_{z=z_m}^{\tau=\tau_n}\frac{\partial^\eta v_{n,\rho}(\tau_n,s)}{\partial\tau^\eta}ds\\
&+\sum_{r=0}^{j-1}\sum_{l=0}^{r}\sum_{q=0}^{i-1}\sum_{\eta=0}^{q}\binom{r}{l}\binom{q}{\eta}\frac{\partial^{q-\eta}}{\partial\tau^{q-\eta}}\left[\frac{\partial^{i-1-q}}{\partial\tau^{i-1-q}}\bigg|_{t=\tau}\left(\frac{\partial^{r-l}}{\partial z^{r-l}}\left[\partial_2^{(j-1-r)}\kappa_3(\tau,z,t,z)\right]\right)\right]_{z=z_m}^{\tau=\tau_n}\\
&\times\frac{\partial^{\eta+l}\hat{v}_{n,m}(\tau_n,z_m)}{\partial\tau^\eta\partial z^l}.
\end{aligned}$$

3 Study of Convergence and Error of the Numerical Method

We consider the space $L^\infty(D)$ with the norm

$$\|\varphi\|_{L^\infty(D)} = \inf\{C \in \mathbb{R} : |\varphi(\tau,z)| \le C \text{ for a.e. } (\tau,z) \in D\} < \infty.$$

The following lemmas will be used in proving the convergence of the presented method.

Lemma 3.1 (*Gronwall-type inequality [2]*) *Let $\omega(\tau,z)$ and $p(\tau,z)$ be non-negative continuous functions in $\Omega = [a,b] \times [c,d]$, and let $p(\tau,z)$ be nondecreasing in each of the variables in Ω and satisfy the following inequality:*

$$\omega(\tau,z) \le p(\tau,z) + \kappa \int_a^\tau \omega(t,z)dt + \kappa \int_c^z \omega(\tau,s)ds + \kappa \int_a^\tau \int_c^z \omega(t,s)dsdt, (\tau,z) \in \Omega,$$

where κ is a positive constant. Then there exists a positive constant ν such that

$$\omega(\tau,z) \le \nu p(\tau,z).$$

Lemma 3.2 (*Discrete Gronwall-type inequality [15]*) *Let $\{k_j\}_{j=0}^n$ be a given non-negative sequence and the sequence $\{\varepsilon_n\}$ satisfy $\varepsilon_0 \le p_0$ and $\varepsilon_n \le p_0 + \sum_{j=0}^{n-1} k_j\varepsilon_j$, $n \ge 1$, with $p_0 \ge 0$. Then*

$$\varepsilon_n \le p_0 \exp\left(\sum_{j=0}^{n-1} k_j\right), \quad n \ge 1.$$

Lemma 3.3 (*Discrete Gronwall-type inequality of two variables [1]*) *Let $\omega_{n,m}$ be a given non-negative sequence, and let b_1, b_2, b_3 and β be independent of h and k and strictly positive. If the sequence $\omega_{n,m}$ satisfies*

$$\omega_{n,m} \le hb_1 \sum_{\xi=0}^{n-1} \omega_{\xi,m} + kb_2 \sum_{\rho=0}^{m-1} \omega_{n,\rho} + hkb_3 \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} \omega_{\xi,\rho} + \beta,$$

for all $n = 0,1,...,N, m = 0,1,...,M$, then

$$\omega_{n,m} \le \beta exp(\gamma(Nh + Mk),$$

where $\gamma = \frac{1}{2}\left(b_1 + b_2 + \sqrt{(b_1+b_2)^2 + 4b_3}\right)$.

Theorem 3.1 *Let g, κ_1, κ_2 and κ_3 be p times continuously differentiable on their respective domains. Then* (4),(5),(7) *define a unique approximation $v \in S_{p-1,p-1}^{(-1)}(\Pi_{N,M})$, and the resulting error function $e(\tau,z) = \omega(\tau,z) - v(\tau,z)$ satisfies*

$$\|e\|_{L^\infty(D)} \le C(h+k)^p,$$

where C is a finite constant independent of h and k.

Proof. Define the error $e(\tau,z)$ on $D_{n,m}$ by $e_{n,m}(\tau,z) = \omega(\tau,z) - v_{n,m}(\tau,z)$ for all $n = 0,1,...,N-1$ and $m = 0,1,...,M-1$.
There exists a constant C independent of h and k such that

$$\|e_{n,m}\|_{L^\infty(D_{n,m})} \le C(h+k)^p$$

for all $n = 0, 1, \dots, N-1$ and $m = 1, \dots, M-1$.
First, let $(\tau, z) \in D_{0,0}$, we obtain from (4),

$$|e_{0,0}(\tau, z)| \leq \sum_{i+j=p} \frac{1}{i!j!} \left\| \frac{\partial^{i+j}\omega}{\partial\tau^i \partial z^j} \right\| h^i k^j.$$

When using a more direct generalization of the procedures utilized in Lemma 3.6 in [10], there exists a positive number $\alpha(p)$ such that $\left\| \frac{\partial^{i+j}\hat{v}_{n,m}}{\partial\tau^i \partial z^j} \right\| \leq \alpha(p)$ for all $n = 0, 1, \dots, N-1$, $m = 0, \dots, M-1$ and $i + j = 0, ..., p$, where $\hat{v}_{0,0}(\tau, z) = \omega(\tau, z)$ for $(\tau, z) \in D_{0,0}$. Hence,

$$|e_{0,0}(\tau, z)| \leq \alpha(p) \sum_{i+j=p} \frac{1}{i!j!} h^i k^j = \underbrace{\frac{\alpha(p)}{p!}}_{C_1} (h+k)^p.$$

Second, let $(\tau, z) \in D_{n,0}$, for all $n = 1, \dots, N-1$, we have from (6),

$$\begin{aligned}
|\omega(\tau, z) - \hat{v}_{n,0}(\tau, z)| \leq & \sum_{\xi=0}^{n-1} h\kappa \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \sum_{\xi=0}^{n-1} hk\kappa \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} \\
& + \kappa \int_{\tau_n}^{\tau} |\omega(t, z) - \hat{v}_{n,0}(t, z)| dt + \kappa \int_0^z |\omega(\tau, s) - \hat{v}_{n,0}(\tau, s)| ds \\
& + \kappa \int_{\tau_n}^{\tau} \int_0^z |\omega(t, s) - \hat{v}_{n,0}(t, s)| ds dt,
\end{aligned}$$

where $\kappa = max\{\|\kappa_i\|_{L^\infty(D)}, i = 1, 2, 3\}$, then by Lemma 3.1

$$\begin{aligned}
|\omega(\tau, z) - \hat{v}_{n,0}(\tau, z)| &\leq \left(\sum_{\xi=0}^{n-1} h\kappa \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \sum_{\xi=0}^{n-1} hk\kappa \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} \right) \nu \\
&\leq \sum_{\xi=0}^{n-1} h \underbrace{\kappa(1+Z)\nu}_{\lambda_1} \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})},
\end{aligned}$$

which implies that

$$\begin{aligned}
\|e_{n,0}\|_{L^\infty(D_{n,0})} &\leq \|\omega - \hat{v}_{n,0}\| + \|\hat{v}_{n,0} - v_{n,0}\| \\
&\leq \sum_{\xi=0}^{n-1} h\lambda_1 \|e_{\xi,0}\|_{L^\infty(D_{\xi,0})} + \frac{\alpha(p)}{p!} (h+k)^p,
\end{aligned}$$

then, by Lemma 3.2, we have

$$\|e_{n,0}\|_{L^\infty(D_{n,0})} \leq \underbrace{\frac{\alpha(p)}{p!} \exp(T\lambda_1)}_{C_2} (h+k)^p.$$

Third, let $(\tau, z) \in D_{n,m}$ for all $n = 0, \dots, N-1$ and $m = 1, \dots, M-1$, we have from (8),

$$\begin{aligned}|\omega(\tau,z) - \hat{v}_{n,m}(\tau,z)| \leq & \sum_{\xi=0}^{n-1} h\kappa\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} k\kappa\|e_{n,\rho}\| + \kappa\int_{\tau_n}^{\tau}\int_{z_m}^{z}|\omega(t,s) - \hat{v}_{n,m}(t,s)|dsdt \\ & + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\kappa\|e_{\xi,\rho}\| + \sum_{\xi=0}^{n-1} hk\kappa\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} hk\kappa\|e_{n,\rho}\| \\ & + \kappa\int_{\tau_n}^{\tau}|\omega(t,z) - \hat{v}_{n,m}(t,z)|dt + \kappa\int_{z_m}^{z}|\omega(\tau,s) - \hat{v}_{n,m}(\tau,s)|ds,\end{aligned}$$

then by Lemma 3.1,

$$\begin{aligned}|\omega(\tau,z) - \hat{v}_{n,m}(\tau,z)| \leq & \sum_{\xi=0}^{n-1} h\underbrace{\kappa(1+k)\nu}_{\lambda_2}\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} k\underbrace{\kappa(1+h)\nu}_{\lambda_3}\|e_{n,\rho}\| \\ + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\underbrace{\kappa\nu}_{\lambda_4}\|e_{\xi,\rho}\|,\end{aligned}$$

which implies that

$$\begin{aligned}\|e_{n,m}\| \leq & \|\omega - \hat{v}_{n,m}\| + \|\hat{v}_{n,m} - v_{n,m}\| \\ \leq & \sum_{\xi=0}^{n-1} h\lambda_2\|e_{\xi,m}\| + \sum_{\rho=0}^{m-1} k\lambda_3\|e_{n,\rho}\| + \sum_{\xi=0}^{n-1}\sum_{\rho=0}^{m-1} hk\lambda_4\|e_{\xi,\rho}\| + \frac{\alpha(p)}{p!}(h+k)^p,\end{aligned}$$

using Lemma 3.3, we obtain

$$\|e_{n,m}\| \leq \underbrace{\frac{\alpha(p)}{p!}\exp(\gamma_3(T+Z))}_{C_3}(h+k)^p$$

such that $\gamma_3 = \frac{1}{2}\left(\lambda_2 + \lambda_3 + \sqrt{(\lambda_2+\lambda_3)^2 + 4\lambda_3}\right)$.

Thus, the proof is completed by taking $C = \max\{C_1, C_2, C_3\}$. $\square$

4 Numerical examples

In this section, we present numerical experiments that assess the performance of the Taylor collocation method (TCM) for solving problems of the form (1) in Example 4.4 and the form (2) in Examples 4.1–4.3. We also compare the TCM results with those obtained using other methods such as the multi-step method [9], Euler-type method, and Trapezoidal method [1], Chelyshkov polynomials method [6], bivariate shifted Legendre functions method [16], and two-dimensional block-pulse functions method [17]. In each example, we compare the TCM solution with the results obtained from previous references. Our numerical experiments were conducted using Maple version 17 and a PC with Intel Core i7-2630QM CPU @2.00 GHz and 8,00 Go of RAM, running MS Windows 7 operating system. We observed that the TCM produces more accurate results than the previous methods.

(τ, z)	$N = M = 10$,$p = 3$	$N = M = 20$,$p = 3$	$N = M = 10$, $p = 4$
$(0.1, 0.1)$	$1.73e-06$	$5.99e-07$	$2.59e-06$
$(0.2, 0.2)$	$1.84e-05$	$5.29e-06$	$2.20e-05$
$(0.3, 0.3)$	$6.23e-05$	$1.68e-05$	$7.02e-05$
$(0.4, 0.4)$	$1.34e-04$	$3.53e-05$	$1.47e-04$
$(0.5, 0.5)$	$2.28e-04$	$5.85e-05$	$2.46e-04$
$(0.6, 0.6)$	$3.29e-04$	$8.35e-05$	$3.52e-04$
$(0.7, 0.7)$	$4.27e-04$	$1.07e-04$	$4.55e-04$
$(0.8, 0.8)$	$5.14e-04$	$1.28e-04$	$5.46e-04$
$(0.9, 0.9)$	$5.85e-04$	$1.45e-04$	$6.21e-04$
$(1.0, 1.0)$	$1.86e-03$	$3.17e-04$	$5.93e-04$

Table 1: Absolute errors function for Example 4.1.

Example 4.1 Consider the linear 2D VIE of the first kind

$$\int_0^\tau \int_0^z (\tau z + 1)\omega(t, s)dsdt = f(\tau, z), \quad \tau, z \in [0, 1]. \tag{9}$$

By differentiating both sides of equation (9), we obtain

$$\omega(\tau, z) = g(\tau, z) - \int_0^\tau \frac{z\omega(t, z)}{\tau z + 1}dt - \int_0^z \frac{\tau\omega(\tau, s)}{\tau z + 1}ds - \int_0^\tau \int_0^z \frac{\omega(t, s)}{\tau z + 1}dsdt, \tag{10}$$

where $g(\tau, z) = \frac{-3\tau^2+(2+3\tau+3\tau z)\tau e^z}{2(1+\tau z)}$, which has the exact solution $\omega(\tau, z) = \tau e^z$.

A comparison between the approximate and exact solutions is shown in Table 1 by applying the TCM to equation (10) at some points with $p = 3, 4$ and $(N, M) = (10, 10), (20, 20)$.

Example 4.2 Consider the linear 2D VIE of the first kind [9]

$$\left(\frac{\tau^2 z^2 + 2\sin(\tau z) - 2\tau z\cos(\tau z)}{2z^2}\right)\sin(z) = \int_0^\tau \int_0^z (\sin(zt) + 1)\omega(t, s)dsdt$$

for $\tau, z \in [0, 1]$, and the exact solution is $\omega(\tau, z) = \tau\cos(z)$. This equation is equivalent to the following linear 2D VIE of the second kind:

$$\omega(\tau, z) = \tau\cos(z) + \frac{\tau^2\sin(z)\cos(\tau z)}{\sin(\tau z) + 1} - \int_0^z \frac{\tau\cos(\tau z)}{\sin(\tau z) + 1}\omega(\tau, s)ds.$$

The numerical results for $p = 4$ and $N = M = 15$ of the TCM and the numerical results obtained by using the multi-step method [9] are compared in Table 2.

Example 4.3 Consider the linear 2D VIE of the first kind [1]

$$f(\tau, z) = \int_0^\tau \int_0^z (\sin(z + t) + \sin(\tau + s) + 3)\omega(t, s)dsdt, \quad \tau, z \in [0, 2],$$

where $g(\tau, z)$ is chosen so that the exact solution is $\omega(\tau, z) = \cos(\tau + z)$.

In Table 3, the numerical results for $p = 3$ and $h = k = 0.1, 0.05$ of the present method (TCM) are compared with the numerical results obtained by using the Euler-type method (EM) and Trapezoidal method (TM) [1], Chelyshkov polynomials method (2D-CPs) [6], bivariate shifted Legendre functions method [16] and two-dimensional block-pulse functions method (2D BPFs) [17].

(τ, z)	multi-steps method	present method
$(2^{-7}, 2^{-7})$	$2.38e-07$	$2.00e-12$
$(2^{-6}, 2^{-6})$	$1.90e-06$	$4.00e-11$
$(2^{-5}, 2^{-5})$	$1.57e-05$	$1.24e-09$
$(2^{-4}, 2^{-4})$	$2.25e-06$	$3.97e-08$
$(2^{-3}, 2^{-3})$	$1.51e-07$	$1.88e-07$
$(2^{-2}, 2^{-2})$	$1.92e-07$	$2.66e-07$
$(2^{-1}, 2^{-1})$	$6.16e-07$	$8.87e-08$

Table 2: Comparison of the absolute errors of Example 4.2.

	TCM	EM [1]	TM [1]	Method in [16]	2D-BPFs [17]
(τ, z)	$h = 0.05$	$h = 0.05$	$h = 0.05$	$M = 4$	$m = 32$
$(1, 1)$	$8.57e-07$	$4.06e-02$	$9.80e-04$	$4.96e-06$	$6.08e-02$
$(1, 2)$	$2.29e-05$	$1.23e-02$	$5.47e-04$	$5.98e-06$	$4.00e-03$
$(2, 1)$	$2.29e-05$	$1.23e-02$	$5.47e-04$	$9.87e-03$	$4.00e-03$
$(2, 2)$	$4.27e-05$	$4.06e-02$	$2.03e-03$	$1.22e-05$	$4.74e-02$
$\|e\|_\infty$	$4.27e-05$	$5.49e-02$	$2.03e-03$	/	/

	TCM		2D-CPs [6]	
(τ, z)	$h = 0.1$	$h = 0.05$	$N = 2$	$M = 4$
$(0.1, 0.1)$	$2.77e-07$	$3.78e-08$	$7.41e-03$	$4.77e-06$
$(0.2, 0.2)$	$9.38e-07$	$1.18e-07$	$4.43e-04$	$2.10e-05$
$(0.3, 0.3)$	$1.63e-06$	$2.03e-07$	$5.40e-03$	$7.20e-06$
$(0.4, 0.4)$	$2.14e-06$	$2.67e-07$	$6.65e-03$	$8.20e-06$
$(0.5, 0.5)$	$2.33e-06$	$2.91e-07$	$4.48e-03$	$6.40e-06$
$(0.6, 0.6)$	$2.09e-06$	$2.62e-07$	$5.43e-05$	$6.90e-06$
$(0.7, 0.7)$	$1.28e-06$	$1.60e-07$	$4.80e-03$	$1.20e-05$
$(0.8, 0.8)$	$2.85e-07$	$3.66e-08$	$7.93e-03$	$5.00e-06$
$(0.9, 0.9)$	$2.80e-06$	$3.58e-07$	$7.08e-03$	$3.00e-05$
$(1, 1)$	$6.85e-06$	$8.59e-07$	$2.77e-04$	$1.80e-06$

Table 3: Comparison of the absolute errors for Example 4.3.

Example 4.4 Consider the nonlinear 2D VIE of the first kind [17]

$$\frac{1}{9}(e^{\tau+z} - e^{\tau+4z} - e^{7\tau+z} + e^{7\tau+4z}) = \int_0^\tau \int_0^z 2e^{\tau+z}\omega^3(t,s)dsdt$$

for $\tau, z \in [0, 1]$, and the exact solution is $\omega(\tau, z) = e^{\tau+2z}$. This equation is equivalent to the following linear 2D VIE of the second kind:

$$u(\tau, z) = g(\tau, z) - \int_0^\tau u(t, z)dt - \int_0^z u(\tau, s)ds - \int_0^\tau \int_0^z u(t, s)dsdt,$$

where $u = \omega^3$. In Table 4, the numerical results for $p = 3$ and $N = M = 64$ of the TCM are compared with the numerical results obtained by using the Chelyshkov polynomials method (2D CPs) [6], bivariate shifted Legendre functions method [16] and two-dimensional block-pulse functions method (2D BPFs) [17].

$(2^{-i}, 2^{-i})$	2D-BPFs [17]	Method in [16]	2D-CPs [6]	Present method
$i = 1$	$1.0e-1$	$2.6e-6$	$3.5e-5$	$6.1e-6$
$i = 2$	$4.6e-2$	$4.6e-6$	$2.0e-6$	$2.6e-6$
$i = 3$	$2.9e-2$	$6.3e-7$	$1.5e-5$	$1.3e-6$
$i = 4$	$2.3e-2$	$1.2e-5$	$1.2e-5$	$7.2e-7$
$i = 5$	$2.0e-2$	$3.8e-6$	$5.9e-5$	$3.7e-7$
$i = 6$	$3.1e-2$	$9.0e-6$	$9.6e-5$	$1.9e-7$

Table 4: Comparison of the absolute errors of Example 4.4.

5 Conclusion

In this paper, the problem expressed in (1) is transformed into a linear 2D VIE of the second kind, which is given by (3). A collocation method using the Taylor polynomials is developed to solve the 2D VIE of the second kind. The convergence and error analysis of this method are investigated, and numerical examples are provided to illustrate its effectiveness and accuracy. The numerical results confirm the theoretical estimates, and comparisons with other methods are presented. This method can be easily generalized and applied to a system of 2D VIEs of the first and second kinds.

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